FAREY GRAPHS, FRIEZE PATTERNS, AND SL_2-TILINGS

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Abstract. This paper explores the connections between hyperbolic geometry, Farey sequences, frieze patterns, and SL_2-tilings. We begin by introducing key concepts in hyperbolic geometry, including the upper half-plane model and the Farey tessellation of the hyperbolic plane. The paper proceeds to discuss Conway-Coxeter friezes and their connection to triangulated polygons. The main result we focus on is a detailed proof of the bijection between (n, m)-antiperiodic SL_2-tilings with positive rectangular domains and certain triples (q, q', M) that was first established by S. Morier-Genoud, V. Ovsienko, and S. Tabachnikov. By exploring this proof, we enhance our understanding of the relation between abstract geometry, number theory, combinatorics, and dynamical systems.

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1. Introduction to Hyperbolic Geometry

Over the past two millennia, many mathematicians have unsuccessfully attempted to derive the parallel postulate from the other four postulates of Euclidean geometry. While such efforts were doomed to fail, they were not entirely without fruit.

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By modifying the elusive fifth postulate to allow for more than one line to be parallel to any given line in a plane, they discovered a variety of useful non-Euclidean geometries. Central among these is hyperbolic geometry.

1.1. **Hyperbolic Distance.** To avoid working with abstract and unwieldy definitions, mathematicians have proposed a variety of models for studying hyperbolic space, such as the Poincaré disk model, Klein disk model, and Minkowski model. In this paper, we will primarily use the upper half-plane model, which associates the hyperbolic plane $\mathbb{H}$ with the upper half of the complex plane $\mathbb{C}$:

$$\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}.$$

The key difference between hyperbolic and Euclidean geometry is the distance metric, which is now renormalized by the y-coordinate:

$$ds_{\mathbb{H}} = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

This has immediate consequences for the shortest paths between points. In contrast to Euclidean geometry, the length of a segment is not independent of its position, which means that Euclidean straight lines do not necessarily minimize distance in $\mathbb{H}$.

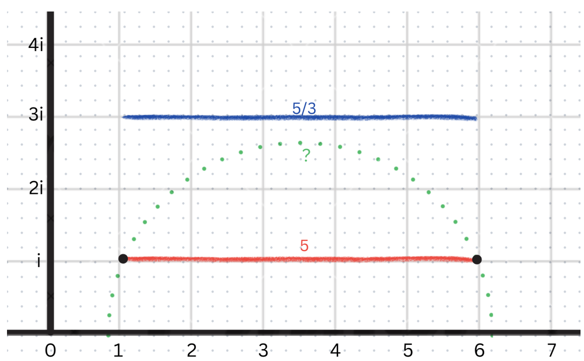


FIGURE 1. The length of a line segment in $\mathbb{H}$ depends on its position.

We will discuss this further in the section on geodesics. Another important property of this metric is that points on the real line $\mathbb{R}$ are not part of the hyperbolic plane, as they are infinitely far from any point in $\mathbb{H}$. Unsurprisingly, ∞ is also infinitely far from any point in $\mathbb{H}$. To ensure that shapes in $\mathbb{H}$ are well-defined, we often work with $\mathbb{R} \cup \{\infty\}$, the so-called *boundary at infinity* of $\mathbb{H}$, which we can visualize as a circle around the hyperbolic plane. Points on $\mathbb{R} \cup \{\infty\}$ are called *ideal points* and can be approached but never reached by curves in $\mathbb{H}$.

Note that hyperbolic geometry is infinitesimally the same as Euclidean geometry (up to scaling), which allows us to retain the familiar definition of angles and compute the length of paths. In Euclidean geometry, which uses the metric

$$ds_{\mathbb{E}} = \sqrt{dx^2 + dy^2},$$

the length of a (parametrized) path $\gamma(t) : [a, b] \to \mathbb{R}^2$, where $\gamma(t) = (x(t), y(t))$, is given by the integral

$$l_{\mathbb{E}}(\gamma) = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt.$$

Taking our new metric into account, it follows that the hyperbolic length of a comparable curve $\gamma(t) : [a, b] \to \mathbb{H}$ is given by the integral

$$l_{\mathbb{H}}(\gamma) = \int_a^b \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)} dt,$$

while the distance between two points v and w in $\mathbb{H}$ is given by

$$d_{\mathbb{H}}(v, w) = \inf\{l_{\mathbb{H}}(\gamma) : \gamma(a) = v, \gamma(b) = w, \text{ and } \gamma : [a, b] \to \mathbb{H} \text{ is piecewise } C^1\}.$$

Definition 1.1. A path $\gamma : [a, b] \to \mathbb{H}$ is called a *geodesic* if it locally minimizes distance, i.e. if for every $x \in [a, b]$, there exists some $\epsilon > 0$ such that

$$d_{\mathbb{H}}(\gamma(c), \gamma(d)) = l_{\mathbb{H}}(\gamma|_{[c,d]})$$

for all $[c, d] \subset [a, b] \cap [x - \epsilon, x + \epsilon]$.

In the hyperbolic plane, a path γ locally minimizes the distance if and only if it globally minimizes the distance, i.e. if $d_{\mathbb{H}}(\gamma(c), \gamma(d)) = l_{\mathbb{H}}(\gamma|_{[c,d]})$ for all $[c, d] \subset [a, b]$. As a consequence of the renormalization of the metric, a geodesic in $\mathbb{H}$ is not necessarily a straight line. In fact, this is rarely the case:

Proposition 1.2. *Geodesics in $\mathbb{H}$ are either vertical lines or semicircles centered at the real axis.*

Proof. Let $v = (c + di)$ and $w = (c_0 + d_0 i)$ be two points in $\mathbb{H}$. First, suppose that v and w lie on the same vertical line L, i.e. that $c = c_0$. Let $\gamma : [a, b] \to \mathbb{H}$ be an arbitrary path connecting v and w such that $\gamma(a) = v$ and $\gamma(b) = w$, given by $\gamma(t) = (x(t), y(t))$ for all $t \in [a, b]$. Let $\gamma_0 : [a, b] \to \mathbb{H}$ be the projection of γ onto the line $x = c$, i.e. let $\gamma_0(t) = (c, y(t))$ for all $t \in [a, b]$. Then we have

$$\begin{aligned} l_{\mathbb{H}}(\gamma) &= \int_a^b \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)} dt \\ &\geq \int_a^b \frac{\sqrt{0^2 + \dot{y}(t)^2}}{y(t)} \\ &= l_{\mathbb{H}}(\gamma_0). \end{aligned}$$

Since geodesics locally minimize distance and equality is attained if and only if $\dot{x}(t) = 0$ for all $t \in [a, b]$ (i.e. if and only if $x(t) = c$ for all $t \in [a, b]$), it follows that the geodesic between v and w is given by a vertical straight line in this case.

Now suppose that v and w do not lie on the same vertical line, i.e. that $c \neq c_0$. From Euclidean geometry, we know that there exists a unique circle C centered at the real axis that passes through both points. Since horizontal translations do not affect the hyperbolic metric, we can assume without loss of generality that this circle is centered at the origin. Let r_0 denote the radius of C. As before, let $\gamma : [a, b] \to \mathbb{H}$ be an arbitrary path connecting v and w such that $\gamma(a) = v$ and $\gamma(b) = w$, given by $\gamma(t) = (x(t), y(t))$ for all $t \in [a, b]$. Let $\gamma_0 : [a, b] \to \mathbb{H}$ be the projection of γ onto the circle C. Using polar coordinates, we have the identity

$$\frac{\sqrt{dx^2 + dy^2}}{y} = \frac{\sqrt{dr^2 + r^2 d\theta^2}}{r \sin\theta} = \frac{\sqrt{r^{-2} dr^2 + d\theta^2}}{\sin\theta}.$$

It follows that

$$\begin{aligned} l_{\mathbb{H}}(\gamma) &= \frac{\sqrt{r^{-2}\dot{r}(t)^2 + \dot{\theta}(t)^2}}{\sin\theta} \\ &\geq \int_a^b \frac{\sqrt{0 + \dot{\theta}(t)^2}}{\sin\theta} \\ &= l_{\mathbb{H}}(\gamma_0). \end{aligned}$$

As before, equality is attained if and only if $\dot{r}(t) = 0$ for all $t \in [a, b]$, i.e. if and only if $r(t) = r_0$ for all $t \in [a, b]$. Since geodesics locally minimize distance, it follows that the geodesic between v and w is given by a semicircle centered at the real axis in this case. □

We also seek to define an analogue of Euclidean triangles:

Definition 1.3. For any $x, y, z \in \mathbb{H}$, the triangle T with vertices x, y, and z is defined to be the union of the three geodesics connecting x, y, and z. If x, y, and z lie on the same geodesic, we say that T is *degenerate*. If x, y, and z lie on the boundary at infinity of $\mathbb{H}$, we say that T is an *ideal triangle*.

1.2. **Isometries.**

Definition 1.4. A map $f : \mathbb{H} \to \mathbb{H}$ is called an *isometry* if for any $v, w \in \mathbb{H}$ we have

$$d_{\mathbb{H}}(f(v), f(w)) = d_{\mathbb{H}}(v, w).$$

As in Euclidean geometry, isometries in $\mathbb{H}$ preserve a variety of useful attributes of sets, making them important tools for understanding the hyperbolic plane. Many of their properties stem directly from their definition and are thus similar to those of Euclidean isometries. From now on, all isometries are understood to be in $\mathbb{H}$ unless stated otherwise.

Property 1.5. Isometries are continuous, and as a consequence:

- Isometries map geodesics to geodesics.
- Isometries preserve angles.
- The composition of two isometries is an isometry.

Example 1.6. Perhaps the two most elementary examples of isometries in the hyperbolic plane are of the form $f(v) = v + c$, where $c \in \mathbb{R}$, and $g(w) = \lambda w$, where $\lambda > 0$. These mappings correspond to Euclidean horizontal translation and Euclidean multiplication by a positive scalar, respectively. It is relatively straightforward to check that f and g act on $\mathbb{H}$ and preserve distances.

To study and classify isometries, it is essential to work with various groups of matrices. Of particular interest is the special linear group:

Definition 1.7. The *special linear group* $SL(2, \mathbb{R})$ is defined as the group of 2×2 matrices with real entries and determinant 1, i.e.

$$SL(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$

Remark 1.8. We oftentimes instead work with $SL(2, \mathbb{Z})$, which is the subgroup of $SL(2, \mathbb{R})$ with only integer entries.

The special linear group acts on $\mathbb{H}$ in the following way: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $z \in \mathbb{H}$, then

$$A(z) = \frac{az+b}{cz+d}.$$

Moreover, $A(\infty) = \frac{a}{c}$ and $A(\frac{-d}{c}) = \infty$ by convention.

Definition 1.9. A mapping of the form $z \mapsto \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad-bc \neq 0$, is called a *Möbius transformation.*

Remark 1.10. All matrices in $SL(2,\mathbb{R})$ are Möbius transformations, and all Möbius transformations preserve circles, angles, and geodesics.

From basic linear algebra, we know that

$$A, B \in SL(2,\mathbb{R}) \implies -A, A^{-1}, AB \in SL(2,\mathbb{R}).$$

While A and $-A$ are different elements of $SL(2,\mathbb{R})$, an interesting property of the corresponding Möbius transformations is that they encode the same map, as

$$A(z) = \frac{az+b}{cz+d} = \frac{-az-b}{-cz-d} = (-A)(z)$$

for all $z \in \mathbb{H}$. For this reason, we normally work with the *projective special linear group*, which is defined in such a way that A and $-A$ are considered the same element:

Definition 1.11.

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I_2\},$$

where I_2 is the 2×2 identity matrix.

Proposition 1.12. *Transformations in $PSL(2,\mathbb{R})$ are isometries of $\mathbb{H}$, i.e.*

$$T \in PSL(2,\mathbb{R}) \implies d_{\mathbb{H}}(T(v), T(w)) = d_{\mathbb{H}}(v, w)$$

for all $v, w \in \mathbb{H}$.

Proof. Let $z \in \mathbb{H}$, and let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Define u as the image of z under T, i.e. let $u := T(z) = \frac{az+b}{cz+d}$. Since $T \in SL(2,\mathbb{R})$, we know that the derivative of $T(z)$ with respect to z is given by

$$T'(z) = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2},$$

and therefore that

$$du = T'(z)dz = \frac{dz}{(cz+d)^2}.$$

Since

$$u = \frac{az+b}{cz+d} = \frac{(az+b)(\overline{cz+d})}{|cz+d|^2} = \frac{acz\overline{z} + adz + bc\overline{z} + bd}{|cz+d|^2},$$

we also know that the the imaginary part of u is given by

$$\Im(u) = \frac{(ad-bc)\Im(z)}{|cz+d|^2} = \frac{\Im(z)}{|cz+d|^2}.$$

Now let $|dz|$ denote $\sqrt{dx^2 + dy^2}$, and note that we have

$$\frac{|du|}{\Im(u)} = \frac{\frac{|dz|}{|cz+d|^2}}{\frac{\Im(z)}{|cz+d|^2}} = \frac{|dz|}{\Im(z)}.$$

It follows that

$$l_{\mathbb{H}}(\gamma) = l_{\mathbb{H}}(T(\gamma))$$

for all smooth curves γ in $\mathbb{H}$, and therefore that

$$d_{\mathbb{H}}(T(v), T(w)) = d_{\mathbb{H}}(v, w)$$

for all $v, w \in \mathbb{H}$. □

Remark 1.13. Note that all isometries in $PSL(2, \mathbb{R})$ are orientation-preserving, as their determinant is positive. In fact, it turns out that $PSL(2, \mathbb{R})$ consists of all the orientation-preserving isometries of $\mathbb{H}$.

2. The Farey Tessellation

One of the most interesting structures in hyperbolic geometry is the Farey graph, which depicts a tiling of the hyperbolic plane. The tessellation and its properties offer many useful insights into SL_2-tilings, a topic we will discuss later. It can also be used to approximate irrational numbers, thereby bridging abstract geometry and classical number theory. A key advantage of this approach is that it allows us to not only compute these approximations but also to visualize them.

2.1. **Construction.** To define and study the graph, we rely on a few important definitions:

Definition 2.1. Two rational numbers $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ are called *Farey neighbors* if

$$|ps - rq| = 1.$$

Remark 2.2. Note that this implies that any two consecutive integers are neighbors, as

$$|n \cdot 1 - (n+1) \cdot 1| = 1$$

for all $n \in \mathbb{N}$.

We define the *Farey sum* of two rationals, denoted by $\oplus$, as follows:

Definition 2.3. If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$, then

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

Two important propositions follow immediately from these definitions:

Proposition 2.4. *If $\frac{p}{q}, \frac{r}{s}$ are neighbors, then so are $\frac{p}{q}, \frac{p}{q} \oplus \frac{r}{s}$ and $\frac{p}{q} \oplus \frac{r}{s}, \frac{r}{s}$.*

Proof. If $\frac{p}{q}$ and $\frac{r}{s}$ are neighbors, then $|ps - rq| = 1$, so

$$|p(q+s) - (p+r)q| = |pq + ps - pq - rq| = |ps - rq| = 1;$$
$$|(p+r)s - r(q+s)| = |ps + rs - rq - rs| = |ps - rq| = 1.$$

□

Proposition 2.5. *If $\frac{p}{q} < \frac{r}{s}$, then $\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}$.*

Proof.

$$\begin{aligned} &\frac{p}{q} < \frac{r}{s} \\ \iff &ps < rq \\ \iff &ps + pq < rq + pq \\ \iff &p(s+q) < q(r+p) \\ \iff &\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s}; \end{aligned} \qquad \begin{aligned} &\frac{p}{q} < \frac{r}{s} \\ \iff &ps < rq \\ \iff &ps + rs < rq + rs \\ \iff &s(p+r) < r(q+s) \\ \iff &\frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}. \end{aligned}$$

□

We can now define the Farey tessellation $\mathcal{F}$ in the complex plane. First, draw a vertical line from the real axis to ∞ at every $n \in \mathbb{Z}$. After this, connect each $n \in \mathbb{Z}$ to $n+1$ with semicircles centered at $\mathbb{R}$. Note that we are drawing arcs between Farey neighbors. We will now proceed iteratively as follows: If $\frac{p}{q}$ and $\frac{r}{s}$ are two rationals on $\mathbb{R}$ connected by an arc, we will draw a semicircle centered at $\mathbb{R}$ between $\frac{p}{q}$ and $\frac{p}{q} \oplus \frac{r}{s}$ and between $\frac{p}{q} \oplus \frac{r}{s}$ and $\frac{r}{s}$.

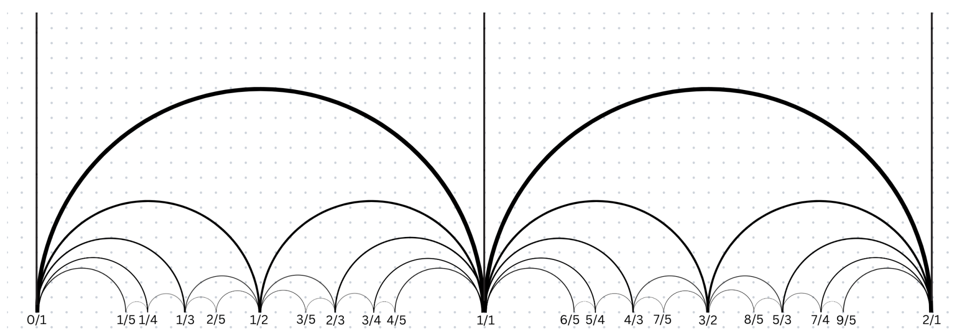


Figure 2. A section of the Farey tessellation after five iterations.

It is easy to see that the lines we have drawn are all geodesics, and therefore that the tiles they demarcate are all (ideal) triangles in $\mathbb{H}$. Note that this includes tiles with two vertical edges, as these edges meet at ∞. Of particular interest is the tile with vertices 0, 1, and ∞, which we call the *basic triangle* Δ. We are now sufficiently equipped to prove that our construction is indeed a tiling of $\mathbb{H}$.

Theorem 2.6. *The triangles of the Farey tessellation cover the hyperbolic plane without their interiors overlapping.*

Proof. The following is a completed version of the argument outlined in [11]. From our construction, it is easy to see that every point in $\mathbb{H}$ is contained in at least one triangle of the tessellation. It therefore remains to prove that the interiors of the triangles do not overlap. First, we want to show that every triangle in $\mathcal{F}$ is the image of Δ under some transformation $T \in SL(2, \mathbb{Z})$. Let $\mathcal{T}$ denote the set of triangles in $\mathcal{F}$, and let Δ' be an arbitrary element of $\mathcal{T}$. Note that every möbius transformation

$$z \mapsto \frac{az+b}{cz+d}$$

satisfies $T(0) = \frac{b}{d}$, $T(1) = \frac{a+b}{c+d}$, and $T(\infty) = \frac{a}{c}$. Since $\Delta' \in \mathcal{T}$, we know that Δ' has three rational vertices of the form $\frac{p}{q}$, $\frac{p+r}{q+s}$, and $\frac{r}{s}$, and therefore that there exists some möbius transformation $T : z \mapsto \frac{pz+r}{qz+s}$ that maps the vertices of Δ to the vertices of Δ'. Without loss of generality, assume that $\frac{p}{q} > \frac{r}{s}$. Since $\frac{p}{q}$ and $\frac{r}{s}$ are connected by an arc in the Farey graph, we know that they are neighbors, and therefore that $ps - rq = 1$. It follows that $T \in SL(2, \mathbb{Z})$. Since T is an orientation preserving-isometry of $\mathbb{H}$ that maps the vertices of Δ to the vertices of Δ', it follows that $T(\Delta) = \Delta'$. Since $\Delta' \in \mathcal{T}$ was arbitrary, it follows that every triangle in $\mathcal{F}$ is the image of Δ under some transformation $T \in SL(2, \mathbb{Z})$.

Now let C° denote the interior of C for all $C \in \mathcal{T}$. We want to show that $A^\circ \cap B^\circ = \emptyset$ for all $A, B \in \mathcal{T}$. Since every triangle in $\mathcal{F}$ is the image of Δ under some transformation $T \in SL(2, \mathbb{Z})$, it suffices to prove that $T_1(\Delta^\circ) \cap T_2(\Delta^\circ) = \emptyset$ for all $T_1, T_2 \in SL(2, \mathbb{Z})$. In fact, it suffices to prove that $\Delta^\circ \cap T(\Delta^\circ) = \emptyset$ for all $T \in SL(2, \mathbb{Z})$. To see this, let $T_1, T_2 \in SL(2, \mathbb{Z})$ be given, and suppose that $\Delta^\circ \cap T(\Delta^\circ) = \emptyset$ for all $T \in SL(2, \mathbb{Z})$. Note that $T_1^{-1}T_2 \in SL(2, \mathbb{Z})$. Since T_1 is a möbius transformation, we know that T_1 is injective, and therefore that

$$\Delta^\circ \cap T_1^{-1}T_2(\Delta^\circ) = 0 \implies T_1(\Delta^\circ) \cap T_1T_1^{-1}T_2(\Delta^\circ) = 0 \implies T_1(\Delta^\circ) \cap T_2(\Delta^\circ) = 0$$

Therefore we want to show that $\Delta^\circ \cap T(\Delta^\circ) = \emptyset$ for all $T \in SL(2, \mathbb{Z})$. Suppose for the sake of contradiction that $T \in SL(2, \mathbb{Z})$ and $\Delta^\circ \cap T(\Delta^\circ) \neq \emptyset$. Let $\frac{p}{q}$, $\frac{p+r}{q+s}$, and $\frac{r}{s}$ denote the vertices of $T(\Delta^\circ)$, and assume without loss of generality that $\frac{p}{q} > \frac{r}{s}$. Since $\Delta^\circ \cap T(\Delta^\circ) \neq \emptyset$, we know that either $\frac{r}{s} < 0 < \frac{p}{q}$ or $\frac{r}{s} < 1 < \frac{p}{q}$. If $\frac{r}{s} < 1 < \frac{p}{q}$, we can apply

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \in SL(2, \mathbb{Z})$$

to send 0 to 1, 1 to ∞, and ∞ to 0, so assume without loss of generality that $\frac{r}{s} < 0 < \frac{p}{q}$. Since $\frac{r}{s} < 0$, we know that r and s have opposite signs. Since $0 < \frac{p}{q}$, we also know that p and q have the same sign. It therefore suffices to consider the following four cases:

- If $r, p, q > 0$ and $s < 0$, then $1 = ps - rq \leq -1 - 1 = -2$
- If $s, p, q > 0$ and $r < 0$, then $1 = ps - rq \geq 1 + 1 = 2$
- If $r > 0$ and $s, p, q < 0$, then $1 = ps - rq \geq 1 + 1 = 2$
- If $s > 0$ and $r, p, q < 0$, then $1 = ps - rq \leq -1 - 1 = -2$

Since all four cases lead to a contradiction, it follows that $T \notin SL(2, \mathbb{Z})$, and therefore that the triangles of the Farey tessellation cover the hyperbolic plane without their interiors overlapping, as desired. □

From the above proof, it also follows immediately that $T(\Delta) \in \mathcal{T}$ for all $T \in SL(2, \mathbb{Z})$, as the tessellation covers the entirety of $\mathbb{H}$ and no transformation in $SL(2, \mathbb{Z})$ 'overlaps' with any others. We will use this result to establish an interesting geometric equivalent of continued fractions.

2.2. Continued Fractions.

Definition 2.7. A *(simple) continued fraction* is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}},$$

where $x \in \mathbb{R}$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{N}$ for all $i \geq 1$. We commonly denote this as $x = [a_0; a_1, a_2, \dots]$ to save space.

Remark 2.8. There are several conventions for dealing with negative numbers, of which the most natural is replacing the first plus with a minus.

It is fairly straightforward to see that every rational number can be written as a finite continued fraction and conversely that every infinite continued fraction is irrational. A more interesting property is that every irrational number can be represented as a unique infinite sequence, as this allows us to find accurate rational approximations of irrational numbers.

Example 2.9. Perhaps the most famous approximation through continued fractions is that of the golden ratio $x = \frac{1+\sqrt{5}}{2}$. Rewriting this equation shows that $x = 1 + \frac{1}{x}$, from which it follows that $x = [1; 1, 1, \ldots]$.

To obtain a similar expansion from the Farey graph, one merely needs to draw a geodesic from any $x \in \mathbb{R}$ to the imaginary axis. Since all the tiles in the graph are ideal triangles, we know that this geodesic will cut each tile it intersects in exactly two places (barring at x itself, if x is a vertex). We will use this to create a *cutting sequence* for x, starting from the imaginary axis. If the two intersected edges of an ideal triangle meet to the left of the geodesic, we add an L to the sequence. If they meet to the right, we add an R. If $x \in \mathbb{Q}$, we can add either an L or an R to the end of the sequence. It turns out that this cutting sequence directly corresponds to the continued fraction expansion of x:

Theorem 2.10. *$x \in \mathbb{R}$ has the cutting sequence $L^{n_0}R^{n_1}L^{n_2}\ldots$ if and only if $x = [n_0; n_1, n_2, \ldots]$.*

Proof. A detailled proof can be found in [11], and is therefore not included here. □

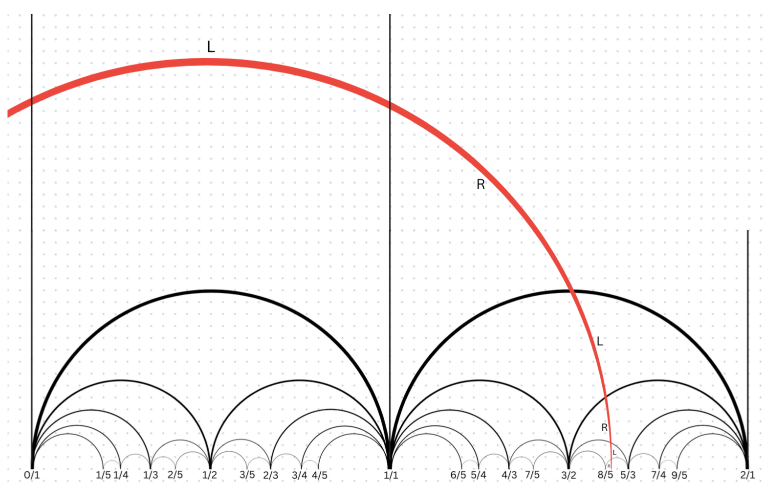


FIGURE 3. The beginning of the cutting sequence for $\frac{1+\sqrt{5}}{2}$.

3. Polygons and Friezes

In the previous section, we primarily focused on hyperbolic triangles and their role in the Farey tessellation. The tessellation also encompasses hyperbolic polygons, which turn out be connected to certain infinite arrays called friezes. Frieze patterns were first introduced by H. S. M. Coxeter in [3] to study the pentagramma mirificum, and have since become useful tools in exploring the relation between geometry, algebra, and combinatorics. In recent years, they have been connected with cluster algebras, Grassmannians, and moduli spaces, yielding promising results in representation theory, category theory, and geometry [8]. The connection between triangulated polygons and friezes is described by a famous result known as the Conway-Coxeter Theorem.

3.1. **Triangulated Polygons.** To discuss this theorem in more detail, we must again start by introducing a few important definitions:

Definition 3.1. The *Farey sequence* of order N, often called a *Farey series*, is the sequence of irreducible rationals $\frac{p}{q} \in [0, 1]$ such that $q \leq N$.

Definition 3.2. A *hyperbolic triangulated polygon* is a finite union of hyperbolic triangles such that any two triangles in the union are connected by a finite sequence of adjacent triangles in the union.

Definition 3.3. A *quiddity* of order n is a sequence of n positive integers $(q_1, \dots, q_n)$ such that each q_i corresponds to the number of incident triangles at the i-th vertex of some triangulated n-gon.

Remark 3.4. Note that every triangulated polygon corresponds to exactly one quiddity (up to cyclic permutation).

A key property of Farey sequences is that they form a cycle in the Farey graph:

Proposition 3.5. *Every pair of consecutive numbers in a Farey sequence is connected by an edge of the Farey graph.*

Proof. Let $N \in \mathbb{N}$ be given, and let $\frac{p}{q}$ and $\frac{r}{s}$ be two consecutive fractions in the Farey series of order N such that $\frac{p}{q} > \frac{r}{s}$. We want to show that $ps - qr = 1$. Suppose for the sake of contradiction that $ps - qr \geq 2$, and note that the area A of the Euclidean triangle T with vertices $(0, 0)$, (p, q), and (r, s) is given by

$$A = \frac{ps - qr}{2}.$$

Following [10], we will obtain a contradiction by using Pick's theorem, which states that the area of any polygon with integer vertices is given by

$$A = i + \frac{b}{2} - 1,$$

where i is the number of integer points in the interior of A and b is the number of integer points on its boundary. Since $ps - qr \geq 2$, it follows that

$$\frac{ps - qr}{2} = i + \frac{b}{2} - 1 \geq 1.$$

Since $i + \frac{b}{2} \geq 2$, we know that there exists some integer coordinate (x, y) that is either in the interior or on the boundary of T. Since $\frac{p}{q}$ and $\frac{r}{s}$ are irreducible, it follows that (x, y) is either on the segment between (p, q) and (r, s) or in the interior of T, and therefore that

$$y \leq \max\{q, s\} \leq N \quad \text{and} \quad \frac{p}{q} \geq \frac{x}{y} \geq \frac{r}{s}.$$

However, this implies that $\frac{p}{q}$ and $\frac{r}{s}$ are not two consecutive fractions in the Farey series of order N, which is a contradiction. Since $ps - qr \ngeq 2$ and $\frac{p}{q} > \frac{r}{s}$, it follows that $ps - qr = 1$, as desired. ∎

We can use this proposition to prove an important corollary that ties Farey sequences to triangulated polygons:

Corollary 3.6. *Every Farey sequence of order $N \geq 2$ forms a triangulated polygon in the Farey graph.*

Proof. We will prove this by induction on N. For the base case $N = 2$, it is easy to see that we have a triangle with vertices $\frac{0}{1}$, $\frac{1}{2}$, and $\frac{1}{1}$. Now suppose that the Farey sequence of order N forms a triangulated polygon in the Farey graph for some $N \in \mathbb{N}$. Note that the Farey sequence of order $N+1$ is created by adding points of the form $\frac{k}{N+1}$ to the Farey sequence of order N. Let $\frac{k_1}{N+1}$ and $\frac{k_2}{N+1}$ be two such points, and note that

$$k_1(N+1) - (N+1)k_2 = (N+1)(k_1 - k_2) \neq 1$$

as $N > 1$ and $|(N+1)(k_1 - k_2)| \geq N$. By Proposition 3.5, it follows that $\frac{k_1}{N+1}$ and $\frac{k_2}{N+1}$ are not consecutive numbers in the Farey sequence, and therefore that every 'new' point z in the Farey sequence of order $N+1$ occurs between two consecutive points x and y in the Farey sequence of order N. By the same proposition, we also know that x and y are joined by an edge, and similarly that z is joined by edges to both x and y. It follows that every Farey sequence of order $N \geq 2$ forms a triangulated polygon in the Farey graph, as desired. □

Perhaps unsurprisingly, there exists a direct correspondence between hyperbolic triangulated polygons and Euclidean triangulated polygons. To obtain a Euclidean triangulated polygon from a hyperbolic one, we will emulate [12] and introduce two alternative models of the hyperbolic plane. As suggested by their names, both the Poincaré disk model and the Klein disk model situate the hyperbolic plane in the interior of the unit disk. The former depicts hyperbolic lines as circular arcs orthogonal to the boundary circle, while the latter represents them as straight line segments (chords) across the disk. To map the upper half-plane model to the Poincaré disk model, we use the transformation

$$z \mapsto \frac{z - i}{z + i}$$

for all $z = x + yi \in \mathbb{H}$, which maps $z = i$ to the origin, the real line to the unit circle, and $z = \infty$ to the point 1 on the boundary of the disk. To map the Poincaré disk to the Klein disk model, we stereographically project onto a hemisphere above the disk and then orthogonally project back onto the disk. These two projections map geodesics in the upper half-plane model to Euclidean straight lines in the Klein disk model, transforming any hyperbolic triangulated polygon into a Euclidean one in the unit disk.

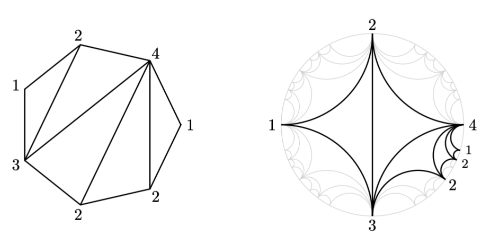


Figure 4. An example of a hyperbolic triangulated polygon in the Poincaré Disk and its Euclidean equivalent, from [12].

In Figure 4, we can clearly see that the Euclidean polygon retains the numbers of triangles incident at each vertex, and therefore has the same quiddity (up to cyclic permutation) as its hyperbolic counterpart. A similar process, albeit inverted, can be used to obtain a hyperbolic triangulated polygon from a Euclidean one.

In 1973, John Conway and Harold Coxeter [2] established a connection between such triangulated Euclidean polygons and frieze patterns.

3.2. **The Conway-Coxeter Theorem.**

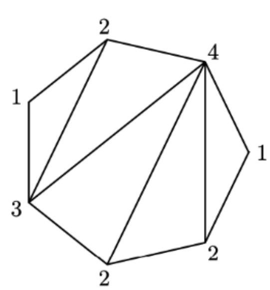
Definition 3.7. An *adjacent* $n \times n$ *minor* of an array is the determinant of an $n \times n$ submatrix of the array whose entries are not separated by any rows or columns.

Definition 3.8. A *frieze pattern* is an infinite array of numbers bounded by a diagonal of 0s followed by a diagonal of 1s, such that every adjacent 2×2 minor in the array is 1. We primarily work with *closed frieze patterns*, which are bounded on both sides by diagonals of 1s and 0s.

Remark 3.9. Note that the unimodular condition implies that every frieze pattern is uniquely determined by its first nontrivial diagonal, i.e. by the diagonal that follows the diagonal of 1s.

Definition 3.10. The *width* of a closed frieze pattern is given by the number of nontrivial diagonals in the frieze, i.e. by the number of diagonals between the two bounding diagonals of 1s.

We seek to prove that every triangulated polygon corresponds to a unique closed integer frieze pattern and vice versa. Since every frieze pattern is completely determined by the entries on its first nontrivial diagonal, it follows that we can obtain a unique n-periodic frieze from an arbitrary triangulated n-gon by placing its quiddity on the first nontrivial diagonal of the frieze.



⋱

0	1	1	3	5	2	1	0								
	0	1	4	7	3	2	1	0							
		0	1	2	1	1	1	1	0						
			0	1	1	2	3	4	1	0					
				0	1	3	5	7	2	1	0				
					0	1	2	3	1	1	1	0			
						0	1	2	1	2	3	1	0		
							0	1	1	3	5	2	1	0	
								0	1	4	7	3	2	1	0

⋱

FIGURE 5. An example of a triangulated heptagon and its corresponding 7-periodic frieze

It turns out that such a frieze is always closed with width $n-3$, as stated in the Conway-Coxeter theorem:

Theorem 3.11 (Conway-Coxeter)**.** *There exists a one-to-one correspondence between triangulated n-gons and closed frieze patterns of width $n-3$ with positive integer entries.*

Proof. The proof of this theorem is well-documented, so we will merely sketch an outline of the argument provided in [5]. First, we claim that the entries (f_i) in any row of a frieze satisfy the equation

$$f_{i+1} = c_i f_i - f_{i-1},$$

where the coefficients (c_i) are given by the entries of the first nontrivial diagonal of the frieze (we will prove this claim later). A corollary of this proposition is that every nontrivial diagonal of a frieze of width $n-3 \geq 1$ is n periodic.

We must now define a few useful tools. It is fairly straightforward to prove that the first nontrivial diagonal (a_i) of any frieze satisfies $a_j = 1$ for some $j \in \mathbb{N}$, which we can use to define the operator

$$T : (\ldots, a_{j-1}, a_j, a_{j+1}, \ldots) \mapsto (\ldots, a_{j-2}, a_{j-1} - 1, a_{j+1} - 1, a_{j+1}, \ldots).$$

Applying this operator to the first nontrivial diagonal of a closed integral frieze of width w turns it into a closed integral frieze of width $w-1$, as it becomes $(w+2)-$periodic, while the inverse procedure T^* has the opposite effect.

We can also prove (by induction) that every triangulated n-gon has at least one vertex that is attached to only one triangle, which we can use to define the procedures V and V^* that respectively remove and add such an exterior vertex.

We can now use induction to prove the theorem. For the base case, it is easy to see that the trivial frieze of width 0 corresponds to a triangle and vice versa.

Now suppose that every frieze of width w corresponds to an $(w+3)$-gon, and let F be a frieze of width $w+1$. We can use T to obtain a frieze of width w, and then V^* to add a vertex to the $(w+3)$-gon that shares its quiddity, thereby obtaining a unique $(n+4)$-gon that corresponds to F.

To go from triangulated n-gons to friezes of width $n-3$, we reverse the above process. Suppose that every n-gon corresponds to some frieze of width $n-3$, and let P be an $(n+1)$-gon. We can use V to remove a vertex from P and obtain an n-gon, and then apply T' to $n-3$ frieze that shares its quiddity, thereby obtaining a unique frieze of width $n-2$ that corresponds to P.

It follows that there exists a one-to-one correspondence between triangulated n-gons and frieze patterns of width $n-3$ with positive integer entries, where the quiddity of every triangulated polygon coincides with the first nontrivial diagonal of the corresponding frieze pattern. □

4. SL_2-TILINGS

SL_2-tilings are an important generalization of frieze patterns. Since their introduction in [1] for the study of cluster algebras, they have had many applications in hyperbolic geometry, dynamical systems, and modular forms. In the next two sections, we will be demonstrating their versatility by fleshing out the correspondence between SL_2-tilings and quiddities that was established in [10].

Definition 4.1. An SL_2-*tiling* is an infinite array of numbers such that every adjacent 2×2 minor in the array is 1.

Note that SL_2-tilings are effectively frieze patterns without bounding diagonals. While such tilings are not necessarily periodic, we can extend n-periodic friezes by continuing antiperiodically after the diagonals of zeroes, leading to an (n,n)-antiperiodic SL_2-tiling. In this paper, we are primarily interested in 'rectangular' antiperiodic tilings.

Definition 4.2. An SL_2-tiling is said to be (n,m)-*antiperiodic* if every row is n-antiperiodic and every column is m-antiperiodic, i.e. if

$$a_{i,j+n} = -a_{i,j};$$
$$a_{i+m,j} = -a_{i,j}$$

for all $i, j \in \mathbb{Z}$.

Definition 4.3. An (n, m)-antiperiodic SL_2-tiling is said to have a *positive rectangular domain* if it contains an adjacent $m \times n$ submatrix of positive integers

$$\begin{array}{ccccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 1 & 2 & 5 & 3 & 1 & -1 & -2 & -5 & -3 & -1 & 1 & \cdots \\
\cdots & 2 & 5 & 13 & 8 & 3 & -2 & -5 & -13 & -8 & -3 & 2 & \cdots \\
\cdots & 3 & 8 & 21 & 13 & 5 & -3 & -8 & -21 & -13 & -5 & 3 & \cdots \\
\cdots & 1 & 3 & 8 & 5 & 2 & -1 & -3 & -8 & -5 & -2 & 1 & \cdots \\
\cdots & -1 & -2 & -5 & -3 & -1 & 1 & 2 & 5 & 3 & 1 & -1 & \cdots \\
\cdots & -2 & -5 & -13 & -8 & -3 & 2 & 5 & 13 & 8 & 3 & -2 & \cdots \\
\cdots & -3 & -8 & -21 & -13 & -5 & 3 & 8 & 21 & 13 & 5 & -3 & \cdots \\
\cdots & -1 & -3 & -8 & -5 & -2 & 1 & 3 & 8 & 5 & 2 & -1 & \cdots \\
\cdots & 1 & 2 & 5 & 3 & 1 & -1 & -2 & -5 & -3 & -1 & 1 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

FIGURE 6. A section of a $(5, 4)$-antiperiodic SL_2-tiling with a positive rectangular domain.

Definition 4.4. An SL_2-tiling is said to be *tame* if every adjacent 3×3 minor is 0.

Lemma 4.5. *Every (n, m)-antiperiodic SL_2-tiling that has a positive rectangular domain is tame.*

Proof. Let T be an arbitrary (n, m)-antiperiodic SL_2-tiling that has a positive rectangular domain. The proposition follows directly from the Desnanot-Jacobi identity, which states that

$$\begin{vmatrix} a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} \\ a_{i,j-1} & a_{i,j} & a_{i,j+1} \\ a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} \begin{vmatrix} a_{i,j} \end{vmatrix} = \begin{vmatrix} a_{i-1,j-1} & a_{i-1,j} \\ a_{i,j-1} & a_{i,j} \end{vmatrix} \begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} - \begin{vmatrix} a_{i,j-1} & a_{i,j} \\ a_{i+1,j-1} & a_{i+1,j} \end{vmatrix} \begin{vmatrix} a_{i-1,j} & a_{i-1,j+1} \\ a_{i,j} & a_{i,j+1} \end{vmatrix}$$

for all $i, j \in \mathbb{Z}$. Since T is (n, m)-antiperiodic and has a positive rectangular domain, we know that $\begin{vmatrix} a_{i,j} \end{vmatrix} \neq 0$ for all $i, j \in \mathbb{Z}$. Since T is an SL_2-tiling, we also know that every adjacent 2×2 minor is 1, and therefore that

$$\begin{vmatrix} a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} \\ a_{i,j-1} & a_{i,j} & a_{i,j+1} \\ a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = 0$$

for all $i, j \in \mathbb{Z}$. □

To prove our main theorem, we also rely on a lemma involving linear difference equations.

Definition 4.6. Let $(c_i)_{i \in \mathbb{Z}}$ be an arbitrary n-periodic sequence of integers. A linear difference equation of the form

$$f_{i+1} = c_i f_i - f_{i-1}$$

is said to be *non-oscillating* if every solution $(f_i)_{i \in \mathbb{Z}}$ is n-antiperiodic and moreover has exactly one sign change in any sequence

$$(f_i, \ldots, f_{i+n}).$$

Remark 4.7. Note that any solution (f_i) to an LDE of the form

$$f_{i+1} = c_i f_i - f_{i-1}$$

can be recovered from any two consecutive elements f_i, f_{i-1} of the solution.

Remark 4.8. For all $x, y \in \mathbb{R}$ and all $i \in \mathbb{Z}$, there exists some solution (f_i) to the LDE such that $f_i = x$ and $f_{i-1} = y$, as x and y can be chosen as the initial conditions of (f_i).

Lemma 4.9. *A linear difference equation of the form $f_{i+1} = c_i f_i - f_{i-1}$, where $(c_i)_{n\in\mathbb{Z}}$ is an arbitrary n-periodic sequence of integers, is non-oscilating if and only if $(c_0, \dots, c_{n-1})$ is a quiddity.*

Proof. Although this lemma was first implied in [3] and later explicitly stated in [10], neither provides a complete argument. This paper presents a rigorous proof of the induction sketched by Coxeter and justifies the subsequent conclusions drawn by Morier-Genoud.

First, we will prove by induction that the entries in any row of a frieze pattern, extended by antiperiodicity, form a solution to an LDE of the form $f_{i+1} = c_i f_i - f_{i-1}$, where the coefficients $(c_i)_{n\in\mathbb{Z}}$ are given by the entries of the first nontrivial diagonal of the frieze. Since the frieze is trivial if $n \leq 3$, we will assume that $n > 3$, and use the following notation:

$$\begin{array}{cccccccccccccc}
\ddots \\
& f_{0,-1} & f_{0,0} & f_{0,1} & f_{0,2} & \cdots & f_{0,j} & \cdots & 1 & 0 \\
& & f_{1,-1} & f_{1,0} & f_{1,1} & f_{1,2} & \cdots & f_{1,j} & \cdots & 1 & 0 \\
& & & f_{2,-1} & f_{2,0} & f_{2,1} & f_{2,2} & \cdots & f_{2,j} & \cdots & 1 & 0 \\
& & & & & & & & & & & & \ddots
\end{array}$$

Note that $f_{i,-1} = 0$, $f_{i,0} = 1$, and $c_i \coloneqq f_{i,1}$ for all $i \in \mathbb{N}$. We claim that, for all $i \in \mathbb{N}$, the entries in the i-th row of the frieze satisfy the relation

$$f_{i,j+1} = c_{i+j} f_{i,j} - f_{i,j-1}$$

for all $j \in \mathbb{N}$. For the base case, let $i \in \mathbb{N}$ be given, and note that we have

$$f_{i,1} = c_i f_{i,0} - f_{i,-1},$$

as $c_i = f_{i,1}$, $f_{i,0} = 1$, and $f_{i,-1} = 0$. Similarly,

$$f_{i+1,1} = c_{i+1} f_{i+1,0} - f_{i+1,-1}.$$

For the induction step, suppose that

$$f_{i,n} = c_{i+n-1} f_{i,n-1} - f_{i,n-2} \quad \text{and} \quad f_{i+1,n} = c_{i+n} f_{i+1,n-1} - f_{i+1,n-2}$$

for some $n \in \mathbb{N}$. We want to show that

$$f_{i,n+1} = c_{i+n} f_{i,n} - f_{i,n-1}.$$

To do this, consider the following section of the frieze pattern:

$$\begin{array}{cccccc}
\cdots & f_{i,n-1} & f_{i,n} & f_{i,n+1} & \cdots \\
\cdots & f_{i+1,n-2} & f_{i+1,n-1} & f_{i+1,n} & \cdots
\end{array}$$

From the definition of a frieze pattern, we know that

$$f_{i,n-1} f_{i+1,n-1} - f_{i,n} f_{i+1,n-2} = 1 \tag{4.10}$$

and that

$$f_{i,n}f_{i+1,n} - f_{i,n+1}f_{i+1,n-1} = 1. \tag{4.11}$$

Substituting

$$f_{i+1,n} = c_{i+n}f_{i+1,n-1} - f_{i+1,n-2}$$

into (4.11), we get

$$f_{i,n}(c_{i+n}f_{i+1,n-1} - f_{i+1,n-2}) - f_{i,n+1}f_{i+1,n-1} = 1.$$

Equating this to (4.10), it follows that

$$f_{i,n}(c_{i+n}f_{i+1,n-1} - f_{i+1,n-2}) - f_{i,n+1}f_{i+1,n-1} = f_{i,n-1}f_{i+1,n-1} - f_{i,n}f_{i+1,n-2}$$
$$\iff f_{i,n}c_{i+n}f_{i+1,n-1} - f_{i,n+1}f_{i+1,n-1} - f_{i,n-1}f_{i+1,n-1} = 0$$
$$\iff f_{i,n}c_{i+n} - f_{i,n+1} - f_{i,n-1} = 0,$$

and therefore that

$$f_{i,n+1} = c_{i+n}f_{i,n} - f_{i,n-1},$$

as desired.

We can now prove that the linear difference equation $f_{i+1} = c_i f_i - f_{i-1}$ being non-oscillating implies that $(c_0, \ldots, c_{n-1})$ is a quiddity. First, we will prove that (c_i) is a sequence of positive integers. Let $f_{i+1} = c_i f_i - f_{i-1}$ be non-oscilating, and suppose for the sake of contradiction that $c_j \leq 0$ for some $j \in \mathbb{Z}$. We will consider the following two cases:

- $\boldsymbol{c_j < 0:}$ By Remark 4.8, we can let $f_j > 0$, and choose f_{j-1} such that $c_j f_j < f_{j-1} < 0$. Since
$$f_{j+1} = c_j f_j - f_{j-1},$$
it follows that $f_{j+1} < 0$, and therefore that there are two sign changes in the sequence (f_{j-1}, f_j, f_{j+1}). Since $n > 3$, it follows that there exists some sequence $(f_i, \ldots, f_{i+n})$ with more than one sign change. This is a contradiction, so $(c_i) \not< 0$ for all $i \in \mathbb{Z}$.
- $\boldsymbol{c_j = 0:}$ By Remark 4.8, we can let $f_{j-1} < 0$, and choose f_j such that $f_j = (c_{j+1} + 1)f_{j+1}$. Since
$$f_{j+1} = c_j f_j - f_{j-1} = -f_{j-1}$$
and $c_{j+1} \not< 0$, it follows that $f_j, f_{j+1} > 0$. Since
$$f_{j+2} = c_{j+1}f_{j+1} - f_j = -f_{j+1},$$
we know that $f_{j+2} < 0$, and therefore that there are two sign changes in the sequence $(f_{j-1}, f_j, f_{j+1}, f_{j+2})$. Since $n > 3$, it follows that there exists some sequence $(f_i, \ldots, f_{i+n})$ with more than one sign change. This is a contradiction, so $(c_i) \neq 0$ for all $i \in \mathbb{Z}$.

To prove that (c_i) is a quiddity, it now suffices to note that the frieze pattern generated by placing (c_i) on the first nontrivial diagonal has positive integer entries. By Theorem 3.11, it follows that (c_i) is a quiddity.

To prove the other direction, first note that all the solutions to an LDE of the form $f_{i+1} = c_i f_i - f_{i-1}$ are generated by any two linearly independent solutions, as it is a homogeneous linear difference equation of the second order. Let $(c_0, \ldots, c_{n-1})$ be a quiddity. By Theorem 3.11, we know that $(c_0, \ldots, c_{n-1})$ forms the first non-trivial diagonal of a unique positive integer frieze. By the above proof, we know that all the rows of this frieze are solutions to our LDE. Since $n > 3$, we know that c_i is not a constant sequence, and therefore that there exist some $i, j \in \mathbb{Z}$ such that $c_i \neq c_j$. Since $(1, c_i)$ and $(1, c_j)$ are linearly independent, it follows that the rows they correspond to are also linearly independent. Now note that any row of an n-periodic positive integer frieze that is extended by antiperiodicity is clearly antiperiodic with exactly one sign change in any sequence $(f_i, \ldots, f_{i+n})$, which implies that rows i and j satisfy the non-oscilating conditions. Since every solution to our LDE can be written as a linear combination of two solutions that satisfy the non-oscilating conditions, it follows that $f_{i+1} = c_i f_i - f_{i-1}$ is non-oscilating, as desired. □

We are now ready to prove our main theorem.

5. Main Theorem

Theorem 5.1. *There exists a bijection between the set of* (n, m)*-antiperiodic* SL_2*-tilings that have a positive rectangular domain and the set of triples* (q, q', M)*, where*

$$q = (q_0, \ldots, q_{n-1}); \qquad q' = (q'_0, \ldots, q'_{m-1})$$

are quiddities of order n *and* m *respectively and* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a* 2×2 *matrix with positive integer entries satisfying*

$$q_0 < \frac{b}{a}, \qquad q'_0 < \frac{c}{a}, \quad \text{and} \quad ad - bc = 1.$$

Proof. The original proof of this theorem in [10] is highly concise, with several key steps either omitted or mentioned only briefly. A more detailed version is presented below.

We will construct a mapping $\mathscr{F}$ between the set of triples satisfying the above conditions and the set of (n, m)-antiperiodic SL_2-tilings that have a positive rectangular domain and prove that this mapping is bijective. Let $t = (q, q', M)$ be an arbitrary triple such that

$$q = (q_0, \ldots, q_{n-1}); \qquad q' = (q'_0, \ldots, q'_{m-1})$$

are quiddities of order n and m respectively and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix with positive integer entries satisfying

$$q_0 < \frac{b}{a}, \qquad q'_0 < \frac{c}{a}, \quad \text{and} \quad ad - bc = 1.$$

5.1. **Construction.** First, we must extend q and q' by letting $q_i = q_{i+n}$ and $q'_i = q'_{i+m}$ for all $i, j \in \mathbb{Z}$. We can now define $\mathscr{F}(t)_{i,j} = a_{i,j}$ by letting

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} \coloneqq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and recursively letting

$$a_{i,j+1} \coloneqq q_j a_{i,j} - a_{i,j-1}; \tag{5.2}$$

$$a_{i+1,j} \coloneqq q'_i a_{i,j} - a_{i-1,j} \tag{5.3}$$

for all $i, j \in \mathbb{Z}$. Note that the two recurrence relations commute, as

$$\begin{aligned}
a_{i+1,j+1} &= q_j a_{i+1,j} - a_{i+1,j-1} \\
&= q_j \left(q'_i a_{i,j} - a_{i-1,j}\right) - \left(q'_i a_{i,j-1} - a_{i-1,j-1}\right) \\
&= q_j q'_i a_{i,j} - q_j a_{i-1,j} - q'_i a_{i,j-1} + a_{i-1,j-1} \\
&= q'_i \left(q_j a_{i,j} - a_{i,j-1}\right) - \left(q_j a_{i-1,j} - a_{i-1,j-1}\right) \\
&= q'_i a_{i,j+1} - a_{i-1,j+1} = a_{i+1,j+1},
\end{aligned}$$

which implies that $\mathscr{F}$ is well-defined.

Moreover, we can prove by induction that $\mathscr{F}(t)$ is an SL_2 tiling. By construction, we know that

$$\det \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} = a_{0,0}a_{1,1} - a_{0,1}a_{1,0} = ad - bc = 1.$$

For the induction step, assume that

$$\det \begin{pmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{pmatrix} = a_{i,j}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j} = 1,$$

and note that

$$\begin{aligned}
&\det \begin{pmatrix} a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j+1} & a_{i+1,j+2} \end{pmatrix} \\
&= a_{i,j+1}a_{i+1,j+2} - a_{i,j+2}a_{i+1,j+1} \\
&= a_{i,j+1}\left(q_{j+1}a_{i+1,j+1} - a_{i+1,j}\right) \\
&- \left(q_{j+1}a_{i,j+1} - a_{i,j}\right) a_{i+1,j+1} \\
&= -a_{i,j+1}a_{i+1,j} + a_{i,j}a_{i+1,j+1} = 1;
\end{aligned}
\qquad
\begin{aligned}
&\det \begin{pmatrix} a_{i+1,j} & a_{i+1,j+1} \\ a_{i+2,j} & a_{i+2,j+1} \end{pmatrix} \\
&= a_{i+1,j}a_{i+2,j+1} - a_{i+1,j+1}a_{i+2,j} \\
&= a_{i+1,j}\left(q'_{i+1}a_{i+1,j+1} - a_{i,j+1}\right) \\
&- \left(q'_{i+1}a_{i+1,j} - a_{i,j}\right) a_{i+1,j+1} \\
&= -a_{i+1,j}a_{i,j+1} + a_{i,j}a_{i+1,j+1} = 1.
\end{aligned}$$

A similar proof shows that he same holds for

$$\det \begin{pmatrix} a_{i,j-1} & a_{i,j} \\ a_{i+1,j-1} & a_{i+1,j} \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} a_{i-1,j} & a_{i-1,j+1} \\ a_{i,j} & a_{i,j+1} \end{pmatrix},$$

and therefore that $\mathscr{F}(t)$ is an SL_2 tiling.

By Lemma 4.9, we know that $\mathscr{F}(t)$ is also (n, m)-antiperiodic, as the rows and columns of $\mathscr{F}(t)$ are the solutions to non-oscilating equations with respectively n-periodic and m-periodic coefficient sequences q and q'.

It remains to show that $\mathscr{F}(t)$ has a positive rectangular domain. Since the rows and columns of $\mathscr{F}(t)$ satisfy the non-oscilating conditions, it suffices to compute the signs of $a_{0,-1}$, $a_{-1,-1}$, and $a_{-1,0}$. Since $a_{0,-1} = q_0 a - b$ and $q_0 < \frac{b}{a}$, it follows that $\frac{a_{0,-1}}{a} < 0$, and therefore that $a_{0,-1} < 0$. Similarly, since $a_{-1,0} = q'_0 a - c$ and $q'_0 < \frac{c}{a}$, it follows that $\frac{a_{-1,0}}{a} < 0$, and therefore that $a_{-1,0} < 0$. Finally, note that $a_{-1,-1} = q_0 q'_0 a - q_0 c - q'_0 b + d$. Since $q_0 < \frac{b}{a}$ and $q'_0 < \frac{c}{a}$, it follows that

$$b(q_0 q'_0 a - q_0 c - q'_0 b + d) = b(d - q'_0 b) - b q_0 (c - q'_0 a) > a q_0 (d - q'_0 b) - b q_0 (c - q'_0 a) = q_0 > 0,$$

and therefore that $q_0 q'_0 a - q_0 c - q'_0 b + d = a_{-1,-1} > 0$. Since $a_{0,-1} < 0$ and $a = a_{0,0} > 0$, it follows from the definition of a non-oscillating equation that $a_{0,1}, \ldots, a_{0,n-1}$ are all positive. Similarly, since $a_{-1,-1} > 0$ and $a_{-1,0} < 0$, it also follows that $a_{-1,1}, \ldots, a_{-1,n-1}$ are all negative. Since $a_{-1,j} < 0$ and $a_{0,j} > 0$ for all $0 \leq j < n$, it finally follows that $a_{0,j}, \ldots, a_{m-1,j}$ are all positive for all $0 \leq j < n$, and therefore that $\mathscr{F}(t)$ has a positive rectangular domain.

Since t was arbitrary, it follows that $\mathscr{F}$ maps the set of triples (q, q', M) satisfying the above conditions to the set of (n, m)-antiperiodic SL_2-tilings that have a positive rectangular domain. We now want to show that $\mathscr{F}$ is bijective.

5.2. **Proof of Injectivity.** To prove that $\mathscr{F}$ is injective, suppose that $\mathscr{F}(q, q', M) = \mathscr{F}(q^*, q'^*, M^*)$. Then clearly $M = M^*$. Moreover, since

$$q_j a_{i,j} - a_{i,j-1} = a_{i,j+1} = q_j^* a_{i,j} - a_{i,j-1};$$
$$q_i' a_{i,j} - a_{i-1,j} = a_{i+1,j} = q_i'^* a_{i,j} - a_{i-1,j}$$

for all $i, j \in \mathbb{Z}$, it follows that $q = q^*$ and $q' = q'^*$, and therefore that $\mathscr{F}$ is injective.

5.3. **Proof of Surjectivity.** We will now prove that this construction is surjective. Let T be an arbitrary (n, m)-antiperiodic SL_2-tiling that has a positive rectangular domain. We want to show that there exists some triple t, satisfying the above conditions, such that $\mathscr{F}(t) = T$. By Lemma 4.5, we know that T is tame. Now let

$$P = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \dots & a_{1,n-1} \\ \dots & \dots & \ddots & \dots \\ a_{m-1,0} & a_{m-1,1} & \dots & a_{m-1,n-1} \end{pmatrix}$$

denote the positive rectangular domain of T. We will start by proving that the ratios between the first two rows of P form a decreasing sequence, i.e. that

$$\frac{a_{0,0}}{a_{1,0}} > \frac{a_{0,1}}{a_{1,1}} > \dots > \frac{a_{0,n-1}}{a_{1,n-1}},$$

and similarly that the ratios between the first two columns of P form a decreasing sequence, i.e. that

$$\frac{a_{0,0}}{a_{0,1}} > \frac{a_{1,0}}{a_{1,1}} > \dots > \frac{a_{m-1,0}}{a_{m-1,1}}.$$

Since T is an SL_2-tiling, we know that

$$a_{0,j}a_{1,j+1} - a_{0,j+1}a_{1,j} = 1$$

for all $j \in \mathbb{Z}$, and therefore that

$$\frac{a_{0,j}}{a_{1,j}} = \frac{a_{0,j+1}}{a_{1,j+1}} + \frac{1}{a_{1,j+1}}$$

for all $j \in \mathbb{Z}$. Since $a_{1,j+1}$ is positive for all $j \leq n-2$, it follows that

$$\frac{a_{0,j}}{a_{1,j}} > \frac{a_{0,j+1}}{a_{1,j+1}}$$

for all $j \leq n-2$, as desired. In a similar manner, we know that

$$a_{i,0}a_{i+1,1} - a_{i,1}a_{i+1,0} = 1$$

for all $i \in \mathbb{Z}$, and therefore that

$$\frac{a_{i,0}}{a_{i,1}} > \frac{a_{i+1,0}}{a_{i+1,1}}$$

for all $i \leq m-2$.

Using this information, we can now prove that the entries of T satisfy the recurrence relations (5.2) and (5.3), where $(q_j)_{j\in\mathbb{N}}$ and $(q_i')_{i\in\mathbb{N}}$ are quiddities of order n and m, respectively. Let $i, j \in \mathbb{Z}$ be given. Since T is tame, we know thatevery adjacent 3×3 minor is 0. It follows that the columns of such matrices are not linearly independent, and therefore that there exist some $\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j} \in \mathbb{C}$ such that

$$\alpha_{i,j} \begin{pmatrix} a_{i-1,j-1} \\ a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix} + \beta_{i,j} \begin{pmatrix} a_{i-1,j} \\ a_{i,j} \\ a_{i+1,j} \end{pmatrix} + \gamma_{i,j} \begin{pmatrix} a_{i-1,j+1} \\ a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = 0.$$

Dropping the $(i-1)$th row, we are left with

$$\gamma_{i,j} \begin{pmatrix} a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = -\beta_{i,j} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} - \alpha_{i,j} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix}.$$

Suppose for the sake of contradiction that $\gamma_{i,j} = 0$. Then

$$\alpha_{i,j} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix} + \beta_{i,j} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} = 0,$$

which implies that

$$\begin{vmatrix} a_{i,j-1} & a_{i,j} \\ a_{i+1,j-1} & a_{i+1,j} \end{vmatrix} = 0.$$

Since every adjacent 2×2 minor is 1, we know that this is a contradiction, and therefore that $\gamma_{i,j} \neq 0$. It follows that we can divide $\alpha_{i,j}$ and $\beta_{i,j}$ by $\gamma_{i,j}$ to get

$$\begin{pmatrix} a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = -\frac{\beta_{i,j}}{\gamma_{i,j}} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} - \frac{\alpha_{i,j}}{\gamma_{i,j}} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix}.$$

Now let $\lambda_{i,j} := -\frac{\beta_{i,j}}{\gamma_{i,j}}$ and $\mu_{i,j} := -\frac{\alpha_{i,j}}{\gamma_{i,j}}$, and note that we have

$$a_{i,j+1} = \lambda_{i,j} a_{i,j} + \mu_{i,j} a_{i,j-1};$$
$$a_{i+1,j+1} = \lambda_{i,j} a_{i+1,j} + \mu_{i,j} a_{i+1,j-1}.$$

Equating these equations to solve for $\mu_{i,j}$ gives

$$\frac{a_{i,j+1} - \mu_{i,j} a_{i,j-1}}{a_{i,j}} = \frac{a_{i+1,j+1} - \mu_{i,j} a_{i+1,j-1}}{a_{i+1,j}}$$
$$\iff a_{i,j+1} a_{i+1,j} - \mu_{i,j} a_{i,j-1} a_{i+1,j} = a_{i+1,j+1} a_{i,j} - \mu_{i,j} a_{i+1,j-1} a_{i,j}$$
$$\iff a_{i,j+1} a_{i+1,j} - a_{i+1,j+1} a_{i,j} = \mu_{i,j} (a_{i,j-1} a_{i+1,j} - a_{i+1,j-1} a_{i,j}).$$

Since T is an SL_2-tiling, we know that $a_{i,j+1} a_{i+1,j} - a_{i+1,j+1} a_{i,j} = -1$ and that $a_{i,j-1} a_{i+1,j} - a_{i+1,j-1} a_{i,j} = 1$, and therefore that

$$\mu_{i,j} = -1.$$

In a similar manner, we can solve for $\lambda_{i,j}$ as follows:

$$\frac{a_{i,j+1} - \lambda_{i,j} a_{i,j}}{a_{i,j-1}} = \frac{a_{i+1,j+1} - \lambda_{i,j} a_{i+1,j}}{a_{i+1,j-1}}$$
$$\iff a_{i,j+1} a_{i+1,j-1} - \lambda_{i,j} a_{i,j} a_{i+1,j-1} = a_{i+1,j+1} a_{i,j-1} - \lambda_{i,j} a_{i+1,j} a_{i,j-1}$$
$$\iff \lambda_{i,j} (a_{i+1,j} a_{i,j-1} - a_{i,j} a_{i+1,j-1}) = a_{i+1,j+1} a_{i,j-1} - a_{i,j+1} a_{i+1,j-1}.$$

Since T is an SL_2-tiling, we know that $a_{i+1,j} a_{i,j-1} - a_{i,j} a_{i+1,j-1} = 1$, and therefore that

$$\lambda_{i,j} = a_{i+1,j+1} a_{i,j-1} - a_{i,j+1} a_{i+1,j-1}.$$

Since the ratios between the first two rows of P form a decreasing sequence, it follows that

$$\frac{a_{i,j-1}}{a_{i+1,j-1}} > \frac{a_{i,j+1}}{a_{i+1,j+1}},$$

and therefore that $\lambda_{i,j} > 0$. Since $(a_{i,j})_{j\in\mathbb{Z}}$ is n-antiperiodic, we also know that $\lambda_{i,j}$ is n-periodic, as

$$\begin{aligned} a_{i,j+n+1} &= \lambda_{i,j+n}a_{i,j+n} - a_{i,j+n-1} \\ \iff -a_{i,j+1} &= \lambda_{i,j+n}(-a_{i,j}) + a_{i,j-1} \\ \iff -(\lambda_{i,j}a_{i,j} - a_{i,j-1}) &= -\lambda_{i,j+n}a_{i,j} + a_{i,j-1} \\ \iff -\lambda_{i,j}a_{i,j} &= -\lambda_{i,j+n}a_{i,j} \\ \iff \lambda_{i,j} &= \lambda_{i,j+n} \end{aligned}$$

for all $i, j \in \mathbb{Z}$. It merely remains to show that $\lambda_{i,j}$ does not depend on i. To do this, we will prove that

$$a_{i+1,j+1}a_{i,j-1} - a_{i,j+1}a_{i+1,j-1} = a_{i,j+1}a_{i-1,j-1} - a_{i-1,j+1}a_{i,j-1}.$$

Since T is tame, we know that

$$\begin{vmatrix} a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} \\ a_{i,j-1} & a_{i,j} & a_{i,j+1} \\ a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = 0.$$

Computing the determinant in two ways, it follows that

$$a_{i-1,j-1}\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} - a_{i-1,j}\begin{vmatrix} a_{i,j-1} & a_{i,j+1} \\ a_{i+1,j-1} & a_{i+1,j+1} \end{vmatrix} + a_{i-1,j+1}\begin{vmatrix} a_{i,j-1} & a_{i,j} \\ a_{i+1,j-1} & a_{i+1,j} \end{vmatrix} = 0$$

and that

$$a_{i+1,j-1}\begin{vmatrix} a_{i-1,j} & a_{i-1,j+1} \\ a_{i,j} & a_{i,j+1} \end{vmatrix} - a_{i+1,j}\begin{vmatrix} a_{i-1,j-1} & a_{i-1,j+1} \\ a_{i,j-1} & a_{i,j+1} \end{vmatrix} + a_{i+1,j+1}\begin{vmatrix} a_{i-1,j-1} & a_{i-1,j} \\ a_{i,j-1} & a_{i,j} \end{vmatrix} = 0.$$

Since every adjacent 2×2 minor is 1, it follows that we have

$$\begin{aligned} &a_{i-1,j-1} - a_{i-1,j}(a_{i,j-1}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j-1}) + a_{i-1,j+1} = 0 \\ &\qquad \iff a_{i,j-1}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j-1} = \frac{a_{i-1,j-1} + a_{i-1,j+1}}{a_{i-1,j}}, \end{aligned}$$

and similarly

$$\begin{aligned} &a_{i+1,j-1} - a_{i+1,j}(a_{i-1,j-1}a_{i,j+1} - a_{i-1,j+1}a_{i,j-1}) + a_{i+1,j+1} = 0 \\ &\qquad a_{i-1,j-1}a_{i,j+1} - a_{i-1,j+1}a_{i,j-1} = \frac{a_{i+1,j-1} + a_{i+1,j+1}}{a_{i+1,j}}. \end{aligned}$$

Therefore we want to show that

$$\frac{a_{i-1,j-1} + a_{i-1,j+1}}{a_{i-1,j}} = \frac{a_{i+1,j-1} + a_{i+1,j+1}}{a_{i+1,j}}.$$

Since T is an SL_2-tiling, we know that

$$\begin{aligned} &\begin{vmatrix} a_{i,j-1} & a_{i,j} \\ a_{i+1,j-1} & a_{i+1,j} \end{vmatrix} = 1 \\ \iff &a_{i,j-1}a_{i+1,j} - a_{i,j}a_{i+1,j-1} = 1 \\ \iff &a_{i,j-1}a_{i+1,j} - 1 = a_{i,j}a_{i+1,j-1} \\ \iff &\frac{a_{i,j-1}a_{i+1,j} - 1}{a_{i,j}} = a_{i+1,j-1} \end{aligned}$$

and similarly that

$$\begin{aligned} \begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} &= 1 \\ \iff a_{i,j}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j} &= 1 \\ \iff a_{i,j}a_{i+1,j+1} &= a_{i,j+1}a_{i+1,j} + 1 \\ \iff \frac{a_{i,j+1}a_{i+1,j} + 1}{a_{i,j}} &= a_{i+1,j+1}. \end{aligned}$$

Therefore

$$\frac{a_{i+1,j-1} + a_{i+1,j+1}}{a_{i+1,j}} = \frac{1}{a_{i+1,j}} \left(\frac{a_{i,j-1}a_{i+1,j} - 1}{a_{i,j}} + \frac{a_{i,j+1}a_{i+1,j} + 1}{a_{i,j}} \right) = \frac{a_{i,j-1} + a_{i,j+1}}{a_{i,j}}.$$

An identical proof shows that

$$\frac{a_{i,j-1} + a_{i,j+1}}{a_{i,j}} = \frac{a_{i-1,j-1} + a_{i-1,j+1}}{a_{i-1,j}},$$

which implies that

$$a_{i+1,j+1}a_{i,j-1} - a_{i,j+1}a_{i+1,j-1} = a_{i,j+1}a_{i-1,j-1} - a_{i-1,j+1}a_{i,j-1},$$

and therefore that $\lambda_{i,j}$ is independent of i. Since $i, j \in \mathbb{Z}$ were arbitrary, it follows that we can let

$$q_j := \lambda_{i,j}$$

for all $j \in \mathbb{Z}$, thereby proving that the rows of T satisfy (5.2). A similar argument shows that the columns of T satisfy (5.3). Moreover, since T is (n, m)-antiperiodic and has a positive rectangular domain, we know that that (5.2) and (5.3) are both non-oscillating, and therefore by Lemma 4.9 that (q_i) and $(q_i)'$ are quiddities of order n and m, respectively.

Finally, it remains to prove that

$$q_0 < \frac{a_{0,1}}{a_{0,0}} \quad \text{and} \quad q'_0 < \frac{a_{1,0}}{a_{0,0}}$$

Since T is (n, m)-antiperiodic and has a positive rectangular domain, we know that $a_{0,-1} < 0$ and $a_{-1,0} < 0$. Since

$$\begin{aligned} a_{0,1} &= q_0 a_{0,0} - a_{0,-1}; \\ a_{1,0} &= q'_0 a_{0,0} - a_{-1,0}, \end{aligned}$$

it follows that $a_{0,1} > q_0 a_{0,0}$ and $a_{1,0} > q'_0 a_{0,0}$, as desired. □

6. Conclusion

In this paper, we provided a detailed proof of the bijection between (n, m)-antiperiodic SL_2-tilings with positive rectangular domains and certain triples (q, q', M), including several steps that have not previously been published. We arrived at this result by proving a series of connections between Farey graphs, quiddities, frieze patterns, and SL_2-tilings, which allowed us to explore the relations between geometry, number theory, combinatorics, and dynamical systems.

This study was primarily motivated by an interest in the applications of non-Euclidean geometries, which crystallized around hyperbolic geometry due to its relevance in dynamical systems. After examining some classical number theory, we established connections with combinatorics through the study of triangulations of polygons, and with algebra through determinant computations and frieze patterns (which can be viewed as complicated collections of 2×2 matrices). Friezes and SL_2-tilings are of particular interest because they bridge geometry, combinatorics, and algebra through their connection to the Fomin-Zelevinsky theory of cluster algebras. They also have some applications in theoretical physics via systems of equations, which we examined briefly in Lemma 4.9.

Building on the applications of frieze patterns detailed in [8], recent papers have focused on developing friezes over finite fields [7], relating them to symplectic geometry [9], and further generalizing the Conway-Coxeter theorem [6]; [4]. Future research could adapt friezes to new areas of geometry, explore further connections with representation and category theory, or perhaps attempt to extend our bijection to SL_k-tilings.

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