RANDOM FRACTALS FROM THE SAMPLE PATHS OF BROWNIAN MOTION

AVERY FOX

ABSTRACT. In this paper, we analyze some of the random fractals obtained from Brownian motion. In particular, we turn to the notion of Hausdorff dimension which provides us with almost sure dimensions of these sets. We first outline relevant Brownian motion results that describe its almost sure behavior, taking a special interest in theorems related to planar Brownian motion. The main section of our paper examines a set of points known as α-cone points, or points when a planar Brownian motion stays within a cone with angle α with a vertex along its path. We prove that α-cone points exist only when $\alpha \geq \pi$ and that for such α the dimension of α-cone points is almost surely $2 - \frac{2\pi}{\alpha}$.

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1. Introduction

Brownian motion is a random continuous function originally intended to model the random movements of particles and therefore has many different applications to many fields including the sciences and economics. For instance, in biology, Brownian motion may be used to calculate the average exit time of a particle from a domain with a reflecting boundary except for a small opening, known as the narrow escape problem. On the other hand, in finance, it may model the fluctuations of stock prices.

The scaling invariance property of Brownian motion is useful when looking at sets obtained from the sample paths of Brownian motion, implying that the Brownian motion has similar geometric structures regardless of scale and can therefore be seen as a random fractal. We may therefore use tools from fractal geometry, part of geometric measure theory, to analyze these sets.

In this paper, we analyze some of the sets obtained from sample paths of Brownian motion by viewing them as random fractals. We assume that the reader has familiarity with real analysis and measure-theoretic probability. We also assume familiarity with basic properties of Brownian motion and stochastic calculus as seen in [2] or [4].

Using the Markov property and the strong Markov property of Brownian motion, we first look at some of their applications that will become relevant in later sections of the paper. We then develop Hausdorff measure and Hausdorff dimension, which is the main tool from geometric measure theory we will use to analyze the fractal nature of sets obtained from the sample paths of Brownian motion. Then, we begin to look at the specific case of planar Brownian motion and some of its typical behaviors, or those events that occur almost surely. For example, we prove Spitzer's Law, Theorem 4.13, to build a foundation for understanding an example of an exceptional set arising from the sample paths of planar Brownian motion. When we say exceptional here, we mean a set comprised of points with a certain property in which a fixed point along the sample path almost surely doesn't have this property. From here, we turn to the main section of our paper which analyzes such a family of exceptional sets arising from planar Brownian motion known as α-cone points for $\alpha \in (0, 2\pi)$ through the lens of Hausdorff dimension. We prove a theorem by Steven Evans, Theorem 5.16, that α-cone points exist only when $\alpha \geq \pi$. Moreover, for such α, the Hausdorff dimension of these points is almost surely $2 - \frac{2\pi}{\alpha}$. We end the paper with a few applications of this theorem to the convex hull of a planar Brownian motion and the intersection of a planar Brownian motion with a line.

2. Applications of the Markov and Strong Markov Properties

This section will cover some theorems in Brownian motion that will become relevant in our discussion later on. We begin by introducing the **reflection principle**, a useful tool in studying Brownian motion.

Theorem 2.1. *Let T be a stopping time and $\{B(t) : t \geq 0\}$ be a standard linear Brownian motion. Then, the process $\{B'(t) : t \geq 0\}$ defined by*

$$B'(t) = B(t)\chi_{t \leq T} + (2B(T) - B(t))\chi_{t \geq T},$$

known as a ***reflected Brownian motion****, is also a standard linear Brownian motion.*

Proof. For finite T, the strong Markov property implies that $\{B(t+T)-B(T) : t \geq 0\}$ and $\{-(B(t+T)-B(T)) : t \geq 0\}$ are Brownian motions. Moreover, both are independent of $\{B(t) : 0 \leq t \leq T\}$. Therefore, the processes $B(t)\chi_{t\leq T}+(B(t+T)-B(T)+B(T))\chi_{t\geq T}$ and $B(t)\chi_{t\leq T}+(-B(t+T)+B(T)+B(T))\chi_{t>T}$ have the same distribution. This implies the theorem as the first processes is $\{B(t) : t \geq 0\}$ and the second is $\{B'(t) : t \geq 0\}$ and because the concatenation we performed made a new continuous and measurable path. □

The reflection principle can be used to analyze the maximum process of a standard linear Brownian motion $\{B(t) : t \geq 0\}$, or the process $\{M(t) : t \geq 0\}$ defined by $M(t) = \sup_{u\in[0,t]} B(u)$, as we do in this next theorem.

Theorem 2.2. *Let $a > 0$. Then, for a standard linear Brownian motion $\{B(t) : t \geq 0\}$,*

$$\mathbb{P}\{M(t) > a\} = 2\mathbb{P}\{B(t) > a\} = \mathbb{P}\{|B(t)| > a\}.$$

Proof. Let $T = \inf\{t \geq 0 : B(t) = a\}$ and consider the Brownian motion $\{B'(t) : t \geq 0\}$ reflected at T. We note that the event $\{M(t) > a\} = \{B(t) > a\} \cup \{M(t) > a, B(t) \leq a\}$ and that this is a disjoint union of events. The event $\{M(t) > a, B(t) \leq a\} = \{B'(t) \geq a\}$ by definition. The reflection principle completes the proof. □

The maximum process is one of several processes we can define derive from Brownian motion. Before we introduce another process, we need to first define a couple of terms.

Definition 2.3. A function $p : [0,\infty) \times \mathbb{R}^d \times \mathcal{B} \to \mathbb{R}$ where $\mathcal{B}$ is the Borel sigma algebra over $\mathbb{R}^d$ is called a **Markov transition kernel** if the following conditions hold.

(1) For all $A \in \mathcal{B}$, $p(\cdot,\cdot,A)$ is a measurable function of (t,x).
(2) For all $t \geq 0$ and $x \in \mathbb{R}^d$, $p(t,x,\cdot)$ is a Borel probability measure on $\mathbb{R}^d$. The integral of a function f with respect to this measure is written as

$$\int f(y)p(t,x,dy).$$

(3) For all $A \in \mathcal{B}$, $x \in \mathbb{R}^d$, and $t,s > 0$, we have that

$$\int_{\mathbb{R}^d} p(t,y,A)p(s,x,dy) = p(t+s,x,A).$$

Definition 2.4. A **Markov process** is an adapted process $\{X(t) : t \geq 0\}$ with respect to a filtration $(\mathcal{F}(t) : t \geq 0)$ and transition kernel p if for all $t \geq s \geq 0$ and for every Borel set $A \subset \mathbb{R}^d$,

$$\mathbb{P}\{X(t) \in A|\mathcal{F}(s)\} = p(t-s, X(s), A).$$

Loosely speaking, $p(t,x,A)$ is the probability that our process takes a value in A at time t given that the process starts at the point x.

Example 2.5. The Markov transition kernel of a d-dimensional Brownian motion is notated by probability measures $\mu(t,x,\cdot)$ and has a density function

$$f(t,x,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(\frac{-|x-y|^2}{2t}\right).$$

As an example of how the Markov transition kernel can be used to analyze Brownian motion, we go back to looking at the maximum process of a standard linear Brownian motion and define a new process using the maximum and the Brownian motion itself.

Theorem 2.6. *Consider the maximum process $\{M(t) : t \geq 0\}$ of a standard linear Brownian motion $\{B(t) : t \geq 0\}$. Then, the process $\{Y(t) : t \geq 0\}$ defined by $Y(t) = M(t) - B(t)$ has the same distribution as a reflected Brownian motion.*

Proof. Fix $s > 0$. We will consider the two processes defined for $t \geq 0$: $B'(t) = B(s+t) - B(s)$ and $M'(t) = \max_{u \in [0,t]} B'(u)$. Note that if we define $\mathcal{F}^+(s) = \bigcap_{t>s} \sigma(B(u) : 0 \leq u \leq t)$, then $Y(s)$ is measurable with respect to $\mathcal{F}^+(s)$, so to show the conclusion, it suffices to show that $Y(s+t)$ has the same distribution as $|Y(s) + B'(t)|$ conditional on $\mathcal{F}^+(s)$. This directly implies that $\{Y(t) : t \geq 0\}$ is a Markov process with the same Markov transition kernel as a reflected Brownian motion, therefore implying that their finite dimensional distributions agree.

First, we have that for $t \geq 0$,

$$M(s+t) = \max(M(s), (B(s) + M'(t)))$$

so we may rewrite

$$Y(s+t) = \max(M(s), (B(s) + M'(t))) - (B(s) + B'(t)) = \max(Y(s), M'(t)) - B'(t).$$

This implies that it suffices to show that for all $y \geq 0$, $\max(y, M'(t)) - B'(t))$ has the same distribution as $|y + B'(t)|$. Fix $y, a \geq 0$ and define

$$P_1 = \mathbb{P}\{y - B'(t) > a\} = \mathbb{P}\{y + B'(t) > a\}$$

where the equality comes from the fact that $\{B'(t) : t \geq 0\}$ has the same distribution as $\{-B'(t) : t \geq 0\}$. We also define

$$P_2 = \mathbb{P}\{y - B'(t) \leq a, M'(t) - B'(t) > a\}$$

so $\mathbb{P}\{\max(y, M'(t) - B'(t)) > a\} = P_1 + P_2$. We also define the process $\{W(u) : 0 \leq u \leq t\}$ by $W(u) = B'(t-u) - B'(t)$, which upon computation can be seen to be a Brownian motion for $0 \leq u \leq t$. Let $M_W(t) = \max_{u \in [0,t]} W(u)$, so $M_W(t) = M'(t) - B'(t)$. Note as well that $W(t) = -B'(t)$ so

$$P_2 = \mathbb{P}\{y + W(t) \leq a, M_W(t) > a\}.$$

Applying the reflection principle to the first time $W(u)$ hits a gives us another Brownian motion W^* and we may rewrite

$$P_2 = \mathbb{P}\{W^*(t) \geq a + y\} = \mathbb{P}\{y + B'(t) \leq -a\}$$

because $\{W^*(t) : t \geq 0\}$ has the same distribution as $\{B'(t) : t \geq 0\}$. Thus, given that $B'(t)$ has a continuous distribution, we get that

$$P_1 + P_2 = \mathbb{P}\{|y + B'(t)| > a\}.$$

We've therefore proven that for all $y \geq 0$, $\max(y, M'(t)) - B'(t))$ has the same distribution as $|y + B'(t)|$, completing our proof. □

This next theorem describes a process known as the **stable subordinator** of index $\frac{1}{2}$.

Theorem 2.7. *For all $a \geq 0$, define stopping times*

$$T_a = \inf\{t \geq 0 : B(t) = a\}.$$

Then, $\{T_a : a \geq 0\}$ is an increasing Markov process with a transition kernel given by the densities

$$p(a,t,s) = \frac{a}{\sqrt{2\pi(s-t)^3}} \exp\left(-\frac{a^2}{2(s-t)}\right) \chi\{s > t\}.$$

Proof. Fix $a \geq b \geq 0$. Note that for all $t \geq 0$,

$$\{T_a - T_b = t\} = \{B(T_{b+s}) - B(T_b) < a - b \text{ for } s < t, B(T_b + t) - B(T_b) = a - b\}.$$

By the strong Markov property, this event is independent of $\{T_d : d \leq b\}$, therefore implying that the process $\{T_a : a \geq 0\}$ has the Markov property. On the other hand, by the reflection principle and the prior results in this section,

$$\mathbb{P}\{T_a - T_b \leq t\} = \mathbb{P}\{T_{a-b} \leq t\} = \mathbb{P}\{\max_{s \in [0,t]} B(s) \geq a - b\} = 2\mathbb{P}\{B(t) \geq a - b\} = 2\int_{a-b}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx.$$

Therefore, by performing the substitution $x = \frac{t}{s}(a-b)$ in the integral, we have the desired transition kernel densities. □

Planar Brownian motion has an additional Markov process, known as a **Cauchy process**.

Theorem 2.8. *Let $\{B(t) : t \geq 0\}$ for $B(t) = (B_1(t), B_2(t))^T$ be a planar Brownian motion. Consider vertical lines defined by $V(a) = \{(x,y) \in \mathbb{R}^2 : x = a\}$ for $a \geq 0$ and let $T(a)$ be the first hitting time of $V(a)$. Then, the process $X(a) = B_2(T(a))$ is a Markov process with Cauchy probability density functions for each fixed $a \geq 0$. A Cauchy probability density function is given by*

$$f(x) = \frac{1}{\pi} \cdot \frac{a}{a^2 + x^2}.$$

Proof. The Markov property of the Cauchy process is implied from the strong Markov property applied with the stopping times $T(a)$ and because for all $a < b$, $T(a) < T(b)$. Note as well that $T(a)$ is independent of the Brownian motion $\{B_2(s) : s \geq 0\}$. Therefore, because of this property and Theorem 2.7, it follows that the density of $B_2(T(a))$ is

$$\int_0^\infty \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) ds.$$

Making the substitution $z = \frac{1}{2s}(a^2 + x^2)$, this integral is equal to

$$\int_0^\infty \frac{a}{\pi(a^2 + x^2)} e^{-z} dz = \frac{a}{\pi(a^2 + x^2)}$$

completing the proof. □

3. Hausdorff Dimension and Initial Brownian Motion Applications

In the next three definitions, we let X be a subset of a metric space equipped with a metric d.

Definition 3.1. The **diameter** of X, which we denote by $|X|$, is defined by $|X| = \sup_{x,y \in X} d(x, y)$.

Definition 3.2. Let $s \geq 0$ and $\delta > 0$. We define

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{n=1}^{\infty} |E_i|^s : X \subset \bigcup_{i=1}^{\infty} E_i, \text{ for all } i \in \mathbb{N} \ |E_i| \leq \delta\right\}.$$

From this, we can define the **s-dimensional Hausdorff measure**, $\mathcal{H}^s$, of X to be

$$\mathcal{H}^s(X) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(X).$$

The limit is well-defined because as δ decreases to 0, there are less possible coverings, meaning that $\mathcal{H}^s_\delta(X)$ can be viewed as a monotonic function. As is, the s-dimensional Hausdorff measure is an outer measure but Carathéodory's extension theorem may be applied to restrict it to a measure. In this paper, we will work with the measure as is, i.e. as an outer measure.

Definition 3.3. The **Hausdorff dimension** of X is

$$\dim(X) = \inf\{s : \mathcal{H}^s(X) = 0\}.$$

For example, for any $d \in \mathbb{N}$, $\dim(\mathbb{R}^d) = d$ and the dimension of the standard middle-thirds Cantor set is $\frac{\log(2)}{\log(3)}$. Interested readers may consult chapter 4 of [13] for explicit proofs.

Theorem 3.4. *Hausdorff dimension satisfies the* **countable stability property**, *meaning that if* $A_1, A_2, ...$ *is a countable sequence of sets with* $A = \bigcup_{n=1}^\infty A_n$, *then* $\sup_{n\in\mathbb{N}} \dim(A_n) = \dim(A)$.

Proof. By monotonocity, $\dim(A) \geq \dim(A_n)$ for all $n \in \mathbb{N}$ so $\sup_{n\in\mathbb{N}} \dim(A_n) \leq \dim(A)$. We also have that if $s > \dim(A_n)$ for all $n \in \mathbb{N}$, then $\mathcal{H}^s(A_n) = 0$ for all $n \in \mathbb{N}$, meaning that $\mathcal{H}^s(A) = 0$, proving that $\sup_{n\in\mathbb{N}} \dim(A_n) \geq \dim(A)$. □

From the definition of Hausdorff dimension, oftentimes, it is more clear and straightforward to find an upper bound on the Hausdorff dimension of a set. Therefore, we will now prove a theorem that gives us a lower bound on a metric space's Hausdorff dimension, called the **mass distribution principle**.

Definition 3.5. Let X be a metric space and let μ be a measure on the Borel sigma algebra on X. We call μ a **mass distribution** on X if

$$0 < \mu(X) < \infty.$$

Theorem 3.6. *Let X be a metric space and let $s \geq 0$. Assume that μ is a mass distribution on X and there are constants $C > 0$ and $\delta > 0$ such that for all closed sets F with $|F| \leq \delta$ we have*

$$\mu(F) \leq C \cdot |F|^s.$$

Then we have that

$$\mathcal{H}^s(X) \geq \frac{\mu(X)}{C} > 0$$

and $\dim(X) \geq s$.

Proof. Let $E_1, E_2, ...$ be a cover of X such that for all $n \in \mathbb{N}$, $|E_n| \leq \delta$. Consider the closures of each of these sets; by definition, for all $n \in \mathbb{N}$, we have that $|E_n| = |\overline{E_n}|$. It follows that

$$0 < \mu(X) \leq \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \mu\left(\bigcup_{n=1}^{\infty} \overline{E_n}\right) \leq \sum_{n=1}^{\infty} \mu\left(\overline{E_n}\right) \leq C \sum_{n=1}^{\infty} |E_n|^s$$

so taking the infimum over all possible covers and considering the limit as δ decreases to 0 gives us the first result. The second result follows from the first result and the definition of Hausdorff dimension. □

We now turn to an application of the mass distribution principle and Theorem 2.6, but first we need to define one additional term.

Definition 3.7. Let $\{B(t) : t \geq 0\}$, $\{M(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$ be as in Theorem 2.6. Then, we say that $t \geq 0$ is a **record time of** $\{B(t) : t \geq 0\}$ if $Y(t) = M(t) - B(t) = 0$.

Lemma 3.8. *Let Z be the set of zeroes of a linear Brownian motion. Then, almost surely,*

$$\dim(Z \cap [0,1]) \geq \frac{1}{2}.$$

Proof. We use the notation from Definition 3.7. Let R be the set of record times of the Brownian motion. By Theorem 2.6, as the distribution of record times is the same as the distribution of zeroes for a linear Brownian motion, it suffices to show that $\dim(R \cap [0,1]) \geq \frac{1}{2}$. By definition of a maximum process, the function $f : [0,1] \longrightarrow \mathbb{R}$ defined by $f(t) = M(t)$ is increasing and almost surely continuous. It therefore induces a measure μ defined by $\mu(a,b] = M(b) - M(a)$ which is supported on the Borel subsets of $[0,1]$, including R, a closed set almost surely. We also have that almost surely, if $\alpha < \frac{1}{2}$, Brownian motion is locally α-Hölder continuous. For each such α, there exists some C_α such that almost surely, for all $0 \leq a \leq b \leq 1$,

$$M(b) - M(a) \leq \max_{0 \leq \epsilon \leq b-a} B(a+\epsilon) - B(a) \leq C_\alpha (b-a)^\alpha.$$

Therefore, by the mass distribution principle, we have that

$$\dim(R \cap [0,1]) \geq \alpha.$$

As this holds for all $\alpha < \frac{1}{2}$, $\dim(R \cap [0,1]) \geq \frac{1}{2}$. □

In fact, we can also prove the reverse inequality.

Lemma 3.9. *Let Z be the set of zeroes of a standard linear Brownian motion. Then, almost surely,*

$$\dim(Z \cap [0,1]) \leq \frac{1}{2}.$$

Proof. We again use the same notation as in Definition 3.7. First, we prove that for all $x, \epsilon > 0$,

$$\mathbb{P}\{\exists t \in (x, x+\epsilon) : B(t) = 0\} \leq \sqrt{\frac{\epsilon}{x+\epsilon}}.$$

To do so, for all $y \in \mathbb{R}$ we define $f(y)$ to be the probability that there's at least one zero in $(x, x+\epsilon)$ given that $B(x) = y$, and fix $x, \epsilon > 0$. The probability $f(y)$ is equal to the probability that there's a zero in $(0, \epsilon)$ given that $B(0) = y$ by the Markov property. Therefore, we obtain that

$$\mathbb{P}\{\exists t \in (x, x+\epsilon) : B(t) = 0\} = \sqrt{\frac{2}{\pi x}} \int_0^\infty f(x) e^{-\frac{y^2}{2x}} dy.$$

By Theorem 2.2, it may be seen that for $y \neq 0$, the probability there's a zero in $(0,t)$ given that the Brownian motion starts at y is equal to

$$\frac{|y|}{\sqrt{2\pi}} \int_0^t z^{-\frac{3}{2}} e^{-\frac{y^2}{2z}} dz.$$

Therefore, plugging this probability in gives us that $\mathbb{P}\{\exists t \in (x, x+\epsilon) : B(t) = 0\} = \frac{2}{\pi} \arccos\left(\sqrt{\frac{x}{x+\epsilon}}\right) \leq 1$. The first part of the proof follows from the fact that $0 \leq \frac{x}{x+\epsilon} \leq 1$. Note that the same upper bound applies for the probability that there's a zero in the closed interval $[x, x+\epsilon]$ and that we may extend this result to closed intervals of the form $[0, \epsilon]$.

We now fix some $n \in \mathbb{N}$ and consider the number of intervals of the form $[\frac{k-1}{2^n}, \frac{k}{2^n}]$ where $k \in \mathbb{N}$ and $k \leq 2^n$ that have a nonempty intersection with Z, which we'll call $Z(n)$. We may verify that the expected number of these intervals is at most

$$\sum_{k=1}^{2^n} \frac{1}{\sqrt{k}} \leq 4 \cdot 2^{\frac{n}{2}}.$$

We therefore have that for $\epsilon > 0$, by the monotone convergence theorem and the geometric series test,

$$\mathbb{E}\left[\sum_{n=1}^{\infty} Z(n) \cdot \left(\frac{1}{2^n}\right)^{\frac{1}{2}+\epsilon}\right] = \sum_{n=1}^{\infty} \frac{\mathbb{E}[Z(n)]}{2^{n(\frac{1}{2}+\epsilon)}} \leq \sum_{n=1}^{\infty} \frac{4 \cdot 2^{\frac{n}{2}}}{2^{\frac{n}{2}} \cdot 2^{n\epsilon}} < \infty.$$

Therefore, we have that

$$\lim_{n\to\infty} \frac{\mathbb{E}[Z(n)]}{2^{n(\frac{1}{2}+\epsilon)}} = 0$$

implying that $\dim(Z \cap [0,1]) \leq \frac{1}{2} + \epsilon$. Sending ϵ to 0 finishes the proof. □

We have therefore proven the following theorem.

Theorem 3.10. *Let Z be the set of zeroes of a linear Brownian motion. Then, almost surely,*

$$\dim(Z \cap [0,1]) = \frac{1}{2}.$$

Another technique for finding the lower bound of the Hausdorff dimension is called the **energy method** and is similar to the mass distribution principle.

Definition 3.11. Let μ be a mass distribution on a metric space (X, m) and let $\alpha \geq 0$. We define the **α-potential** of $x \in X$ with respect to μ to be

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{m(x,y)^\alpha}$$

and the **α-energy** of μ to be

$$I_\alpha(\mu) = \int \phi_\alpha(x) d\mu(x) = \int\int \frac{d\mu(x)d\mu(y)}{m(x,y)^\alpha}.$$

The energy method replaces the condition that the measure of a set is bounded above by a constant multiplied by a power of its diameter; instead, it uses the idea that if the α-energy of μ is finite, then the mass is spread so that everywhere the concentration of mass is sufficiently small and therefore may overcome the integrand's singularity.

Theorem 3.12. *Let $\alpha \geq 0$ and let μ be a mass distribution on a metric space (X, m). For every $\epsilon > 0$, we then have*

$$H_\epsilon^\alpha(X) \geq \frac{\mu(X)^2}{\int\int_{m(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{m(x,y)^\alpha}}$$

meaning that if $I_\alpha(\mu) < \infty$, $H^\alpha(X) = \infty$ and $\dim(X) \geq \alpha$.

Proof. Let $\epsilon, \delta > 0$ and let $\{A_n\}_{n=1}^{\infty}$ be a pairwise disjoint covering of X that have diameters less than ϵ but also

$$\sum_{n=1}^{\infty} |A_n|^\alpha \leq H_\epsilon^\alpha(X) + \delta.$$

We then have

$$\int\int_{m(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{m(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \int\int_{A_n\times A_n} \frac{d\mu(x)d\mu(y)}{m(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^\alpha}.$$

In addition, we have that

$$\sum_{n=1}^{\infty} \mu(A_n) \geq \mu(X).$$

Therefore, by the Cauchy-Schwarz inequality,

$$\mu(X)^2 \leq \sum_{n=1}^{\infty} |A_n|^\alpha \sum_{n=1}^{\infty} \frac{\mu(A_n)^2}{|A_n|^\alpha} \leq (H_\epsilon^\alpha(X) + \delta) \int\int_{m(x,y)<\epsilon} \frac{d\mu(x)d\mu(y)}{m(x,y)^\alpha}.$$

We can let δ decrease to 0 and divide by the integral to get the desired inequality specific for ϵ. Letting ϵ decrease to 0, we have that the integral converges to 0 when $\mathbb{E}[I_\alpha(\mu)]$ is finite, meaning that $H^\alpha(X)$ is infinite and $\dim(X) \geq \alpha$. □

The next lemma we prove is known as **Frostman's lemma** and can be thought of as a converse to the mass distribution principle. We only will prove the lemma in the case of compact sets as this is all we need for this paper, but the same statement holds for all Borel subsets of $\mathbb{R}^d$.

Lemma 3.13. *Let X be a compact subset of $\mathbb{R}^d$. Then, the following conditions are equivalent:*

(1) $\mathcal{H}^s(X) > 0$.
(2) There exists a probability Borel measure μ *supported on* X *such that for all* $x \in \mathbb{R}^d$ *and* $r > 0$, $\mu(B(x,r)) \le cr^s$ *for some positive constant* $c \in \mathbb{R}$*, where* $B(x,r)$ *is the ball around* x *of radius* r*.*

Proof. ($\Longleftarrow$) We cover X by balls $\{A_n\}_{n=1}^{\infty}$, each with radius r_n. We then have that

$$\sum_{n=1}^{\infty} |A_i|^s \ge \sum_{n=1}^{\infty} r_n^s \ge \frac{1}{c}\sum_{n=1}^{\infty} \mu(A_n) \ge \frac{1}{c}\mu(X) > 0.$$

($\Longrightarrow$) We fix $\delta > 0$ such that $\mathcal{H}^s_\delta(X) > 0$. Consider the function $g : C(X) \longrightarrow \mathbb{R}$ where $C(X)$ is the space of continuous functions on X as defined here:

$$g(f) = \inf\{\sum_{i\in I} c_i|A_i|^s : I \text{ is at most countable such that for all } i \in I,$$
$$0 < c_i < \infty, A_i \subset X, |A_i| < \delta, \sum_{i\in I} c_i\chi_{A_i} \ge f\}.$$

Observe that by definition, for all $f \in C(X)$, $g(f) \ge 0$ and g is sublinear. We can therefore use the Hahn-Banach theorem to extend the linear functional $c \longrightarrow cg(1)$ where $c \in \mathbb{R}$ from the subspace of constant functions to a linear functional $L : C(X) \longrightarrow \mathbb{R}$ satisfying $L(1) = g(1)$ and $-g(-f) \le L(f) \le g(f)$ for all $f \in C(X)$. By the Riesz representation theorem, there exists a Radon measure μ such that $L(f) = \int f d\mu$ for all $f \in C(X)$. Note that $\mu(X) = L(\chi_X) = H^s_\delta(X)$ and that for all $f \in C(X)$, $L(f) \ge 0$, so $\mu \ge 0$ and we can therefore normalize it by $g(\chi_X)$ to get a probability measure supported on X. Moreover, we can also verify that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x,r)) = \sup\left\{\int f d\mu : f \in C(X), 0 \le f \le \chi_{B(x,r)}\right\}$$
$$\le \sup\{g(f) : f \in C(X), 0 \le f \le \chi_{B(x,r)}\} \le (2r)^s$$

up to rescaling by a normalization constant, giving the desired inequality. □

We end this section by proving **Kaufman's Dimension Doubling Theorem** and two lemmas that lead up to it, which is useful when looking at the image or preimage of a set under a Brownian motion and how a set's dimension is affected. We use the approach from [16].

Lemma 3.14. *Let* $f : X_1 \longrightarrow X_2$ *be surjective and locally* α*-Hölder continuous with constant* C*. Then for any* $s \ge 0$,

$$\mathcal{H}^s(X_2) \le C^s\mathcal{H}^{\alpha s}(X_1)$$

and thus, $\dim(X_2) \le \frac{1}{\alpha}\dim(X_1)$.

Proof. Assume that $H^{\alpha s}(X_1) < \infty$. Let $\epsilon, \delta > 0$. We will cover X_1 with countably many sets $A_1, A_2, ...$ each with diameter at most δ such that

$$\sum_{i=1}^{\infty} |A_i|^{\alpha s} \le \mathcal{H}^{\alpha s}(X_1) + \epsilon.$$

Note as well that X_2 is covered by $f(A_1), f(A_2), ...$ and by definition, $|f(A_i)| \le C|A_i|^s \le C\delta^s$ for all $i \in \mathbb{N}$. This implies that

$$\sum_{i=1}^{\infty} |f(A_i)|^s \le C^s \sum_{i=1}^{\infty} |A_i|^{\alpha s} \le C^s\mathcal{H}^{\alpha s}(X_1) + C^s\epsilon.$$

Sending ϵ to 0 proves the result for α-Hölder continuous functions and the countable stability of Hausdorff dimension finishes the proof for locally α-Hölder continuous functions. □

Lemma 3.15. *Let* $\{B(t) : t \ge 0\}$ *be a planar Brownian motion. For all* $n \in \mathbb{N}$*, we will let* $U_n(x) \subset \mathbb{R}^2$ *be an open ball of radius* $\frac{1}{2^n}$ *centered at* x*. We define the event*

$$A_n = \left\{\exists U_n(x) \text{ such that } \#k \in \mathbb{N}, k \le 4^n : \left[\frac{k-1}{4^n}, \frac{k}{4^n}\right] \cap B^{-1}(U_n(x)) \neq \emptyset \ge n^4\right\}.$$

Then, $\mathbb{P}\{\limsup_{n\to\infty} A_n\} = 0$*, meaning only finitely many such events may occur.*

Proof. First, we note that for sufficiently large $n \in \mathbb{N}$ and for $t, s \in [0,1]$ with $|t-s| \le \frac{1}{4^n}$, Lévy's modulus of continuity and the triangle inequality imply that for all $c > 1$,

$$|B(t) - B(s)| \le 2c\sqrt{2 \cdot 4^{-n}\log(4^n)} \le 3\sqrt{n}2^{-n}.$$

So, if $U \subset \mathbb{R}^2$ with radius $\frac{1}{2^n}$ centered at $x \in \mathbb{R}^2$ intersects $B(\frac{k}{4^n})$ for some $k \in \mathbb{N}$ with $k \leq 4^n$, and for all $t \in \left[\frac{k-1}{4^n}, \frac{k}{4^n}\right]$,

$$\left| B\left(\frac{k}{4^n}\right) - B(t) \right| < 3\sqrt{n}2^{-n} + 2^{-n} \leq 2\sqrt{n}\frac{1}{2^{n+1}}.$$

Thus, it suffices to consider the sequence of events

$$A'_n = \{\exists \text{ neighborhood } U \text{ in } \mathbb{R}^2 \text{ with radius } \frac{2\sqrt{n}}{2^{n-1}} \text{ such that } \#k \in \mathbb{N}, k \leq 4^n : B\left(\frac{k}{4^n}\right) \in B^{-1}(U) \geq n^4\}$$

as $A_{n+1} \subset A'_n$ for sufficiently large $n \in \mathbb{N}$. Now, let $P_n = \{k_1, ..., k_n\}$ consist of n distinct integers in $[1, 4^n]$. Let P'_n be the set of all such possible partitions. Without loss of generality, let $k_1 < k_2 < ... < k_n$. Then, we have that for some constant c, by definition of the transition kernel, for any $i \in \mathbb{N}$ with $i \leq n-1$,

$$\mathbb{P}\left\{\left| B\left(\frac{k_{i+1}}{4^n}\right) - \frac{k_i}{4^n} \right| < \frac{4\sqrt{n}}{2^n}\right\} = \mathbb{P}\left\{|B(1)| < \frac{4\sqrt{n}}{\sqrt{k_{i+1} - k_i}}\right\} \leq \frac{16cn}{k_{i+1} - k_i}.$$

Thus, the probability this event holds for all $i \in \mathbb{N}$ with $i \leq n-1$ by independence is at most

$$(16cn)^{n-1}\Pi_{i=1}^{n-1}(k_{i+1} - k_i)^{-1}.$$

Therefore, it may be seen by definition of A'_n that

$$\mathbb{P}\{A'_n\} \leq \frac{n}{\binom{n^4}{n}} \sum_{P_n \in P'_n} \Pi_{i=1}^{n-1}(k_{i+1} - k_i)^{-1}.$$

Upon computation and by Stirling's formula, it may be seen that for some small (i.e. less than 1) constant C,

$$\mathbb{P}\{A'_n\} \leq C^n \cdot n^{-n+\frac{1}{2}}.$$

The Borel-Cantelli lemma finishes the proof. ☐

Theorem 3.16. *Let $\{B(t) : t \geq 0\}$ be a planar Brownian motion. Then, almost surely, for any set $A \subset [0, 1]$,*

$$\dim(B(A)) = 2\dim(A).$$

In fact, the theorem holds for all dimensions $d \geq 2$ and for arbitrary sets $A \subset [0, \infty)$. Note that Theorem 3.10 implies that the theorem cannot hold in dimension $d = 1$.

Proof. We've already proven the upper bound in Lemma 3.14, so we just need to prove a lower bound for all sets with dimension greater than 0.

Lemma 3.15 implies the existence of some large $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ and all neighborhoods $U \subset \mathbb{R}^2$ with radius 2^{-n},

$$\mathbb{P}\left\{\#k \in \mathbb{N}, k \leq 4^n : \left[\frac{k-1}{4^n}, \frac{k}{4^n}\right] \cap B^{-1}(U) \neq \emptyset \leq n^{2d}\right\} = 1.$$

Let $n \in \mathbb{N}$ be such that $n \geq N$ and let $A_1, A_2, ...$ be a cover of squares of $B(A)$ where $A \subset [0, 1]$ and the diameter of each square is between $\frac{1}{2^{n+1}} < |A_i| < \frac{1}{2^n}$. Therefore, for each A_i, we may cover $B^{-1}(A_i)$ with at most n^{2d} intervals of length $\frac{1}{4^n}$. Now, let $0 < \beta < \alpha < \dim(A)$, so $\mathcal{H}^\alpha(A) = \infty$. Therefore, there exists some $c, C > 0$ such that

$$0 < c \leq \sum_{i=1}^{\infty} n^4 \cdot 4^{-\alpha n} \leq C \sum_{i=1}^{\infty} |A_i|^{2\alpha} \leq C \sum_{i=1}^{\infty} |A_i|^{2\beta}.$$

This therefore implies that $\mathcal{H}^{2\beta}(B(A)) > 0$, implying that $\dim(B(A)) \geq 2\beta$ with probability 1. Sending β to $\dim(A)$ finishes the proof. ☐

4. Typical Behaviors of Planar Brownian Motion

Before diving into atypical points on a Brownian curve, we first look at the typical behavior of planar Brownian motion. It is therefore sometimes more convenient to use complex representation of the plane, so we can write a planar Brownian motion $B(t) = B_1(t) + iB_2(t)$ where $\{B_1 : t \geq 0\}$ and $\{B_2 : t \geq 0\}$ are two independent linear Brownian motions. Our approach in this section follows that of [12] and [15].

Definition 4.1. Consider continuous real-valued stochastic processes $\{X(t) : t \geq 0\}$, and $\{Y(t) : t \geq 0\}$. Fix $t \geq 0$ and let $\delta(n)$ be the mesh size of a partition $0 = t_0 \leq t_1 \leq ... \leq t_n = t$. Then, we define the **quadratic variation** of X to be

$$\langle X, X \rangle_t = \lim_{\delta(n)\downarrow 0} \sum_{i=1}^{n} (X(t_i) - X(t_{i-1}))^2$$

and the **covariation** of X and Y to be

$$\langle X, Y \rangle_t = \lim_{\delta(n)\downarrow 0} \sum_{i=1}^{n} (X(t_i) - X(t_{i-1}))(Y(t_i) - Y(t_{i-1})),$$

where the limits, if they exist, are defined using convergence in probability.

In particular, we can define these processes for linear Brownian motions as it can be shown that the limits exist. For more discussion on quadratic variation and covariation, interested readers may consult [15].

Definition 4.2. An adapted stochastic process $\{X(t) : 0 \leq t \leq T\}$ is called a **local martingale** if there exist stopping times T_n that almost surely increase to T such that for all $n \in \mathbb{N}$, $\{X(\min(t, T_n)) : t \geq 0\}$ is a martingale.

Lemma 4.3. *Let $M(t)$ be a continuous local martingale. Let $0 \leq a \leq b$. Then, M is constant on $[a, b]$ if and only if the quadratic variation of M is constant on $[a, b]$.*

Proof. On one hand, we see that if M is constant on $[a, b]$, then by definition its quadratic variation is constant on $[a, b]$. On the other hand, for nonnegative $q \in \mathbb{Q}$, we may see that $N(t) = M(t + q) - M(q)$ is a $\mathcal{F}_{t+q}$ local martingale and the process $\langle N, N \rangle_t = \langle M, M \rangle_{t+q} - \langle M, M \rangle_q$ is increasing. Then, $T_q = \inf\{s > 0 : \langle N, N \rangle_s > 0\}$ is a stopping time with respect to $\mathcal{F}_{t+q}$. The optional stopping theorem implies that for the stopped local martingale N^{T_q},

$$\langle N^{T_q}, N^{T_q} \rangle = \langle N, N \rangle^{T_q} = \langle M, M \rangle_{q+T_q} - \langle M, M \rangle_q = 0,$$

which implies that M is almost surely constant on $[q, q + T_q]$. Therefore, as any constant interval is the closure of a countable union of such intervals, the proof is finished. □

Local martingales are closely related to the theory of time changes, which we now briefly explore.

Definition 4.4. A **time change** T is a family $\{T_s\}_{s\geq 0}$ of stopping times such that the maps $s \to T_s$ are almost surely increasing and right-continuous.

Definition 4.5. Let T be a time change. A process X is **T-continuous** if X is constant on each interval $[T_{s-}, T_s]$ where $T_s = \inf\{t : X(t) > s\}$ and $T_{s-} = \inf\{t : X(t) \geq s\}$.

The following theorem is attributed to Lester Dubins and Gideon E. Schwarz, as well as K.E. Dambis, and is therefore known as the **Dubins-Schwarz theorem** or the **Dambis-Dubins-Schwarz theorem**.

Theorem 4.6. *Let $M(t)$ be a continuous local martingale such that almost surely, $M(0) = 0$ and $\lim_{t\to\infty} \langle M, M \rangle_t = \infty$. For $t \geq 0$, define $T_t = \inf\{s : \langle M, M \rangle_s > t\}$. Then, for all $t \geq 0$, $B(t) = M(T_t)$ is a Brownian motion adapted to the filtration $\mathcal{F}(T_t)_{t\geq 0}$. Furthermore, $M(t) = B(\langle M, M \rangle_t)$.*

The Brownian motion obtained from this theorem may be referred to as the **DDS Brownian motion of M**.

Proof. We define the time change T using the family $\{T_t\}_{t\geq 0}$ of stopping times. So, the proposed filtration is in fact a filtration and the stopping times in our family are stopping times adapted for the filtration. We also have that because $\lim_{t\to\infty} \langle M, M \rangle_t = \infty$ that $T_t < \infty$ for all $t \geq 0$. Lemma 4.3 implies that M is T-continuous. Thus, it may be seen that B is a continuous local martingale with respect to $\mathcal{F}(T_t)_{t\geq 0}$ and $\langle B, B \rangle_t = \langle M, M \rangle_{T_t} = t$. Paul Lévy's characterization theorem implies that B is a Brownian motion adapted to $\mathcal{F}(T_t)_{t\geq 0}$.

For the second part, we may see that $B(\langle M, M \rangle_t) = M(T_{\langle M,M \rangle_t})$ and because of how M is constant on level stretches of $\langle M, M \rangle$ it may also be seen that $M(T_{\langle M,M \rangle_t}) = M(t)$, completing the proof. □

We now prove an important property of Brownian motion, known as **conformal invariance**. Recall that a conformal mapping is a holomorphic map with nonvanishing derivative. This theorem tells us that under a conformal function, a Brownian motion still remains a Brownian motion.

Theorem 4.7. *Consider a complex Brownian motion $\{B(t) : t \geq 0\}$ starting at $x \in \mathbb{C}$. Let $U \subset \mathbb{C}$ be open and containing x, and $f : U \longrightarrow \mathbb{C}$ be holomorphic. Let $T_U = \inf\{t \geq 0 : B(t) \notin U\}$. Then, there exists a complex Brownian motion $\{B'(t) : t \geq 0\}$ started at $f(B(0))$ such that for $C(t) = \int_0^t |f'(B(s))|^2 ds$ for all $t \in [0, T_u)$,*

$$f(B(t)) = B'(C(t)).$$

Note that for any point $z' \in U$, we may see that f is locally tangent to the map $z \to f(z') + f'(z')(z - z')$.

Proof. We break up $f = g + ih$, so that g and h are both harmonic on U, and $B(t) = B_1(t) + iB_2(t)$. By the multivariable Itô formula, we have

$$g(B(t)) = g(B(0)) + \int_0^t \frac{\partial g}{\partial x} B(s) dB_1(s) + \int_0^t \frac{\partial g}{\partial y} B(s) dB_2(s)$$

and

$$h(B(t)) = h(B(0)) + \int_0^t \frac{\partial h}{\partial x} B(s) dB_1(s) + \int_0^t \frac{\partial h}{\partial y} B(s) dB_2(s)$$

implying that the processes $X(t) = g(B(t))$ and $Y(t) = h(B(t))$ are continuous local martingales on the stochastic interval $[0, T_u)$. Recalling the Cauchy-Riemann equations and by considering the differentials of the quadratic variation processes, we can show that $\langle X, X\rangle_t = \langle Y, Y\rangle_t = \int_0^t |f'(B(s))|^2 ds = C(t)$ and $\langle X, Y\rangle_t = 0$.

Now, for $t \geq 0$, let $T_t = \inf\{s \geq 0 : C(s) > t\}$ and define $B'(t) = f(B(T_t)) = g(B(T_t)) + ih(B(T_t))$. By Theorem 4.6, $B'(t)$ is a complex Brownian motion adapted to the filtration $(\mathcal{F}(T_t))_{t\geq 0}$. Note that for all $t \geq 0$, $B'(C(t)) = f(B(T_{C(t)}))$ and that $T_{C(t)} = t$, therefore finishing the proof. □

As a corollary, we can prove that a planar Brownian motion almost surely never hits a fixed point. Sometimes, this property is referred to as **polarity**.

Corollary 4.8. *Let $\{B(t) : t \geq 0\}$ be a planar Brownian motion started at $x \in \mathbb{R}^2$. Let $y \in \mathbb{R}^2$ such that $y \neq x$. Then,*

$$\mathbb{P}\{\exists t > 0 \text{ such that } B(t) = y\} = 0.$$

Proof. Again, we will use complex notation. It suffices to prove the case when our Brownian motion starts at 1 and show that it almost surely won't cross the origin by symmetry. Let $\{X(t) : t \geq 0\}$ be a complex Brownian motion started at the origin. By Theorem 4.7,

$$\exp(B(t)) = B'(C(t))$$

where

$$C(t) = \int_0^t \exp(2X_1(s)) ds,$$

X_1 is the real part of X, and $\{B'(t) : t \geq 0\}$ is a complex Brownian motion started at 1. Note that $\lim_{t\to\infty} C(t) = \infty$ almost surely, which implies that almost surely,

$$\{B'(t) : t \geq 0\} = \{\exp(B(t)) : t \geq 0\}$$

meaning they have the same paths. But as an exponential function is never equal to 0, it is implied that $\{B'(t) : t \geq 0\}$ almost surely never hits the origin. Replacing $\{B'(t) : t \geq 0\}$ with an arbitrary complex Brownian motion $\{B(t) : t \geq 0\}$ starting at 1, which we can do because they'll have the same distribution, the conclusion follows. □

This next theorem is the **skew-product representation** of a planar Brownian motion and is useful for representing a planar Brownian motion in polar coordinates.

Theorem 4.9. *Let $\{B(t) : t \geq 0\}$ be a planar Brownian motion such that $B(0) = 1$. Then, there exist two independent linear Brownian motions, $\{X_1(t) : t \geq 0\}$ and $\{X_2(t) : t \geq 0\}$ such that*

$$B(t) = \exp(X_1(H(t)) + iX_2(H(t)))$$

for all $t \geq 0$, where

$$H(t) = \int_0^t \frac{ds}{|B(s)|^2} = \inf\left\{u \geq 0 : \int_0^u \exp(2X_1(s)) ds > t\right\}.$$

Proof. First, note that $H(t)$ is well defined by Corollary 4.8 for all $t \geq 0$. Let $X(t) = X_1(t) + iX_2(t)$ be a planar Brownian motion started at 0. By Theorem 4.7, there exists a planar Brownian motion $\{B'(t) : t \geq 0\}$ such that

$$\exp(X(t)) = B'(C(t))$$

where $C(t) = \int_0^t \exp(2X_1(s))ds$. Let $H'(t) = \int_0^t \frac{ds}{|B'(s)|^2}$, so $H(t)$ is the inverse of $C(t)$. Therefore, for $t = H(s)$,

$$B'(s) = \exp(X_1(H(s)) + iX_2(H(s))),$$

which is the desired result except for the fact that $B'(s)$ should be replaced by $B(s)$. For all $t \geq 0$, we'll let $s_t = \inf\{s \geq 0 : \int_0^s \frac{1}{|B(u)|}du > t\}$, and define $Y_1(t) = \log|B(s_t)|$ and $Y_2(t) = \arg(B(s_t))$. Note that Theorem 4.7 is equivalent to saying that if we define these three terms for all $t \geq 0$, then Y_1 and Y_2 are independent linear Brownian motions. Moreover, they are deterministic functions of B so their joint distribution only depends on B. So, it sufficed to start with a planar Brownian motion and show that the resulting process was also a planar Brownian motion as we previously demonstrated. □

The process $H(t)$ is sometimes referred to as a random clock and is deeply related to winding numbers, which is the next topic we examine in this paper. In loose terms, the winding number of a point may be thought of as the number of times a planar Brownian motion winds around itself. We first prove a lemma that we'll use in the rest of this section which is sometimes known as **Laplace's Method**.

Lemma 4.10. *Let $t > 0$ and $f : [0,t] \longrightarrow \mathbb{R}$ be continuous. Then,*

$$\lim_{a\uparrow\infty} \frac{1}{a} \ln \int_0^t \exp(af(u))du = \max_{x\in[0,t]} f(x).$$

Proof. Let $\max_{x\in[0,t]} f(x) = y$ and assume that this maximum is achieved at $x \in [0,t]$. Then, we have that

$$\lim_{a\uparrow\infty} \frac{1}{a} \ln \int_0^t \exp(af(u))du \leq \lim_{a\uparrow\infty} \frac{1}{a} \ln \int_0^t \exp(a \cdot y)du = y.$$

On the other hand, letting $\epsilon > 0$, we can find some $\delta > 0$ such that for all $z \in [0,t]$ where $|x - z| < \delta$, $f(z) \geq y - \epsilon$. We therefore have that

$$\lim_{a\uparrow\infty} \frac{1}{a} \ln \int_0^t \exp(af(u))du \geq \lim_{a\uparrow\infty} \frac{1}{a} \ln \int_{[x-\frac{\delta}{2},x+\frac{\delta}{2}]\cap[0,t]} \exp(a(y-\epsilon))du = y - \epsilon.$$

Because this holds for all $\epsilon > 0$, it follows that the limit as a approaches ∞ is at least y, so combined with our prior proof, the conclusion follows. □

When analyzing the random clock $H(t)$, for some applications, we need to be able to control the asymptotic behavior of $H(t)$. As such, we prove the below theorem which is used to prove many asymptotic results of planar Brownian motion. We already observed a way to express $H(t)$ in terms of part of a planar Brownian motion in Theorem 4.9, but this theorem helps relate $H(t)$ to the first hitting time of a level.

Theorem 4.11. *Let the processes $H(t)$ and $X_1(t)$ be defined as in Theorem 4.9. For all $a > 0$, define a Brownian motion $Y_a(t) = \frac{X_1(a^2t)}{a}$. Moreover, fix a level b and let $T_{(a,b)} = \inf\{t \geq 0 : Y_a(t) = b\}$ be its first hitting time of b. Then, for all $\epsilon > 0$,*

$$\lim_{t\to\infty} \mathbb{P}\left\{\left|\frac{4H(t)}{(\log(t))^2} - T_{(\frac{1}{2}log(t),1)}\right| > \epsilon\right\} = 0.$$

Proof. Fix $\epsilon > 0$. Without loss of generality, because of scaling, we assume that $X_1(0) = 0$. Note that for all $\delta > 0$, we have

$$\lim_{\epsilon\downarrow 0} \mathbb{P}\{T_{(\frac{1}{2}\log(t),1+\epsilon)} - T_{(\frac{1}{2}\log(t),1-\epsilon)} > \delta\} = \lim_{\epsilon\downarrow 0} \mathbb{P}\{T_{(1,1+\epsilon)} - T_{(1,1-\epsilon)} > \delta\} = 0.$$

Therefore, it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}\left\{\frac{4H(t)}{(\log(t))^2} > T_{(\frac{1}{2}log(t),1+\epsilon)}\right\} = \lim_{t\to\infty} \mathbb{P}\left\{\frac{4H(t)}{(\log(t))^2} > T_{(\frac{1}{2}log(t),1-\epsilon)}\right\} = 0. \tag{4.12}$$

In this proof, we will only prove that

$$\lim_{t\to\infty} \mathbb{P}\left\{\frac{4H(t)}{(\log(t))^2} > T_{(\frac{1}{2}\log(t),1+\epsilon)}\right\} = 0,$$

as the proof for showing the other part of (4.12) follows similar logic. Throughout this proof, we sometimes make the substitution $a = a(t) = \frac{1}{2}\log(t)$. First, we note that

$$\left\{\frac{4H(t)}{(\log(t))^2} > T_{(\frac{1}{2}\log(t),1+\epsilon)}\right\} = \left\{\int_0^{a^2T_{(a,1+\epsilon)}} \exp(2X_1(u))du < t\right\} = \left\{\frac{1}{2a}\int_0^{a^2T_{(a,1+\epsilon)}} \exp(2X_1(u))du < 1\right\}.$$

We also know that

$$\frac{1}{2a}\log\int_0^{a^2T_{(a,1+\epsilon)}} \exp(2X_1(u))du = \frac{\log(a)}{a} + \frac{1}{2a}\log\int_0^{T_{(a,1+\epsilon)}} \exp(2aY_a(u))du$$

which is equal in distribution to

$$\frac{\log(a)}{a} + \frac{1}{2a}\log\int_0^{T_{(1,1+\epsilon)}} \exp(2aX_1(u))du.$$

By Lemma 4.10, almost surely,

$$\lim_{a\uparrow\infty}\frac{1}{2a}\log\int_0^{T_{(1,1+\epsilon)}} \exp(2aX_1(u))du = \sup_{s\in[0,T_{(1,1+\epsilon)}]} X_1(s) = 1+\epsilon.$$

Therefore, almost surely,

$$\lim_{a\uparrow\infty}\mathbb{P}\left\{\left|\frac{\log(a)}{a} + \frac{1}{2a}\log\int_0^{T_{(1,1+\epsilon)}} \exp(2aX_1(u))du - (1-\epsilon)\right| > \epsilon\right\} = 0$$

implying the desired equality. □

We now come to the **asymptotic law of winding numbers**, also known as **Spitzer's Law**.

Theorem 4.13. *Let the processes $H(t)$ and $X_2(t)$ be defined as in Theorem 4.9. For all $t \geq 0$, we define $\theta(t) = X_2(H(t))$. Then, for all $x \in \mathbb{R}$,*

$$\lim_{t\to\infty}\mathbb{P}\left\{\frac{2}{\log(t)}\theta(t) \leq x\right\} = \int_{-\infty}^{x}\frac{dy}{\pi(1+y^2)}$$

meaning that the law of $\frac{2\theta(t)}{\log(t)}$ converges to the symmetric Cauchy distribution.

First, we note that θ effectively represents the angle of a planar Brownian motion and in its (almost surely) continuous representation. Hence, this is the reason why we call this theorem the asymptotic law of winding numbers.

Proof. For $a > 0$, we define the process $\{Y_a(t) : t \geq 0\}$ by $Y_a(t) = \frac{X_2(a^2t)}{a}$. Therefore, we have that

$$\frac{\theta(t)}{a} = \frac{X_2(H(t))}{a} = Y_a\left(\frac{H(t)}{a^2}\right).$$

By Theorem 4.11, for $a = \frac{1}{2}\log(t)$, we have

$$\lim_{t\to\infty}\mathbb{P}\left\{\left|\frac{2\theta(t)}{\log(t)} - Y_a(T_{(a,1)})\right| > \epsilon\right\} = 0$$

where $T_{(a,1)}$ is defined as in Theorem 4.11. Note that $Y_a(T_{(a,1)})$ is a random variable whose law doesn't depend on a, so passing to Theorem 2.8 says it has a Cauchy distribution. □

Spitzer's Law gives us information on the behavior of a Brownian motion within a small period of time. We assume in this discussion that our planar Brownian motion starts at the origin. Note that in this case, the definition of $\theta(t)$ doesn't make sense, as this represents the angular component of a planar Brownian motion. But, by Corollary 4.8 and the Markov property, for every $0 < \epsilon < 1$, we may consider the behavior of $\theta(t)$ on $[\epsilon, 1]$ as ϵ decreases to 0. By the scaling property of Brownian motion, the variation of $\theta(t)$ on $[\epsilon, 1]$ is equal in distribution to the variation of $\theta(t)$ on $[1, \frac{1}{\epsilon}]$. Therefore, Theorem 4.13, the Markov property, and the Blumenthal zero-one law imply that, informally, almost surely, a Brownian motion performs an infinite number of windings in both directions around its starting point in any unit of time. Note that by the Markov property, this idea can be extended to any point along the Brownian curve.

5. Cone Points

In the prior section, we described the typical behavior of points along a Brownian curve in the plane. In this section, we take a closer look at an example of atypical behavior.

First, we consider a brief example to show that there are exceptions to Spitzer's Law.

Example 5.1. We define

$$y = \min\{x : (x, 0) \in B[0, 1]\}.$$

Assume for the sake of contradiction that the Brownian motion performs windings around $(y, 0)$. Then, the Brownian motion must cross a line of the form $\{(z, 0) : z < y\}$, which contradicts our definition of y being the minimum x-value achieved on this slice.

Definition 5.2. Let $\{B(t) : t \geq 0\}$ be a planar Brownian motion, $\alpha \in (0, 2\pi)$ and $\zeta \in [0, 2\pi)$. We define the closed **cone** with angle α and direction ζ as a subset of $\mathbb{R}^2$:

$$W[\alpha, \zeta] = \left\{ re^{i(\theta-\zeta)} : |\theta| \leq \frac{\alpha}{2}, r \geq 0 \right\}.$$

The **dual** of a cone $x + W[\alpha, \zeta]$ is the cone $x + W[2\pi - \alpha, \zeta + \pi]$. A point $x = B(t)$ for $t \in [0, 1]$ is an α **two-sided cone point** if there exists some $\epsilon > 0$ and $\zeta \in [0, 2\pi)$ such that

$$B(0, 1) \cap N_\epsilon(x) \subset x + W[\alpha, \zeta]$$

where $N_\epsilon(x)$ is the ball around x of radius ϵ.

In this definition, we use the notation two-sided cone point, as there is another related topic, one-sided cone points, which is discussed in more depth in [12], for example. But, we will only look at two-sided cone points in this paper, so as there's no confusion, we will now adopt the notation of **cone point** when describing such points. Note that cone points are atypical, in the sense that only a finite number of windings around them can occur by definition.

Given that this behavior is atypical, it is natural to wonder exactly how large this set of points is and does it depend on the value of α? We will now explore this, following the approaches of [12] and [14]. We begin with proving an upper bound through a series of lemmas.

Definition 5.3. Let $\{B(t) : t \geq 0\}$ be a planar Brownian motion. We fix an angle $\alpha \in (0, 2\pi)$ and a direction $\zeta \in [0, 2\pi)$. Let $\epsilon > 0$ and for $\delta \in (0, \epsilon)$, let

$$T_\delta(z) = \inf\{s \geq 0 : B(s) \in N_\delta(z)\}$$

and

$$S_{\delta,\epsilon}(z) = \inf\{s \geq T_{\frac{\delta}{2}}(z) : B(s) \notin N_\epsilon(z)\}.$$

Then, $z \in \mathbb{R}^d$ is a (δ, ϵ) **approximate cone point** if

(1) $B(0, T_\delta(z)) \subset z + W[\alpha, \zeta]$

(2) $B(T_{\frac{\delta}{2}}(z), S_{\delta,\epsilon}(z)) \subset z + W[\alpha, \zeta]$.

One of the most evident aspects that distinguish between approximate cone points and cone points is that approximate cone points need not lie on the path of the Brownian motion. Throughout the rest of this section, we fix an angle α and a direction ζ. The first lemma we prove includes a reference to what is known as the **heat equation** and a solution of it, so we provide some necessary background on this before proving the first lemma.

Definition 5.4. Let $U \subset \mathbb{R}^d$ be either the entire space ($\mathbb{R}^d$) or open and bounded. Let $u : (0, \infty) \times U \longrightarrow [0, \infty)$ be twice differentiable. Then, u solves the heat equation with heat dissipation rate $V : U \longrightarrow \mathbb{R}$ and initial condition $f : U \longrightarrow [0, \infty)$ on U if

(1) $\lim_{x \to y, t \downarrow 0} u(t, x) = f(y)$ for $y \in U$

(2) $\lim_{x \to y, t \to a} u(t, x) = 0$ for $y \in \partial U$

(3) $\partial_t u(t, x) = \frac{1}{2}\Delta_x u(t, x) + V(x)u(t, x)$ on $(0, \infty) \times U$ where Δ_x is the Laplacian in the x variable.

Using this, we now prove the following lemma, which will aid us in proving the probability that a point is a (δ, ϵ) approximate cone point. When proving an upper bound on the Hausdorff dimension of cone points, the relationship between cone points and approximate cone points will be explored.

Lemma 5.5. *There exist positive constants $c < C$ such that for all $\delta > 0$, the following properties hold.*

(1) For all $z \in \mathbb{R}^2$, $\mathbb{P}\{B(0, T_\delta(z)) \subset z + W[\alpha, \zeta]\} \leq C(\frac{\delta}{|z|})^{\frac{\pi}{\alpha}}$.

(2) For all $z \in \mathbb{R}^2$ *such that* $0 \in z + W[\frac{\alpha}{2}, \zeta]$, $\mathbb{P}\{B(0, T_\delta(z)) \subset z + W[\alpha, \zeta]\} \geq c(\frac{\delta}{|z|})^{\frac{\pi}{\alpha}}$.

Note that we may consider the two intervals $[0, T_\delta(z)]$ and $[T_{\frac{\delta}{2}}(z), 1]$ separately by the strong Markov property.

Proof. We can rewrite $z = |z|e^{i\theta}$ in order to apply the skew-product representation to the Brownian motion $\{z - B(t) : t \geq 0\}$ so that we may write

$$B(t) = z - R(t)\exp(i\theta(t))$$

for all $t \geq 0$, $R(t) = \exp(W_1(H(t))$ and $\theta(t) = W_2(H(t))$ where $\{W_1(t) : t \geq 0\}$ and $\{W_2(t) : t \geq 0\}$ are two independent linear Brownian motions started at $\log(|z|)$, respective to θ, and a strictly increasing time change $\{H(t) : t \geq 0\}$ depending on $\{W_1(t) : t \geq 0\}$. Using the previously defined notation related to approximate cone points, we can see that $\inf\{s \geq 0 : R(s) \leq \delta\} = T_\delta(z)$ and that $H(T_\delta(z)) = \inf\{u \geq 0 : W_1(u) \leq \log(\delta)\}$. Therefore,

$$\{B(0, T_\delta(z)) \subset z + W[\alpha, \zeta]\} = \left\{|W_2(u) + \pi - \zeta| \leq \frac{\alpha}{2} \text{ for all } u \in [0, H(T_\delta(z)]\right\}.$$

The latter event here is the event that a linear Brownian motion started in θ stays within the interval $[\zeta - \pi - \frac{\alpha}{2}, \zeta - \pi + \frac{\alpha}{2}]$ up until $H(T_\delta(z))$. By a separation of variables approach, we may yield the following solution of the heat equation for domain $(0, a)$:

$$u(t, x) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \exp\left(-\frac{(2n+1)^2\pi^2}{2a^2}t\right) \sin\left(\frac{(2n+1)\pi x}{a}\right).$$

We may also show that for any linear Brownian motion $\{B'(t) : t \geq 0\}$ and for any $\sigma > 0$, $\{\exp(\sigma B(t) - \frac{\sigma^2 t}{2}) : t \geq 0\}$ is a martingale, which may be used in conjunction with the optional stopping theorem to show that if we have $a, \lambda > 0$ and $A_a = \inf\{t \geq 0 : B'(t) = a\}$, then

$$\mathbb{E}[e^{-\lambda A_a} : B'(0) = 0] = e^{-a\sqrt{2\lambda}}.$$

Combining these remarks,

$$\begin{aligned}
&\mathbb{P}\{|W_2(u) + \pi - \zeta|\} \leq \frac{\alpha}{2} \text{ for all } u \in [0, H(T_\delta(z))] \\
&= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi(\frac{\alpha}{2} + \zeta - \pi - \theta)}{\alpha}\right) \mathbb{E}\left[\exp(-\frac{(2k+1)^2\pi^2}{2\alpha^2}H(T_\delta(z)))\right] \\
&= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi(\frac{\alpha}{2} + \zeta - \pi - \theta)}{\alpha}\right) \left(\frac{\delta}{|z|}\right)^{(2k+1)\frac{\pi}{\alpha}}.
\end{aligned}$$

The upper bound from the lemma follows directly from this final equality if $|z| \leq 2\delta$ and otherwise, an upper bound may be given by

$$\left(\frac{\delta}{|z|}\right)^{\frac{\pi}{\alpha}} \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} 2^{-2k\frac{\pi}{\alpha}}.$$

On the other hand, if $\frac{\delta}{|z|}$ is bounded from below, then Brownian scaling can give us a lower bound of the desired form. Otherwise, by the assumption that $0 \in z + W[\frac{\alpha}{2}, \zeta]$, $|\theta + \pi - \zeta| \leq \frac{\alpha}{4}$ the sin term for $k = 0$ is bounded below by $\sin(\frac{\pi}{4})$. Therefore, a lower bound may be given by

$$\left(\frac{\delta}{|z|}\right)^{\frac{\pi}{\alpha}} \left[\frac{4}{\pi}\sin\left(\frac{\pi}{4}\right) - \sum_{k=1}^{\infty} \frac{4}{(2k+1)\pi}\left(\frac{\delta}{|z|}\right)^{2k\frac{\pi}{\alpha}}\right]$$

which is bounded away from zero for small $\frac{\delta}{|z|}$. ☐

By similar logic to the proof of Lemma 5.5, we may also prove the below lemma, which is omitted in this paper due to the similarity. Throughout the rest of this section, we will use the notation

$$S_\epsilon^{(t)}(z) = \inf\{s > t : B(s) \notin N_\epsilon(z)\}.$$

Lemma 5.6. *There are constants* $C > c > 0$ *such that for all* $\delta \in (0, \epsilon)$ *the following properties hold.*

(1) For all $x, z \in \mathbb{R}^2$ *with* $|x - z| = \frac{\delta}{2}$, $\mathbb{P}\{B(0, S_\epsilon^{(0)}(z)) \subset z + W[\alpha, \zeta] : B(0) = x\} \leq C(\frac{\delta}{\epsilon})^{\frac{\pi}{\alpha}}$.

(2) For all $x, z \in \mathbb{R}^2$ *with* $|x - z| = \frac{\delta}{2}$ *and* $x - z \in W[\frac{\alpha}{2}, \zeta]$, $\mathbb{P}\{B(0, S_\epsilon^{(0)}(z)) \subset z + W[\alpha, \zeta] : B(0) = x\} \geq c(\frac{\delta}{\epsilon})^{\frac{\pi}{\alpha}}$.

We are now in a position to prove an upper bound on the probability that a point is a (δ, ϵ) approximate cone point.

Lemma 5.7. *There is a constant $C_0 > 0$ such that for any $z \in \mathbb{R}^2$,*

$$\mathbb{P}\{z \text{ is a } (\delta,\epsilon) \text{ approximate cone point}\} \leq C_0|z|^{-\frac{\pi}{\alpha}}\epsilon^{-\frac{\pi}{\alpha}}\delta^{\frac{2\pi}{\alpha}}.$$

Proof. We apply the strong Markov property with the stopping time $T_{\frac{\delta}{2}}(z)$ to get that the probability of this event is at least

$$\mathbb{E}[[\chi_{B(0,T_\delta(z))\subset z+W[\alpha,\zeta]}]\mathbb{P}\{B(0,S^{(0)}_\epsilon(z))\subset z+W[\alpha,\zeta] : B(0)=T_{\frac{\delta}{2}}(z)\}]\leq C^2\left(\frac{\delta}{|z|}\right)^{\frac{\pi}{\alpha}}\left(\frac{\delta}{\epsilon}\right)^{\frac{\pi}{\alpha}}$$

by Lemma 5.5 and Lemma 5.6. Therefore, by selecting $C_0 = C^2$, the conclusion follows. □

Before returning to cone points, we will now prove the Hausdorff dimension of approximate cone points.

Lemma 5.8. *Let $M(\alpha,\zeta,\epsilon)$ be the set of all (δ,ϵ) approximate cone points in $\mathbb{R}^2$ for all $\delta > 0$. Then, almost surely,*

(1) if $\alpha \in (0,\pi)$ then $M(\alpha,\zeta,\epsilon) = \emptyset$.
(2) if $\alpha \in [\pi,2\pi)$ then $\dim\,(M(\alpha,\zeta,\epsilon)) \leq 2-\frac{2\pi}{\alpha}$.

Proof. First, we note that $z \in M(\alpha,\zeta,\epsilon)$ if and only if there is some $t > 0$ such that $z = B(t)$ and $B(0,t) \subset z + W[\alpha,\zeta]$. We also note that $B(t, S^{(t)}_\epsilon(z)) \subset z + W[\alpha,\zeta]$. Let A be a compact cube of unit side length that doesn't contain the origin. Note that it suffices to show that $M(\alpha,\zeta,\epsilon)\cap A = \emptyset$ if $\alpha \in (0 \in \pi)$ and that $\dim(M(\alpha,\zeta,\epsilon)) \cap A \leq 2-\frac{2\pi}{\alpha}$ for $\alpha \in (\pi,2\pi)$. Consider a half-open dyadic subcube of A with side lengths $\frac{1}{2^k}$ for some $k \in \mathbb{N}$, $D \in D_k$, where D_k is the family of all cubes of the form $x + \Pi_{i=1}^2[\frac{k_i}{2^k}, \frac{k_i+1}{2^k})$ and $k_1, k_2 \in \{0,1,...,2^k-1\}$. Let $D \subset D^*$ be a concentric ball around D with radius $(1+\sqrt{2})2^{-k}$. Throughout the rest of the proof, we will use the notation $x = x(D)$ to define the focal point of D where x is chosen so that

(1) if $\alpha < \pi$, the tip of the cone $x + W[\alpha,\zeta]$ has boundary halflines tangent to D^*,
(2) or if $\alpha > \pi$, x is the point on the cone whose dual has boundary halflines tangent to D^*.

It may be shown that for all $\epsilon > 0$ and $\alpha \in [0,2\pi)$, there is some $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq k_0$, $D \in D_k$, and $y \in D$, $N_{\frac{\epsilon}{2}}(x) \subset N_\epsilon(y)$ and $y + W[\alpha,\zeta] \subset x + W[\alpha,\zeta]$. We can also find constants $0 < c_1 < C_1$ such that

(1) $N_{C_1 2^{-k}}(y) \subset N_{C_1^2 2^{-k}}(x)$,
(2) $N_{c_1 C_1 2^{-k}}(x) \subset N_{\frac{1}{2}C_1 2^{-k}}(y)$,
(3) and $|x-y| < c_1 C_1 2^{-k}$.

Therefore, if we choose k sufficiently large and if $D \in D_k$ contains a $(C_1 2^{-k},\epsilon)$ approximate cone point, then the focal point of D must satisfy

(1) $B(0, T_{C_1^2 2^{-k}}(x)) \subset x + W[\alpha,\zeta]$
(2) and $B(T_{c_1 C_1 2^{-k}}(x), S_{c_1 C_1 2^{-k},\frac{\epsilon}{2}}(x)) \subset x + W[\alpha,\zeta]$.

We may use similar logic to Lemma 5.7 to find a positive constant C_2 such that

$$\mathbb{P}\{D \text{ contains a } (C_1 2^{-k},\epsilon) \text{ approximate cone point}\} \leq C_2|x|^{-\frac{\pi}{\alpha}}\epsilon^{-\frac{\pi}{\alpha}}2^{-k\frac{2\pi}{\alpha}}.$$

Let $k_1 \in \mathbb{N}$ such that $k_1 \geq k_0$ and that $|x(D)|$ is bounded away from zero for all $D \in D_k$ for $k \geq k_1$ and $k \in \mathbb{N}$. Then, we may find a positive constant C_3 such that for all $k \in \mathbb{N}$ with $k \geq k_1$,

$$\mathbb{P}\{D \text{ contains a } (C_1 2^{-k},\epsilon) \text{ approximate cone point}\} \leq C_3 2^{-k\frac{2\pi}{\alpha}}.$$

We may therefore finish the proof of the first statement. For $\alpha \in (0,\pi)$,

$$\mathbb{P}\{M(\alpha,\zeta,\epsilon) \neq \emptyset\} \leq \sum_{D\in D_k}\mathbb{P}\{D \text{ contains a } (C_1 2^{-k},\epsilon) \text{ approximate cone point}\} \leq C_3 2^{2k}2^{-k\frac{2\pi}{\alpha}}$$

which converges to 0 as k approaches ∞. On the other hand, if $\alpha \in (\pi,2\pi)$, we can cover $M(\alpha,\zeta,\epsilon)\cap A$ with the collection of subcubes $D \in D_k$ that contain a $(C_1 2^{-k},\epsilon)$ approximate cone point. Let $\gamma > 2-\frac{2\pi}{\alpha}$.

Then, computing the γ dimensional Hausdorff measure of this set, we get that

$$\begin{aligned}&\mathbb{E}\left[\sum_{D\in D_k} 2^{-k\gamma+\frac{1}{2}\gamma}\chi_{D \text{ contains a } (C_1 2^{-k},\epsilon) \text{ approximate cone point}}\right]\\ &\leq 2^{\frac{\gamma}{2}}\sum_{D\in D_k} 2^{-k\gamma}\mathbb{P}\{D \text{ contains a } (C_1 2^{-k},\epsilon) \text{ approximate cone point}\}\\ &\leq C_3 2^{k(2-\frac{2\pi}{\alpha}-\gamma)}\end{aligned}$$

which goes to 0 as k goes to ∞, proving the other part of the lemma. □

Observe that the result of this lemma is very similar to the desired result regarding the upper bound of the Hausdorff dimension of α-cone points. We now relate our discussion of approximate cone points to cone points and in doing so prove the desired upper bound.

Theorem 5.9. *Let $\{B(t) : 0 \leq t \leq 1\}$ be a planar Brownian motion. Almost surely, α-cone points exist for $\alpha \geq \pi$ and don't exist for $\alpha < \pi$. For $\alpha \in [\pi, 2\pi)$,*

$$\dim\{x \in \mathbb{R}^2 : x \text{ is an } \alpha \text{ cone point}\} \leq 2 - \frac{2\pi}{\alpha}.$$

Proof. Let $\delta > 0$ and let $z \in \mathbb{R}^2$ be an α-cone point. Then, we may find a rational number $q \in [0, 1)$, choose $\zeta \in [0, 2\pi)$ such that ζ is rational, and rational $\epsilon > 0$ such that for some $t \in (q, 1)$, $z = B(t)$ and

(1) $B(q, t) \subset z + W[\alpha + \delta, \zeta]$
(2) $B(t, S_\epsilon^{(t)}(z)) \subset z + W[\alpha + \delta, \zeta]$.

First, we consider when $\alpha < \pi$. By Lemma 5.8, the set is empty under any choice of rationals if $\alpha+\delta < \pi$, so by choosing δ small enough, this would ensure there are no α-cone points with probability 1. On the other hand, if $\alpha \geq \pi$, by Lemma 5.8 and the countable stability of Hausdorff dimension, almost surely, there's an upper bound of $2 - \frac{2\pi}{\alpha+\delta}$ for the upper bound of α-cone points, so the second conclusion follows from the fact that δ is arbitrary. □

We'll now prove a lower bound, again using a series of lemmas and theorems. Similar to our proof of the upper bound, our approach requires us to find a link between cone points and approximate cone points. Moreover, we will use a cover of our set similar to that in Theorem 3.10, in the sense of looking at nested sequences of sets. This is the motivation behind the main step of our proof, Theorem 5.12. The theorem itself isn't specific to cone points and approximate cone points, but gives us some conditions for which the intersection of such a nested sequence and any closed subset of a cube have at least a certain Hausdorff dimension with positive probability. Our proof finishes by relating this theorem to the Hausdorff dimension of approximate cone points and then using Blumenthal's zero-one law to help connect this result to the Hausdorff dimension of cone points. First, we prove a lemma which is used in the proof of our key step.

Lemma 5.10. *Let μ be a probability measure on $\mathbb{R}^d$ such that there exists some $C, \lambda > 0$ where for all $x \in \mathbb{R}^d$ and $r > 0$, $\mu(N_r(x)) \leq Cr^\lambda$. Then, for all $\beta \in (0, \lambda)$, there exists some $C_1 > 0$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,*

$$\int_{N_r(x)} |x - y|^{-\beta}\mu(dy) \leq C_1 r^{-\lambda-\beta},$$

in particular implying that

$$\int\int |x - y|^{-\beta} d\mu(x)d\mu(y) < \infty.$$

Proof. Fix $x \in \mathbb{R}^d$ and $r > 0$. By Fubini's theorem, we have that

$$\begin{aligned}\int_{N_r(x)} |x - y|^{-\beta}\mu(dy) &= \int_0^\infty \mu\{y \in N_r(x) : |x - y|^{-\beta} > s\}ds\\ &\leq \int_{r^{-\beta}}^\infty \mu(N_{s^{-\frac{1}{\beta}}}(x))ds + Cr^{\lambda-\beta}\\ &\leq C\int_{r^{-\beta}}^\infty s^{-\frac{\lambda}{\beta}}ds + Cr^{\lambda-\beta}\end{aligned}$$

so evaluating the improper integral implies the first part of the lemma. We also obtain the inequality

$$\int\int |x - y|^{-\beta} d\mu(x)d\mu(y) \leq \int d\mu(x)\int_{N_1(x)} |x - y|^{-\beta} d\mu(x) + 1 \leq C_1 + 1 < \infty.$$

implying the second part of the lemma. □

We now prove the theorem which is at the heart of showing the lower bound of the Hausdorff dimension of the set of α-cone points, but we first need to introduce one more relevant definition.

Definition 5.11. The **distance** between two subsets of $\mathbb{R}^d$ X and Y, denoted $\mathrm{dist}(X, Y)$, is defined as

$$\mathrm{dist}(X, Y) = \inf\{|x - y| : x \in X, y \in Y\}.$$

Theorem 5.12. *Fix some $x \in \mathbb{R}^d$ and consider the cube $x+[0,1)^d$. Let D_k be the family of all cubes of the form $x + \Pi_{i=1}^{2}[\frac{k_i}{2^k}, \frac{k_i+1}{2^k})$ and $k_1, k_2 \in \{0, 1, ..., 2^k - 1\}$. Also, define $D = \bigcup_{k=1}^{\infty} D_k$. Moreover, let $\{Z(I) : I \in D\}$ be a collection of random variables each taking values in $\{0, 1\}$ and $A = \bigcap_{k=1}^{\infty} \bigcup_{I \in D_k, Z(I)=1} I$. Assume that for all $I, J \in D$ with $I \subset J$ that if $Z(I) = 1$, then $Z(J) = 1$. Furthermore, assume that there exist positive constants γ, c, C such that*

(1) $c|I|^{\gamma} \leq \mathbb{E}[Z(I)] \leq C|I|^{\gamma}$ for all $I \in D$

(2) and $\mathbb{E}[Z(I)Z(J)] \leq C|I|^{2\gamma}\mathrm{dist}(I, J)^{-\gamma}$ for all $I, J \in D_k$ with $\mathrm{dist}(I, J) > 0$ and $k \geq 1$.

Then, if $\lambda > \gamma$, for any closed subset of the cube B such that $\mathcal{H}^{\lambda}(B) > 0$, there is some $p > 0$ such that

$$\mathbb{P}\{\dim(A \cap B) \geq \lambda - \gamma\} \geq p.$$

Proof. Fix some $\lambda > \gamma$ and some closed subset B of the cube with $\mathcal{H}^{\lambda}(B) > 0$. It suffices to show that there exists some $p > 0$ such that for all $\beta \in (0, \lambda - \gamma)$ that the event there is some positive measure μ on $A \cap B$ such that its β energy $I_{\beta}(\mu) < \infty$ has probability of at least p of occurring. This is because we can then apply the energy method to show that $\dim(A \cap B)$ is at least β.

First, by Frostman's lemma, there is some Borel probability measure ν on B and a positive constant C_1 such that if $D \subset \mathbb{R}^d$ is a Borel set, $\nu(D) \leq C_1|D|^{\lambda}$. For all $n \in \mathbb{N}$, we will define

$$A_n = \bigcup_{I \in D_n, Z(I)=1} I.$$

Then, for $n \in \mathbb{N}$, we can define a Borel measure μ_n supported on B defined as

$$\mu_n(D) = 2^{n\gamma}\nu(D \cap A_n).$$

Condition 1 implies that

$$\mathbb{E}[\mu_n(A_n)] = 2^{n\gamma} \sum_{I \in D_n} \nu(I)\mathbb{E}[Z(I)] \geq cd^{\frac{\gamma}{2}} \sum_{I \in D_n} \nu(I) = cd^{\frac{\gamma}{2}}.$$

Recall that for every cube I there are 3^d cubes J with $\mathrm{dist}(I, J) = 0$. This coupled with condition 2 implies that

$$\begin{aligned}\mathbb{E}[\mu_n(A_n)^2] &= 2^{2n\gamma} \sum_{I \in D_n} \sum_{J \in D_n} \mathbb{E}[Z(I)Z(J)]\nu(I)\nu(J) \\ &\leq Cd^{\gamma} \sum_{I \in D_n} \sum_{J \in D_n : \mathrm{dist}(I,J)>0} \mathrm{dist}(I, J)^{-\gamma}\nu(I)\nu(J) \\ &\quad + C_1 3^d \sqrt{d}^{\lambda} 2^{2n\gamma - n\lambda} \sum_{I \in D_n} \mathbb{E}[Z(I)]\nu(I).\end{aligned}$$

We also have that for all $x \in I$ and $y \in J$ such that $\mathrm{dist}(I, J) > 0$, $|x - y| \leq (1 + 2\sqrt{d})\mathrm{dist}(I, J)$. Consequently, we know that

$$\mathbb{E}[\mu_n(A_n)^2] \leq C((1 + 2\sqrt{d})d)^{\gamma} \int\int |x - y|^{-\gamma} d\nu(x)d\nu(y) + C3^d\sqrt{d}^{\lambda} d^{\frac{\gamma}{2}} = C_2 < \infty$$

where finiteness comes from Lemma 5.10.

Now, we fix $\beta \in (0, \lambda - \gamma)$. Again because of Lemma 5.10 which gives us a constant C_3 from the first statement and the fact that $\mathrm{dist}(I, J)^{-\gamma} \leq (3\sqrt{d})^{\gamma}|x-y|^{-\gamma}$ for $x \in I$ and $y \in J$, we get that

$$\begin{aligned}
\mathbb{E}[I_\beta(\mu_n)] &= 2^{2n\gamma} \sum_{I,J \in D_n} \mathbb{E}[Z(I)Z(J)] \int_I d\nu(x) \int_J d\nu(y)|x-y|^{-\beta} \\
&\leq Cd^{\gamma} \sum_{I \in D_n} \sum_{J \in D_n : \mathrm{dist}(I,J)>0} \mathrm{dist}(I,J)^{-\gamma} \int_I d\nu(x) \int_J d\nu(y)|x-y|^{-\beta} \\
&+ Cd^{\frac{\gamma}{2}} 2^{n\gamma} \sum_{I \in D_n} \sum_{J \in D_n : \mathrm{dist}(I,J)=0} \int_I d\nu(x) \int_J d\nu(y)|x-y|^{-\beta} \\
&\leq C_4 + CC_3 d^{\frac{\gamma}{2}} 2^{n\gamma} (3\sqrt{d}2^{-n})^{\lambda-\beta} \sum_{I \in D_n} \nu(I) \\
&\leq C_4 + CC_3 d^{\frac{\gamma}{2}} (3\sqrt{d})^{\lambda-\beta}.
\end{aligned}$$

This calculation therefore implies that $\mathbb{E}[I_\beta(\mu_n)]$ is uniformly bounded in n, so there's some $k(\beta)$ such that $\mathbb{E}[I_\beta] \leq k(\beta)$. So, we may find $l(\beta) > 0$ such that

$$\mathbb{P}\{I_\beta(\mu_n) \geq l(\beta)\} \leq \frac{k(\beta)}{l(\beta)} \leq \frac{c^2}{8C_2}.$$

We recall the Paley-Zygmund inequality from measure-theoretic probability: if X is a non-negative random variable with $\mathbb{E}[X^2] < \infty$, then for any $t \geq 0$, $\mathbb{P}\{X > t\} \geq \frac{\mathbb{E}[X-t]^2}{\mathbb{E}[X^2]}$ (a proof may be found in section 2 of chapter 6 in [2]). This implies that

$$\mathbb{P}\left\{\mu_n(A_n) > \frac{c}{2}\right\} \geq \mathbb{P}\left\{\mu_n(A_n) > \frac{1}{2}\mathbb{E}[\mu_n(A_n)]\right\} \geq \frac{1}{4}\frac{\mathbb{E}[\mu_n(A_n)]^2}{\mathbb{E}[\mu_n(A_n)^2]} \geq \frac{c}{4C_2}$$

meaning that

$$\mathbb{P}\left\{\mu_n(A_n) > \frac{c}{2}, I_\beta(\mu_n) < l(\beta)\right\} \geq p = \frac{c^2}{8C_2}.$$

By Fatou's lemma, we have that

$$\mathbb{P}\left\{\mu_n(A_n) > \frac{c}{2}, I_\beta(\mu_n) < l(\beta) \text{ infinitely often}\right\} \geq \liminf_{n\to\infty} \mathbb{P}\left\{\mu_n(A_n) > \frac{c}{2}, I_\beta(\mu_n) < l(\beta)\right\} \geq p.$$

We end the proof by choosing a subsequence where μ_n converges to a measure μ, which ensures that μ is supported by A and the μ measure of our cube is at least $\frac{c}{2}$. Therefore, for all $\epsilon > 0$, along the subsequence,

$$\int\int_{|x-y|>\epsilon} |x-y|^{-\beta} d\mu(x)d\mu(y) = \lim \int\int_{|x-y|>\epsilon} |x-y|^{-\beta} d\mu_n(x)d\mu_n(y) \leq \lim I_\beta(\mu_n) \leq l(\beta)$$

so taking the limit as ϵ decreases to 0 gives us the desired conclusion that $I_\beta(\mu) \leq l(\beta)$. □

Throughout the next two proofs, we fix an angle $\alpha \in (\pi, 2\pi)$ and a cube $A = x_0 + [0,1]^2 \subset W[\frac{\alpha}{2}, 0]$ so that A doesn't contain the origin. We will consider the classes D and D_k of compact dyadic subcubes, where D_k is the family of all cubes of the form $x + \Pi_{i=1}^2[\frac{k}{2^k}, \frac{k_i+1}{2^k}]$ and $k_1, k_2 \in \{0, 1, ..., 2^k - 1\}$ and D is the union of all compact dyadic subcubes in A, for side lengths equal to $\frac{1}{2^k}$ for some $k \in \mathbb{N} \cup \{0\}$. Let $R > 2$ such that $A \subset N_{\frac{R}{2}}(0)$ and for all $k \in \mathbb{N}$, we define

$$r_k = R - \sum_{j=1}^{k} \left(\frac{1}{2}\right)^j.$$

Observe that this quantity is always at least $\frac{R}{2}$ by our choice of R. For a subcube $I \in D_k$ with center z, we will define $Z(I) = 1$ if z is a $(2^{-k}, r_k)$ approximate cone point with direction π and otherwise, $Z(I) = 0$. Note that by definition of r_k, we have that if $I \subset J$ and $Z(I) = 1$, then $Z(J) = 1$. Therefore, we've set up a connection between the terminology in Theorem 5.12 and approximate cone points that we can begin to use to derive a lower bound.

Lemma 5.13. *There exist positive, finite constants $c_1 < C_1$ such that for any cube $I \in D$,*

$$c_1|I|^{\frac{2\pi}{\alpha}} \leq \mathbb{P}\{Z(I) = 1\} \leq C_1|I|^{\frac{2\pi}{\alpha}}.$$

Proof. Lemma 5.7 implies the existence of the upper bound here. On the other hand, we note that if $z \in A$ and $\delta > 0$, then

$$\inf_{|x-z|=\delta} \mathbb{P}\left\{B(T_{\frac{\delta}{2}}(z)) \in z + W\left[\frac{\alpha}{2}, \pi\right] : B(0) = x\right\} = \inf_{|x|=1} \mathbb{P}\left\{B(T_{\frac{1}{2}}(0)) \in W\left[\frac{\alpha}{2}, \pi\right] : B(0) = x\right\} > 0.$$

We will let this constant be equal to c_0. Therefore, if $I \in D_k$ and z is the center of I, for $\delta = 2^{-k}$, by Lemma 5.5 and Lemma 5.6,

$$\begin{aligned}
&\mathbb{P}\{Z(I) = 1\} \\
&\geq \mathbb{E}[(\chi_{B(0,T_\delta(z))\subset z+W[\alpha,\pi]})\mathbb{E}[(\chi_{B(T_{\frac{\delta}{2}}(z))\in z+W[\frac{\alpha}{2},\pi]}) \\
&\times \mathbb{P}\{B(0, S^0_{r_k}(z)) \subset z + W[\alpha, \pi] : B(0) = B(T_\delta(z))\} : B(0) = B(T_\delta(z))]] \\
&\geq c_0 c^2 \delta^{\frac{2\pi}{\alpha}} (R|z|)^{-\frac{\pi}{\alpha}}
\end{aligned}$$

for some constant $c > 0$. Because $|z|$ is bounded away from infinity, a lower bound follows. □

Lemma 5.14. *There is a positive, finite constant C_1 such that for any subcubes $I, J \in D_k$ for $k \in \mathbb{N}$,*

$$\mathbb{E}[Z(I)Z(J)] \leq C_1 |I|^{\frac{4\pi}{\alpha}} \operatorname{dist}(I, J)^{-\frac{2\pi}{\alpha}}.$$

Proof. Fix subcubes I and J and let their centers be z_I and z_J, respectively. Let $z = |z_I - z_J|$ and $\delta = 2^{-k}$. The first case we must consider is when $z > 2\delta$. By the strong Markov property, Lemma 5.5, Lemma 5.6, we have that

$$\begin{aligned}
&\mathbb{E}[Z(I)Z(J)\chi_{T_{\frac{\delta}{2}}(z_I)<T_{\frac{\delta}{2}}(z_J)}] \\
&\leq \mathbb{E}[(\chi_{B(0,T_\delta(z_I))\subset z_I+W[\alpha,\pi]})\mathbb{E}[(\chi_{B(0,S^{(0)}_{\frac{z}{2}}(z_I))\subset z_I+W[\alpha,\pi]}) \\
&\times \mathbb{E}[(\chi_{B(0,T_\delta(z_J))\in z_J+W[\alpha,\pi]})\mathbb{P}\{B(0, S^{(0)}_{r_k}(z_J)) \subset z_J + W[\alpha, \pi] : B(0) = B(T_{\frac{\delta}{2}}(z_J))\} : B(0) = B(T_{\frac{z}{2}}(z_J))] \\
&: B(0) = B(T_{\frac{\delta}{2}}(z_I))]] \\
&\leq C^4 \left(\frac{\delta}{|z_I|}\right)^{\frac{\pi}{\alpha}} \left(\frac{\delta}{z}\right)^{\frac{2\pi}{\alpha}} \left(\frac{2\delta}{R}\right)^{\frac{\pi}{\alpha}} \\
&\leq \frac{C_1}{2} |I|^{\frac{4\pi}{\alpha}} \operatorname{dist}(I, J)^{-\frac{2\pi}{\alpha}}
\end{aligned}$$

for $C_1 = 2^{\frac{\pi}{\alpha}+1}(\max(C^4, 1))$. Otherwise, we have that $z \leq 2\delta$. In this case, we have that

$$\begin{aligned}
&\mathbb{E}[Z(I)Z(J)\chi_{T_{\frac{\delta}{2}}(z_I)<T_{\frac{\delta}{2}}(z_J)}] \\
&\leq \mathbb{E}[(\chi_{B(0,T_\delta(z_I))\subset z_I+W[\alpha,\pi])})\mathbb{P}\{B(0, S^{(0)}_{r_k}(z_J)) \subset z_J + W[\alpha, \pi] : B(0) = B(T_{\frac{\delta}{2}}(z_J))\}] \\
&\leq C^2 (\frac{\delta}{|z_I|})^{\frac{\pi}{\alpha}} (\frac{2\delta}{R})^{\frac{\pi}{\alpha}} \\
&\leq \frac{C_1}{2} |I|^{\frac{4\pi}{\alpha}} \operatorname{dist}(I, J)^{-\frac{2\pi}{\alpha}}.
\end{aligned}$$

Note that similar logic holds to produce the same upper bound estimate when we reverse the roles of I and J, so the conclusion follows by adding the two estimates. □

We can now finish proving the lower bound, as we have the necessary tools to apply Theorem 5.12 in the prior two lemmas.

Theorem 5.15. *Let $\{B(t) : 0 \leq t \leq 1\}$ be a planar Brownian motion. Almost surely, for $\alpha \in [\pi, 2\pi)$,*

$$\dim\{x \in \mathbb{R}^2 : x \textit{ is an } \alpha \textit{ cone point}\} \geq 2 - \frac{2\pi}{\alpha}.$$

Proof. The lower bound holds by definition if $\alpha = \pi$. Otherwise, if we define A' to be the set in Theorem 5.12 that is obtained from our choice of $\{Z(I) : I \in C\}$, then we can see that $A' \subset A$" where

$$A" = \{B(t) : t > 0, B(0, S^{(t)}_{\frac{R}{2}}(B(t))) \subset B(t) + W[\alpha, \pi]\}.$$

By Theorem 5.12, $\dim(A") \geq 2 - \frac{2\pi}{\alpha}$ with positive probability. Now, let $\delta \in (0, \frac{1}{2})$ and $r > 0$. We define an increasing sequence of stopping times $\{t^{(\delta)}_n\}_{n\in\mathbb{N}}$ such that $t^{(\delta)}_1 = 0$ and for $k \in \mathbb{N} \setminus \{1\}$,

$$t^{(\delta)}_k = S^{(t^{(\delta)}_{k-1})}_{\delta R}(B(t^{(\delta)}_{k-1})).$$

Note that if we now define for $k \in \mathbb{N} \setminus \{1\}$,

$$A_k^{(\delta)} = \{B(t) : t_{k-1}^{(\delta)} \le t \le t_k^{(\delta)}, B(t_{k-1}^{(\delta)}, S_{\frac{R}{2}}^{(t)}(B(t))) \subset B(t) + W[\alpha, \pi]\},$$

then $A" \subset \bigcup_{k=1}^{\infty} A_k^{(\delta)}$. Let $\beta < 2 - \frac{2\pi}{\alpha}$. As the events $\{\dim(A_k^{(\delta)}) \ge \beta\}$ have the same probability, the probability cannot be zero for otherwise there would be a contradiction on the previously found dimension of the lower bound of $A"$. So, there exists some positive $p_R^{(\delta)}$ such that

$$\mathbb{P}\{\dim(B(t) : 0 \le t \le S_{\delta r}^{(0)}(0), B(0, S_{\frac{R}{2}}^{(0)}(0)) \subset B(t) + W[\alpha, \pi]) \ge \beta\} \ge p_R^{(\delta)}.$$

A scaling argument can then be applied to show that $p_R^{(\delta)}$ doesn't depend on R, meaning that by Blumenthal's zero-one law, $p_R^{(\delta)} = 1$ for all $\delta > 0$ and $R > 0$. Letting β increase to $2 - \frac{2\pi}{\alpha}$ implies that almost surely,

$$\dim\{B(t) : 0 \le t \le S_{\delta R}^{(0)}(0), B(0, S_{\frac{R}{2}}^{(0)}(0)) \subset B(t) + W[\alpha, \pi]\} \ge 2 - \frac{2\pi}{\alpha}.$$

This holds for any $\delta > 0$ and $R > 0$, so by choosing $R > 0$ and $\delta > 0$ such that $S_{\frac{R}{2}}^{(0)}(0) > 1$ and $S_{\delta R}^{(0)}(0) < 1$, we may finish the proof. □

In summary, we proved the following in this section, known as an example of a Hausdorff dimension spectrum.

Theorem 5.16. *Let $\{B(t) : 0 \le t \le 1\}$ be a planar Brownian motion. Almost surely, α cone points exist for $\alpha \ge \pi$ and don't exist for $\alpha < \pi$. For $\alpha \in [\pi, 2\pi)$,*

$$\dim\{x \in \mathbb{R}^2 : x \text{ is an } \alpha \text{ cone point}\} = 2 - \frac{2\pi}{\alpha}.$$

6. Further Directions

We end the paper with a brief look at some of the directions studies of the dimensions of cone points lead to, beginning with applications to the convex hull of a planar Brownian motion and then to a line's intersection with a planar Brownian motion.

Theorem 6.1. *Let $\{B(t) : 0 \le t \le 1\}$ be a planar Brownian motion. Then, almost surely, its convex hull H has a differentiable boundary.*

Proof. It suffices to show that for all $x \in \partial H$ that there doesn't exist a cone with vertex x and opening $\alpha < \pi$ that contains H. This is because we may see that the supporting hyperplanes are unique at all $x \in \partial H$ which implies the conclusion. We have by Spitzer's theorem that $B(0)$ and $B(1)$ almost surely don't have this property. For other points $x \in \partial H$, if these have the property, then we must have that $x = B(t)$ for some $t \in (0, 1)$ and that t is an α-cone point, a contradiction to Theorem 5.16, proving the result. □

Theorem 6.2. *Let $\{B(t) : 0 \le t \le 1\}$ be a planar Brownian motion. Then, its convex hull H has no isolated extreme points and if E denotes the set of extreme points,* $\dim(E) = 0$.

Proof. As extreme points of convex hulls of continuous curves belong to the curve itself, we have by Theorem 5.16 that as E is contained in the set of all π-cone points that $\dim(E) = 0$. On the other hand, the Krein-Milman theorem implies that if x is an isolated extreme point, then H is the convex hull of $\{x\} \cup (H \setminus N_\epsilon(x))$ for some $\epsilon > 0$. This implies H has a corner at x, contradicting Theorem 6.1. □

Finally, we turn to the first intersection of a line with a Brownian motion.

Theorem 6.3. *For $y \in \mathbb{R}$, let $L_y = \{x + iy : x \in \mathbb{R}\}$ and let $x(y) = \sup\{x : x + iy \in B[0, 1]\}$ where $\sup \emptyset = -\infty$. Then, almost surely, for almost every $y \in \mathbb{R}$ with respect to Lebesgue measure, $x(y) = -\infty$ or for all $\theta > 0$*

$$\{x(y) + iy + re^{iu} : r > 0, |u| < \theta\} \cap B[0, 1] \ne \emptyset.$$

Visually, $x(y) + iy$ is the first hitting point of the restriction of the Brownian motion to $[0, 1]$ of a particle coming from infinity on the line L_y. The result tells us that this point, if it exists, usually isn't a cone point. While the proof is omitted here, a similar result holds for all directions.

Proof. By Theorem 5.16, for all $\theta > 0$, the dimension of all cone points with angle $2\pi - \theta$ is less than 1 almost surely. So, the dimension of its projection of $x + iy$ onto y is less than 1 almost surely making its Lebesgue measure 0 almost surely. We can therefore take a sequence of angles $\{\theta_n\}_{n=1}^{\infty}$ decreasing to 0. It follows that the Lebesgue measure of the unions of such projections is 0 almost surely, implying the theorem. □

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