TOTALLY SYMMETRIC SETS

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Abstract. Totally symmetric sets are a construct in group theory first introduced by Kordek and Margalit in [4] to aid in their study of homomorphisms between braid groups. In this paper, we discuss some interesting properties of totally symmetric sets. We also construct an upper bound on the totally symmetric sets of the affine group in $\mathbf{R}^2$.

Contents

1. Introduction

Totally symmetric sets are a construct in group theory first introduced by Kordek and Margalit in [4] to aid in their study of homomorphisms between braid groups. Since then, totally symmetric sets have been used to study various other groups as well as in their own right.

Definition 1.1. Let G be a group. Let $X \subseteq G$ have cardinality n. Then X is totally symmetric if for all $\sigma \in \Sigma_n$ and for all $x_i \in X$, there exists some $g_\sigma \in G$ such that $g_\sigma x_i g_\sigma^{-1} = x_{\sigma(i)}$.

Informally, X is totally symmetric if all possible permutations of the elements of X are attainable via conjugation by some element in G. Consider the braid group $G = B_n$ generated by the half-twists $\{\sigma_1, \cdots, \sigma_{n-1}\}$ on n strands. Then the set of odd half-twists (or interchangeably, even half-twists) $\{\sigma_1, \sigma_3, \cdots, \sigma_m\}$, where m is either $n-1$ or $n-2$ is totally symmetric. Every permutation $\tau \in \Sigma_{n/2}$ can be achieved by conjugation by some $\sigma'_i \in B_{n/2}$. Conceptually, this groups together (slightly under, depending on whether n is odd or even) every pair of strands to construct $B_{n/2}$ from B_n. Then, for $\sigma'_i \in B_{n/2}$ and $\sigma_j \in \{\sigma_1, \cdots, \sigma_m\} \subset B_n$, the conjugation $\sigma'_i \sigma_j \sigma'^{-1}_i$ entails grouping the strands, performing the half-twists, and then ungrouping the strands to give the final permutation (see Figure 1 for a visualization).

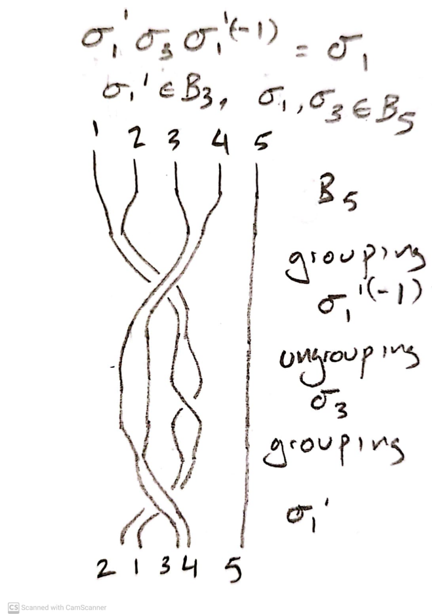

FIGURE 1. A visualization of the conjugation $\sigma_1'\sigma_3\sigma_1'^{-1} = \sigma_1$

Totally symmetric sets have two essential properties that make them useful tools for exploring and classifying homomorphisms. First, collision implies collapse. Second, the image of a totally symmetric set under a group homomorphism is always totally symmetric. Both lemmas originally appear in [4], and are the most important properties of totally symmetric sets.

Lemma 1.2. *Let $f : G \to H$ is a group homomorphism and let $X \subseteq G$ be totally symmetric. If $|X| = n$, then $|f(X)| = n$ or 1*

Proof. If $|f(X)| < n$, then there must exist some $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. Let $x_3 \in X$ also. Then there exists some $g \in G$ such that g conjugates (x_1, x_2, x_3) to (x_1, x_3, x_2). We then have

$$f(x_1x_3^{-1}) = f(gx_1x_2^{-1}g^{-1}) = f(g)f(x_1x_2^{-1})f(g)^{-1} = e.$$

Therefore, $f(x_1) = f(x_3)$, so $|f(X)| = 1$. □

Lemma 1.3. *If $f : G \to H$ is a group homomorphism and $X \subseteq G$ is totally symmetric, then $f(X)$ is totally symmetric.*

Proof. Let $|X| = n$ and $\sigma \in \Sigma_n$. Then from Lemma 1.2, $|f(X)| = n$ or 1. If $|f(X)| = 1$, then $f(X)$ is trivially totally symmetric. If $|f(X)| = n$, let $x_i \in X$. Since X is totally symmetric, there exists $g_\sigma \in G$ such that $g_\sigma x_i g_\sigma^{-1} = x_{\sigma(i)}$. Thus, $f(g_\sigma x_i g_\sigma^{-1}) = f(g_\sigma)f(x)_i f(g_\sigma)^{-1} = f(x_{\sigma(i)}) = f(x)_{\sigma(i)}$. Any permutation of $\{f(x_i)\} \in f(X)$ can thus be achieved via conjugation, so $f(X)$ is totally symmetric. □

By taking a totally symmetric set in G and classifying its image in H under the group homomorphism $f : G \to H$, we are thus able to study f. See [1] and [4] for more on how these properties have been used to classify homomorphisms on the braid group.

2. Totally Symmetric Sets in Isom($\mathbf{R}^2$)

We first classify the maximal totally symmetric set of translations in Isom($\mathbf{R}^2$). Let $g \in$ Isom($\mathbf{R}^2$). Then g is of the form $A \cdot T_u$, where $A \in O(n)$ and T_u is a translation. Let T_x be any translation. Then

$$gT_ug^{-1} = AT_uT_xT_{-u}A^{-1} = AT_xA^{-1} = T_{A(x)},$$

since any two translations commute.

A set containing a single translation is trivially totally symmetric, since only one permutation exists, and it is realized via conjugation by the identity element.

Lemma 2.1. *Let $\vec{a} \neq \vec{b}$ have the same magnitude. Then a set of two translations $\{T_a, T_b\}$ is totally symmetric.*

Proof. We can conjugate $\{T_a, T_b\}$ to $\{T_{k\cdot e_1}, T_z\}$ where k is a constant. Then let A be the matrix representing the reflection across the bisector between the two translations. We then have $T_{A(k\cdot e_1)} = T_z$ and $T_{A(z)} = T_{k\cdot e_1}$, so any set of two translations is totally symmetric. □

Lemma 2.2. *If a set of three translations of equal magnitude $\{T_a, T_b, T_c\}$ is totally symmetric, then $\{\vec{a}, \vec{b}, \vec{c}\}$ form the axes of a regular triangle centered at the origin.*

Proof. We can conjugate $\{T_a, T_b, T_c\}$ to $\{T_{k\cdot e_1}, T_z, T_w\}$ where k is a constant. If this set is totally symmetric, we should be able to fix any one of the translations while swapping the other two.

First fix $T_{k\cdot e_1}$ while swapping T_z and T_w. Let $A \cdot T_v \in$ Isom($\mathbf{R}^2$) be an element that swaps T_z and T_w. For $A(k \cdot \vec{e}_1) = k \cdot \vec{e}_1$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} ak \\ ck \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}.$$

We must then have $A = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. Then for $A\vec{z} = \vec{w}$, we have

$$\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + bz_2 \\ dz_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Similarly, we get $\begin{pmatrix} w_1 + bw_2 \\ dw_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ using the equation $A\vec{w} = \vec{z}$. Then, solving for b and d gives $b = 0$ and $d = \pm 1$, so $A = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

Now consider $A\vec{z} = \vec{w}$. We have

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ \pm z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Since $\vec{z}$ and $\vec{w}$ are distinct, we know that $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ -w_2 \end{pmatrix}$.

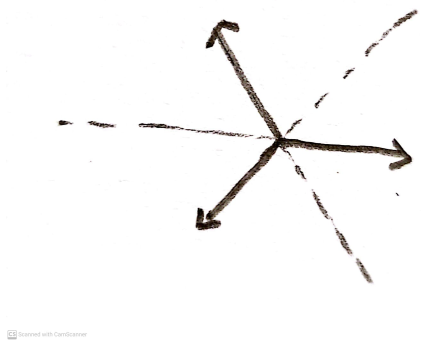

FIGURE 2. Relative positions of a set of three totally symmetric translation vectors.

Now fix T_w and swap $T_{k\cdot e_1}$ and T_z. For $A(k\cdot\vec{e}_1)=\vec{z}$ and $A\vec{z}=k\cdot\vec{e}_1$, we must have $A=\begin{pmatrix} z_1 & z_1-\frac{1}{z_1} \\ z_2 & -z_1 \end{pmatrix}$. Then $A\vec{w}=\vec{w}$ in combination with the relation found above gives us $\begin{pmatrix} z_1 & z_1-\frac{1}{z_1} \\ z_2 & -z_1 \end{pmatrix}\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}=\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}$. Solving the resulting equations,

$$z_1^2 - z_2(z_1-\frac{1}{z_1}) = z_1$$
$$2z_1z_2 = -z_2$$

gives us $z_1=-\frac{1}{2}$. Then, since the magnitudes of both $\vec{z}$ and $\vec{w}$ are 1, we have $\vec{z}=\begin{pmatrix} -\frac{1}{2} \\ z_2 \end{pmatrix}$ and $\vec{w}=\begin{pmatrix} -\frac{1}{2} \\ -z_2 \end{pmatrix}$. Using the Pythagorean Theorem gives $z_2=\frac{\sqrt{3}}{2}$.

Therefore, there is only one set of three totally symmetric translations, consisting of translations along the axes of an equilateral triangle. □

Theorem 2.3. *The largest totally symmetric set of translations in Isom($\mathbf{R}^2$) has cardinality 3*

Proof. From Lemmas 2.1 and 2.2, we know that sets of translations with cardinality 2 or 3 can be totally symmetric in Isom($\mathbf{R}^2$).

There cannot be a larger totally symmetric set of translations in $\mathbf{R}^2$. If we add a fourth vector, for instance, we find by similar calculation that three of the four translation vectors must have the same x-component. However, as mentioned in Lemma 2.1, swapping two translations requires reflecting over their bisector. In the three-vector case, the translation vectors are spaced such that the bisector of any pair of vectors lies on the same line as the remaining vector (see Figure 2). As such, reflection fixes that third vector.

Adding a fourth vector with the same x-component results in a bisector that *does not* lie on the line containing the remaining vector. Therefore, reflecting over this bisector results in an entirely new configuration of translations, rather than a permutation of the existing configuration (see Figure 3 for an example).

Therefore, a set of four translations in $\mathbf{R}^2$ cannot be totally symmetric.

□

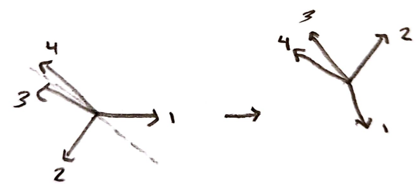

FIGURE 3. An example of attempted permutation of a set of four translation vectors.

This extends to higher dimensions. By Proposition 6.8 in [3], we find that for Isom($\mathbf{R}^n$), the maximal totally symmetric set of translations has cardinality $n+1$ and represents translations of equal magnitude along the axes of the n-simplex (see section 3.1).

3. Totally Symmetric Sets in Aff($\mathbf{R}^2$)

The affine group Aff($\mathbf{R}^n$) is the set of all affine transformations in $\mathbf{R}^n$. It is comprised of maps $\mathbf{R}^n \to \mathbf{R}^n$ such that $x \mapsto Ax + b$ where $A \in GL_n(\mathbf{R})$ and $b \in \mathbf{R}^n$. Elements of Aff($\mathbf{R}^n$) can be represented as ordered pairs (A, b). Define $p((A, b)) = A$. This simply gives the linear transformation component of the map.

Let $f : x \mapsto Ax + z$ and $g : x \mapsto Bx + w$. Then $g^{-1} : x \mapsto B^{-1}x - B^{-1}w$. Conjugating f by g thus gives

$$gfg^{-1} = g(AB^{-1}x - AB^{-1}w + z) = BAB^{-1}x - BAB^{-1}w + Bz + w.$$

Note that since p is a homomorphism, $p(gfg^{-1}) = BAB^{-1} = p(g)p(f)p(g)^{-1}$.

Lemma 3.1. *If $\{(A_1, b_1), \cdots, (A_k, b_k)\} \subset$ Aff($\mathbf{R}^n$) is totally symmetric, then $\{A_1, \cdots, A_k\} \subset GL_n(\mathbf{R})$ is totally symmetric.*

Proof. Let $\sigma \in \Sigma_k$. Since $\{(A_i, b_i)\}$ is totally symmetric, there exists a $g \in$ Aff($\mathbf{R}^n$) such that $g(A_i, b_i)g^{-1} = (A_{\sigma(i)}, b_{\sigma(i)})$. As noted above, $A_{\sigma(i)} = p(g(A_i, b_i)g^{-1}) = p(g)A_i p(g)^{-1}$. Thus, there exists an element $h = p(g) \in GL_n(\mathbf{R})$ such that $hA_ih^{-1} = A_{\sigma(i)}$. □

Totally symmetric sets in $GL_n(\mathbf{R})$ have already been classified; we can thus use this result along with the following results by Caplinger and Salter [3] to aid in our classification of totally symmetric sets in Aff($\mathbf{R}^n$)

Theorem 3.2. *If $\{A_1, \cdots, A_k\} \subset GL_n(\mathbf{R})$ is totally symmetric, then $k \leq n+1$.*

Theorem 3.3 (Caplinger-Salter)**.** *If $\{A_1, \cdots, A_{n+1}\} \subset GL_n(\mathbf{R})$ is totally symmetric, then it is conjugate to the simplex construction.*

3.1. **The Simplex Construction.** The simplex construction, as defined in [3], is a totally symmetric set in $GL_n(\mathbf{R})$ based on the vertices $\{v_1, \cdots, v_{n+1}\} \subset \mathbf{R}^n$ of the n-simplex centered on the origin.

Lemma 3.4. *Let V be a vector space, and $W_1, W_2 \in V$ be complementary subspaces of V. Choose unique $\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{0\}$. Then the map $A : V \to V$ defined by $\vec{w_1} + \vec{w_2} \mapsto \lambda_1\vec{w_1} + \lambda_2\vec{w_2}$ is linear and unique, satisfying $E^A_{\lambda_1} = W_1$ and $E^A_{\lambda_2} = W_2$.*

We can thus construct a set of linear maps $\{A_1, \cdots, A_{n+1}\}$ by using Lemma 3.4 and the subspaces W_{i1}=span(v_i) and $W_{i2} = W_{i1}^{\perp}$.

Lemma 3.5. *For a linear map A and a transformation B, $E_{\lambda}^{BAB^{-1}} = B \cdot E_{\lambda}^{A}$.*

Proof. First let $\vec{v} \in E_{\lambda}^{BAB^{-1}}$. Then $BAB^{-1}\vec{v} = \lambda\vec{v}$, so $AB^{-1}\vec{v} = B^{-1}\lambda\vec{v}$. Let $\vec{w} = B^{-1}\vec{v}$. Then $A\vec{w} = \lambda\vec{w}$, so $\vec{w} \in E_{\lambda}^{A}$. Since $\vec{v} = B\vec{w}$, we have $\vec{v} \in BE_{\lambda}^{A}$. Thus, $E_{\lambda}^{BAB^{-1}} \subset BE_{\lambda}^{A}$.

Now let $\vec{v} \in B \cdot E_{\lambda}^{A}$. Then $\vec{v} = B\vec{w}$ for some $\vec{w}$ such that $A\vec{w} = \lambda\vec{w}$. Then $AB^{-1}\vec{v} = \lambda B^{-1}\vec{v}$, so $BAB^{-1}\vec{v} = \lambda\vec{v}$. Therefore, $\vec{v} \in E_{\lambda}^{BAB^{-1}}$. Thus, $BE_{\lambda}^{A} \subset E_{\lambda}^{BAB^{-1}}$, so $BE_{\lambda}^{A} = E_{\lambda}^{BAB^{-1}}$ □

Theorem 3.6. *The set $\{A_1, \cdots, A_{n+1}\}$ generated via the simplex construction is totally symmetric.*

Proof. Let $\sigma \in \Sigma_{n+1}$, and let $B \in O(n)$ be an orthogonal transformation such that $B(v_i) = B(v_{\sigma(i)})$. Such a transformation must exist, because $\{v_1, \cdots, v_{n+1}\}$ is totally symmetric. By construction, A_i are diagonalizable. We then know by Lemma 3.5 that $E_{\lambda_1}^{BA_iB^{-1}} = BE_{\lambda_1}^{A_i} = B \cdot \text{span}(v_i) = \text{span}(v_{\sigma(i)})$. Then since B is orthogonality-preserving, $E_{\lambda_2}^{BA_iB^{-1}} = \text{span}(v_{\sigma(i)})^{\perp}$. Then because all A_i are unique by construction, we have $BA_iB^{-1} = A_{\sigma(i)}$. □

3.2. Upper Bounds on Totally Symmetric Sets in Aff($\mathbf{R}^2$).

Consider the set $\{(A_1, b_1), (A_2, b_2), \cdots, (A_k, b_k)\} \subset \text{Aff}(\mathbf{R}^2)$ and assume that it is totally symmetric. Then by Lemma 3.1, $\{A_i\}$ is totally symmetric. We have three possible cases:

(1) If all $A_1, \cdots, A_k$ are distinct, then since $\{A_i\}$ is totally symmetric, $k \leq n+1$ by Theorem 3.2 [3].
(2) By collision-implies-collapse (Lemma 1.3), if all A_i are not distinct, they must be equal.
   (a) If $A = I$, then we have a set of translations. As proven in Section 2, $k \leq n + 1$.
   (b) If all A_i are equal and not the identity, then we will show that $k < n+1$.

By the Jordan Decomposition Theorem, all $n \times n$ matrices can be reduced to the Jordan Canonical Form. In the case of 2×2 matrices, this is either the form $\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$ or $\begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix}$. We will show that the largest possible totally symmetric set in either case has cardinality 2.

Lemma 3.7. *Let $A \in GL_2(\mathbf{R})$. Any set of two affine transformations $\{(A, b_1), (A, b_2)\}$ in $\mathbf{R}^2$ is totally symmetric.*

Proof. First conjugate A to its Jordan Canonical Form. Without loss of generality, we will continue to use A, b_1, and b_2 to refer to the conjugated versions of these values.

Consider the case of $A = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$. Since all A are the same, we know that $BAB^{-1} = A$, so B must commute with A. We know that the set of matrices that commute with A are all diagonal. Thus, B has the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.

Now we want to choose $\vec{w}$ such that $-BAB^{-1}\vec{w} + Bb_1 + \vec{w} = (I - A)\vec{w} + Bb_1 = b_2$, swapping b_1 and b_2. We have

$$\begin{aligned}
(I - A)\vec{w} + B\vec{b}_1 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \\
&= \begin{pmatrix} 1-m & 0 \\ 0 & 1-n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} b_{11}x \\ b_{12}y \end{pmatrix} \\
&= \begin{pmatrix} (1-m)w_1 + b_{11}x \\ (1-n)w_2 + b_{12}y \end{pmatrix} \\
&= \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}.
\end{aligned}$$

Then $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-m}(b_{21} - b_{11}x) \\ \frac{1}{1-n}(b_{22} - b_{12}y) \end{pmatrix}$. Therefore, any set $\{(A, b_1), (A, b_2)\}$ for $A = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$ is totally symmetric.

Now consider $A = \begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix}$. Here, commuting matrices have the form $B = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$.

Now we want to choose $\vec{w}$ such that $-BAB^{-1}\vec{w} + Bb_1 + \vec{w} = (I - A)\vec{w} + Bb_1 = b_2$, swapping b_1 and b_2. We have

$$\begin{aligned}
(I - A)\vec{w} + B\vec{b}_1 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix} \right) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix} \\
&= \begin{pmatrix} 1-n & -1 \\ 0 & 1-n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} b_{11}x + b_{12}y \\ b_{12}x \end{pmatrix} \\
&= \begin{pmatrix} (1-n)w_1 - w_2 + b_{11}x + b_{12}y \\ (1-n)w_2 + b_{12}x \end{pmatrix} \\
&= \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}.
\end{aligned}$$

Then $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-n}(b_{21} - b_{11}x - b_{12}y) + \frac{1}{(1-n)^2}(b_{22} - b_{12}x) \\ \frac{1}{1-n}(b_{22} - b_{12}x) \end{pmatrix}$. Therefore, any set $\{(A, b_1), (A, b_2)\}$ for $A = \begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix}$ is totally symmetric. □

Lemma 3.8. *Let $A \in GL_2(\mathbf{R})$. Any set of three affine transformations $\{(A, b_1), (A, b_2), (A, b_3)\}$ in $\mathbf{R}^2$ where $A \neq I$ cannot be totally symmetric.*

Proof. First conjugate A to its Jordan Canonical Form. For simplicity's sake, we will continue to use A, b_1, b_2, and b_3 to refer to the conjugated versions of these values.

Consider the case of $A = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}$ such that $A \neq I$. As above, B has the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. We want to choose $\vec{w}$ to swap $\vec{b}_1$ and $\vec{b}_2$ while fixing $\vec{b}_3$. Fixing $\vec{b}_3$ means that

$$\begin{aligned}
(I-A)\vec{w} + B\vec{b}_3 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix} \\
&= \begin{pmatrix} 1-m & 0 \\ 0 & 1-n \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} b_{31}x \\ b_{32}y \end{pmatrix} \\
&= \begin{pmatrix} (1-m)w_1 + b_{31}x \\ (1-n)w_2 + b_{32}y \end{pmatrix} \\
&= \begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix}.
\end{aligned}$$

Then $\vec{w} = \begin{pmatrix} \frac{1}{1-m}(b_{31} - b_{31}x) \\ \frac{1}{1-n}(b_{32} - b_{32}y) \end{pmatrix}$.

Then swapping $\vec{b}_1$ and $\vec{b}_2$ gives

$$\begin{aligned}
(I-A)\vec{w} + B\vec{b}_1 &= \begin{pmatrix} 1-m & 0 \\ 0 & 1-n \end{pmatrix}\begin{pmatrix} \frac{1}{1-m}(b_{31} - b_{31}x) \\ \frac{1}{1-n}(b_{32} - b_{32}y) \end{pmatrix} + \begin{pmatrix} b_{11}x \\ b_{12}y \end{pmatrix} \\
&= \begin{pmatrix} b_{31} - b_{31}x + b_{11}x \\ b_{32} - b_{32}y + b_{12}y \end{pmatrix} \\
&= \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}
\end{aligned}$$

Now fix $\vec{b}_1$ and swap $\vec{b}_2$ and $\vec{b}_3$. By similar calculation, we get $\vec{w} = \begin{pmatrix} \frac{1}{1-m}(b_{11} - b_{11}x) \\ \frac{1}{1-n}(b_{12} - b_{12}y) \end{pmatrix}$ and $\begin{pmatrix} b_{11} - b_{11}x + b_{21}x \\ b_{12} - b_{12}y + b_{22}y \end{pmatrix} = \begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix}$. Combining the resulting equations with those from the previous calculations allows us to substitute for b_{21}, giving

$$\begin{aligned}
b_{11} - b_{11}x + b_{21}x &= b_{31} \\
b_{11} - b_{11}x + b_{31} - b_{31}x^2 + b_{11}x^2 &= b_{31} \\
b_{11}x^2 - b_{11}x + b_{11} &= b_{31}x^2 - b_{31}x + b_{31} \\
b_{11} &= b_{31}.
\end{aligned}$$

Similar calculations give $b_{12} = b_{32}$, so $\vec{b}_1 = \vec{b}_3$. Thus, the set can have cardinality at most 2.

Now consider the case of $A = \begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix}$. As above, B has the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. We want to choose $\vec{w}$ to swap $\vec{b}_1$ and $\vec{b}_2$ while fixing $\vec{b}_3$. Fixing $\vec{b}_3$ means that

$$\begin{aligned}(I-A)\vec{w} + B\vec{b}_3 &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} n & 1 \\ 0 & n \end{pmatrix}\right)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix} \\ &= \begin{pmatrix} 1-n & -1 \\ 0 & 1-n \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} b_{31}x + b_{32}y \\ b_{32}x \end{pmatrix} \\ &= \begin{pmatrix} (1-n)w_1 - w_2 + b_{31}x + b_{32}y \\ (1-n)w_2 + b_{32}x \end{pmatrix} \\ &= \begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix}.\end{aligned}$$

Then $\vec{w} = \begin{pmatrix} \frac{1}{1-n}(b_{31} - b_{31}x - b_{32}y) + \frac{1}{(1-n)^2}(b_{32} - b_{32}x) \\ \frac{1}{1-n}(b_{32} - b_{32}x) \end{pmatrix}$.

Then swapping $\vec{b}_1$ and $\vec{b}_2$ gives

$$\begin{aligned}(I-A)\vec{w} + B\vec{b}_1 &= \begin{pmatrix} 1-n & -1 \\ 0 & 1-n \end{pmatrix}\begin{pmatrix} \frac{1}{1-n}(b_{31} - b_{31}x - b_{32}y) + \frac{1}{(1-n)^2}(b_{32} - b_{32}x) \\ \frac{1}{1-n}(b_{32} - b_{32}x) \end{pmatrix} + \begin{pmatrix} b_{11}x + b_{12}y \\ b_{12}x \end{pmatrix} \\ &= \begin{pmatrix} b_{31} - b_{31}x - b_{32}y + b_{11}x + b_{12}y \\ b_{32} - b_{32}x + b_{12}x \end{pmatrix} \\ &= \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}\end{aligned}$$

Now fix $\vec{b}_1$ and swap $\vec{b}_2$ and $\vec{b}_3$. By similar calculation, we find

$$\vec{w} = \begin{pmatrix} \frac{1}{1-n}(b_{11} - b_{11}x - b_{12}y) + \frac{1}{(1-n)^2}(b_{12} - b_{12}x) \\ \frac{1}{1-n}(b_{12} - b_{12}x) \end{pmatrix}$$

and

$$\begin{pmatrix} b_{11} - b_{11}x - b_{12}y + b_{21}x + b_{22}y \\ b_{12} - b_{12}y + b_{22}y \end{pmatrix} = \begin{pmatrix} b_{31} \\ b_{32} \end{pmatrix}.$$

Combining the resulting equations with those from the previous calculations allows us to substitute for b_{21} and b_{22}, giving

$$\begin{aligned} b_{11} - b_{11}x - b_{12}y + b_{21}x + b_{22}y &= b_{31} \\ b_{11} - b_{11}x - b_{12}y + b_{22}y &= b_{31} - b_{21}x \\ b_{11} - b_{11}x - b_{12}y + b_{32}y - b_{32}xy + b_{12}xy &= b_{31} - b_{31}x + b_{31}x^2 + b_{32}xy - b_{11}x^2 - b_{12}xy \\ b_{11} - b_{11}x + b_{11}x^2 - b_{12}y + 2b_{12}xy &= b_{31} - b_{31}x + b_{31}x^2 - b_{32}y + 2b_{32}xy \\ b_{11} &= b_{31} \end{aligned}$$

Similar calculations give $b_{12} = b_{32}$, so $\vec{b}_1 = \vec{b}_3$. Thus, the set can have cardinality at most 2.

Thus, a set of three affine transformations in $\mathbf{R}^2$ with the same, non-identity matrix component cannot be totally symmetric. □

3.3. **Bound Sharpness.**

Consider the totally symmetric set $\{(A_1, b_1), (A_2, b_2), \cdots, (A_k, b_k)\} \subset \mathrm{Aff}(\mathbf{R}^2)$. We have found that in all cases, $k \leq n+1$. We now want to classify the cases in which $k = n+1$. When $k = n+1$, we know that either

(1) all $A_i = I$. In this case, we have a set of translations along the axes of a regular triangle as per Section 2. Or,
(2) all A_i are distinct. In this case, we know both that $\{A_i\} \subset GL_2(\mathbf{R})$ is totally symmetric (Lemma 3.1) and that $\{A_i\}$ is in the simplex construction (Theorem 3.3). But how does this construction translate to the affine group?

There exists a homomorphism $f : \mathrm{Aff}(\mathbf{R}^n) \to GL_{n+1}(\mathbf{R})$ by $(A, b) \mapsto \begin{pmatrix} A & \vec{b} \\ 0 & 1 \end{pmatrix}$, where A is an $n \times n$ matrix and b is a $n \times 1$ vector. We thus know that $\mathrm{Aff}(\mathbf{R}^n) \subset GL_{n+1}(\mathbf{R})$ under this homomorphism. Using Theorem 3.2, we find that if $X \subset \mathrm{Aff}(\mathbf{R}^n) \subset GL_{n+1}(\mathbf{R})$ is totally symmetric, then $|X| \leq n+2$.

In the case of $\mathrm{Aff}(\mathbf{R}^2)$, then, we have $X \subset \mathrm{Aff}(\mathbf{R}^2) \subset GL_3(\mathbf{R})$ and $|X| \leq n+2 = 4$. From Theorem 3.3, we know that if $|X| = 4$, it must be in the 4-simplex configuration. However, can a 4-simplex exist in $\mathrm{Aff}(\mathbf{R}^2)$?

Lemma 3.9. *The eigenvalues of the block matrix* $G = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ *are the combined eigenvalues of* A *and* D.

Proof. Let λ be an eigenvalue of G. Then $\det(K - \lambda I) = 0$. We know that

$$\begin{aligned} \det(G - \lambda I) &= \det\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} - \lambda I\right) \\ &= \det \begin{pmatrix} A - \lambda I & B \\ 0 & D - \lambda I \end{pmatrix} \\ &= \det(A - \lambda I) \cdot \det(D - \lambda I). \end{aligned}$$

Therefore, if $\det(G - \lambda I) = 0$, we must have either $\det(A - \lambda I) = 0$ or $\det(D - \lambda I) = 0$. Thus, λ is an eigenvalue of either A or D.

Now let λ be an eigenvalue of A. Since we know that $\det(A - \lambda I) = 0$, we must have $\det(G - \lambda I) = 0$ from above. The same holds for eigenvalues of D. Thus, the eigenvalues of G are the combined eigenvalues of A and D. □

Lemma 3.10. *Aff*$(\mathbf{R}^n)$ *does not contain an* $n+2$*-simplex*

Proof. Assume that $X \subset \mathrm{Aff}(\mathbf{R}^n)$ is totally symmetric, in the simplex configuration, and $|X| = n+2$. Define a function $f : \mathrm{Aff}(\mathbf{R}^n) \to GL_n(\mathbf{R})$ by $\begin{pmatrix} A & \vec{b} \\ 0 & 1 \end{pmatrix} \mapsto A$.

Then $|f(x)|$ is either $n+2$ or 1 by collision-implies-collapse. Since $f(X)$ must be totally symmetric, we have $|f(X)| \leq n+1$ by Theorem 3.2. Thus, $|f(X)| = 1$, so all A are identical. We thus have $X = \left\{ \begin{pmatrix} A & \vec{b}_i \\ 0 & 1 \end{pmatrix} \right\}$.

From Lemma 3.9, we know that all $x \in X$ have the same eigenvalues: those of A and 1. Notably, the transformations composing in the simplex construction are unique, and thus have different eigenvalues. Therefore, X cannot be an $n+2$-simplex. □

We therefore know that $\mathrm{Aff}(\mathbf{R}^2)$ cannot contain a 4-simplex. Thus, any totally symmetric set $X \subset \mathrm{Aff}(\mathbf{R}^2)$ has cardinality less than or equal to 3.

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