MAPPING CLASS GROUPS THROUGH A KALEIDOSCOPE

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ABSTRACT. The mapping class group of a surface is the group of homeomorphisms of the surface up to homotopy. In this paper we provide a brief introduction to the theory and explore connections along the way.

CONTENTS

1. INTRODUCTION

The study of symmetry is ubiquitous in mathematics. Given a mathematical object X, a symmetry can be described as a function $f : X \to X$ that preserves a certain structure on X. If X carries multiple structures, we can study all of these symmetries in unison, or restrict to as few as we choose. In this way, a richer understanding of all types of symmetries of X leads to a richer overall understanding of X.

The objects whose symmetries we will investigate in this paper are *orientable surfaces*, or 2-dimensional manifolds with a continuous assignment of orientation for each tangent space. One natural kind of symmetry of an oriented surface is an *orientation-preserving homeomorphism* of our surface: a continuous bijection with continuous inverse that preserves the orientation on tangent spaces. The collection of all orientation-preserving homeomorphisms of a surface S form a group, denoted $\mathrm{Homeo}^+(S)$. This group is large – it is always uncountable. One way to see this is by noting that for any two points x and y on a surface, there is always an orientation-preserving homeomorphism that takes x to y. One could visualize this as pushing the point x to point y, and dragging along any part of the surface with it. Elements of the *mapping class group* of a surface S, denoted $\mathrm{Mod}(S)$, are defined to be orientation-preserving homeomorphisms up to homotopy; $\mathrm{Mod}(S)$ is a quotient of $\mathrm{Homeo}^+(S)$. Taking this quotient allows us to compute and work more concretely with $\mathrm{Mod}(S)$. We can study the action on S via curves, compute the isomorphism types for $\mathrm{Mod}(S)$ for some surfaces, and even show that $\mathrm{Mod}(S)$

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is finitely generated (see Section 3). The mapping class group $\mathrm{Mod}(S)$ is our protagonist in this paper.

In studying $\mathrm{Mod}(S)$, we will interact with many different areas of mathematics: geometric group theory, algebraic topology, and hyperbolic geometry, to name a few. This paper is divided into two sections. The first part of each section is dedicated to proving a certain result or giving a detailed outline of a construction. The latter parts, titled *excursions*, are intended to be "peeks through the kaleidoscope". There, we showcase a few applications of the theory developed just before, albeit with less detail. The word "excursion" will almost certainly be reductive in many cases.

Almost all of the material is taken from *A Primer on Mapping Class Groups* by Farb and Margalit [2]. In exchange for being concise, a lot of very interesting mathematics must be omitted. We encourage the interested reader to read the corresponding sections of Farb and Margalit [2] themselves.

2. Preliminaries & first examples

In this section, we define the mapping class group $\mathrm{Mod}(S)$ and compute a few examples. As we will see, one can gain a lot of information about $\mathrm{Mod}(S)$ via the action on simple closed curves lying in S. In the process, we also introduce important techniques we will repeatedly use to study this action, such as the change of coordinates principle, and the notion of geometric intersection number.

We begin by ironing out once and for all the class of surfaces (2-dimensional real manifolds) we will consider in this paper, starting with a fundamental result:

Theorem 2.1 (Classification of surfaces)**.**
Any closed, connected, orientable surface is homeomorphic to the connected sum of a 2-dimensional sphere with $g \geq 0$ tori. Any compact, connected, orientable surface can be obtained by removing $b \geq 0$ open discs from a closed surface.

For a proof of this theorem, see Thomassen [5]. We may also consider the surfaces obtained by puncturing compact, connected, orientable surfaces. These are obtained by removing n points from the interior of S.

Notation 2.2. We will abbreviate a surface S with genus g, b boundary components, and n punctures as $S^b_{g,n}$. If the surface has no boundary, we typically supress b and just write $S_{g,n}$.

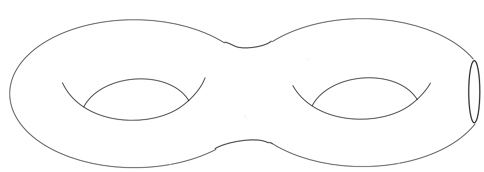

FIGURE 1. A surface homeomorphic to $S^1_{2,0}$.

These quantities determine the homeomorphism type of our surface. The *Euler characteristic* of a surface $S^b_{g,n}$ is

$$\chi(S^b_{g.n}) = \sum_{i=0}^{2}(-1)^i\beta_i,$$

the alternating sum of the Betti numbers. It turns out that the Euler characteristic can also be defined purely through the genus, number of boundary components, and punctures:

$$\chi(S^b_{g.n}) = 2 - 2g - (b+n),$$

and since homology is homeomorphism invariant, any three of the four quantities $\chi(S^b_{g,n})$, g, n, and b determine the surface up to homeomorphism. This fundamental result is indispensable as it allows for induction. For example, both of the proofs of finite generation mentioned in Section 3 are a double induction on genus and the number of punctures.

We now define the mapping class group $\mathrm{Mod}(S)$ of a surface S. Let $\mathrm{Homeo}^+(S, \partial S)$ be the group of orientation-preserving homeomorphisms that fix the boundary ∂S pointwise. We can topologize this group with the compact-open topology.

Definition 2.3. The *mapping class group* $\mathrm{Mod}(S)$ of a surface S is defined to be $\pi_0(\mathrm{Homeo}^+(S, \partial S))$. An element of $\mathrm{Mod}(S)$ is called a *mapping class*.

Note that in our definition above, elements of $\mathrm{Mod}(S)$ are required to fix the boundary pointwise, but are allowed to permute the punctures. As this is equivalent to taking the quotient by the connected component of the identity, this turns $\mathrm{Mod}(S)$ into a discrete topological group. Though an element of $\mathrm{Mod}(S)$ defines a class of homeomorphisms, we may abuse notation to sometimes mean a homeomorphism representing that class. We remark that there are equivalent definitions:

$$\begin{aligned}\mathrm{Mod}(S) &= \pi_0(\mathrm{Homeo}^+(S, \partial S)) \\ &= \mathrm{Homeo}^+(S, \partial S)/\text{isotopy}\end{aligned}$$

The equivalence of these definitions is nontrivial and relies on a number of results specific to surfaces that allow one to upgrade homotopies to isotopies. We refer the reader to Section 1.4 of [2] for a complete discussion.

We begin computing $\mathrm{Mod}(S)$ for a few surfaces: the closed disc, the thrice-punctured sphere, and finally the torus. First we have the closed disc, which will turn out to be an important example.

Lemma 2.4 (Alexander trick)**.** *The group* $\mathrm{Mod}(D^2)$ *is trivial.*

Proof. Let ϕ be a homeomorphism that fixes the boundary pointwise. After identifying D^2 with the closed unit disc, we define

$$F(x,t) = \begin{cases}(1-t)\phi(\frac{x}{1-t}) & 0 \le |x| < 1-t \\ x & 1-t \le x \le 1\end{cases},$$

with $F(x,1) = id_{D^2}$, the identity map. □

This function constructs an isotopy that undoes any twisting as time goes on. This proof also holds for the once-punctured disc, as we can choose to center the puncture at the origin. We can also see that $\mathrm{Mod}(S_{0,1}) = 1$, after identifying the punctured sphere with $\mathbb{R}^2$, which is homeomorphic to D^2. Since every homeomorphism of the sphere can be chosen to fix a point by post-composing with a map using the isotopy extension theorem [4], we can apply the previous example to see that $\mathrm{Mod}(S^2) = 1$. The Alexander trick is very useful in more general situations:

one can cut more complicated surfaces up and then use the Alexander trick. To compute our next example, $S_{0,3}$, we introduce two more important concepts.

Definition 2.5. A *simple closed curve* is an injective embedding $S^1 \to S$. A closed curve is *essential* if it is not homotopic to a point, a puncture, or a boundary component. An *arc* is an injective embedding $I \to S$.

Understanding how simple closed curves behave under the action of $\mathrm{Mod}(S)$ is important. We will use them to verify the nontriviality of an important class of elements of $\mathrm{Mod}(S)$ called *Dehn twists*. It turns out that a finite number of Dehn twists generate $\mathrm{Mod}(S_{g,n})$ for $g \geq 1$ and $n \geq 0$; Section 3 is dedicated to proving this result.

Definition 2.6. Let α be an essential closed curve, and let S be a surface. The surface $S - \alpha$ is a surface obtained by *cutting along* α, and comes with a homeomorphism h between two of its boundary components such that

(1) $(S - \alpha)/(x \sim h(x))$ is homeomorphic to S
(2) the image of the boundary components under the quotient is α

One can define cutting S along an arc analogously.

Theorem 2.7. *Let* $S_{0,3}$ *denote the thrice-punctured sphere, or alternatively, the sphere with three marked points. Then* $\mathrm{Mod}(S_{0,3}) \cong \mathfrak{S}_3$*, the symmetric group on the three elements*

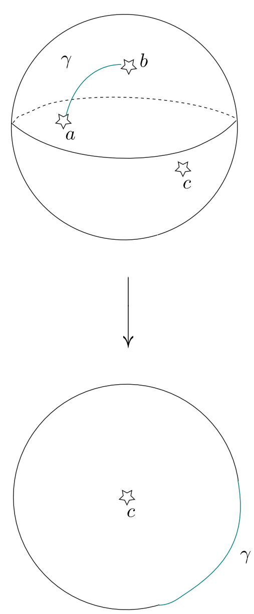

FIGURE 2. Cutting $S_{0,3}$ along γ.

Proof. We sketch the argument, thinking about the punctures as marked points. Denote the marked points by a, b, and c. The action on $\{a, b, c\}$ is clearly transitive, so it suffices to check injectivity. That is, any element of $\mathrm{Mod}(S_{0,3})$ that fixes the three punctures is homotopic to the identity.

Let ϕ be a homeomorphism representing such an element. Let γ be an arc connecting a and b. Since ϕ fixes the three marked points, it follows that ϕ fixes γ too (Proposition 2.2 in [2]). We then cut $S_{0,3}$ along γ to obtain a once-marked disc as in Figure 2. Since ϕ induces a map on the once-marked disc, and we know the mapping class group of a once-marked disc is trivial, ϕ induces a homeomorphism isotopic to the identity on the once-marked disc. We can then extend the homotopy to all of $S_{0,3}$, since ϕ must induce a map that fixes the boundary of the once-marked disc. For a complete proof see Proposition 2.3 in [2]. □

Even though the proof of this statement is not complete, we hope that the procedure is clear – by a clever choice of curve or arc, we can cut to reduce to cases we are more familiar with. We continue with our study of simple closed curves on surfaces by first defining two more important kinds of curves.

Definition 2.8. A simple closed curve α is *nonseparating* if $S - \alpha$ is connected. A simple closed curve β is *separating* if $S - \beta$ is not connected.

If S is closed, then a curve β is separating if and only if the class $[\beta]$ is vanishing in homology. That is, it bounds some subsurface. The next result is one that is extremely useful and allows us to prove certain properties simply by drawing out a picture.

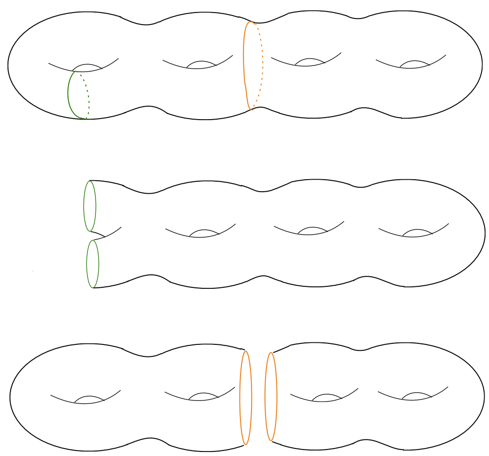

FIGURE 3. Curves and their corresponding cut surfaces. The green curve is nonseparating, while the orange curve is separating.

Lemma 2.9 (Change of coordinates principle)**.** *Let S be a surface. Let α and β be two simple closed curves. There is an orientation-preserving homeomorphism $\phi : S \to S$ such that $\phi(\alpha) = b$ if and only if the cut surfaces $S - \alpha$ and $S - \beta$ are homeomorphic.*

Proof. If there exists such an orientation-preserving homeomorphism, this extends to one of cut surfaces after canonically identifying the (possibly disconnected) boundary components. Now suppose that $S - \alpha$ and $S - \beta$ are homeomorphic. Let ψ be such a homeomorphism. The map ψ naturally respects the distinguished boundary components arising from cutting along α or β, so precomposing and postcomposing ψ with the identifications gives the desired homeomorphism. If ψ was orientation-reversing, we can postcompose again with another orientation-reversing homeomorphism. □

At once, we have an extremely important corollary: the action of $\mathrm{Mod}(S)$ is transitive on nonseparating curves. This is because any cutting S along any nonseparating curve will decrease the genus by one, and increase the number of boundary components by two. Since the number of punctures stays the same, the classification of surfaces guarantees that the any two surfaces obtained by cutting along a nonseparating curve are homeomorphic. The transitivity of this action helps us prove a key lemma in in Section 3.

The examples of $\mathrm{Mod}(S)$ we computed thus far have relied on the Alexander trick, which is a useful tool in understanding when a mapping class is trivial. The next few definitions are devoted to building up tools that allow us to verify that a mapping class is nontrivial.

Definition 2.10. Let a and b be free (unbased) homotopy classes of simple closed curves. The *geometric intersection number* $i(a, b)$ is defined to be $i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a, \beta \in b\}$, the minimal number of intersection points taken over all curves in each class.

Immediately, we can see that $i(a, a) = 0$, and importantly that $i(a, b) > 1$ implies that a and b represent distinct classes. A natural worry that arises from this definition is that one may not know when a number of intersections is minimal. This next result gives us one solution.

Definition 2.11. Two transverse simple closed curves α and β in a surface S form a *bigon* if there is an embedded disc in S whose boundary is the union of an arc of α and an arc of β, with the arcs intersecting in two points.

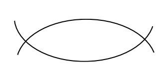

FIGURE 4. A bigon.

Lemma 2.12 (Bigon criterion)**.** *Let α and β be two transverse simple closed curves in a surface S. The number of intersection points is minimized if and only if they do not form a bigon.*

Proof. We omit the full proof for space. The result relies on the following lemma: if transverse simple closed curves α and β do not form any bigons, then in the universal cover of S, any pair of lifts of α and β intersect in at most one point. By assuming $\chi(S) \leq 0$, we know that the universal cover of S must be homeomorphic to $\mathbb{R}^2$ by the uniformization theorem. From there, we can use the Jordan curve theorem and the Brouwer fixed point theorem to finish the proof of the lemma. The proof of the theorem uses the classification of the isometry group of the hyperbolic plane. See Lemma 1.8 and Proposition 1.7 in [2]. □

We end the section by stating an important classical result: the mapping class group of the torus, which turns out to be the modular group: $\mathrm{SL}_2(\mathbb{Z})$. Based on this result, Fricke called the mapping class group the extended modular group, hence the notation $\mathrm{Mod}(S)$ chosen by Farb and Margalit [2].

Theorem 2.13. *Let T^2 be the torus. Then there exists an isomorphism $\phi : \mathrm{Mod}(T^2) \to \mathrm{SL}_2(\mathbb{Z})$.*

Proof. After choosing a representative, an element $\phi \in \mathrm{Mod}(T^2)$ is a homeomorhism of T^2, so it induces an automorphism on homology. This gives us a map $\mathrm{Mod}(T^2) \to \mathrm{GL}_2(\mathbb{Z})$. Via an explicit calculation of algebraic intersection numbers on a torus (see Section 1.2.3 in [2]) we get a map into $\mathrm{SL}_2(\mathbb{Z})$. We can then argue bijectivity of this map through covering space theory. For full proofs, see Theorem 7.2 in [6] or Theorem 2.5 in [2]. □

This result serves as the base case for the inductive proof of finite generation of $\mathrm{Mod}(S)$ for general surfaces. It can be shown that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ generate $\mathrm{SL}_2(\mathbb{Z})$, and after fixing a basis on $\mathbb{R}^2$, we can visualize $\phi(A)$ and $\phi(B)$ on the torus. We label two important curves on the torus in FIGURE: the green meridian curve and the red longitudinal one. It is easiest to visualize $\phi(A)$ and $\phi(B)$ through the action on these curves. Up to a choice of basis, $\phi(A)$ twists the a portion of the green curve around the orange curve once, as is shown in FIGURE. Similarly, $\phi(B)$ twists the a portion of the orange curve around the green curve in FIGURE.

The two elements $\phi(A)$ and $\phi(B)$ are examples of a more general kind of mapping class called a *Dehn twist*, which we formally define in Section 3. As mentioned earlier, A and B generate $\mathrm{SL}_2(\mathbb{Z})$, so we get the remarkable fact that $\phi(A)$ and $\phi(B)$ generate $\mathrm{Mod}(T^2)$.

2.1. **Excursions.** We touch on algebraic intersection numbers, symplectic representations, and the Torelli group, and then discuss a more powerful version of the Alexander trick, called the Alexander method.

Algebraic intersection number and symplectic representations. Instead of counting the number of unsigned intersection points through the geometric intersection number, we can do a signed count.

Definition 2.14. Let α and β be a pair of transverse, oriented, simple closed curves in S. The *algebraic intersection number* is the sum of the indices of the intersection points of α and β.

This definition makes sense for homology classes of closed curves, and thus gives a skew-symmetric alternating bilinear form on $H_1(S_g; \mathbb{Z})$. Now, we also know that there is a map $\psi : \mathrm{Mod}(S_g) \to \mathrm{Aut}(H_1(S_g; \mathbb{Z})) \cong \mathrm{GL}_{2g}(\mathbb{Z})$, as homeomorphisms induce automorphisms on homology. It is not so bad to show that the map is well-defined. What is more interesting is that the image of ψ must land in $\mathrm{SL}_{2g}(\mathbb{Z})$, as the homeomorphism must be orientation-preserving and preserve $H_1(S_g; \mathbb{Z})$ in $H_1(S_g; \mathbb{R})$. Since algebraic intersection number is also homeomorphism invariant, it means that the image $\psi(\mathrm{Mod}(S_g))$ actually lies in $\mathrm{Sp}_{2g}(\mathbb{Z})$, the group of integral symplectic matrices of dimension $2g$. The map $\psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is called the *symplectic representation* of the mapping class group.

It can be shown that this representation is surjective, and should be thought of as a first linear approximation of the mapping class group. This representation has a large, infinite-index kernel $\mathcal{I}(S_g)$ called the *Torelli group*. Many basic properties are still unknown for the Torelli group, such as finite presentability. It is still an active area of research today. We direct the interested reader to Chapter 6 in [2].

The Alexander method. The Alexander method bootstraps the idea used in the Alexander trick to be applicable in a more general setting. That is, we reduce the complexity of a certain orientation-preserving homeomorphism by arguing that it is determined by its action on certain curves and arcs.

Theorem 2.15 (Alexander method.)**.** *Let S be a compact surface possibly with marked points, and let $\phi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S with the following properties.*

(1) The γ_i are in pairwise minimal position.
(2) The γ_i are pairwise nonisotopic.
(3) For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, $\gamma_k \cap \gamma_j$ is empty.

Now, if there is a permutation σ of $\{1, \ldots, n\}$ so that $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\phi(\bigcup \gamma_i)$ is isotopic to $\bigcup \gamma_i$ relative to ∂S.

If we take $\bigcup \gamma_i$ as a graph Γ in S, with vertices being the intersection points and the endpoints of arcs, then the composition of ϕ with this isotopy gives an automorphism ϕ_* of Γ.

Suppose further that $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ has a nontrivial power isotopic to the identity.

For a proof of the theorem, see Section 2.8 in [2]. Though lengthy to state, this theorem is incredibly useful. It is used to show that for $g \geq 3$, the center of the mapping class group $Z(\mathrm{Mod}(S_g)) = 1$ is trivial. It can also be used in tandem with the bigon criterion (and finite presentability) to show that the mapping class group has solvable word problem. We direct the interested reader to Chapter 3 in [2].

3. Finite generation

In this section, we prove that $\mathrm{Mod}(S_{g,n})$ is finitely generated by Dehn twists about nonseparating curves for $g \geq 1$ and $n \geq 0$, a cornerstone result. There are two proofs of this result in [2], and they both follow a similar recipe. Both proofs use double induction on the genus and number of punctures. To attack the inductive step on the genus, we use a lemma that has a geometric group theoretic flavor. For the inductive step on the punctures, we use a versatile result called the *Birman exact sequence*, which relates $\mathrm{Mod}(S)$ and $\pi_1(S)$.

We have seen examples of Dehn twists in the example of the torus above. We formally define them here.

Definition 3.1. Let α be a simple oriented closed curve in S. Let N be a regular neighborhood of α. Let $A = S^1 \times [0, 1]$ denote the annulus. Let $T : A \to A$ be defined as $T(\theta, t) = (\theta + 2\pi t, t)$. Choose ϕ to be an orientation-preserving homeomorphism $\phi : A \to N$. Define

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & if x \in N \\ x & \text{if } x \notin N. \end{cases}$$

The map T_α is called the *Dehn twist about* α.

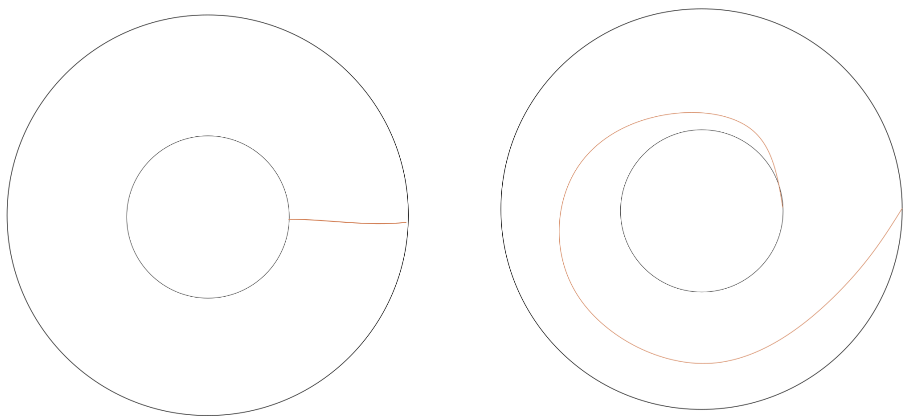

FIGURE 5. The image of a curve under a Dehn twist.

Now that we have defined Dehn twists, we check that they are nontrivial mapping classes. The proof utilizes the machinery developed above. We first use the change of coordinates principle to greatly simplify the curves about which we twist. Then, by calculating their geometric intersection number we notice that a curve b intersects its image $T_\alpha(b)$ once, which then implies the Dehn twist was indeed nontrivial (see 6).

Lemma 3.2. *Let α be a simple oriented closed curve in S that is not homotopic to a puncture or point on S. Then T_α is a nontrivial element of $\mathrm{Mod}(S)$.*

Proof. Since α can either be separating or nonseparating, there are two cases to check. Suppose α is separating. Then, applying the change of coordinates principle, we can find a simple closed curve β as in Figure 6 with $i(a, b) = 1$. We can then check using the bigon criterion that $i(b, T_\alpha(b)) = 1$, which means that the curves b and $T_\alpha(b)$ are distinct.

The separating case is similar. By again using the change of coordinates principle, one can take the separating curve to be slicing the surface between two genera. One can then select a curve b that goes through the genera, so we have $i(a, b) = 2$. Then, we can again draw out $T_\alpha(b)$ and use the bigon criterion to check that $i(b, T_\alpha(b)) = 4$. □

Of the two proofs of finite generation in [2], one constructs a $K(\pi, 1)$-space with a finite 2-skeleton, which allows us to jump to finite presentability. The other, which is outlined in this paper, allows us to directly check if Dehn twists about a

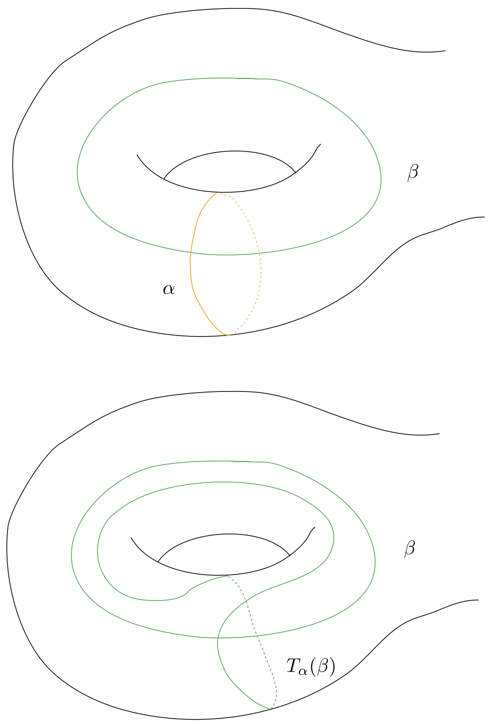

FIGURE 6. Dehn twist about a nonseparating curve

given set of curves in our surface S generate $\mathrm{Mod}(S)$. See the excursions section below for descriptions of explicit sets of generators.

Next, we prove the Birman exact sequence, which involves two maps. Let the pair (S, x) denote the surface S with a marked point x which lies in the interior of S. There is a map $\mathcal{F} : \mathrm{Mod}(S, x) \to \mathrm{Mod}(S)$ by forgetting that x was marked. This map is naturally surjective, and it turns out that the kernel of $\mathcal{F}$ is exactly $\pi_1(S, x)$, the fundamental group of S with basepoint x.

To help motivate this result, we discuss the map $\mathcal{P} : \pi_1(S, x) \to \mathrm{Mod}(S)$, descriptively named the push map. A loop a is a map $a : [0, 1] \to S$. We can think about this as an isotopy of points, and again by the isotopy extension theorem, this can be extended to an ambient isotopy of the whole surface S. Since a loop has the property that $a(0) = a(1)$,taking the ambient isotopy at $t = 1$ gives a homeomorphism ϕ of S. Taking the isotopy class of ϕ, we are left with a mapping class. In this way, a loop based at x can be thought of as the trail left behind x as it moves around on our surface S. It is important to note that this informal discussion does not give a well-defined map. We would want homotopic loops to give isotopic homeomorphisms, and to make sure that this does not depend on the choice of ambient isotopy.

Before proceeding to the proof of the Birman exact sequence, it is helpful to have a more concrete visual image of the image of a homotopy class of loops under the push map.

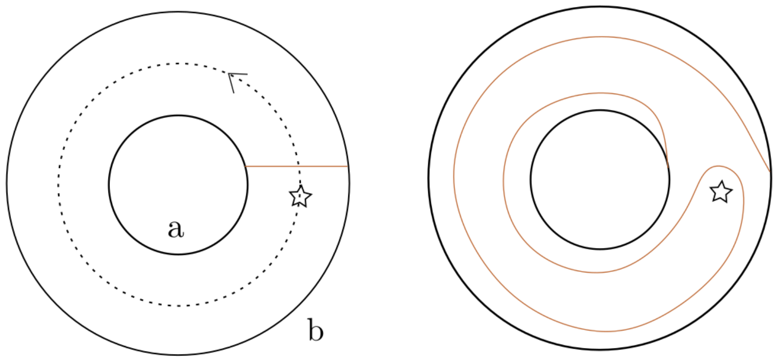

FIGURE 7. The image of a push map realized as Dehn twists.

Lemma 3.3. *Let α be a simple loop in a surface S representing an element of $\pi_1(S)$. Then $\mathcal{P}([\alpha]) = T_a T_b^{-1}$, the product of two Dehn twists. Here a and b are isotopy classes of the boundary of the regular neighborhood about α (see 7).*

For a more complete discussion, see Section 4.2.2 in [2]. In addition to providing useful visual intuition, this result is also used in the proof of finite generation when inducting on the number of punctures, as we want to lift elements of the kernel to products of Dehn twists.

Lemma 3.4 (Birman exact sequence)**.** *Let S be a surface with $\chi(S) < 0$ possibly with punctures or boundary. Let (S, x) be the surface obtained by marking a point x in the interior of S. The following sequence is exact.*

$$1 \to \pi_1(S, x) \xrightarrow{\mathcal{P}} \mathrm{Mod}(S, x) \xrightarrow{\mathcal{F}} \mathrm{Mod}(S) \to 1$$

Proof. We claim there is a fiber bundle $\mathrm{Homeo}^+(S) \xrightarrow{\mathcal{E}} S$ with fiber $\mathrm{Homeo}^+(S, x)$, the subgroup fixing x pointwise, with the map $\mathcal{E}$ being the evaluation at x. To verify this, we must show that for any neighborhood U of S, there is some homeomorphism from $U \times \mathrm{Homeo}^+(S, x)$ to $\mathcal{E}^{-1}(U)$.

Let V be a neighborhood of x homeomorphic to a disc. Since for any two points x and y on S, we can find a homeomorphism taking x to y, we choose $v \in V$ without loss of generality. Given a point $v \in V$, we can find some ϕ_v with $\phi_v(x) = u$. This assignment gives us a homeomorphism $U \times \mathrm{Homeo}^+(S, x) \to \mathcal{E}^{-1}(V)$ via $(v, \psi) \mapsto \phi_v \circ \psi$. The inverse of this map is $\psi \mapsto (\psi(x), \phi_{\psi(x)}^{-1} \circ \psi)$.

Now, the relevant part of the associated long exact sequence in homotopy groups is

$$\pi_1(\mathrm{Homeo}^+(S)) \to \pi_1(S) \to \pi_0(\mathrm{Homeo}^+(S, x)) \to \pi_0(\mathrm{Homeo}^+(S)) \to \pi_0(S).$$

It is a result of Hamstrom [3] that $\pi_1(\mathrm{Homeo}^+(S)) = 1$ in this case (see Theorem 1.14 in [2] for a precise statement), and since S is connected, $\pi_0(S) = 1$, so we are finished. The maps in the long exact sequence are exactly those described above. For a proof of the long exact sequence of homotopy groups arising from a fiber bundle, see Section 4.2 in [1]. □

The main result of the section is actually slightly stronger than previously mentioned. We will prove that the *pure mapping class group* of a surface $S_{g,n}$ with $g \geq 1$ and $n \geq 0$ is generated by finitely many Dehn twists about nonseparating curves.

Definition 3.5. The *pure mapping class group* $\mathrm{PMod}(S_{g,n})$ is the subgroup of the $\mathrm{Mod}(S_{g,n})$ that fixes all punctures individually.

It is important that the Birman exact sequence still holds for the pure mapping class group – the proof is almost identical.

Next, we prove the aforementioned lemma that uses fundamental ideas from geometric group theory to aid us in inducting on the genus. This result urges us to construct such a complex for $\mathrm{Mod}(S)$, which we do below.

Lemma 3.6. *Suppose that a group G acts by simplicial automorphisms on a connected, 1-dimensional simplicial complex X. Suppose further that G acts transitively on the vertices of X and that it also acts transitively on pairs of vertices of X that are connected by an edge. Let v, w be two adjacent vertices and let $h \in G$ such that $h(w) = v$. Then G is generated by h and $\mathrm{Stab}_G(v)$, the stabilizer of v in G.*

Proof. Let H denote the subgroup generated by h and $\mathrm{Stab}_g(v)$. Let $g \in G$ be arbitrary. Since X is connected, there is a path of vertices $v = v_0, v_1, \ldots, v_n = g(v)$. Since the action of G is transitive on vertices, there is some $g_i \in G$ such that $g_i(v) = v_i$. We can take g_0 to be the identity. Now, we show that $g_{i+1} \in H$ by induction. Consider the edge $(v_i, v_{i+1}) = (g_i(v), g_{i+1}(v))$. If we multiply by g_i^{-1}, we get the edge $(v, g_1^{-1} g_{i+1}(v))$. By transitivity on edges, there is some $r \in G$ such that $(rv, rg_1^{-1}g_{i+1}(v)) = (v, w)$. This shows that $r \in \mathrm{Stab}_G(v)$, and so $r \in H$. By assumption, we have that $h(w) = v$, so $hrg_1^{-1}g_{i+1}(v) = v$. This shows that $hrg_1^{-1}g_{i+1} \in \mathrm{Stab}_G(v)$, and since h,r, and g_i^{-1} are in H, so is g_{i+1}. □

Definition 3.7. Let $\widehat{\mathcal{N}}(S_{g,n})$ denote the 1-dimensional simplicial complex defined with vertices corresponding to isotopy classes of nonseparating simple closed curves in $S_{g,n}$. Two vertices a and b are connected by an edge if $i(a, b) = 1$.

Lemma 3.8. *If $g \geq 2$ and $n \geq 0$, then $\widehat{\mathcal{N}}(S_{g,n})$ is connected.*

Proof. This relies on the connectedness of another similarly constructed simplicial complex, called the curve complex. The curve complex is defined identically to $\widehat{\mathcal{N}}(S_{g,n})$, except that two vertices are joined by an edge if their geometric intersection number is 0. For a proof of the connectedness of the curve complex, see Section 4.1 in [2].

If we assume connectedness of the curve complex, we get a path of vertices $a = p_0, p_1, \ldots, p_n = b$ with $i(p_i, p_{i+1}) = 0$. By the change of coordinates principle, we can find a nonseparating curve d_i such that $i(p_i, d_i) = 1 = i(p_i, d_{i+1})$. The path $a = p_0, d_0, p_1, d_1, \ldots, d_{n-1}, p_n = b$ then gives us the desired connecting path in $\widehat{\mathcal{N}}(S_{g,n})$. □

We can visualize the curves d_i as the orange curves in Figure 8. Now that we have constructed a complex, we check that the action of $\mathrm{Mod}(S)$ complies with the previous lemma.

Lemma 3.9. *Let $g \geq 2$ and $n \geq 0$. The action of $\mathrm{Mod}(S_{g,n})$ on $\widehat{\mathcal{N}}(S_{g,n})$ satisfies the assumptions in Lemma 3.6.*

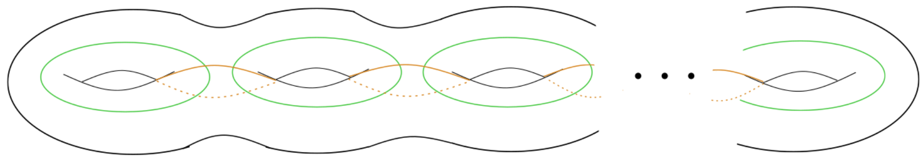

FIGURE 8. Linking the curves.

Proof. By Lemma 3.8, we have that $\widehat{\mathcal{N}}(S_{g,n})$ is connected. The change of coordinates principle says that given two pairs of isotopy classes of curves (a, b) and (a', b'), with $i(a, b) = 1 = i(a', b')$, there is some orientation-preserving homeomorphism ϕ such that $\phi(a) = a'$ and $\phi(a') = b'$. That is, ϕ represents a class in $\mathrm{Mod}(S_{g,n})$ such that $\phi((a, b)) = (a', b')$. Thus, the action is transitive. □

Finally, we state a lemma that elucidates the kernel of a map between the mapping class groups of a surface S and the corresponding cut surface $S - a$. When combined with our earlier lemma, this result will allow us to complete the inductive step on the genus.

Lemma 3.10. *Let a be an isotopy class of essential simple closed curves. There is a well defined homomorphism*

$$\zeta : \mathrm{Stab}_{\mathrm{Mod}(S)}(a) \to \mathrm{Mod}(S - a)$$

with kernel $\langle T_a \rangle$, where $S - \alpha$ is the surface obtained by cutting S along α.

Proof. We omit this proof for space as it relies on a number of results involving explicit calculations with Dehn twists. For a proof see Section 3.6 in [2]. □

With that, we are finally ready to state and prove the main result of the section.

Theorem 3.11. *Let $S_{g,n}$ be a surface with $g \geq 1$ and $n \geq 0$. The group $\mathrm{PMod}(S_{g,n})$ is finitely generated by Dehn twists about nonseparating simple closed curves.*

Proof. We induct on the number of punctures and the genus. We have already seen that $S_{1,0} = T^2$ and $S_{1,1}$ have mapping class groups that are finitely generated by Dehn twists about nonseparating simple closed curves.

First, we induct on the number of punctures. Assume that $n \geq 1$, since we have already verified the case of $S_{1,1}$, the once-punctured torus. With this extra assumption, we calculate to see that $\chi(S_{g.n}) < 0$, so the Birman exact sequence holds. Assume that $\mathrm{PMod}(S_{g,n})$ is finitely generated and consider $\mathrm{PMod}(S_{g,n+1})$. Applying Lemma 3.4, we have that the map $\mathcal{F} : \mathrm{PMod}(S_{g,n+1}) \to \mathrm{PMod}(S_{g.n})$ is surjective with kernel $\pi_1(S_{g,n})$. By assumption, $\mathrm{PMod}(S_{g,n})$ is finitely generated, and by surjectivity, we can choose a lift of each of the generators of $\mathrm{PMod}(S_{g,n})$. It remains to show that we can generate $\pi_1(S_{g,n})$ by Dehn twists about nonseparating simple closed curves. Using the classificiation of surfaces, it can be shown that $\pi_1(S_{g,n})$ is finitely generated by simple nonseparating loops. By Lemma 3.3, each of these can be written in terms of Dehn twists about nonseparating simple closed curves. Thus, we get a finite generating set for $\mathrm{PMod}(S_{g,n+1})$ by taking the Dehn twists corresponding to the generators of $\pi_(S_{g,n})$, and by lifting each generator of $\mathrm{PMod}(S_{g,n})$.

Next, we induct on genus. Assume $\mathrm{PMod}(S_{g-1})$ is finitely generated by Dehn twists about nonseparating simple closed curves and consider $\mathrm{PMod}(S_g)$. The cases of $g = 0$ and $g = 1$ have been taken care of, so we can assume that $g \geq 2$, which means that Lemma 3.9 holds. As in the proof of Lemma 3.9, let a and b be two isotopy classes of nonseparating simple closed curves with $i(a, b) = 1$. Since $T_bT_a(b) = a$, we get that $\mathrm{Mod}(S_g)$ is generated by T_a, T_b, and $\mathrm{Stab}_{\mathrm{Mod}(S_g)}(a)$. Thus, it suffices to show that $\mathrm{Stab}_{\mathrm{Mod}(S_g)}(a)$, hereby denoted by $\mathrm{Stab}(a)$, is finitely generated by Dehn twists about nonseparating simple closed curves. Let $\vec{a}$ be a endowed with a chosen orientation. It suffices to show that $\mathrm{Stab}(\vec{a})$ is finitely generated as $T_bT_a^2T_b$ switches the orientation of $\vec{a}$. By Lemma 3.10, we get that $\mathrm{PMod}(S_g - a) \cong \mathrm{Stab}(\vec{a})/\langle T_a \rangle$, but $S_g - a$ is homeomorphic to a surface with genus one less than S_g, so by the inductive hypothesis, we have that $\mathrm{PMod}(S_g - a)$ is finitely generated by Dehn twists about nonseparating simple closed curves. Since this map is naturally surjective, by lifting as we did before, we get that $\mathrm{Stab}(\vec{a})$ is finitely generated by Dehn twists about nonseparating simple closed curves. □

3.1. **Excursions. More with Dehn twists.** There is a lot that can be done with Dehn twists. For instance, they are used in understanding the induced maps on mapping class groups given an inclusion of surfaces. In fact it is possible to totally classify all possible relations between two Dehn twists. See Section 3.4 in [2].

Humphries' generators. In 1979, Humphries proved that $2g + 1$ curves suffice to generate $\mathrm{Mod}(S_g)$. Using the aforementioned symplectic representation one can show that this result is sharp in that no fewer curves suffice. For more information see Chapters 4 and 6 in [2].

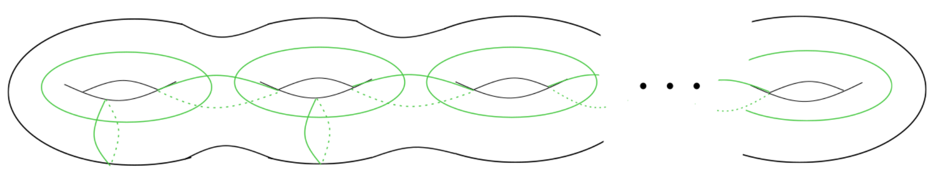

FIGURE 9. The $2g + 1$ Humphries generators.

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