INTRODUCING MODULI SPACES: TWO EXAMPLES.

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Abstract. The purpose of this paper is to introduce and motivate moduli spaces through a few key examples. These are the moduli space of elliptic curves and the moduli space of triangles. We also show that the moduli space of acute triangles is equivalent to the moduli space of elliptic curves. This paper side-steps much of the heavy scheme-theoretic machinery that is used in the study of moduli spaces, and rather seeks to introduce the reader to intuition that will prepare them for rigorous study of moduli problems and stacks.

Contents

1. Introduction

The goal of this paper is to motivate the construction of moduli spaces through a few key examples — the moduli space of complex elliptic curves and the moduli space of triangles. Moduli spaces are spaces whose points correspond to isomorphism classes of objects. Their purpose is to 'classify' objects geometrically in some way. They are inspired by a compelling collection of motivating principles, but their technical study tends to rely the heavy machinery of stacks. Nonetheless, moduli spaces and their study have some inherent appeal — the core intuition behind the construction is compelling even without the formalisms that typify its

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study. In this paper, I will give an accessible introduction to the topic in language that is accessible to students with some background in basic topology, complex and real analysis, and general awareness of what a category is. Some background in algebraic geometry may be useful but is not required.

In Section 2, I will discuss what it means for objects to vary continuously in a family. This will motivate the definitions of coarse and fine moduli spaces, which are defined formally in the next section.

In Section 3, I will construct the moduli space of elliptic curves or smooth cubics, $M_{1,1}$, by relating complex elliptic curves to complex lattices.

In Section 4, I will introduce the moduli spaces of triangles, an important example that helps illustrate the differences between coarse and fine moduli spaces.

In Section 5, we show that the moduli space of acute triangles is related to the moduli space of elliptic curves — and in particular that acute triangles up to similarity correspond to isomorphism classes of elliptic curves.

In Section 6, we introduce the reader to some problems that remain open in the study of moduli spaces.

2. Varying Smoothly in a Family

Given a collection of objects and some notion of similarity between these objects, a natural question to ask is how objects in this collection differ 'modulo' the particular notion of similarity that we choose. In other words, how to classify these objects taking their similarities into account.

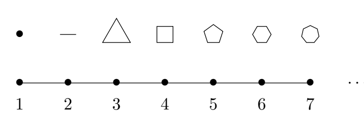
A family is a collection of objects. One of the nice properties of moduli spaces is that they preserve the structure of continuous families of objects (introduced in Definition 2.2). In this section, we will define continuous families and motivate the definitions of moduli spaces.

We begin with a discrete example of a family of objects.

Example 2.1. We consider polygons in euclidean geometry under the notion of similarity given by their n-gon classification.

In particular, we study polygons with the understanding that two polygons are the same if they have the same number of sides. All triangles are similar, all quadrilaterals are similar, etc. We will refer as *similarity classes* for the remainder of this discussion.

There is a natural parameterization of similarity classes of polygons via the natural numbers — with the degenerate monogon being parameterized by 1. Like many of the other objects we study via their encoding in moduli spaces, there are degenerate polygons. For many moduli spaces, it is the boundary of these spaces that encode degenerate objects.



Definition 2.2. A *Continuous Family* is a family of objects $\mathscr{F}$ which is parameterized by the topological space T. We usually write this as $\mathscr{F}/T$. For each $t \in T$, we denote the object in the family to which this points corresponds $\mathscr{F}_t$. We will usually require that the map $T \to \mathscr{F}$ which defines the parameterization is continuous.

For instance, Example 2.1 fails to qualify as a continuous family because the corresponding map is not continuous. Even though the polygons in this example look like they satisfy some kind of continuity, there isn't a well defined topology of the set {polygons} to make the corresponding map $T \to \mathscr{F}$ (above it's $\mathbb{Z} \to$ {polygons} continuous.

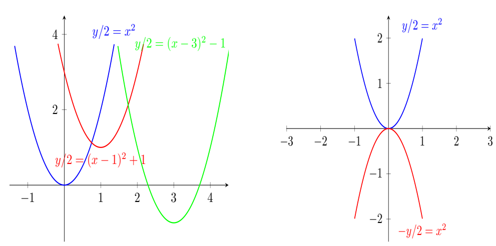
This is a rather imprecise notion of what it means for objects to vary continuously in a family. We will later introduce a more explicit definition for continuous families of triangles (Definition 4.2). However, in the interest of developing intuition for what exactly it will mean to vary continuously in a family in this section, Definition 2.2 will suffice.

Example 2.3. The objects that we study are parabolas in $\mathbb{R}^2$ which are given by equations of the form

$$ay = (x-b)^2 + c \tag{2.4}$$

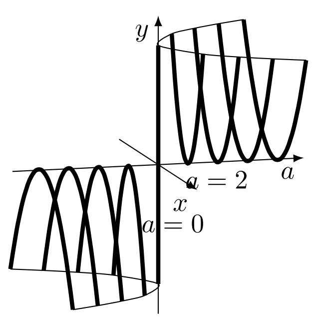
with $a, b, c \in \mathbb{R}$. For $a \neq 0$, this curve looks like a normal parabola lying in the plane. We will consider two parabolas to be 'similar' if they are the same up to translation in $\mathbb{R}^2$ – namely, if they are characterized by the same choice of a.

Under translation, the three parabolas on the right would be similar, however the two parabolas on the left would not be since they are oriented differently (a has a different sign).



Since similar parabolas have the same a value, these objects may be parameterized by choice of 'a'. Since the parabolas we consider live in $\mathbb{R}^2$, In particular, they may be parameterized by the real line $\mathbb{R}$ with a degenerate parabola at $a = 0$.

FIGURE 1. A continuous family of parabolas. Adapted from a figure in [9] p. 90



We may view the parameterization space in Example 2.3 informally as a motivating example for the study of moduli spaces (introduced formally in the next Section 2.1). However, this example is a naïve classification space and not a moduli space. This is becuase all parabolas are isomorphic, and the notion of 'similarity' used for true moduli spaces is always isomorphism.

For a more exhaustive an informal discussion of continuous families, their properties, and how this motivates certain aspects of the definition of Coarse and Fine Moduli Spaces, see [2].

So far we have discussed only notions of 'geometric similarity' — translation invariance, number of sides, etc. However, our notion of two objects being similar need not only be geometric in nature — as we will see in the case of elliptic curves, where we describe two curves as similar if there is a group homomorphism.

2.1. **Defining Moduli Spaces.** There are two notions of moduli space that we work with — fine and coarse moduli spaces. We first introduce the moduli map of a family.

Definition 2.5. If $\mathscr{F}/T$ is a continuous family of parameterized by a topological space T, we define the *moduli map* of $\mathscr{F}$ to be the continuous map $T \to M$ which gives rise to the pasteurization (M may be either a coarse or fine moduli space. Definitions to follow).

With this definition being established, we can define the fine moduli space.

Definition 2.6. A *fine moduli space* is a space M such that

(i) the points of M are in one-to-one correspondence with isomorphism classes of the objects we are studying, which are the objects in $\mathscr{F}$.
(ii) for every family $\mathscr{F}/T$, the associated moduli map $T \to M$ (which maps the point $t \in T$ to the isomorphism class of the family member $\mathscr{F}$) is continuous.
(iii) every continuous map from a space T to M is the moduli map of some family parameterized by T (equivalently M parametrizes a family $\mathscr{M}$, whose moduli map is the identity id_M),
(iv) if two families have the same moduli map, they are isomorphic families.

Definition 2.7. A *coarse moduli space* is a space M such that the first two conditions of Definition 2.6 are satisfied and moreover M carries the finest topology making condition (ii) true.

Remark 2.8. A standard construction for a coarse moduli space might proceed as follows:

- Begin with the set of all isomorphism classes of the objects that are considered and take this to be M, the space studied
- Endow M with the finest topology such that all moduli maps of continuous families are continuous. For example, by taking the topology on the set to be formed by taking the union of all topologies which satisfy this property.

This gives a coarse moduli space. Moreover, this construction gives the *only* coarse moduli space for the objects considered (up to homomorphism).

Proposition 2.9. *Any space satisfying the first three conditions of Definition 2.6 is a coarse moduli space*

Proof. See [2] p. 21. □

In particular, every fine moduli Space is a coarse moduli space. The converse of Proposition 2.9 does not hold. (See [2] p.25-29 for an example).

Remark 2.10. If we think intuitively, and pretend that the notion of 'similarity' used in Definition 2.6 of fine moduli spaces is that used in Example 2.3 with the continuous family of parabolas, then the family satisfies properties (i)-(iv) somewhat trivially.

(i) The space is constructed so that points on the real line correspond to similarity class of parabolas.
(ii) We require all continuous families $\mathscr{F}/T$ to be defined by a continuous mapping, $T \to \mathscr{F}$. By taking this notion of continuity precisely to mean that for an arbitrary family $\mathscr{F}/T$ the mapping $T \to \mathbb{R}$ is continuous, this space is constructed to satisfy this property.
(iii) Likewise, the space is constructed to satisfy (iii) under the notion of continuity used in the preceding bullet.
(iv) If $\mathscr{F}/T$ and $\mathscr{G}/T$ are arbitrary families of parabolas parameterized by the same moduli map $T \to \mathbb{R}$, then $\mathscr{F}$ and $\mathscr{G}$ must be similar families, since the map gives rise to the parameterization.

3. Elliptic Curves

In this section we construct the moduli space of elliptic curves. This construction relies on a correspondence between isomorphism classes of elliptic curves and complex tori. This correspondence is described by the Uniformization Theorem (Corollary 3.21).

Over the course of this section, we will work up to the Uniformization Theorem before introducing the moduli space of elliptic curves.

3.1. Defining Elliptic Curves.

The objects that we study are *elliptic curves* $E/\mathbb{C}$. An elliptic curve $E/\mathbb{C}$ is the set of solutions of a *Weierstrass equation* of the form

$$E : y^2 = x^3 + Ax + B \tag{3.1}$$

for scalars A and B in $\mathbb{C}$. It just so happens that these are precisely the smooth projective curves of genus one having a distinguished point $\mathcal{O}$ at infinity. For rigorous treatment see [10] p.42-51. Associated with any curve are the quantities

$$\Delta(E) = -16(4A^3 + 27B^2) \qquad \text{and} \qquad j(E) = -1728\frac{(4A)^3}{\Delta}.$$

We refer to these as the discriminant and the j-invariant of an elliptic curve.

Proposition 3.2. *(a) The curve given by a Weierstrass equation is non-singular and has no cusps or points of self intersections if and only if $\Delta = 0$. In particular, a Weierstrass equation of the form in (3.1) defines a (non-degenerate) elliptic curve if, and only if, its discriminant is non-zero.*

(b) Two elliptic curves over $\mathbb{C}$ are isomorphic (see Definition 3.4, to follow) if and only if they both have the same j-invariant.

Proof. See [10] p. 46 where these results are shown for elliptic curves defined over an arbitrary algebraically closed field. □

FIGURE 2. Three representative examples of elliptic curves with non-zero discriminants. Adapted from [10]

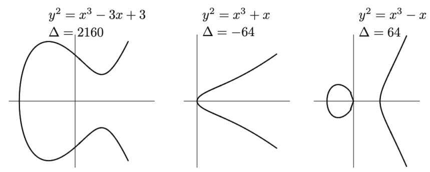
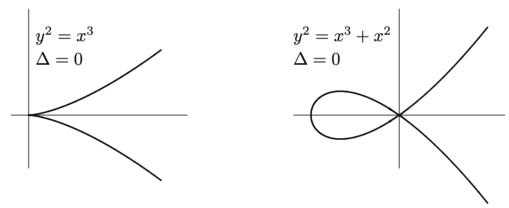


FIGURE 3. Two elliptic curves with discriminant zero. Adapted from [10]



The degenerate elliptic curves are precisely those which have discriminant zero. Examples appear in Figure 3.

Next we define what it means for elliptic curves to be isomorphic. This notion of isomorphism relies on the fact that elliptic curves satisfy a group law, since two elliptic curves will be isomorphic if they have a map between then which preserved the group structure.

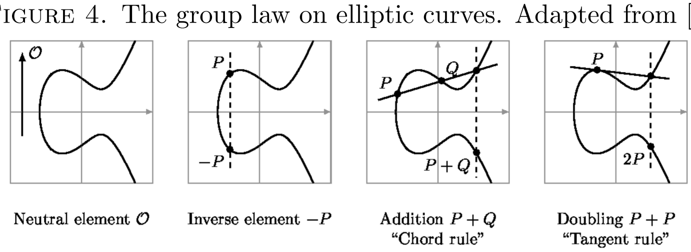
Proposition 3.3. *Every elliptic curve E with distinguished point $\mathcal{O}$ at infinity has the structure of a group with identity $\mathcal{O}$.*

The group law is defined geometrically as follows. For two points P and Q on an elliptic curve E, draw the line that intersects P and Q. This line will meet E at a third point (possibly the point at infinity) which we call R. We then take

$$P + Q = -R.$$

Proof. See [10] p. 51. ∎

FIGURE 4. The group law on elliptic curves. Adapted from [4]



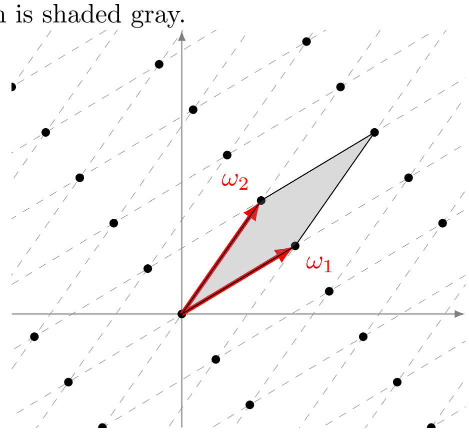
Next, we state the definition of an isomorphism of elliptic curves.

Definition 3.4. An isomorphism between two elliptic curves E and E' over $\mathbb{C}$ is an *isogeny* (group homomorphism) such that there exists an isogeny $g : E' \to E$ with the compositions $g \circ f$ and $f \circ g$ being the identity maps.

3.2. **Lattices.** In this section we introduce complex lattices and some of their properties. We begin by defining lattices in the complex plane. This will allow us to classify elliptic curves.

Definition 3.5. We define a lattice $\Lambda = [\omega_1, \omega_2]$ in $\mathbb{C}$ to be the additive subgroup $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ of $\mathbb{C}$ where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.

FIGURE 5. The lattice generated by ω_1, ω_2 in $\mathbb{C}$. The fundamental paralelogram is shaded gray.



Taking the quotient of the the complex plane modulo a complex lattice Λ yields a one dimensional complex torus, $\mathbb{C}/\Lambda$. In particular, where the *fundamental parallelogram* of a lattice $\Lambda = [\omega_1, \omega_2]$ is the parallelogram with vertices at 0, ω_1, ω_2, and $(\omega_1 + \omega_2)$. We may identify points on the fundamental parallelogram with the points of the complex torus $\mathbb{C}/\Lambda$. See Figure 5.

Remark 3.6. Two lattices Λ_1, Λ_2 are *homothetic* if there exists some $\alpha \in \mathbb{C}^*$ such that $\Lambda_1 = \alpha\Lambda_2$ – i.e. they are equivalent under scaling.

Moreover, any lattice $[\omega_1, \omega_2]$ over $\mathbb{C}$ is homothetic to a lattice of the form $[1, \tau]$ over $\mathbb{C}$ where τ lies in the upper half plane, $\mathbb{H}$, and takes the form $\tau = \pm\frac{\omega_2}{\omega_1}$ taking the sign to be such that $\tau > 0$.

Homothety is an equivalence relation between latices on $\mathbb{C}$.

We will see shortly that every elliptic curve $E(\mathbb{C})$ is isomorphic to a 1-dimensional complex torus $\mathbb{C}/\Lambda$ for some lattice Λ (The Uniformization Theorem, Corollary 3.21). Moreover, elliptic curves, up to isomorphism, are equivalent to lattices, up to homothety, as categories. Before stating this theorem, however, we will first introduce the Weierstrass $\wp$-function, from which we may derive the correspondence between complex latices and complex elliptic curves.

Definition 3.7. The *Weierstrass $\wp$-function* relative to a lattice Λ over $\mathbb{C}$ is defined by the series

$$\wp(z) := \wp(z; L) := \frac{1}{z^2} + \sum_{w \in \Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

The Weierstrass $\wp$ function and its derivative have many nice properties. Some of these appear in the following theorem.

Theorem 3.8. *(a) The series defining the Weierstrass $\wp$-function converges absolutely and uniformly on every compact subset of $\mathbb{C} \setminus \Lambda$.*

(b) The series defines a meromorphic function on $\mathbb{C}$ having a double pole at each lattice point and no other poles.

(c) Moreover, the same is true of its derivative

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

except this function has a triple pole at each lattice point.

In particular, both $\wp(z)$, $\wp'(z)$ are holomorphic at every $z \notin \Lambda$.

(d) The Weierstrass function $\wp$ is an even function and satisfies the condition

$$\wp(z+\omega) = \wp(z), \qquad \textit{for all } \omega \in \Lambda, z \in \mathbb{C}.$$

Proof. See [10] p.165. □

In fact, we call all meromorphic functions on $\mathbb{C}$ satisfying the condition given in part (d) *elliptic functions*. We may think of elliptic functions as functions over $\mathbb{C}/\Lambda$ for lattice Λ. This is because it is invariant under translation, by elements of Λ, so it's well defined on $\mathbb{C}/\Lambda$. Then we may equivalently state Theorem 3.8(d) as *The Weierstrass function $\wp(z)$ is an elliptic function.* Where Λ is a lattice, we denote the set of all elliptic function by $\mathbb{C}(\Lambda)$.

Proposition 3.9. *A holomorphic elliptic function, which is to say an elliptic function with no zeros, is constant.*

Proof. See [10] p.161. This result follows from Liouville's Theorem. □

3.3. Elliptic Curves from Lattices.

Using the Weierstrass $\wp$ function constructed in the previous section, we will show how a complex lattice gives rise to an elliptic curve.

We begin by deriving the Laurent series expansion for $\wp(z)$ around $z = 0$, from which we will deduce a differential equation ((3.12)) which is the key link between $\wp(z)$ and elliptic curves.

Lemma 3.10. *If Λ is a lattice over $\mathbb{C}$ and $k \in \mathbb{Z}_{>2}$, the* Einstein Series *of weight k is the series*

$$G_k(\Lambda) = \sum_{\omega \in \Lambda^*} \omega^{-k}.$$

The Laurent series for $\wp(z)$ around $z = 0$ is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}$$

Proof. First, we recall a familiar power series expansion:

$$\frac{1}{(1-x)^2} = (1 + x + x^2 + \cdots)^2 = \sum_{n=0}^{\infty} (n+1) x^n,$$

for all $|x| < 1$. However, if $|z| < |\omega|$ we deduce the following equivalence:

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega} = \frac{1}{\omega^2}\left(\frac{1}{1-z/\omega)^2} - 1\right) = \sum_{n=1}^{\infty}(n+1)\frac{z^n}{\omega^{n+2}}.$$

Substituting this formula into the series for $\wp(z)$ and reversing the order of summation (by absolute convergence) gives

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \\
&= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\sum_{n=1}^{\infty}\frac{(n+1)z^n}{\omega^{n+2}} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)z^n\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{n+2}} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)z^n\sum_{n=1}^{\infty}(n+1)G_{n+2}(\Lambda) \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)z^n\sum_{n=1}^{\infty}(2n+1)G_{2n+2}(\Lambda)z^{2n}
\end{aligned}$$

In the last step we are using the fact that $\wp(z)$ is an even function (Theorem 3.8), so the coefficients of the odd terms are 0 and we can sum over 2n rather than n. ☐

Finally we may introduce the differential equation which defines the correspondence between complex elliptic curves and Lattices over $\mathbb{C}$.

Theorem 3.11. *The function $\wp(z)$ satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda), \tag{3.12}$$

where $g_2(\Lambda) := 60G_4(\Lambda)$ and $g_3(\Lambda) := 140G_6(\Lambda)$.

Proof. By applying Lemma 3.10, we begin by computing the first few terms of the Laurent series expansion for $\wp(z)$ and $\wp'(z)$ at $z = 0$.

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + 3G_4(\Lambda)z^2 + 5G_6(\Lambda)z^4 + \cdots \\
\wp(z)^3 &= \frac{1}{z^2} + \frac{9G_4(\Lambda)}{z^2} + 15G_6(\Lambda) + \cdots \\
\wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_4(\Lambda)}{z^2} - 80G_6(\Lambda) + \cdots
\end{aligned}$$

Next, Let

$$f(x) = \wp'(z)^2 - 4\wp(z)^3 + 60G_4(L)\wp(z) + 140G_6(\Lambda).$$

By computing the Laurent series expansion for $f(z)$ at $z_0 = 0$ we see that the function satisfies $f(0) = 0$ and moreover is holomorphic at $z = 0$.

However, f is an elliptic function relative to Λ (it follows from their definition that a linear combination of elliptic functions is elliptic). Then applying Theorem 3.8 f is a holomorphic on $z \notin \Lambda$, and in fact is holomorphic on all of $\mathbb{C}$ since earlier in this proof we have shown $f(0)$ is holomorphic at $z = 0$.

However, Proposition 3.9 asserts that a holomorphic elliptic function is constant. Since $f(0) = 0$, then f is uniformly 0, and the proof is complete. □

In particular, the formula from the preceding theorem ((3.12)) allows us to derive an elliptic curve E_Λ from the lattice Λ by taking $x := \wp(z)$ and $y := \wp'(z)$ and defining E_Λ as the complex elliptic curve defined by the following Weierstrass equation:

$$E_\Lambda : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda).$$

Indeed, E_Λ is an elliptic curve and:

Theorem 3.13. *For any complex lattice Λ over $\mathbb{C}$, the discriminant $\Delta(E_\Lambda)$ is non-zero, and this is sufficient to show that the curve $E_\Lambda/\mathbb{C}$ is an elliptic curve.*

Moreover, the map

$$\phi : \mathbb{C}/\Lambda \to E_\Lambda(\mathbb{C}), \qquad z \mapsto (\wp(z), \wp'(z))$$

is a group isomorphism where the group structure on Λ is defined by addition in $\mathbb{C}$.

Proof. See [10] p. 170. □

In particular, complex lattices give rise to elliptic curves over $\mathbb{C}$. This is the first half of the Uniformization theorem. The second half is showing lattices up to homothety are equivalent to elliptic curves with the same j-invariant.

3.4. **Lattices and the j-invariant.** We are now equipped to prove the uniformization theorem.

Theorem 3.14. *If Λ_1 and Λ_2 are complex lattices, then they are homothetic if and only if $j(E_{\Lambda_1}) = j(E_{\Lambda_2})$.*

Proof. Consider complex lattices Λ_1, Λ_2 and $\alpha \in \mathbb{C}^*$ such that $\Lambda_2 = \alpha\Lambda_1$. Notice:

$$g_2(\Lambda_2) = 60 \sum_{\omega \in \Lambda_2^*} \omega^{-4} = 60 \sum_{\omega \in \Lambda^*} (\alpha\omega)^{-4} = \alpha^{-4} g_2(\Lambda_1).$$

Similarly $g_3(\Lambda_2) = \alpha^{-6} g_3(\Lambda_1)$. Notice that the elliptic curve defined by

$$y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

for any lattice Λ is isomorphic to the elliptic curve

$$y^2 = x^3 + (-\frac{g_2(\Lambda)}{4})x + (-\frac{g_3(\Lambda)}{4})$$

(Substituting $y = \frac{y'}{2}$ in the first equation yeilds the second, and this clearly an isomorphism). This latter curve is in the form of (3.1), and so in calculating the j invariant we take $A_1 = -\frac{g_2(\Lambda_1)}{4}$ and $B_1 = -\frac{g_3(\Lambda_1)}{4}$ while defining A_2 and B_2 similarly. This yields

$$\begin{aligned}
j(E_{\Lambda_2}) &= 1728\frac{4A_2^3}{-16(4A_2^3+27B_2^2)} \\
&= -1728\frac{(g_2(\Lambda_2))^3}{16((g_2(\Lambda_2))^3+27(g_3(\Lambda_2)^2} \\
&= -1728\frac{(\alpha^{-4}g_2(\Lambda_1))^3}{16((\alpha^{-4}g_2(\Lambda_2))^3+27(\alpha^{-6}g_3(\Lambda_2)^2} \\
&= 1728\frac{4A_1^3}{-16(4A_1^3+27B_1^2)} \\
&= j(E_{\Lambda_1}).
\end{aligned}$$

□

Corollary 3.15. *Complex lattices Λ_1 and Λ_2 are homothetic if and only if the corresponding elliptic curves E_{Λ_1} and E_{Λ_2} are isomorphic.*

Proof. This corollary is a direct consequence of the preceding result (Theorem 3.14) and Proposition 3.2.b. □

Recall that every complex lattice is homothetic to a lattice of the form $[1, \tau]$ with τ a point in the upper half plane $\mathbb{H}$ (Remark 3.6). This allows us to define the *j-function*.

Definition 3.16. The *j-function* is the function $j : \mathbb{H} \to \mathbb{C}$ which is defined by

$$j(\tau) := j(E_{[1,\tau]}).$$

Two lattices, under the j-function, output the same value when they give rise to isomorphic elliptic curves.

This prompts us to investigate which τ lying in the upper half plane give the same output values under the j-function. To this end, we introduce the modular group.

Definition 3.17. The modular group, or $\mathrm{SL}_2(\mathbb{Z})$ is the following group

$$\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

which consists of linear fractional transformations of the upper half plane $\mathbb{H}$ which have the following form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

Lemma 3.18. *For any $\tau_1, \tau_2 \in \mathbb{H}$ we have $j(\tau_1) = j(\tau_2)$ if and only if $\tau_1 = \gamma\tau_2$ for some $\gamma \in SL_2(\mathbb{Z})$*

Proof. See [11]. □

To determine the moduli space of elliptic cuves, we need only determine the moduli space of lattices in $\mathbb{C}$ per Corollary 3.15. The preceding lemma (Lemma 3.18) suggests that to define the moduli space of complex lattices, we may look to the complex plane modulo the action of the modular group $\mathbb{C}/\mathrm{SL}_2(\mathbb{Z})$. Consider the following set:

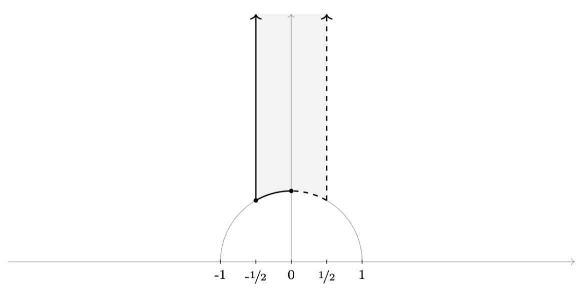
Lemma 3.19. *The set $\mathcal{F}$ is a fundamental domain for $\mathbb{H}/SL_2(\mathbb{Z})$.*

$$\mathcal{F} = \{\tau \in \mathbb{H} : re(\tau) \in [-\frac{1}{2}, \frac{1}{2}) \text{ and } |\tau| \geq 1, \text{ such that } |\tau| > 1 \text{ if } re(\tau) > 0\}.$$

That is to say, every point in $\mathbb{H}$ may be obtained by acting on a point in $\mathcal{F}$ by $SL_2(\mathbb{Z})$. (This is what being a fundamental domain means).

Proof. See [11]. □

Figure 6. Fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Z})$.



This set, $\mathcal{F}$, parameterizes complex latices up to homothety, since the j function is invariant under $\mathrm{SL}_2(\mathbb{Z})$ action.

This leads to the following theorem and a corollary, the uniformization theorem.

Theorem 3.20. *The restriction of the j-function to $\mathcal{F}$ defines a bijection from $\mathcal{F}$ to $\mathbb{C}$.*

Proof. The injectivity of the j-function follows from Lemma 3.18 and Lemma 3.19. Surjectivity remains to be proven. We have

$$g_2(\tau) = 60 \sum_{n,m\in\mathbb{Z},(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4} = 60\Big(2\sum_{m=1}^{\infty}\frac{1}{m^4} + \sum_{n,m\in\mathbb{Z},n\neq 0}\frac{1}{(m+n\tau)^4}\Big).$$

The second sum tends to 0 as $\tau \to \infty$. THen we have:

$$\lim_{\mathrm{im}(\tau)\to\infty} g_2(\tau) = 120\sum_{m=1}^{\infty} m^{-4} = 120\zeta(4) = 120\frac{\pi^4}{90} = \frac{4\pi^3}{3}$$

where $\zeta(s)$ is the Riemann zeta function. Similarly,

$$\lim_{\mathrm{im}(\tau)\to\infty} g_3(\tau) = 280\zeta(6) = 280\frac{\pi^6}{945} = \frac{8\pi^6}{27}.$$

Thus,

$$\lim_{\mathrm{im}(\tau)\to\infty} \Delta(\tau) = \big(\frac{4}{3}\pi^4\big)^3 - 27\big(\frac{8}{27}\pi^6\big)^2 = 0.$$

This also explains the coefficients 60 and 140 in the definitions of g_2 and g_3 in ((3.12)).

Since $\Delta(\tau)$ is in the denominator of the quantity $j(\tau)$, the quantity $j(\tau) = -1728(g_2(\tau)^3)/\Delta(\tau)$ is unbounded as $\text{im}(\tau) \to \infty$.

In particular, the j function is non-constant and thus holomorphic on $\mathbb{H}$ (see [11]). It follows from the open mapping theorem that $j(\mathbb{H})$ is an open subset of $\mathbb{C}$.

We also claim that $j(\mathbb{H})$ is a closed subset of $\mathbb{C}$. Let $\jmath(\tau_1), j(\tau_2), \ldots$ be an arbitrary convergent sequence in $j(\mathbb{H})$, converging to $\omega \in \mathbb{C}$. Since the j-function is $\text{SL}_2(\mathbb{Z})$ invariant by Lemma 3.18, we may assume the τ_n all lie in $\mathcal{F}$. The sequence $\text{im}(\tau_1), \text{im}(\tau_2), \ldots$ must be bounded. This follows from the fact that $j(\tau) \to \infty$ as $\tau \to \infty$ but the sequence $j(\tau_1), j(\tau_2), \ldots$ converges; it follows that the τ_n all lie in the compact set

$$\Omega = \{\tau : \text{re}(\tau) \in [-1/2, 1/2], \text{im}(\tau) \in [1/2, B]$$

where B is some bound on the sequence $\text{im}(\tau_1), \text{im}(\tau_2), \ldots$.

There is thus a sub-sequence of the τ_n that converges to some $\tau \in \Omega \subset \mathbb{H}$. The j-function is holomorphic, hence continuous, so $j(\tau) = \omega$. It follows that the open set $j(\mathbb{H})$ contains all its limit points and is therefore closed.

The fact that the non-empty set $j(\mathbb{H}) \subseteq \mathbb{C}$ is both open and closed implies that $j(\mathbb{H}) = \mathbb{C}$, since $\mathbb{C}$ is connected. It follows that $j(\mathcal{F}) = \mathbb{C}$, since every element of $\mathbb{H}$ is equivalent under $\text{SL}_2(\mathbb{Z})$ action to an element of $\mathcal{F}$ (Lemma 3.19) and the j-function is invariant under $\text{SL}_2(\mathbb{Z})$ action (Lemma 3.18). $\square$

A direct consequence:

Corollary 3.21 (The Uniformization Theorem)**.** *For every elliptic curve $E/\mathbb{C}$ there exists a lattice Λ such that $E = E_\Lambda$.*

Proof. Pick $E/\mathbb{C}$ and pick $\tau \in \mathbb{H}$ such that $j(\tau) = j(E)$ by Theorem 3.20. Then $j(E_{[1,\tau]}) = j(E)$ and thus, by Proposition 3.2 it follows that $E_{[1,\tau]}$ and E are isomorphic. $\square$

3.5. **The Moduli Space of Elliptic Curves.** In Theorem 3.20 we showed that points in $\mathbb{H}/\text{SL}_2(\mathbb{Z})$ correspond to isomorphism classes of elliptic curves. We will denote this quotient by $M_{1,1}$. This follows the standard convention of naming the coarse moduli space of compact Riemann surfaces of genus g with n marked points.

Proposition 3.22. *$M_{1,1}$ is a coarse moduli space (Definition 2.7).*

Proof. See [8] p.12. $\square$

4. Triangles

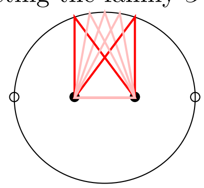
In this section we formally introduce the moduli space of triangles (up to similarity).

4.1. **Continuous families of triangles.** We have already discussed continuous families of parabolas, and how these suggest some of the basic motivation for moduli space constructions. Now, we introduce a more precise notion of continuous families of triangles. But first, we introduce an example which will guide our discussion in this subsection. (The following exmaple is borrowed from [2] and is used in a proof of a later result Proposition 4.4)

Example 4.1. Consider the continuous family of triangles $\mathscr{F}$ (shown in red in the figure below) which we define as follows. First, consider two pins which lie 1/2 distance apart and suppose that we have a string of length 2.

By taking a pen and our loop of string, we may draw an ellipse around these two points. Moreover, at any point while drawing this ellipse, the tip of our pen and the two pins together form a (possibly degenerate) triangle. The triangle is *degenerate* when the tip of our pen is in line with the two pins and the three lines which form the triangle coincide. There are two such points, and these are marked with open dots.

FIGURE 7. constructing the family $\mathscr{F}$, adapted from [2].



The family $\mathscr{F}$ that we consider are those triangles which are formed when our pen lies directly above the left pin, and move the pen continuously to the right until it lies directly above the right pin. The family begins with a 3:4:5 right triangle of lengths 1/2, 2/3, and 5/6 and ends with a congruent 3:4:5 triangle. These are the oriented acute triangles of perimeter length 2 whose shortest side is of length 1/2. We may think of this family of triangles being parameterized by the interval [-1/4, 1/4]. Let $\mathscr{F}_t$ for $t \in [-1/4, 1/4]$ corresponds to the triangle for which the x-coordinate of the tip of the pen lies at t.

What makes Example 4.1 compelling as an example of a continuous family is that moving continuously through the family parameterizes triangles whose side-lengths change in a continuous manner.

To better reflect the particular properties of triangles, we will introduce a more specific notion of continuous families of triangles which is tailored to their study (Definition 4.2).

Definition 4.2. A *continuous family of triangles* parameterized by the topological space T consists of a degree 3 covering map $T' \to T$ and a continuous map $a : T' \to \mathbb{R}_{>0}$. The data $\mathscr{F} = (T', a)$ has to satisfy the triangle inequalities for all $t \in T$. More precisely, for $t \in T$ there are three points in T' lying over t. Call these t'_1, t'_2, t'_3. We require that the corresponding positive real numbers, $a(t'_1), a(t'_2)$, and $a(t'_3)$ satisfy

$$a(t'_1) + a(t'_2) > a(t'_3)$$
$$a(t'_1) + a(t'_3) > a(t'_2)$$
$$a(t'_2) + a(t'_3) > a(t'_1)$$

In particular, the three positive real values that are associated with a point $t \in T$ — $a(t'_1), a(t'_2)$ and $a(t'_3)$ — correspond to the side-lengths of the triangle in the family which is associated with this point.

Consider two continuous families of triangles $\mathscr{F}/T$ and $\mathscr{G}/T$ parameterized by the same space T, where $\mathscr{F} = (T', a)$ and $\mathscr{G} = (T'', b)$. We say that these families are *isomorphic* with the isomorphism $\phi : \mathscr{F} \to \mathscr{G}$ which consists of a homeomorphism of covering spaces $f : T' \to T''$ such that $a = b \circ f$.

It is a basic property of families of mathematical objects that they can be *pulled back* via any map to the parameter space. In the case of triangles in particular, we work with the space M (Definition 4.3) and the family of triangles which it parameterizes, $\mathscr{M}$. Let $\gamma : [0,1] \to M$ be a continuous map. We pull back the family $\mathscr{M}$ by the map γ to obtain a family parameterized by $[0,1]$ which is denoted by $\gamma^*\mathscr{M}$. Moreover, this family is defined in such a way that

$$(\gamma^*\mathscr{M})_t = \mathscr{M}_{\gamma(t)}, \qquad \text{for all } t \in [0,1].$$

4.2. **Introducing the Moduli Space of Triangles.** Our goal in this section is to construct a space that describes all triangles up to similarity (isomorphism).

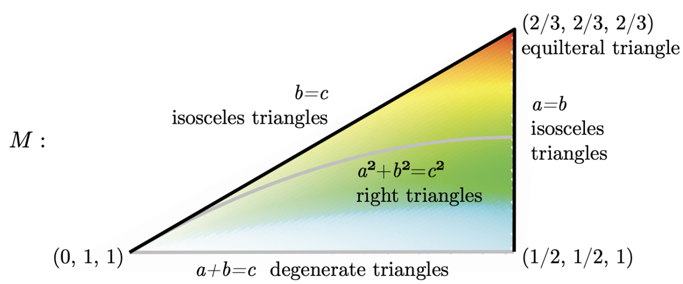
Since all triangles, under scaling, are similar to a triangle of perimeter length 2, perhaps it is sufficient to consider a space parameterizing all non-oriented triangles of perimeter length 2 taking isometry (translation, scaling, rotation, and reflection) to be our notion of similarity. At this point we might ask what properties the triangles parameterized by this space obey. Suppose that we take the convention that all triangles consist of sides of length a, b, c where $a \leq b \leq c$. A space parameterizing such triangles might be the following construction which we call M.

Definition 4.3. We define M, a topological space parameterizing all non-oriented triangles of perimeter length 2 to be the following subset of $\mathbb{R}^3$:

$$M = \{(a,b,c) \in \mathbb{R}^3 : a \leq b \leq c, a+b+c = 2, c < a+b\}.$$

This space, M, is shown in the figure below.

FIGURE 8. The moduli space M of triangles. Adapted from [2].



Every point (a,b,c) in M corresponds to a triangle with side-lengths a, b, and c. Every triangle is similar to one of these. Moreover, different points in M correspond to triangles which are non-similar. The space M has two boundary lines which consist of isosceles triangles and a third boundary which is not included, and consists of degenerate triangles, in which the three lines which define the triangle coincide and all vertices lie on the same line.

Now, we return to the definition of coarse and fine moduli spaces. In particular, we will see that M is a coarse moduli space but not a fine moduli space.

Proposition 4.4. *The space M as defined in Definition 4.3 satisfies properties (i), (ii), and (iii) but not (iv) in Definition 2.6.*

Namely, we may view M as a coarse moduli space (but not a fine one). (See Proposition 2.9)

Proof. All figures in this proof are adapted from [2].

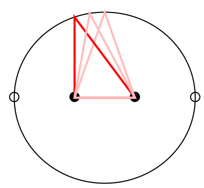
M is constructed to satisfy condition (i).

(ii) Consider an arbitrary continuous family of triangles $\mathscr{F}/T$ given by T'/T and $a : T' \to \mathbb{R}_{>0}$. We want to show that the associated moduli map $T \to M$ is continuous. This follows from properties of the covering map and the continuity of a. See [2] p.18 for full proof.

(iii) We have already shown that via the notion of pullback, a continuous map $\gamma : [0,1] \to M$ gives rise to a family over $[0,1]$ whose moduli map is γ. We can do the same thing for any map $f : T \to M$, from an arbitrary space T to M. We can use f to pull back the family $\mathscr{M}$ to a family $f^*\mathscr{M}$ and this family has a moduli map f. Thus Condition (iii) is satisfied.

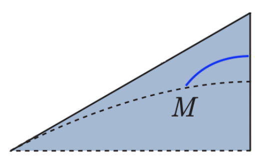
(iv) Consider the family $\mathscr{F}/I$ introduced in Example 4.1.

We may similarly construct another family $\mathscr{G}$ by a similar process, which is equal to the family $\mathscr{F}$ over the interval $[-1/4, 0]$, but then goes back to the starting point once it reaches the isosceles triangle in the middle.

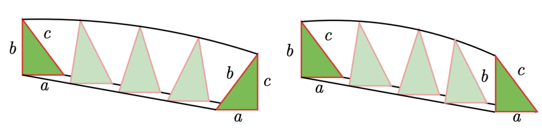


For a value $t \in [0, 1/4]$ if we consider the third point of the corresponding triangle $\mathscr{G}_t$ (defined by the tip of the pen and not the two pins), then the x-coordinate of this point is $-t$.

Now notice that $\mathscr{G}$ has the same moduli map as the family $\mathscr{F}$.



However, $\mathscr{F}$ and $\mathscr{G}$ are not isomorphic. This is easy to see if we label the sides of the initial triangle with a, b, and c in such a way that $a < b < c$, and then label all the following triangles in the respective families in a continuous way. The triangles at the 'ends' of each of the two families respectively are labeled differently, and so on.



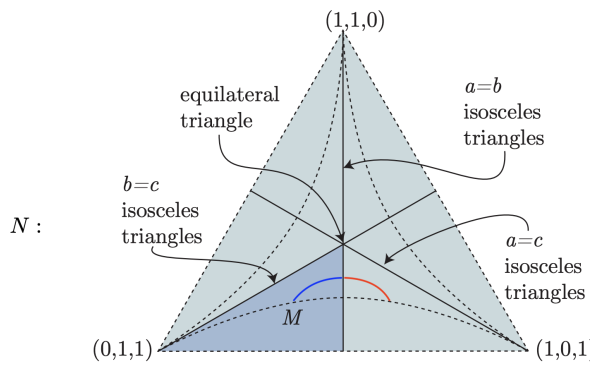
$\square$

In particular, the preceding proof shows that the existing construction is still lacking in the sense that it is does not allow us to differentiate certain families of triangles Example 4.1 which include isosceles triangles. One space that does capture this behavior is the following.

Definition 4.5. Let N parameterize all triangles of side length a, b, c. This space is defined by

$$N := \{(a, b, c) \in \mathbb{R} : a + b + c = 2, a, b, c < 1.\}$$

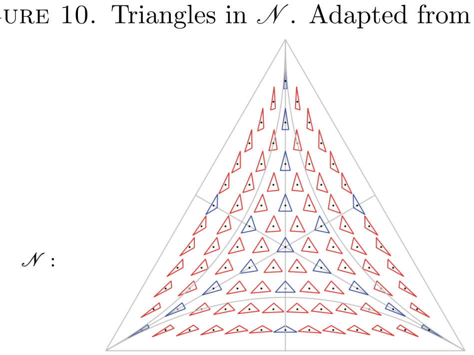
FIGURE 9. The space N. The part of the space parameterizing $\mathscr{G}$ and $\mathscr{F}$ are shown in blue and red respectively. Adapted from [2].



Unlike members of the family $\mathscr{M}$ parameterized by M, members of the family $\mathscr{N}$ parameterized by N have their vertices labeled in a consistent way. If we return to the example given in the proof of Proposition 4.4, the families $\mathscr{F}$ and $\mathscr{G}$ are parameterized by different moduli maps.

However, N has one major disadvantage with respect to our original aim which was to build a space which parameterized triangles up to similarity. Most similar triangles appear multiple times in the family parameterized by the space — scalene triangles appear six times, isosceles triangles appear three times, and the equilateral triangle appears once. The number of times that a triangle appears is inversely proportional to its symmetries. See Figure 10.

FIGURE 10. Triangles in $\mathscr{N}$. Adapted from [2].



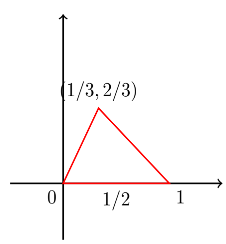
In fact, N is the fine moduli space of triangles with labeled vertices (or equivalently labeled sides) up to similarity. However, similarity of labeled triangles is not a notion of isometry. Then N is not a 'true moduli space', since it relies on a notion of similarity other than isometry. To define a true moduli space of triangles is difficult and requires us to resolve the issue of repeating similar triangles. This involves taking group actions into consideration and is done in [2], involving 'stacky' techniques.

5. Relating the Moduli Space of Acute Triangles and the space of Curves

Having now introduced the moduli spaces of triangles and the space of curves, there is a wonderful way of relating these two constructions.

First recall that the moduli space of elliptic curves up to isomorphism is equivalent to the moduli space of complex lattices up to homothety. Then to show that the moduli space of acute triangles (up to similarity) is equivalent to the moduli space of elliptic curves, it suffices to find a similar correspondence between acute triangles and complex lattices.

Much like how a complex lattice Λ may be uniquely identified with a point $\tau \in \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ such that Λ is homothetic to $[1, \tau]$ (Lemma 3.18), acute triangles may also be identified with points in $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. In particular, every acute triangle with labeled vertices — a specified first, second, and third vertex — may be identified with a similar triangle that has a point at 0, a point at 1, and another point lying somewhere in the upper half plane.



Now we define the space consisting of all points in the upper half plane which define acute triangles.

Definition 5.1. We define $T \subset \mathbb{H}$ as follows:

$$T := \left\{ z \in \mathbb{C} : \mathrm{im}(z) > 0, 0 < \mathrm{re}(z) < 1, |z - \frac{1}{2}| > \frac{1}{2} \right\}.$$

In particular, points in T correspond to (labeled) acute triangles up to similarity. In Figure 11, T is made up of the six colored regions.

The resemblance between T and N (Definition 4.5) from the previous section is not coincidental. The boundaries of purple and yellow regions correspond to isosceles triangles. The point where all six colored regions meet corresponds to the equilateral triangle. The following figure highlights the regions of N which correspond to acute triangles.

Now, we want to show how acute triangles may be related to elliptic curves. To answer this question, we recall the group $\mathrm{GL}_2(\mathbb{Z})$ which consists of invertible 2×2 integer matrices. $\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by

FIGURE 11. The fundamental domains of $\mathrm{GL}_2(\mathbb{Z})$ with the region T shown in color. Figure due to [1].

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

the same way as the modular group $\mathrm{SL}_2(\mathbb{Z})$ which is a subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

In Figure 11, each of the light and dark colored regions correspond to a fundamental domain for the action of $\mathrm{GL}_2(\mathbb{Z})$ over the upper half plane. Moreover, elements of $\mathrm{SL}_2(\mathbb{Z})$ map light regions to light regions and dark ones to dark ones. Then the union of any light region and dark one sharing an edge forms a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$.

Since T is the union of six fundamental domains for $\mathrm{GL}_2(\mathbb{Z})$, it is also the union of three fundamental domains for $\mathrm{SL}_2(\mathbb{Z})$. Then there is a map

$$p : T \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$$

which is three-to-one. However, this map is not surjective, but if we take the closure of T inside $\mathbb{H}$ we get

$$\overline{T} = \left\{ z \in \mathbb{C} : \mathrm{im}(z) > 0, 0 \le \mathrm{re}(z) \le 1, |z - \frac{1}{2}| \ge \frac{1}{2} \right\}$$

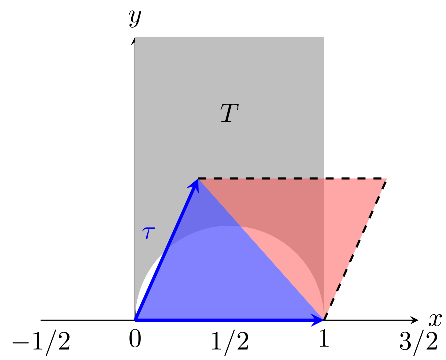
which includes the boundary points of the original space, which correspond to right triangles. Then p extends to an onto map

$$p : \overline{T} \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}).$$

The existence of this map suggests that we can identify oriented right triangles with isomorphism classes of complex elliptic curves. However, there is a nice way by which we can understand the correspondence between acute triangles and elliptic curves more directly — by developing a correspondence between acute triangles and complex latices of the form $[1, \tau]$.

Take any acute (or right) triangle in the complex plane. Scale, orient, and translate it so that one side lies on the line from $x = 0$ to $x = 1$. Shown in blue in Figure 12. Now rotate the triangle by 180° about its midpoint. Join this rotated triangle to the original one as shown in Figure 12. What is formed is the fundamental parallelogram for a complex lattice $[1, \tau]$ which, as we saw in the uniformization theorem (Corollary 3.21), corresponds to an isomorphism class of complex elliptic curves (up to $SL_2(\mathbb{Z})$ action).

Figure 12. Lattice from acute triangle



6. Further Questions

Investigating moduli spaces and their properties is an active area of ongoing research. In this section we will state several such open problems.

Problem 6.1. *What does the the moduli space of n-gons up to similarity look like (without redundancies)?*

This was stated as an open problem in [6] in 2015, however progress was made in 2023 in [7], which gives local descriptions of the space $S(n)$ of positvely oriented simple n-gons up to oriented similarity.

There has been a great deal of interest in the properties of the Deligne-Mumford stack $\mathcal{M}_{g,n}$ which parameterizes families of smooth curves of genus g with n ordered, distinct marked points over an Algebraically closed field K. We will always require that $2g-2+n > 0$. This is coarsely represented by the moduli space $M_{g,n}$. You will recall that the moduli space of elliptic curves over $\mathbb{C}$ — namely the moduli space of genus 1 curves with 1 marked point — was denoted $M_{1,1}$. Certain basic questions remain open about the geometry of these moduli spaces.

Problem 6.2. *We know that $M_{g,n}$ is a quasi-projective variety of dimension $3g - 3 + n$.*

However, put impressionistically, we still don't know "how close" $M_{g,n}$ is to being affine or projective.

For rigorous treatment of this open problem see [3].

Indeed, $\mathcal{M}_{g,n}$ is a rich example. Another open question is the following.

Problem 6.3. *What is the maximal dimension of sub-varieties in $\mathcal{M}_{g,n}$: i.e. what's the largest sub-variety the moduli space contains?*

For an excellent paper which explores this topic see [5].

Finally, another moduli space problem is the problem of attempting to classify all algebraic varieties up to birational equivalence (varieties that are isomorphic outside a small area). This led to the Minimal Model Program which is an extremely technical but very rich moduli space problem.

Acknowledgments

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