BROWNIAN MOTION, HAUSDORFF DIMENSION, AND DIMENSION DOUBLING

TOM DUPOR

Abstract. In this paper, we will introduce Brownian motion and Hausdorff dimension. In addition, we will describe ways to help find the Hausdorff dimension and introduce properties of Brownian motion useful in determining its Hausdorff dimension. Finally, we will end with a proof of Kaufman's doubling theorem, a surprising result regarding the Hausdorff dimension of subsets of Brownian motion.

Contents

1. Introduction

Brownian motion is probably the foremost example of a continuous random function and is often taught as an introduction to probability theory. Not just a learning tool, Brownian motion is used in many fields. To name a few, in finance, Brownian motion is used to help model stock prices and in physics, it is used to understand the movement of particles. In Section 2, we will define and give a proof of Brownian motion. After its construction, we will prove properties of Brownian motion that are important in computing its Hausdorff dimension. For Section 2, all the background that is needed is a general understanding of mean, variance, expectation, probability distributions, and independence as well as knowledge of the Borel-Cantelli lemma.

Hausdorff dimension is a generalization of one of the earliest concepts introduced to a student in geometry, dimensions, to a different class of objects. Using Hausdorff dimension helps to determine the size of interesting objects including Brownian motion. In Section 3, we will introduce Hausdorff dimension without any reference to Brownian motion or any probabilistic language. Only the most fundamental ideas in measure theory are needed for this section, such as the definition of metric spaces and what a measure is as well as a notion of what a fractal is.

Date: September 16, 2024.

Hausdorff dimension is very useful, however it is often unworkable from the definition. So often, one uses known bounds of Hausdorff dimension to find the Hausdorff dimension of an object. Many bounds for the Hausdorff dimension have been found and using these bounds makes finding the Hausdorff dimension of an object simpler. In Section 4, we provide proofs for both an upper and two lower bounds of Hausdorff dimension. We then bring Brownian motion together with Hausdorff dimension and use the image of the Cantor set under Brownian motion as an example on how to work with these bounds.

In Section 5, we arrive at the highlight of this paper, a surprising result about how Brownian motion changes the dimension of sets. We will prove that for subsets of the real line, taking the image of that set under d-dimensional Brownian motion, $d \geq 3$, with probability one, exactly doubles the Hausdorff dimension of that set. This result is Kaufman's Doubling Theorem.

2. Brownian Motion

Brownian motion at its most basic is a continuous random function. It is often thought of in terms of time, with the time run being the domain of the Brownian motion. Thus, using t as the variable for Brownian motion is standard. At any time, t, you have a value indicated by $B(t)$. Before we construct Brownian motion, we require a normal distribution, which we will now define.

Definition 2.1. A random variable X is normally distributed with mean μ and variance σ^2 if

$$\mathbb{P}[X > x] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{\frac{(u-\mu)^2}{2\sigma^2}}\, du.$$

We want Brownian motion to look like what one thinks a continuous random function would look like. So, we define Brownian motion to have constraints that conform to the idea of what a continuous random function should be. It is mean zero; meaning for random t, $B(t)$ will be positive or negative with equal probability. While the distance between the Brownian motion and zero, the variance, depends only on how long the Brownian motion has been running. This means Brownian motion does not blow up on small intervals and tends to get further from zero the longer it is run. While the independence of all intervals of Brownian motion ensures that the function behaves the same on all intervals. Formally writing out these properties gives the definition of Brownian motion.

Definition 2.2. A (standard) linear Brownian motion is a function $B : [0, \infty) \to \mathbb{R}$ such that

- $B(0) = 0$,
- $B(t_2) - B(t_1)$ is a random variable with normal distribution, mean zero, and variance $t_2 - t_1$,
- For all $0 \leq t_1 \leq ... \leq t_n$, $B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$ are independent random variables,
- With probability one, B is continuous.

One can not just define a function and say it exists. So in order to further study Brownian motion, we must first show that a function fulfilling all of these properties does exist.

Theorem 2.3. *Standard linear Brownian motion exists*

Proof. First, we will construct Brownian motion on $[0,1]$. This will be done by first defining a random function on countably many points. In order to complete the construction we will linearly interpolate between those points.

Let $\mathcal{D}_n = \{\frac{k}{2^n} : k \in \mathbb{N}, 0 \le k \le 2^n\}$ and $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$. Let $(\omega, \mathscr{A}, \mathbb{P})$ on which a collection $\{Z_t : t \in \mathcal{D}\}$ of independently and normally distributed random variables, with expectation zero and variance one, is defined.

Define $B(0) = 0$ and $B(1) = Z_1$. For each $n \in \mathbb{N}$ define $B(d), d \in \mathcal{D}_n$ such that

(1) vectors $(B(d) : d \in \mathcal{D}_n)$ and $(Z_t : t \in \mathcal{D} - \mathcal{D}_n)$ are independent,
(2) for all $r < s < t$ in $\mathcal{D}_n$, $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$ and is independent of $B(s) - B(r)$.

We will use induction to show that such a construction is possible. For $n = 0$ we have $\mathcal{D}_0 = \{0,1\}$ and $B(0) = 0, B(1) = Z_1$ which satisfies the above properties.

Now assume we have satisfied the properties for $n-1$. Define $B(d)$, for $d \in \mathcal{D}_n - \mathcal{D}_{n-1}$ by

$$B(d) = \frac{B(d+2^{-n}) + B(d-2^{-n})}{2} + \frac{Z_d}{2^{\frac{n+1}{2}}}.$$

Note that for all k such that $\frac{k}{2^n} \in \mathcal{D}_n - \mathcal{D}_{n-1}$, k is odd. Therefore $d - 2^{-n}, d + 2^{-n} \in \mathcal{D}_{n-1}$ so this value is well-defined and the first summand is independent of the second. By construction, $d - 2^{-n}, d + 2^{-n} \in \mathcal{D}_n$ and $(Z_d : d \in \mathcal{D}_n - \mathcal{D}_{n-1})$ is independent of $(Z_t : t \in \mathcal{D} - \mathcal{D}_n)$. So $(B(d) : d \in \mathcal{D}_n)$ and $(Z_t : t \in \mathcal{D} - \mathcal{D}_n)$ are independent and the first condition is satisfied.

To verify the second property we will look at the difference between $B(d)$ and its nearest neighbors in $\mathcal{D}_n$, $B(d-2^{-n})$ and $B(d+2^{-n})$.

$$B(d) - B(d-2^{-n}) = \frac{B(d+2^{-n}) - B(d-2^{-n})}{2} + \frac{Z_d}{2^{\frac{n+1}{2}}}. \tag{2.4}$$

The two summands are independent of each other for the same reason as above. By construction, each summand has an expectation of zero and

$$\begin{aligned}\mathrm{Var}\left(\frac{B(d+2^{-n}) - B(d-2^{-n})}{2}\right) &= \mathbb{E}\left[\left(\frac{B(d+2^{-n}) - B(d-2^{-n})}{2}\right)^2\right] \\ &= \frac{1}{4}\mathrm{Var}(B(d+2^{-n}) - B(d-2^{-n})) = 2^{-n-1}\end{aligned}$$

and

$$\mathrm{Var}\left(\frac{Z_d}{2^{\frac{n+1}{2}}}\right) = \frac{\mathrm{Var}(Z_d)}{2^{n+1}} = 2^{-n-1}.$$

This means that $B(d) - B(d-2^{-n})$ has expectation zero and variance 2^{-n}. $B(d + 2^{-n}) - B(d)$ can be shown to have the same expectation zero and variance 2^{-n} in a similar way. Since $B(d)$ and $B(d+2^{-n})$ and $B(d-2^{-n})$ are independent for $d \in \mathcal{D}_n - \mathcal{D}_{n-1}$, this means that $B(d) - B(d-2^{-n})$ and $B(d+2^{-n}) - B(d)$ are independent for $d \in \mathcal{D}_n - \mathcal{D}_{n-1}$. For a full proof of why we can relate the independence, expectation, and variance of summands to the sum, see [2, Corollary II.3.4].

In order to confirm construction, we need to show that $B(d) - B(d-2^{-n})$ and $B(d+2^{-n}) - B(d)$ are independent for all $d \in \mathcal{D}_n - \{0\}$. To show this, we need to show that $B(d) - B(d-2^{-n})$ and $B(d+2^{-n}) - B(d)$ are independent for $d \in \mathcal{D}_{n-1}$. So we will look at an interval separated by some $d \in \mathcal{D}_{n-1}$. To do this, we will choose $d \in \mathcal{D}_j$ with minimal j such that that property is fulfilled. This means that $[d-2^{-n}, d] \subset [d-2^{-j}, d]$ and $[d, d+2^{-n}] \subset [d, d+2^{-j}]$. Now by assumption,

$B(d)-B(d-2^{-j})$ and $B(d+2^{-j})-B(d)$ are independent. Since $B(d)-B(d-2^{-n})$ and $B(d+2^{-n})-B(d)$ are constructed from $B(d)-B(d-2^{-j})$ and $B(d+2^{-j})-B(d)$ respectively and disjoint set of random variables $(Z_t : t \in \mathcal{D}_n)$, $B(d)-B(d-2^{-n})$ and $B(d+2^{-n})-B(d)$ are independent.

Now that we have defined a function on the set of dyadic points, we now need to extend it to all points in [0,1]. This will be done by linearly interpolating between all points in $\mathcal{D}_n$. Formally

$$F_0(t) = \begin{cases} Z_1 & t=1 \\ 0 & t=0 \\ \text{Linear between 0 and 1} \end{cases}$$

and $n > 0$

$$F_n(t) = \begin{cases} \frac{Z_n}{2^{\frac{n+1}{2}}} & t \in \mathcal{D}_n - \mathcal{D}_{n-1} \\ 0 & t \in \mathcal{D}_{n-1} \\ \text{Linear between consecutive points in } \mathcal{D}_n \end{cases}.$$

These functions are always continuous on [0,1]. By induction, we will show that for $d \in \mathcal{D}_n$

$$B(d) = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d). \tag{2.5}$$

This is true for $n = 0$. Assume the equation holds for $n-1$. Let $d \in \mathcal{D}_n - \mathcal{D}_{n-1}$. Then for all $0 \le i \le n-1$, F_i is linear on $[d-2^{-n}, d+2^{-n}]$

$$\sum_{i=0}^{n-1} F_i(d) = \sum_{i=0}^{n-1} \frac{F_i(d-2^{-n}) + F_i(d+2^{-n})}{2} = \frac{B(d-2^{-n}) + B(d+2^{-n})}{2}$$

with the last step being by our induction assumption.

Since $F_n(d) = \frac{Z_n}{2^{\frac{n+1}{2}}}$ and for $m > n$, $F_m(d) = 0$, we get that (2.5) is true.

We must also show that the $B(d)$ exists for all values. This is done by showing that $\sum_{i=0}^{\infty} F_i(d)$ is uniformly convergent. Uniform convergence implies existence because the dyadic points, $\mathcal{D} = \bigcup_{i=0}^{\infty} \mathcal{D}_i$, are dense in [0,1].

Since Z_d has a normal distribution, for $c > 0$ and large n,

$$\mathbb{P}\{|Z_d| \ge c\sqrt{n}\} \le \exp(\frac{-c^2 n}{2}).$$

This means

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{P}\{\text{there exists } d \in \mathcal{D}_n \text{ such that } |Z_d| \ge c\sqrt{n}\} &\le \sum_{n=0}^{\infty} \sum_{d \in \mathcal{D}_n} \mathbb{P}\{|Z_d| \ge c\sqrt{n}\} \\ &\le \sum_{n=0}^{\infty} 2^{n+1} \exp(\frac{-c^2 n}{2}). \end{aligned}$$

Setting $c > \sqrt{2 \log 2}$ means

$$\exp(\frac{-c^2 n}{2}) < 2^{-n}.$$

So the series converges for such a c. Fix $c > \sqrt{2 \log 2}$. By the Borel-Cantelli lemma, there exists, with probability one, a finite and random N such that for all $n \ge N$

and $d \in \mathcal{D}_n$, $|Z_d| < c\sqrt{n}$. So for all $n \geq N$,

$$||F_n||_\infty = \min\{M \in [0,\infty] : |F_n| \geq M \text{ almost everywhere}\} < c\sqrt{n}2^{\frac{-n}{2}} \to 0. \tag{2.6}$$

This means that with probability one,

$$\sum_{i=0}^{\infty} F_i(d) = B(d)$$

is uniformly convergent on [0,1]. Now we must show that the properties of Brownian motion hold for this generalization of $B(d)$ to all points in [0,1]. We have already seen it is continuous and $B(0) = 0$.

Now, we must show for any sequence $t_1, t_2, ..., t_n, ...$ with $t_n \in \mathbb{R}$, for all n, $B(t_n) - B(t_{n-1})$ are independent random variables with normal distribution, expectation zero, and variance $t_n - t_{n-1}$. This will be done by constructing a limit of dyadic points to represent every point in $[0, 1]$ and then finding the expectation and covariance of the point based on that sequence.

First note that $\mathcal{D} = \bigcup_{i=0}^{\infty} \mathcal{D}_i$ is dense in [0,1]. So for any sequence $t_1 \leq t_2 \leq .. \leq t_n$ in [0,1]. There exists sequence $t_{1,k} \leq t_{2,k} \leq ... \leq t_{n,k}$ in $\mathcal{D}$ such that $\lim_{k\to\infty} t_{i,k} = t_i$.

From the continuity of B, for $1 \leq i \leq n-1$

$$B(t_{i+1}) - B(t_i) = \lim_{k\to\infty} B(t_{i+1,k}) - B(t_{i,k}).$$

So the random variable $B(t_{i+1}) - B(t_i)$ on [0,1] can be found be taking a limit on the set of $\mathcal{D}$.

It is shown in [2, Proposition II.3.7] that taking the limit of expectation and covariance of $B(t_{i+1,k}) - B(t_{i,k})$ gives us the expectation and covariance of $B(t_{i+1}) - B(t_i)$.

$$\lim_{k\to\infty} \mathbb{E}[B(t_{i+1,k}) - B(t_{i,k})] = 0$$

$$\begin{aligned}\lim_{k\to\infty} \operatorname{Cov}(B(t_{i+1,k}) - B(t_{i,k}), B(t_{j+1,k}) - B(t_{j,k})) &= \lim_{k\to\infty} \mathbb{1}_{\{i=j\}} B(t_{i+1,k}) - B(t_{i,k}) \\ &= \mathbb{1}_{\{i=j\}} t_{i+1} - t_i.\end{aligned}$$

So our construction on [0,1] has the same properties as our definition of Brownian motion. Now we need to extend it to [0,∞). First, we define a function $\text{int}(t)$ such that $\text{int}(t) = \max\{x : x \leq t, x \in \mathbb{Z}\}$. Then, since the construction of B is random, we can choose independent $B_1, B_2, ...$ such that $B_i : [0,1] \to \mathbb{R}$ and define $B : [0,\infty) \to \mathbb{R}$ such that

$$B(t) = B_{\text{int}(t)}(t - \text{int}(t)) + \sum_{i=1}^{\text{int}(t)} B_i(1).$$

This function satisfies all the constraints to qualify as Brownian motion. So Brownian motion does exist. ☐

For most of this paper, we want Brownian motion to have a range in dimensions higher than 1. Constructing higher dimensional Brownian motion is simple.

Remark 2.7. To construct an n-dimensional Brownian motion, $B : [0,\infty) \to \mathbb{R}^n$, simply let $B(t) = (B_1(t), B_2(t), ..., B_n(t))$ where $B_1, B_2, ..., B_n$ are independent linear Brownian motions.

Now that we know Brownian motion exists, we can describe some properties of it. The first property to be proved is Brownian scaling. Roughly, it is useful as it shows that Brownian motion looks no matter the range that one runs it over.

Theorem 2.8. *(Brownian Scaling) Let $B(t)$ be a standard Brownian motion, and $a > 0$. Then $\frac{1}{a}B(a^2t)$ is also a standard Brownian motion.*

Proof. Going down the properties of Brownian motion, we can see that

- $\frac{1}{a}B(a^2 0) = 0$,
- $\frac{1}{a}B(a^2t_2) - \frac{1}{a}B(a^2t_1)$ is a random variable with normal distribution, mean zero, and variance $(\frac{1}{a})^2(a^2t_2 - a^2t_1) = t_2 - t_1$,
- For all $0 \leq t_1 \leq ... \leq t_n$, $B\frac{1}{a}(a^2t_1), \frac{1}{a}B(a^2t_2) - \frac{1}{a}B(a^2t_1), ..., \frac{1}{a}B(a^2t_n) - \frac{1}{a}B(a^2t_{n-1})$ are independent random variables,
- With probability one, $\frac{1}{a}B(a^2t)$ is continuous.

So $\frac{1}{a}B(a^2t)$ is a standard Brownian motion. □

The next property is the continuity of Brownian motion. By definition, Brownian motion is continuous. However, we can be more specific and determine the strength of the continuity of Brownian motion.

In order to determine how strong the continuity is, we introduce α-Hölder continuity. α-Hölder continuity compares the distance of the interval of the domain and the length of the range. By raising the distance of the domain to α and comparing it to the range, we can determine how small the function makes the domain.

Definition 2.9. A function, $f : [0, \infty) \to \mathbb{R}^d$ is **locally α -Hölder continuous** at $x > 0$ if there exists $c > 0$ such that for all $\epsilon > 0$

$$|f(x) - f(y)| \leq c|x - y|^{\alpha} \qquad \text{for all } |y - x| < \epsilon,\ y \geq 0.$$

To determine when Brownian motion is α-Hölder continuous, we will use the following theorem which puts a bound on the image of Brownian motion.

Theorem 2.10. *For any given T, there exists random $C > 0$ such that, with probability one*

$$|B(t + h) - B(t)| \leq C\sqrt{h \log(1/h)}$$

for all $t \in [0, T]$ and small h.

Proof. We will use the F_n introduced in the construction of Brownian motion, where it is the linear interpolation between points defined on the dyadic set. Note,

$$||F_n'||_\infty \leq \frac{2||F_n||_\infty}{2^{-n}}.$$

This is because the derivative of a function with local maximums a fixed length apart that are linearly connected must be less than two times the absolute maximum of that function divided by the fixed length.

By (2.6), we see that for any $c > \sqrt{2 \log 2}$ there exists random N such that for all $n > N$,

$$\frac{2||F_n||_\infty}{2^{-n}} \leq c\sqrt{n}2^{\frac{n}{2}}.$$

From this for any $\mathscr{L} \in \mathbb{N}$,

$$|B(t + h) - B(t)| \leq \sum_{i=0}^{\infty} |F_i(t + h) - F_i(t)| \leq \sum_{i=0}^{\mathscr{L}} h||F_i'||_\infty + \sum_{i=\mathscr{L}+1}^{\infty} 2||F_i||_\infty.$$

The last inequality is obtained by combining two different bounds of $|F_i(t+h) - F_i(t)|$, the first is obtained through the mean-value theorem and the second from the definition of the L^∞ norm.

By (2.6), for all $\mathscr{L} > n$

$$\sum_{i=0}^{\mathscr{L}} h||F_i'||_\infty + \sum_{i=\mathscr{L}+1}^{\infty} 2||F_i||_\infty$$

is bounded by

$$h\sum_{n=0}^{N} ||F_n'||_\infty + 2ch \sum_{n=N}^{\mathscr{L}} \sqrt{n}2^{\frac{n}{2}} + 2c \sum_{n=\mathscr{L}+1}^{\infty} \sqrt{n}2^{\frac{-n}{2}}.$$

Since $||F_n'||_\infty$ is bounded by a constant, we can choose h sufficiently small so that the first term in the sum is less than $\sqrt{h\log(1/h)}$. Based on this h, we can choose $\mathscr{L}$ such that $2^{-\mathscr{L}} \leq h \leq 2^{-\mathscr{L}+1}$ and $h > N$. Plugging in $\mathscr{L}$ to the summands in the second and third term yields a bound of some constant time $\sqrt{h\log(1/h)}$. Thus,

$$|B(t-h) - B(t)| \leq C\sqrt{h\log(1/h)}.$$

□

Corollary 2.11. *For all $\alpha < \frac{1}{2}$, Brownian motion is α- Hölder continuous.*

Proof. Let $C > 0$ be as in Theorem 2.10. Then there exists a fixed h' such that for all $h < h'$,

$$|B(t+h) - B(t)| < C\sqrt{h\log(1/h)}.$$

Since h has an upper bound, we can find $\epsilon > 0$ such that the above equation is bounded by

$$C\sqrt{hh^\epsilon} < Ch^\alpha.$$

This result can easily be applied to higher-dimensional Brownian motion. □

$\alpha = \frac{1}{2}$ is the limit of Hölder continuity for Brownian motion, as

Proposition 2.12. *With probability one, for $\alpha > \frac{1}{2}$, at every point Brownian motion isn't locally α- Hölder continuous*

Proof. A full proof can be found in [2, Theorem 1.13]. □

3. Hausdorff Dimension

The first place that any math student is likely to see the use of dimension is in elementary geometry. There it is used to identify the correct measure to use when describing the space an object takes up. If an object is two-dimensional the area would be the only non-trivial measure. The volume of any two-dimensional object is zero. While taking the one-dimensional measure of the whole of any two-dimensional object is ∞. So the useful measure is the only one that gives a non-trivial answer to what is the mass of the object.

A helpful property of dimensions that follows from that is the ability of dimension to differentiate the size of two objects. Just by the dimensions of the objects, one can tell that a cube occupies more space, is bigger, than a square.

Thus, it would be useful in determining the size of Brownian motion and other fractal sets if there was a way to bring the notion of dimension to these sets. This can be done through the construction of a fractal dimension. There are many different constructions of fractal dimensions but the one to be discussed here is the Hausdorff dimension.

To define the Hausdorff dimension, one must first define the Hausdorff measure.

Definition 3.1. For some metric space A and $E \subset A$ let

$$\mathcal{H}^{\alpha}_{\epsilon}(E) = \inf \{\sum_{i=1}^{\infty} |E_i|^{\alpha} : E \subset \bigcup_{i=1}^{\infty} E_i \text{ , and for all } \ i, |E_i| < \epsilon\}$$

The α**-Hausdorff Measure** of E is defined as

$$\mathcal{H}^{\alpha}(E) = \lim_{\epsilon \downarrow 0} \mathcal{H}^{\alpha}_{\epsilon}(E)$$

The Hausdorff measure takes both the number of sets covering the object and the size of those coverings into account. The smaller the coverings, the quicker the Hausdorff measure reaches zero as α increases.

Proposition 3.2. *If $\epsilon \geq \delta$, then*

$$\mathcal{H}^{\alpha}_{\epsilon}(E) \leq \mathcal{H}^{\alpha}_{\delta}(E).$$

Proof. Since $\epsilon \geq \delta$,

$$\inf\{\sum_{i=1}^{\infty} |E_i|^{\alpha} : \text{for all } \ i, |E_i| < \epsilon\} \leq \inf\{\sum_{i=1}^{\infty} |E_i|^{\alpha} : \text{for all } \ i, |E_i| < \delta\}.$$

By definition of Hausdorff measure,

$$\mathcal{H}^{\alpha}_{\epsilon}(E) \leq \mathcal{H}^{\alpha}_{\delta}(E)$$

□

Corollary 3.3. *The α-Hausdorff measure exists*

Proposition 3.4. *Let E be a metric space and $0 < \alpha < \beta < \infty$. Then*

a) If $\mathcal{H}^{\alpha}(E) < \infty$, then $\mathcal{H}^{\beta}(E) = 0$,
b) If $\mathcal{H}^{\beta}(E) > 0$, then $\mathcal{H}^{\alpha}(E) = \infty$.

Proof. If for all i, $|E_i| < \epsilon$, then

$$\sum_{i=1}^{\infty} |E_i|^{\beta} = \sum_{i=1}^{\infty} |E_i|^{\alpha} |E_i|^{\beta-\alpha} \leq \epsilon^{\beta-\alpha} \sum_{i=1}^{\infty} |E_i|^{\alpha}.$$

Taking the inf, we get

$$\mathcal{H}^{\beta}_{\epsilon}(E) \leq \epsilon^{\beta-\alpha} \mathcal{H}^{\alpha}_{\epsilon}(E)$$

a) Taking $\epsilon \downarrow 0$, since $\beta - \alpha \geq 0$ and $\mathcal{H}^{\alpha}(E) < \infty$

$$\mathcal{H}^{\beta}(E) \leq 0$$

b)

$$\mathcal{H}^{\beta}(E) \leq \epsilon^{\beta-\alpha} \mathcal{H}^{\alpha}(E) \implies \epsilon^{\alpha-\beta} \mathcal{H}^{\beta}(E) \leq \mathcal{H}^{\alpha}(E)$$

Taking $\epsilon \downarrow 0$, since $\alpha - \beta \leq 0$ and $\mathcal{H}^{\beta}(E) > 0 \implies \mathcal{H}^{\alpha}(E) = \infty$

□

From Proposition 3.4 we get that for each metric space, there is at most one α that gives a non-trivial α-Hausdorff measure. In elementary Euclidean geometry, the n-dimensional measure for an n-dimensional object is the only measure that gives a non-trivial answer. Analogous to this, the Hausdorff dimension, β, of an object is defined so that the solution to the α-Hausdorff measure can only be non-trivial when $\alpha = \beta$.

Definition 3.5. The **Hausdorff dimension** of a metric space E is defined as

$$\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha(E) > 0\}$$

As noted previously, the Hausdorff measure reaches zero quicker for finer coverings. This confirms the intuition that the smaller the coverings of an object, the smaller its dimension should be.

4. Upper and Lower Bounds of Hausdorff Dimension

Hausdorff dimension is difficult to calculate directly for most sets. Therefore, knowing the upper and lower bounds of the Hausdorff dimension is helpful in determining the dimension of a set.

The simplest way to determine a upper/lower bound is to use the fact that you are taking an inf or sup to determine the Hausdorff dimension. If any α for some E gives a $\mathcal{H}^\alpha(E) = 0$ then the α is a upper bound for the Hausdorff dimension. If any β gives $\mathcal{H}^\beta(E) < 0$, then β is a lower bound.

However, it is easier to show an upper bound in this manner.

Proposition 4.1. *If there exists a countable covering, E_i, of E such that $\sum_{i=1}^\infty |E_i|^\alpha = 0$ and for all $\epsilon > 0$, $|E_i| < \epsilon$ for all i then dim$(E) < \alpha$.*

Proof. Since the Hausdorff measure takes the inf,

$$0 \leq \mathcal{H}^\alpha_\epsilon(E) \leq \sum_{i=1}^\infty |E_i|^\alpha = 0.$$

Since this holds for all ϵ,

$$\dim(E) = \inf\{\beta : \mathcal{H}^\beta(E) = 0\} \leq \alpha.$$

□

Example 4.2. When talking about fractals or dimensions of fractals, one of the first examples often used is the Cantor set, denoted by $\mathcal{T}^\infty$. If the reader is not familiar with the Cantor set, then [4] is a good and sufficient introduction. For this example, we will use the image of the Cantor set under Brownian motion, which is the set of Brownian motion with times in the Cantor set.

Let $\mathcal{T}^n$ be the nth stage of building the Cantor set and V_i be the ith interval of $\mathcal{T}^n$. By Corollary 2.11 with $\alpha < \frac{1}{2}$,

$$\sum_{i=1}^n B(V_i)^{\frac{\log 4}{\log 3}} \leq C \sum_{i=1}^n |V_i|^{\frac{\alpha \log 4}{\log 3}} \leq C 2^n \left(\frac{1}{3^{n 2\alpha}}\right)^{\frac{\log 2}{\log 3}} = C\left(\frac{2}{2^{2\alpha}}\right)^n.$$

For all $\alpha < \frac{1}{2}$, the last term goes to zero as $n \to \infty$. Therefore taking $n \to \infty$, $\dim(B(\mathcal{T}^\infty)) \leq \frac{\log 4}{\log 3}$

A similar method to find the lower bound would be much harder as one could not just find the a covering, E_i, with $\sum_{i=1}^{\infty} |E_i|^\alpha = \infty$ but one must also show that the covering is the inf. However, there are still ways to determine the lower bound of the Hausdorff dimension of a set. They are more complicated than finding the upper bound of a dimension but they are very useful.

We will need to introduce a definition before we can discuss how to find the lower bound.

Definition 4.3. A measure, γ, on the Borel subsets of set E is called a **mass distribution** if $0 < \gamma(E) < \infty$

Theorem 4.4. *Let E be a metric space and $\alpha > 0$. If there exists a mass distribution γ on E, $\delta > 0$ and $C > 0$ such that*

$$\gamma(V) < C|V|^\alpha$$

for all closed sets with $|V| < \delta$, then

$$\mathcal{H}^\alpha(E) > \frac{\gamma(E)}{C} > 0$$

and dim$(E) > \alpha$.

Proof. Let $U_1, U_2, ...$ be a cover of E and for all i, $|U_i| < \delta$. The closure, V_i, of U_i is also a cover of E with $|V_i| < \delta$ for all i.

$$0 < \gamma(E) \leq \gamma(\bigcup_{i=1}^{\infty} U_i) \leq \gamma(\bigcup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{\infty} \gamma(V_i) < \sum_{i=1}^{\infty} C|V|^\alpha$$

By taking the inf over all covers of E and letting $\delta \downarrow 0$

$$0 < \frac{\gamma(E)}{C} < \mathcal{H}^\alpha(E)$$

and by definition of Hausdorff dimension, $\dim(E) \geq \alpha$. □

What the above theorem is saying is that if one knows a positive lower bound of the diameter of each set raised to the α whose union covers E, this means that all sums of α coverings will be greater than 0. This means that α is a lower bound of the Hausdorff dimension.

Example 4.5. Continuing the use of the image of the Cantor set under Brownian motion, we will now look at a d-dimensional Brownian motion with $d > 2$. We can define a random mass distribution, μ, in a way similar to the Cantor measure, so at the nth-stage of the construction of the Cantor set, μ gives a weight of $\frac{1}{2^n}$ to the set of Brownian motions with the domain that is an interval, V_i in the nth stage of the Cantor set, $\mathcal{T}^n$. Then we have by Proposition 2.12 for $\alpha > \frac{1}{2}$,

$$|B(V_i)| \geq C|\frac{1}{3^n}|^{\frac{\alpha \log 4}{\log 3}} \geq C|\frac{1}{2^n}| = C\mu|V_i|.$$

This means that $\dim B(\mathcal{T}^\infty) \geq \frac{\log 4}{\log 3}$.

By this and Example 4.2, we can see that the dimension of the image of the Cantor set under Brownian motion is twice the dimension of the Cantor set.

We can use mass distributions in a different way, in order to find another lower bound of the Hausdorff dimension.

Definition 4.6. Suppose (E, ρ) is a metric space and $\alpha \geq 0$. Then the α**-potential** of a point $x \in E$ with respect to mass distribution, γ is defined as

$$\phi_\alpha(x) = \int \frac{d\gamma(y)}{\rho(x,y)^\alpha}$$

Definition 4.7. Suppose (E, ρ) is a metric space and $\alpha \geq 0$. Then the α**-energy** with respect to mass distribution, γ is defined as

$$I_\alpha(\gamma) = \int \phi_\alpha(x) d\gamma(x) = \int\int \frac{d\gamma(y)d\gamma(x)}{\rho(x,y)^\alpha}$$

Roughly, what energy and potential do is measure how spread apart the mass distribution is. If a small distance has a large mass distribution then the energy and potential will be very high. Whereas if the mass distribution gives little weight to each range, then the energy and potential will be small.

Theorem 4.8. *Let $\alpha > 0$ and γ be a mass distribution on metric space E. Then for all $\epsilon > 0$*

$$\mathcal{H}_\epsilon^\alpha(E) \geq \frac{\gamma(E)}{\int\int_{\rho(x,y)<\epsilon} \frac{d\gamma(y)d\gamma(x)}{\rho(x,y)^\alpha}}.$$

Thus if $I_\alpha(\gamma) < \infty$, then dim$(E) \geq \alpha$.

Proof. Suppose that $\{A_n : n = 1, 2, ..\}$ is a pairwise-disjoint covering of the set E such that $|A_n| < \epsilon$ for all n. Then

$$\int\int_{\rho(x,y)<\epsilon} \frac{d\gamma(y)d\gamma(x)}{\rho(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \int\int_{A_n \times A_n} \frac{d\gamma(y)d\gamma(x)}{\rho(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \frac{\gamma(A_n)^2}{|A_n|^\alpha}.$$

By the definition of mass distribution, there exists $0 < C < \infty$ depending only on $\gamma(E)$ such that

$$1 \leq C\gamma(E).$$

Fix such a C, by Cauchy-Schwarz's inequality,

$$C\gamma(E) \leq C\sum_{n=1}^{\infty} \gamma(A_n) \leq C^2 (\sum_{n=1}^{\infty} \gamma(A_n))^2 = C^2 (\sum_{n=1}^{\infty} |A_n|^{\frac{\alpha}{2}} \frac{\gamma(A_n)}{|A_n|^{\frac{\alpha}{2}}})^2$$
$$\leq C^2 \sum_{n=1}^{\infty} |A_n|^\alpha \sum_{n=1}^{\infty} \frac{\gamma(A_n)^2}{|A_n|^\alpha} \leq C^2 \mathcal{H}_\epsilon^\alpha(E) \int\int_{\rho(x,y)<\epsilon} \frac{d\gamma(y)d\gamma(x)}{\rho(x,y)^\alpha}.$$

If $I_\alpha(\gamma) < \infty$ then $\int\int_{\rho(x,y)<\epsilon} \frac{d\gamma(y)d\gamma(x)}{\rho(x,y)^\alpha}$ goes to zero as $\epsilon \to 0$. This means that $\mathcal{H}_\epsilon^\alpha(E)$ goes to ∞. Since C is finite, $\alpha \leq \dim(E)$. □

5. Kaufman's Doubling Theorem

The image of the Cantor set under Brownian motion is not unique in its relation to the dimension of the Cantor set. In fact, for every subset of $\mathbb{R}$, the dimension of the original set is half the dimension of the Brownian motion of that set. This result is Kaufman's doubling theorem and is what we will prove in this section.

First, we will prove a weaker version of Kaufman's theorem.

Theorem 5.1. Mckean's Doubling Theorem
Let $A \subset [0, \infty)$ *be a closed subset and let* $\{B(t) : t \in A\}$ *be a* d*-dimensional Brownian motion, then*

$$\dim B(A) = \min(d, 2 \dim A)$$

Proof. We will show that the dimension of Brownian motion is bounded above by the quantity. Suppose that $\dim A < \beta$. Then there exists a covering $A_1, A_2, ...$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$ and $\sum_{i=1}^{\infty} |A_i|^{\beta}$. Then $B(A_1), B(A_2), ...$ is a covering of $B(A)$. By Hölder continuity, for $\alpha < \frac{1}{2}$

$$\sum_{i=1}^{\infty} |B(A_i)|^{\frac{\beta}{\alpha}} \leq C^{\frac{\beta}{\alpha}} \sum_{i=1}^{\infty} |A_i|^{\beta} < C^{\frac{\beta}{\alpha}} \epsilon$$

This goes to zero, so by Theorem 3.1, $\dim B(A) \leq 2\beta$.

It is more involved to show the lower bound and a full proof can be found in [2, Theorem 4.37]. □

Mckean's theorem generalizes the example of image of the Cantor set under Brownian motion that we used in the previous section. It shows that for all fixed sets, Brownian motion doubles the dimension of the original set. However we often want to talk about all sets, which includes sets that depend on the Brownian motion, and that is what Kaufman's theorem allows us to do.

In order to prove Kaufman's doubling theorem, and generalize Mckean's theorem, we need to prove two lemmas. But first we need to introduce some definitions.

Definition 5.2. $\mathbb{P}_z\{x\}$ is the probability that x occurs for a Brownian motion started at z.

Definition 5.3. $\lceil x \rceil = \min(t : x \leq t,\ t \in \mathbb{Z})$

Lemma 5.4. *Let* $Q \subset \mathbb{R}^d, d \geq 3$ *be a cube centered at* x *such that for all* $y \in Q, |y - x| \leq 2r$ *and let* $B(t)$ *be a* d*-dimensional Brownian motion. Define*

$$\tau_1^Q = \inf[t \geq 0 : B(t) \in Q]$$

$$\tau_{k+1}^Q = \inf[t \geq 0 : B(\tau_k^Q + r^2) \in Q] \qquad k \geq 1$$

with $\inf[\emptyset] = \infty$*. Then there exists* $0 < \theta < 1$ *depending only on dimension such that*

$$\mathbb{P}_z\{\tau_{n+1}^Q < \infty\} \leq \theta^n$$

for all $z \in Q$, $n \in \mathbb{N}$.

Proof. It is equivalent to show that

$$\mathbb{P}_z\{\tau_{n+1}^Q = \infty | \tau_n^Q < \infty\} \geq 1 - \theta.$$

The term on the left is bounded below by

$$\mathbb{P}_z\{\tau_{n+1}^Q = \infty \mid \tau_n^Q < \infty, |B(\tau_n^Q + r^2) - x| > 2r\} \mathbb{P}_z\{|B(\tau_n^Q + r^2) - x| > 2r \mid \tau_n^Q < \infty\}.$$

The first term in this product is non-zero by the transience of Brownian motion for $d \geq 3$. The second term is bounded below by

$$\mathbb{P}_z\{B(\tau_n^Q + r^2) - B(\tau_n^Q) \notin Q \mid \tau_n^Q < \infty\}.$$

By the strong Markov property, $B(\tau_n^Q + r^2) - B(\tau_n^Q)$ is a standard Brownian motion independent of τ_n^Q. By Brownian scaling, these probabilities hold regardless of r.

Because $\mathbb{P}_z\{\tau_{n+1}^Q = \infty | \tau_n^Q < \infty\} \geq 1-\theta$ is bounded below by a positive non-zero, it must be bounded by 1-θ for some θ. $\square$

Lemma 5.5. *Let $\mathscr{C}_m$ be the set of d-dimensional dyadic cubes with side length 2^{-m} contained within d-dimensional cube $[-\frac{1}{2}, \frac{1}{2}]^d$. With probability one, there exists random variable C such that for all m and all $Q \in \mathscr{C}_m$, $\tau^Q_{\lceil mC\rceil+1} = \infty$.*

Proof. By Lemma 5.4,

$$\sum_{m=1}^{\infty} \sum_{Q\in\mathscr{C}_m} \mathbb{P}_z\{\tau^Q_{\lceil mc\rceil+1}\} \leq \sum_{m=1}^{\infty} (2^d\theta^c)^m.$$

Choose c such that $2^d\theta^c < 1$, then

$$\sum_{m=1}^{\infty} (2^d\theta^c)^m < \infty.$$

By the Borel-Cantelli lemma, for all but a finitely many m, $\tau^Q_{\lceil cm\rceil+2} = \infty$ for all $Q \in \mathscr{C}_m$. Since there are finitely many exceptional cases, we can choose random $C > c$ to handle those cases. $\square$

What these two lemmas do is to give a finite universal bound to the length of time that a certain Brownian motion spends within a cube. This bound depends only on the length of the dyadic cube.

Theorem 5.6. Kaufman's Doubling Theorem
Let $B(t)$ be an d-dimensional Brownian motion with $d \geq 3$. Then for all $A \subset [0, \infty)$

$$2 \dim A = \dim B(A)$$

with probability one.

Proof. First we can prove $\dim B(A) \leq 2 \dim A$. By the α-Hölder continuity of Brownian motion, for all $\alpha < \frac{1}{2}$

$$|B(x) - B(y)| \leq c|x-y|^{\alpha} \implies |B(x) - B(y)|^{\frac{\dim(A)}{\alpha}} \leq c|x-y|^{\dim A}.$$

Let A_n be any cover of A such that $A \subset \bigcup_{i=1}^{\infty} A_i$. By the definition of Hausdorff dimension,

$$\sum_{i=1}^{\infty} |B(A_i)|^{\frac{\dim(A)}{\alpha}} \leq \sum_{i=1}^{\infty} c|A_i|^{\dim A} < \infty \implies \dim B(A) \leq 2 \dim A.$$

Now we only have to prove that $\dim B(A) \geq 2 \dim A$. This will be done by fixing a d-dimensional cube, $[-L, L]^d$, and showing that for all subsets, S, of this cube with probability one, one has

$$2 \dim B^{-1}S \leq \dim S. \tag{5.7}$$

By scaling it is sufficient to prove this for $L = \frac{1}{2}$.

Fix a path $B^* = \{B(t) : t \geq 0\}$ satisfying the Lemma 5.5 for constant $C > 0$. Let $S = B^* \cap [-\frac{1}{2}, \frac{1}{2}]^d$. If $\beta > \dim S$ and $\epsilon > 0$, then there exists covering of S by dyadic cubes $\{Q_j : j \in \mathbb{N}\} \subset \bigcup_{m=1}^{\infty} \mathscr{C}_m$ such that

$$\sum |Q_j|^{\beta} < \epsilon \implies \sum_{m=1}^{\infty} N_m 2^{-m\beta} < \epsilon \tag{5.8}$$

where N_m is the number of cubes of Q_j that are elements of $\mathscr{C}_m$.

Take the inverse images of these cubes, Q_j under $B(t)$. This gives one a covering of $B^{-1}(S)$. Since $B(t)$ satisfies Lemma 5.5, this means that the inverse of each cube contains at most Cm intervals of length r^2 where r^2 depends only on the dimension d and m. So for each m, the inverse cover of $B^{-1}(S)$ contains at most $N_m Cm$ intervals of length r^2.

Since in Lemma 5.4 the diameter is given by $2r$, the diameter of a dyadic cube is calculated to be $\sqrt{d}2^{-m}$. This means that $r^2 \leq d2^{-2m}$.

For $\gamma > \beta$, one has

$$\mathcal{H}_{\infty}^{\frac{\gamma}{2}}(B^{-1}(S)) \leq \sum_{m=1}^{\infty} CmN_m(d2^{-2m})^{\frac{\gamma}{2}} \leq Cd^{\frac{\gamma}{2}} \sum_{m=1}^{\infty} mN_m 2^{-m\beta}$$

By (5.8) this can be made arbitrarily small. Thus, $\dim B^{-1}(S) \leq \frac{\gamma}{2}$ for all $\gamma > \beta > \dim S$, proving (5.7).

Now, take $S = B(A) \cap [-L, L]^d$ for large L to get $\dim B(A) \geq 2 \dim A$. □

Acknowledgments

Thank you to my mentor Professor Ewain Gwynne for his help in deciding on and understanding this topic and for his very helpful notes on how to make this paper the best it could be. I would also like to thank Professor Gregory Lawler for his talks on Brownian motion which served as my inspiration to study the topic and Professor J. Peter May for organizing the REU in which this paper was written.

References

[1] Gregory F. Lawler. Random Walk and the Heat Equation.
[2] Peter Mörters and Yuval Peres. Brownian Motion.
[3] Erin Pearse. An Introduction to Dimension Theory and Fractal Geometry: Fractal Dimensions and Measures.
[4] Shailesh A Shirali. Fractal Dimension and the Cantor Set.