AN INTRODUCTION TO THE RING OF FRACTIONS

DILLON GRANNIS

Abstract. I construct the ring of fractions in a slightly different manner from the standard method. Typically, the ring of fractions is constructed as the equivalence classes of certain formal fractions. This can be done at two levels of generality. I present the standard equivalence relation for the less general version and piggyback off of it to handle the more general case. I hope that some may find this modified approach to the ring of fractions and my explanation of its "minimality" to be intuitive.

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Conventions

R will always be a commutative ring with identity. All ring homomorphisms will preserve identity. I use the term extension for injective ring homomorphisms.

1. Introduction

In a ring R, it can be convenient to have multiplicative inverses. For example, if a^{-1} exists, then a has the cancellation property: $ab = ac$ implies $b = c$. (We simply multiply both sides by a^{-1}.) But often we can cancel a even if a^{-1} doesn't exist. It's allowed in fact exactly when a is regular (not a zero-divisor). That's because multiplication by a is linear and thus injective if and only if it has trivial kernel. And by definition, its kernel is trivial exactly when a is regular. In this case, we can cancel a from both sides of an equation as if multiplying by a^{-1} although a need not actually have an inverse. It turns out, furthermore, that there is a larger ring in which a is a unit. Suppose on the other hand that a is a zero divisor. No extension can turn a zero divisor into a unit. But we can still find a (necessarily non-injective) ring homomorphism under which a is mapped to a unit. More generally, for any multiplicatively closed set of elements S, there is a "unique" "minimal" ring in which S becomes units. It is called the ring of fractions.

2. Ring of Fractions as an Extension

The most fundamental example of the ring of fractions is the construction of the rationals.

Construction 2.1. Taking for granted the construction of $\mathbb{Z}$, we consider the fractions with integer entries and non-zero denominator. We let $\mathbb{Q}$ denote the equivalence classes of these fractions under the equivalence relation

$$\frac{a}{b} \sim \frac{c}{d} \text{ if } ad = bc.$$

Multiplication is defined component wise:

$$\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}.$$

Addition is defined by finding common denominator and then adding the numerators:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

The rationals are then the ring $(\mathbb{Q}, +, *)$. This process can naturally be generalized:

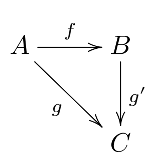
Definition 2.2. A set will be called regular – this is my terminology – if all its elements are regular.

Definition 2.3. A set is multiplicatively closed if it is closed under finite products (including the empty product, so it contains 1).

For the rest of this section, $S \subset R$ will be multiplicatively closed and regular. Now we proceed exactly as we did for the rationals except we use R for set of allowed numerators and S for denominators. We denote the equivalence classes by $S^{-1}R$ and call $(S^{-1}R, +, *)$ the "ring of fractions". It comes with a natural embedding which I denote by $f_S : R \to S^{-1}R$, given by $f_S(r) = \frac{r}{1}$.

Exercise 2.1. Show that $(S^{-1}R, +, *)$ is well defined. This requires showing that $\sim$ is an equivalence relation, that $+, *$ are well defined on the equivalence classes and that $(S^{-1}R, +, *)$ satisfies the ring axioms. Furthermore show that f_S is an embedding and that for all $s \in S$, $\frac{1}{s} = f_S(s)^{-1}$. The construction of the rationals then follows as a special case by taking $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus \{0\}$.

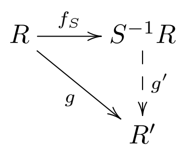
Now I'll argue that $f_S : R \to S^{-1}R$ is 'minimal'. But what does it mean for a homomorphism to minimal? Minimal means 'doing' as little as possible. The larger the kernel, the more 'information' destroyed. If the size of the kernel is fixed, then the larger the codomain, the more 'information' created. Isomorphisms are therefore minimal homomorphisms; no information is created or destroyed. One way to make this notion precise is through factoring. We say that $g : A \to C$ factors through $f : A \to B$ if there exists $g' : B \to C$ such that $g = g' \circ f$ (where all 3 maps are homomorphisms). In other words, there exists a g' such that the following diagram commutes:



(A diagram is said to commute when all paths which start at the same node and end at the same node are equal. In this paper, where all diagrams have rings for

nodes and ring homomorphisms for edges, two paths are said to be equal when the ring homomorphisms produced by composing all homomorphisms along each path are the same. So the diagram above commutes if and only if $g = g' \circ f$.)
If g' is unique, we say g factors uniquely through f and call g' the induced map. If we have a collection of homomorphisms H who all have domain R, we could call some $f \in H$ minimal if all other homomorphisms in H uniquely factor through it. This forces both the kernel and codomain to be small as desired. First, the existence of a factoring of g through f implies that $\ker g$ contains $\ker f$. Thus f has the smallest kernel in H. Second, the uniqueness of such factorings precludes the possibility that the codomain of f contains "extra stuff" unrelated to $f(R)$. For example if h is the natural inclusion into $R[X]$, then no g would factor uniquely through it because X could be sent anywhere (by the universal property of polynomial rings).
Under this notion, isomorphisms are minimal among all homomorphisms because every homomorphism factors uniquely through them. And in that same sense f_S is minimal among homomorphisms turning S into units:

Theorem 2.4. *Every other ring homomorphism sending S to units factors uniquely through f_S. In other words, suppose that $g : R \to R'$ is a homomorphism sending S to units. Then there's a unique ring homomorphism g' completing the commutative diagram:*



Furthermore g' is given by $g'(\frac{a}{b}) = g(a)g(b)^{-1}$.

Proof. $\frac{a}{b} = \frac{a}{1}(\frac{b}{1})^{-1} = f(a)f(b)^{-1}$ by Exercise 2.1. If the desired g' exists, we must therefore have $g'(\frac{a}{b}) = g(a)g(b)^{-1}$. But from this alone I claim g' is well defined, which is to say that this formula for g' agrees on all fractions in the same equivalence class:

$$\frac{a}{b} = \frac{c}{d} \in S^{-1}R \implies ad = bc \implies g(a)g(d) = g(b)g(c)$$

since g is a ring homomorphism. And,

$$b, d \in S \implies g(b), g(d) \text{ are units, so we get } g(a)g(b)^{-1} = g(c)g(d)^{-1},$$

as desired. Thus g' is well defined.
Now we show that g' is a homomorphism, which follows from algebraic manipulations:
Additivity:

$$\begin{aligned} g'(\frac{a}{b} + \frac{c}{d}) = g'(\frac{ad+bc}{bd}) &= g(ad+bc)g(bd)^{-1} \\ &= [g(a)g(d) + g(b)g(c)]g(b)^{-1}g(d)^{-1} = g(a)g(b)^{-1} + g(c)g(d)^{-1} \\ &= g'(\frac{a}{b}) + g'(\frac{c}{d}). \end{aligned}$$

Multiplicativity:

$$g'(\frac{a}{b} * \frac{c}{d}) = g'(\frac{ac}{bd}) = g(ac)g(bd)^{-1} = g(a)g(c)g(b)^{-1}g(d)^{-1} = g'(\frac{a}{b}) * g'(\frac{c}{d})$$

as desired. Thus g' is a homomorphism, completing the proof. □

This theorem is the justification that the ring of fractions is minimal. In the next chapter, I will give a generalization of this theorem and another theorem justifying the "uniqueness" of the ring of fractions.

Remark 2.5. Let R be integral domain. Then $S := R \setminus \{0\}$ is multiplicatively closed and regular. $S^{-1}R$ is a field because $(\frac{a}{b})^{-1} = \frac{b}{a}$ for every $\frac{a}{b} \in S^{-1}R$. It is called the field of fractions of R.

Remark 2.6. If S consists entirely of units, then the extension $f_S : R \to S^{-1}R$ is trivial (by which I mean an isomorphism). We already know f_S is injective by Exercise 2.1. Observing that $\frac{a}{b} = f_S(ab^{-1})$ for any $\frac{a}{b} \in S^{-1}R$ shows surjectivity. If, on the other hand, the extension is an isomorphism, then S must consist entirely of units since an isomorphism can't change unit status.

3. General Ring of Fractions

In this section, we still require that S is multiplicatively closed but drop the condition that it is regular. This generalization costs us the injectivity of f_S since a zero divisor can't become regular under an extension. But there's still a "unique" "minimal" ring homomorphism sending S to units.

Definition 3.1. I call $a, b \in R$ co-zero divisors if $ab = 0$ (even if a or b is zero). This is my term.

From now on, let $B_S := \{k \in R | \exists s \in S \text{ s.t. } ks = 0\}$ – a.k.a all co-zero divisors of the elements of S.

Lemma 3.2. *Suppose $f : R \to R'$ is a homomorphism under which S becomes regular. Then f sends B_S to 0.*

Proof.

$$k \in B_S \implies 0 = ks \text{ for some } s \in S \implies 0 = f(ks) = f(k)f(s) \implies f(k) = 0,$$

since $f(s)$ is regular in R'. □

Remark 3.3. It immediately follows that if S contains a zero divisor, any ring homomorphism sending all $s \in S$ to units is non-injective.

But, if we quotient out the co-zero divisors, S becomes regular and remains multiplicatively closed:

Lemma 3.4. *B_S is an ideal. Let $\pi_S : S \to R/B_S$ be the natural projection. Then $\pi_S(S)$ is regular and multiplicatively closed.*

Proof.
B_S is an ideal:

Closed under addition:

$$\begin{aligned} a, b \in B_S &\implies ax = by = 0 \text{ for some } x, y \in S \\ &\implies (a + b)xy = 0 \text{ with } xy \in S \text{ since } S \text{ is multiplicatively closed} \\ &\implies a + b \in B_S. \end{aligned}$$

Closed under multiplication by any element in R:

$$a \in B_S, r \in R \implies ax = 0 \text{ for some } x \in S \implies (ra)x = 0 \implies ra \in B_S.$$

Non-empty:

$$0 * 1 = 0 \implies 0 \in B_S.$$

Thus B_S is an ideal.
$\pi_S(S)$ is regular: Suppose that $\bar{r}\bar{x} = 0 \in R/B_S$ for some $r \in R, x \in S$. Then $rx = a$ for some $a \in B_S$. $a \in B_S$ implies there's some $y \in S$ such that $ay = 0$. Thus $rxy = ay = 0$. But then since $xy \in S$, we get that $r \in B_S$ so $\overline{r} = 0$ as desired. Thus $\overline{x}$ isn't a zero divisor for any $x \in S$.
$\pi_S(S)$ is multiplicatively closed: $\pi_S(1) = 1 \in \pi_S(S)$. And $\pi_S(x), \pi_S(y) \in \pi_S(S) \implies \pi_S(x)\pi_S(y) = \pi_S(xy) \in \pi_S(S)$. □

Therefore, if we first quotient out the co-zero divisors, we can then apply the extension constructed in the previous section, producing the following diagram:

$$R \xrightarrow{\pi_S} R/B_S \xrightarrow{f_{\pi_S(S)}} \pi_S(S)^{-1}(R/B_S).$$

We then denote the right most ring by $S^{-1}R$ and call it the ring of fractions. We denote the composite map by f_S, which turns S into units because $f_{\pi_S(S)}$ turns $\pi_S(S)$ into units.

Remark 3.5. When S is regular, this definition of the ring of fractions agrees with the one in the previous section. The only difference is an initial quotienting by 0.

Remark 3.6. If we have some homomorphism $R \to R'$ and $a \in R$, we will often abuse notation by saying "$a \in R'$", referring to the image of a in R'. In that sense, for any $a \in R, s \in S$ we get $a \in R/B_S, s \in \pi_S(S)$ and thus we can say "$\frac{a}{b} \in S^{-1}R$" to denote $\frac{\pi_S(a)}{\pi_S(s)}$. Using this notation, we get

$$\frac{a}{b} = \frac{c}{d} \in S^{-1}R \iff \frac{\pi_S(a)}{\pi_S(b)} = \frac{\pi_S(c)}{\pi_S(d)} \iff \pi_S(ad - bc) = 0.$$

This occurs exactly when $(ad - bc)s = 0$ for some $s \in S$. In fact, this is how the ring of fractions is usually constructed: the same formal fractions and operations as I give in Construction 2.1 except with the modified equivalence relation:

$$\frac{a}{b} \sim \frac{c}{d} \text{ if } (ad - bc)s = 0 \text{ for some } s \in S.$$

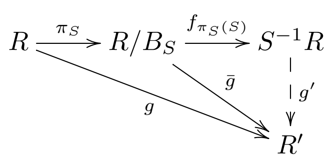
This equivalence relation generalizes to situations were S is non-regular. Of course, the constructions are equivalent.

Remark 3.7. Since $f_{\pi_S(S)}$ is injective, the kernel of f_S is $\ker \pi_S = B_S$. Thus f_S is injective exactly when S is regular. Furthermore, its the 0 map when $1 \in B_S$, which happens exactly when $0 \in S$.

We get the following analogue of Theorem 2.4:

Theorem 3.8. *The Universal Property of the Ring of Fractions*
f_S sends S to units. Let $g : R \to R'$ be another map sending S to units. Then g factors uniquely through f_S. Furthermore the induced map is given by $g'(\frac{a}{b}) = g(a)g(b)^{-1}$.

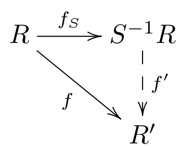
Proof.
Recall that $f_S := f_{\pi_S(S)} \circ \pi_S$. Thus it sends S to units because $f_{\pi_S(S)}$ sends $\pi_S(S)$ to units by Exercise 2.1.
By Lemma 3.2, $\ker g$ contains B_S. Hence, by the universal property of the quotient ring, g factors uniquely through π_S. Let $\bar{g} : R/B_S \to R'$ be the induced map (i.e. $g = \bar{g} \circ \pi_S$). I summarize the situation with the following diagram:



We want to show that there is a unique g' commuting with g. (When I say g' commutes with g, I mean that the homomorphism produced by composing maps along a path from R to R' does not depend on whether the path taken includes g' or g.) Because π_S is surjective and $\bar{g}$ commutes with g, g' commutes with g if and only if it commutes with $\bar{g}$. (Check this.) Theorem 2.4 proves that there's a unique g' commuting with $\bar{g}$ with the formula described. g' therefore is also the unique map commuting with g, completing the proof. ∎

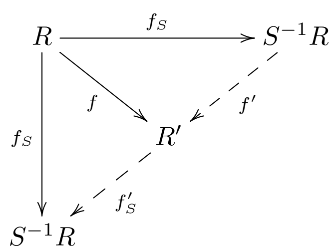
A universal property is meant to characterize some object completely up to isomorphism – often by asserting the property of the object which is most helpful. The above theorem asserts the universal property of the ring of fractions. I now show that it is truly a universal property, by which I mean that $S^{-1}R$ is the unique ring (up to isomorphism) possessing this universal property:

Theorem 3.9. *Uniqueness of the Ring of Fractions*
Let $f : R \to R'$ be a ring homomorphism with the universal property of the ring of fractions. In other words, the assertions about f_S in Theorem 3.8 also apply to f. Then there is a unique isomorphism f' such that the following diagram commutes:

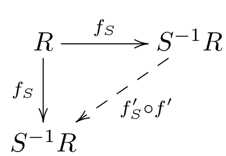


Furthermore, f' is given by $f'(\frac{a}{b}) = f(a)f(b)^{-1}$.

Proof. Since f_S satisfies the universal property of the homomorphism into the ring of fractions and f sends S to units, it follows immediately from Theorem 3.8 that there is a unique f' completing the diagram above which is given by $f'(\frac{a}{b}) = f(a)f(b)^{-1}$. Now we just need to show f' is an isomorphism. Observe that the roles of f and f_S are symmetric here. So there is also a (unique) f'_S such that $f_S = f'_S \circ f$. We can combine these facts into the below commutative diagram:



There may appear to be redundancy, since $S^{-1}R$ appears twice. We could collapse the top right and bottom left nodes into one but then the diagram wouldn't, a priori, be commutative. So I've chosen to write $S^{-1}R$ twice but I could have just easily wrote R' twice – they are still symmetric in this situation. Now omit R' from the commutative diagram and observe that we have found a way in which f_S factors through itself:



But, by the universal property, f_S factors through itself uniquely and therefore $f'_S \circ f'$ must be the identity. By symmetry, $f' \circ f'_S$ is also the identity. This proves that f' is an isomorphism. □

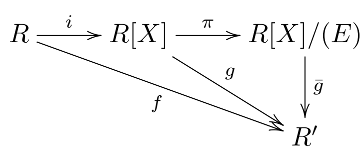
Remark 3.10. Suppose we have an arbitrary subset $A \subset R$ and want a minimal homomorphism on R under which A become units. Let S be the multiplicative closure of A, a.k.a all finite products of elements in A. Any map sending A to units will also sends S to units because products of units are also units. Thus any such map will factor uniquely through f_S. Since f_S also sends A to units, it is therefore the "minimal" homomorphism sending A to units.

4. Applications of the Universal Property

The universal property is meant to characterize an object by its most useful property. One has to use the details of the construction of an object in order prove its universal property. But the hope is that, from that point on, you can largely forget the details of the construction and use the universal property. I provide some examples where using the universal property simplifies an argument:

Theorem 4.1. *Let $X = \{X_s\}_{s \in S}$ be the set of indeterminates indexed by S and $E = \{1 - sX_s\}_{s \in S}$. Then $R[X]/(E) \cong S^{-1}R$.*

Proof.
Let $i : R \to R[X]$ be the inclusion and $\pi : R[X] \to R[X]/(E)$ the natural projection. Then there is a natural homomorphism $\pi \circ i : R \to R[X]/(E)$, which I will show has the universal property given in Theorem 3.8.
First, S are sent to units because s has inverse X_s in $R[X]/(E)$. In particular, $1 - sX_s \in E$ implies $sX_s = 1$ in $R[X]/(E)$.
To show the second part of the universal property, we consider some homomorphism $f : R \to R'$ sending S to units and show that it factors uniquely through $\pi \circ i$. Momentarily, I'll produce maps g and $\bar{g}$ such that the following diagram commutes:



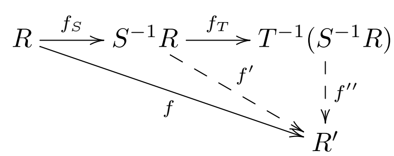
By the universal property of polynomial rings, there is a unique ring homomorphism $g : R[X] \to R'$ which is f on R and sends X_s to $f(s)^{-1}$. Since g is f on R, g commutes with f. Since g sends X_s to $f(s)^{-1}$, it necessarily kills (E). Thus, by the universal property of quotient rings, there is a unique homomorphism $\bar{g}$ commuting with g as pictured above. Furthermore since g commutes with f, it follows that $\bar{g}$ commutes with f. (Check this.) Thus, there exists a factoring of f through $\pi \circ i$ (namely the one given by $\bar{g}$). Now I show that it is unique:
Suppose there is a second map $\bar{g}' : R[X]/(E) \to R'$ commuting with f in the diagram above. First, because it commutes with f, it must send $s^{-1} = X_s \in R[X]/(E)$ to $f(s)^{-1} \in R'$, which means it commutes with g on $X \subset R[X]$. Second, since g also commutes with f, $\bar{g}'$ must commute with g on $R \subset R[X]$. But, since R, X generate $R[X]$, $\bar{g}'$ must commute with g on all of $R[X]$. But, as argued earlier, $\bar{g}$ is the unique map commuting with g in this way so we must have $\bar{g}' = \bar{g}$, showing uniqueness of the factoring.
Thus $\pi \circ i : R \to R[X]/(E)$ has the universal property, implying $R[X]/(E) \cong S^{-1}R$ by Theorem 3.9. □

There is a good intuition for this result. As argued, $f_S : R \to S^{-1}R$ is the "minimal" homomorphism sending S to units. $R[X]/(E)$ meets that criteria quite well: We decide we will denote the inverse of s by X_s. Saying $X_s = s^{-1}$ is the same as saying $1 - sX_s = 0$, which is exactly the information carried in E. Thus, $R[X]/(E)$ is the ring defined by taking R and appending an element X_s for each $s \in S$ with only the information that it is the inverse of s. It may seem redundant to add in so many elements. For example, what if $s \in S$ already has some inverse $s^{-1} \in R$? Well then $s^{-1}(1 - sX_s) = s^{-1} - X_s \in E$, hence $X_s = s^{-1} \in R[X]/(E)$.

In other words, any redundancy is ultimately quotiented out. This notion of the ring of fractions is seldom used because the idea of fractions is more intuitive than appending variables and quotienting like this.

Theorem 4.2. *Let $T \subset S^{-1}R$ be a multiplicative set. Let U be the multiplicative set generated by S and the preimage of T in R. Then $T^{-1}(S^{-1}R)$ and $U^{-1}R$ are canonically isomorphic.*

Proof. There's a natural homomorphism $f_T \circ f_S : R \to S^{-1}R \to T^{-1}(S^{-1}R)$, which I will show has the universal property of f_U:
First, any element of U is a product of elements in S and T and thus is a product of units in $T^{-1}(S^{-1}R)$. Hence U become units in $T^{-1}(S^{-1}R)$ as desired.
To show the second part of the universal property, we consider some homomorphism $f : R \to R'$ sending U to units and show that it factors uniquely through $f_T \circ f_S$. Momentarily, I'll produce maps f' and f'' such that the following diagram commutes:



Since U become units in R, so do S. Thus, by the universal property of $S^{-1}R$, we get a unique map f' commuting with f as depicted above. Because it commutes with f, f' must send elements of T to units. (Check this.) So the universal property of $T^{-1}(S^{-1}R)$ produces a unique f'' commuting with f' as depicted above. Since f' commutes with f, it must also commute with f. This shows existence of a factoring.
Now I show it is the unique factoring: If there is another such map $g : T^{-1}(S^{-1}R) \to R'$ commuting with f, then, since f' commutes with f also, they must agree on the image of R in $T^{-1}(S^{-1}R)$. But because applying a homomorphism respects products and taking multiplicative inverse, then they must also agree on everything generated by the image of R in $T^{-1}(S^{-1}R)$ under multiplication and multiplicative inverse, which is all of $T^{-1}(S^{-1}R)$. Hence $g = f''$, showing uniqueness.
Then we may apply Theorem 3.9 to produce the unique isomorphism commuting with f_U and $f_T \circ f_S$. □

Remark 4.3. A note on canonical isomorphisms:
In general, a map is canonical if it is the "natural" or "standard" map. For example, when a is an ideal of R, the map $R \to R/a$ of $x \to x + a$ is called the canonical projection because this map is a projection and its the obvious and natural map. f_S is the canonical homomorphism from R into $S^{-1}R$. The map isomorphism described above in Theorem 4.2 and given explicitly in Exercise 4.1 is also canonical. A canonical map usually refers to a map which is part of a larger family of maps. The maps in these two examples fit that description. For example, homomorphisms into rings of fractions are a family of maps indexed by R and S because each of these maps is specified by a ring and multiplicative subset. When there would otherwise be ambiguity, the term canonical is used to single out the map which is the most pervasive in a given general situation. For example, suppose I said "let f be the canonical homomorphism from $\mathbb{C}$ into its field of fractions". Since $\mathbb{C}$ is a field, it is its own field of fractions. $\mathbb{C}$ has multiple ring homomorphisms

into itself, most notably the identity and complex conjugation. But, if I say "the canonical homomorphism into its field of fractions", I'm referring to the family of homomorphisms $f_S : R \to S^{-1}R$. So the canonical homomorphism would be the member of this family which corresponds with $R = \mathbb{C}$ and $S = \mathbb{C} \setminus \{0\}$, a.k.a the identity as opposed to complex conjugation.

Exercise 4.1. Show that the canonical isomorphism in the direction $T^{-1}(S^{-1}R) \to U^{-1}R$ is given by $\phi(\frac{a/b}{c}) = \frac{a}{bc}$. Use the fact that the isomorphism commutes with f_U and $f_T \circ f_S$.

5. Conclusion

I have provided introductory explanations to fundamental ideas in algebra such as commutative diagrams, factoring through, universal properties, and canonical maps. My main goal, however, was to present an alternative construction to the general ring of fractions as a quotient and then an extension. (I describe the standard approach in Remark 3.6.) My non-rigorous justification that $f_S : R \to S^{-1}R$ is "minimal" is also my own. My exercises and theorems, however, are quite common. In particular, the universal property of the ring of fractions is fundamental. With it in mind, you may proceed to other texts without being at a disadvantage for having seen a different construction, since all constructions of the ring of fractions are united by sharing the same universal property. On that note, I omit a great number of properties of the ring of fractions as well as its applications. I hope my introduction will provide a good ground work that will be helpful while learning more about it in other texts.

Acknowledgments

I would like to express my enormous gratitude to Professor Madhav Nori who introduced me to these ideas in his algebra course and kindly mentored my through the REU. His lectures, answers to my questions and thoughtful guidance has been of crucial importance to my understanding of this beautiful subject.

6. bibliography

Below are the primary texts from which I learned about the ring of fractions. Altman and Kleiman in particular motivated my use of universal properties.

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[2] A. Altman and S. Kleiman. *A Term of Commutative Algebra*. Worldwide Center of Mathematics, 2013.