TOPOLOGY, GEOMETRY, AND DYNAMICAL SYSTEM OF TORUS

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Abstract. This expository paper explores the torus through the lenses of topology, geometry, and dynamical systems. We begin by examining the fundamental group of the torus, providing a foundation for understanding its topological properties. The concept of the flat torus is then introduced. We also solve the Gauss Circle Problem, uncovering hidden geometric structures within the torus. The study continues with investigating geodesic flows, leading to surprising results about periodicity and density. Finally, we prove Weyl's Equidistribution Theorem, demonstrating the uniform distribution of orbits of the shifting map.

Contents

1. Introduction

The torus, a shape as familiar as a donut yet as rich in complexity as the most intricate mathematical concepts, holds a unique place in the study of mathematics. It's a shape that offers insights that span across topology, geometry, and dynamical systems. This paper explores the torus from multiple perspectives, unraveling some of its mysteries and revealing its significance in various areas of mathematics.

We start by looking at the fundamental group of the torus, a group that helps us to understand its loops and paths and gain a deeper understanding of the torus's topological structure.

From there, we move to the flat tori, characterized by having zero curvature. The flat tori aren't just a mathematical curiosity; they play a significant role in both theoretical mathematics and practical applications.

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We then dive into the Gauss Circle Problem, an ancient and intriguing question connecting number theory and geometry. Our solution to this problem doesn't just solve a puzzle; it adds another layer of understanding to the torus by uncovering some of its hidden geometric structures.

Next, we prove a theorem that leads to surprising results about geodesic flow on the torus. They prove to be either periodic or dense.

Finally, we delve into Weyl's Equidistribution Theorem, a result that beautifully connects number theory and dynamical systems. We demonstrate how sequences that might seem having no pattern at first glance actually distribute uniformly over the torus.

This paper is more than just a series of mathematical discussions—it's a journey through the interconnected world of topology, geometry, and dynamics, with the torus as our guide. This paper invites you to explore the beauty and complexity of the torus in a way that's both accessible and enlightening.

2. Fundamental Group

Let S^1 denote the unit circle. A torus of dimension n, considered as a topological space, is defined to be the product $T^n = S^1 \times \ldots \times S^1$ (with n factors). For $n = 1$, the torus T^1 is simply S^1, which is a circle. The 2-dimensional torus T^2 is given by $S^1 \times S^1$, which can be visualized as a donut-shaped topologically, a surface formed by the Cartesian product of two circles. More generally, the n−dimensional torus represents a space formed by the Cartersian product of n circles. Hence, a torus is a topological space where each dimension is a circle.

A fundamental concept in algebraic topology is the fundamental group. The fundamental group of a topological space X at a chosen base point $x_0 \in X$, denoted $\pi_1(X, x_0)$, is a group that captures information about the space's shape and the possible loops within it.

The fundamental group is the set of all equivalence classes of loops based at the point x_0. Two loops belong to the same class if they are homotopic, meaning they can be continuously deformed into each other without tearing or breaking.

If the space is path-connected, then $\pi_1(X, x_0)$ is independent of the choice of the base point x_0.

Definition 2.1. A topological space X is said to be path-connected if, for any two points in the space, there exists a continuous path between them.

There is a well-known theorem about the fundamental groups of product spaces, see [Arm].

Theorem 2.2. *If X, Y are path-connected spaces, then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

For any two points on the circle S^1, you can always find a continuous path on the circle that connects these points. So, by Definition 2.1, S^1 is path-connected. On the other hand, we can see that the homotopy class of a closed curve on S^1 is determined by its winding number, consequently, $\pi_1(S^1) = \mathbb{Z}$. Therefore, by Theorem 2.2, we have $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.

3. Flat torus

We can equip a torus with a flat metric, as follows. Consider T^1, which is a circle. It can be constructed by identifying the endpoints of an interval in $\mathbb{R}$ and obtaining the metric from the interval. See Figure 1.

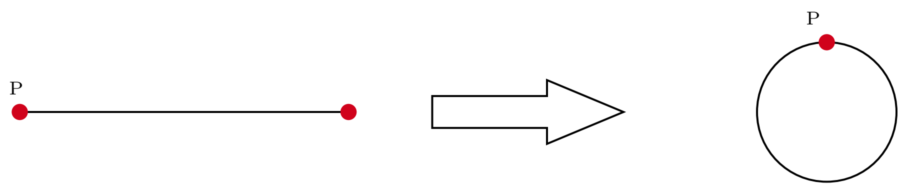

Figure 1. Constructing a 1-dim Torus

Now consider T^2. It can be constructed by identifying the opposite edges of a parallelogram in $\mathbb{R}^2$. See Figure 2. Similarly, this construction induces a flat metric on T^2 from the Euclidean metric on the parallelogram.

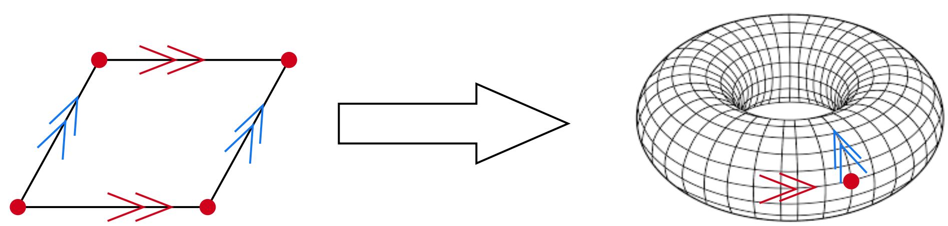

Figure 2. Constructing a 2-dim Torus

Finally, consider T^3, which we can't visualize because it can only be embedded in $\mathbb{R}^n$ when $n > 3$. It is constructed by identifying opposite sides of a parallelepiped and identifying all vertices (red dots in the figure) as one point on the torus. See Figure 3.

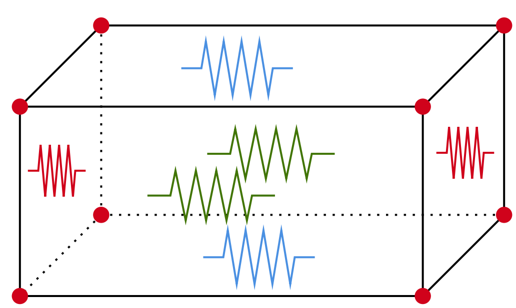

Figure 3. Building Block of a 3-dim Torus

Similarly, we can put a flat metric on T^n in higher dimensions by considering a parallelepiped in $\mathbb{R}^n$.

This construction gives a nice description of the universal cover of a flat torus X. Let P be the n–dimensional parallelepiped used to construct X. We can tile $\mathbb{R}^n$ by copies of P. Each tile is a fundamental domain, and the lattice of vertices corresponds to the preimages of a fixed point on X under the universal cover map. For example, when P is the square $[0,1] \times [0,1]$, the universal cover is $\mathbb{R}^2$ tiled by squares whose vertices have integer coordinates, see Figure 4.

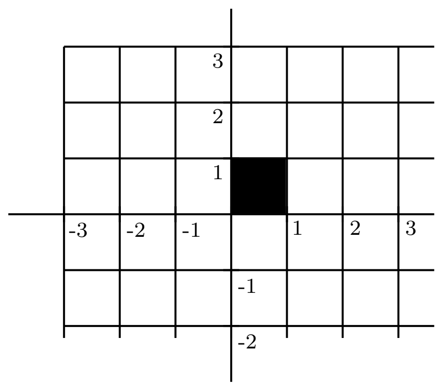

FIGURE 4. Universal Cover of P

4. COUNTING CLOSED GEODESICS[1]

In mathematics, the Gauss circle problem is a problem in determining how many lattice points are in a circle. The first result came from Carl Friedrich Gauss in 1834, hence its name. For reference see [Guy].

4.1. Gauss Circle Problem.

Problem 4.1. *Consider a circle in $\mathbb{R}^2$ with center at the origin and radius $r \geq 0$. How many lattice points, points (m,n) where $m,n \in \mathbb{Z}$, are in the circle? Call this number $N(r)$. In other words, $N(r)$ represents the number of closed geodesics starting from an arbitrarily chosen origin whose lengths are $\leq r$.*

We find an estimate for $N(r)$.

Theorem 4.1. *We have*

$$\lim_{r\to\infty} \frac{N(r)}{r^2} = \pi.$$

Proof. We will solve this problem on $\mathbb{R}^2$, the universal cover of the flat torus $X = \mathbb{R}^2/\mathbb{Z}^2$. A geodesic on X is the image of a line on $\mathbb{R}^2$. A closed geodesic corresponds to a line from the origin to a lattice point. On the other hand, each lattice point is the center of a one-by-one square, and these area 1 squares tile $\mathbb{R}^2$ without overlapping. Consider the tiles whose centers are the lattice points in the disk of radius r. They cover the disk of radius $r-1$ and are inside the disk of radius $r+1$. Therefore, we have $\pi(r-1)^2 \leq N(r) \leq \pi(r+1)^2$, as required. □

[1] A geodesic is the shortest path between two points on a manifold.

4.2. **Primitive Circle Problem.** Now, we will explore an extension of the regular Gauss Circle Problem.

Problem 4.2. *Consider a circle in* $\mathbb{R}^2$ *with center at the origin and radius* $r \geq 0$. *How many lattice points, points* (m, n) *where* $m, n \in \mathbb{Z}$, *are in the circle such that* m *and* n *are coprime? Call this number* $M(r)$. *In other words,* $M(r)$ *represents the number of closed geodesics with different angles with respect to the x-axis and have lengths* $\leq r$. *(In this scenario we are no longer considering closed geodesics that follow the same path but have different lengths.)*

Theorem 4.2. *We have* $\lim_{r\to\infty} \frac{M(r)}{r^2} = \frac{\pi}{6}$.

Proof. To solve this, we will use the same setup as the previous problem, and then expand on it. So, using our equation $N(r)$ for the number of lattice points in a circle of radius r, we will multiply it by P_0, which represents the probability of any two integers being coprime. $P_0 = \frac{6}{\pi^2}$ so overall we get $\sim \frac{6}{\pi} r^2$ points.[2] □

5. Geodesic flow

Geodesic flows describe the free motion of points on manifolds. Consider a manifold M. Given $x \in M$, and a unit verctor v at x, there exists a unique geodesic starting from x in the direction v.

A geodesic on a flat torus is the image of a line in $\mathbb{R}^2$ under the covering map. For the following theorem, we will consider the flat torus X.

Theorem 5.1. *Any geodesic on* X *is either periodic or dense.*

To prove this theorem, we consider rational and irrational slopes separately. We show that the dichotomy in Theorem 5.1 depends on the slope of this line.

Proposition 5.2. *If the angle between the line and the x-axis is rational then the geodesic is periodic.*

Proof. Observe X from its universal cover $\mathbb{R}^2$, where lattice points of $\mathbb{Z} \oplus \mathbb{Z}$ (points (m, n) where $m, n \in \mathbb{Z}$) represent the preimages of a point, as shown in Figure 5. Consider a line in $\mathbb{R}^2$ as the preimage of the geodesic. Without loss of generality, we can assume that this line passes through the origin.

If θ, the slope of the line, is rational then θ can be rewritten as $\theta = \frac{n}{m}$ for $n, m \in \mathbb{Z}$ and the point $(x, x\theta) = (m, n)$ is on the line for $x = m$. Therefore, the geodesic returns to the image of the origin in X, which means it is periodic. □

For the irrational slope case, we use the following result.

Lemma 5.3. *Let* α *be an irrational real number. For any* $\epsilon > 0$, *there are integers* $r > 0, l$ *such that* $|r\alpha - l| < \epsilon$.

Proof. We want to show for some integer r, $r\alpha$ is ϵ close to 0 mod 1. Therefore, we consider numbers $\{r\alpha : r \in \mathbb{Z}\}$ on circle $S^1 = \mathbb{R}/\mathbb{Z}$ representing the fractional parts of the numbers.

Consider integer $n > 1/\epsilon$. Among the points $\alpha, 2\alpha, \ldots, n\alpha$ mod 1 on the circle, there exist $k\alpha, k'\alpha$ for $k, k' \in \mathbb{N}$ whose difference mod 1 can be expressed as follows:

$$|(k - k')\alpha| = |k\alpha - k'\alpha| \equiv \delta \mod 1$$

[2]Notation $\sim$ means their ratio goes to 1 as $r \to \infty$

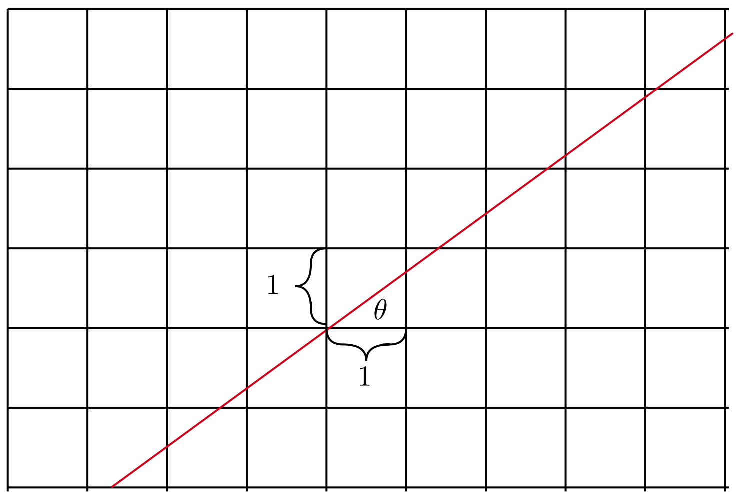

FIGURE 5. Universal Cover of Torus

for some δ representing the fractional part of $|k\alpha - k'\alpha|$, such that:

$$\delta < \frac{1}{n} < \epsilon$$

In other words, there exists $l \in \mathbb{Z}$ s.t.

$$|r\alpha - l| < \epsilon,$$

for $r = k - k'$. □

Proposition 5.4. *If θ is irrational, then the geodesic is dense.*

Proof. Given $\epsilon > 0$, from Lemma 5.3, we get the two following observations:

- Consider $\alpha = \theta$. There are integers r, l such that the vertical displacement between a point $(r, r\theta)$ on the line and a lattice point (r, l) is $\epsilon_y \in (0, \epsilon)$.
- Now, consider $\alpha = \frac{1}{\theta}$. There are integers r, l such that the horizontal displacement between a point $(\frac{r}{\theta}, r)$ on the line and a lattice point (l, r) is $\epsilon_x \in (0, \epsilon)$.

Note that r, l can be negative. Now, we want to show that for any point P in X, the geodesic line reaches the $\epsilon-$neighborhood of P. Consequently, the geodesic line is dense.

Let t_x, t_y be the time it takes, when we move along the line, to get ϵ_y, ϵ_x displacement from a lattice point in the vertical and horizontal directions, respectively.

Consider the lattice $\epsilon_y\mathbb{Z} \oplus \epsilon_x\mathbb{Z}$. Let $(\epsilon_x m, \epsilon_y n)$ be a point of the lattice, which is $\epsilon-$close to the point P. See Figure 6.

The line can reach the point $(\epsilon_x m, \epsilon_y n)$ in $t_x m + t_y n$ time. Therefore, we showed for any point P in X, the geodesic can get ϵ-close to it after some time, as required. □

Proof of Theorem 5.1. Follows from Proposition 5.2 and Proposition 5.4. □

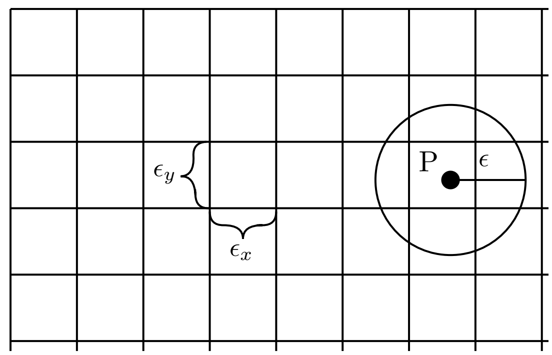

FIGURE 6. ϵ-displacements

6. EQUIDISTRIBUTION

In this section, we will prove Weyl's Equidistribution Theorem by utilizing the theory of the Fourier series. Weyl's equidistribution theorem is a fundamental result in mathematics, particularly in number theory and dynamical systems, that describes how sequences distribute uniformly across the unit circle. Its significance lies in connecting arithmetic properties with uniform distribution, which has far-reaching applications in areas such as Diophantine approximation, cryptography, and numerical methods.

Consider a flat torus X. As we explained, it can be constructed by a parallelepiped in $\mathbb{R}^n$. An important dynamical system on X is shifting points by a vector $v = (\theta_1, \dots, \theta_n)$. We can see that an orbit either is periodic or its closure is a sub-torus in X. Moreover, the Kronecker-Weyl theorem states that the orbit points are equidistributed in their closure. Weyl's theorem is a particular case of this result for $n = 1$. For a reference see [Bai]. The proof that follows is from [SS].

In this proof, we will use the following known result in the theory of the Fourier series, which we will not prove here:

Corollary 6.1. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.* [SS]

Before proceeding, we define trigonometric polynomials:

Definition 6.2. $P(x)$ is a trigonometric polynomial if it can be expressed as $P(x) = c_k e^{2\pi k i} + \dots + c_j e^{2\pi j i}$, where $k, j \in \mathbb{Z}$.

Theorem 6.3. *Weyl's equidistribution theorem: If θ is irrational, then the sequence of fractional parts $\{k\theta : k \in \mathbb{N}\}$ is equidistributed in S^1.*

To prove this, we arbitrarily select two points a and b on the circle S^1 and show that as our sequence goes to infinity, the proportion of total points that fall within $[a, b]$ is equal to the distance between a and b.

To capture the fraction of points within the interval $[a, b]$, we define the characteristic function $\chi_{[a,b]} : \mathbb{R} \to \mathbb{R}$ as follows:

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \mod 1 \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Thus, for any $n \in \mathbb{N}$, we have:

$$\#\{1 \le k \le n : \langle k\theta \rangle \in [a,b]\} = \sum_{k=1}^{n} \chi_{[a,b]}(k\theta)$$

We want to show that as n goes to infinity, the following holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[a,b]}(k\theta) = b - a$$

Or equivalently:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{[a,b]}(k\theta) = \int_{S^1} \chi_{[a,b]}(x)\, dx \tag{1}$$

More generally, this can be expressed as:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\theta) = \int_{S^1} f\, dx,$$

where $f = \chi_{[a,b]}$. We use the following results in the proof.

Lemma 6.4. *(1) holds for $f_k(x) = e^{2\pi i k x}$ when k is a nonnegative integer.*

Proof. This holds for $f_0(x) = 1$ as **LHS** $= \frac{n}{n} = 1$ and **RHS** $= x\big|_0^1 = 1$.
For $f_k(x) = e^{2\pi i k x}$ for $k \neq 0$, we have:
RHS:
We can solve for $\int_{S^1} f$ like so:

$$\begin{aligned}\int_0^1 e^{2\pi i k x}\, dx &= \frac{e^{2\pi i k x}}{2\pi i k}\Big|_0^1 \\ &= \frac{e^{2\pi i k} - 1}{2\pi i k} \\ &= \frac{1-1}{2\pi i k} \\ &= 0\end{aligned}$$

LHS:
The LHS is given by, $\lim_{n\to\infty} \frac{A}{n}$ where

$$A = a^0 + a + \ldots + a^m + \ldots + a^n$$

for $a = e^{2\pi i k\theta}$. So, the LHS is equal to the limit of

$$\frac{1 - a^{n+1}}{n(1-a)} = \frac{1 - e^{(n+1)(2\pi i k\theta)}}{n(1 - e^{2\pi i k\theta})}$$

as n goes to infinity and we can see that its absolute value is:

$$\le \lim_{n\to\infty} \frac{2}{c \cdot n} = 0.$$

Hence, we have shown that both sides are zero and as a result (1) holds for all functions f_k. □

Lemma 6.5. *If two functions f, g satisfy (1) then so does $Af + Bg$ for any $A, B \in \mathbb{C}$.*

Proof. Let f and g be two functions that satisfy (1). For some $A, B \in \mathbb{C}$:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} Af(k\theta)}{n} = \int_{S^1} Af$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} Bg(k\theta)}{n} = \int_{S^1} Bg$$

By linearity of limits and integrals, adding these two equations gives us:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} Af(k\theta)}{n} + \frac{\sum_{k=1}^{n} Bg(k\theta)}{n} = \int_{S^1} Af + \int_{S^1} Bg$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} Af(k\theta) + Bg(k\theta)}{n} = \int_{S^1} Af + Bg$$

Thus, $Af + Bg$ also satisfies (1). □

Remark: As a direct result of this lemma we note that when f is a finite combination of trigonometric polynomials, then (1) holds.

Lemma 6.6. *Assume that a sequence of functions f_n converges absolutely to function f. If (1) holds for f_n's, it holds for f too.*

Proof. Let f be any continuous function on the circle of period 1. Then by Corollary 6.1 we can choose a trigonometric Polynomial P s.t. for $\epsilon > 0$, $\sup_{x\in\mathbb{R}} |f(x) - P(x)| < \frac{\epsilon}{3}$. Then by Lemmas 6.4 and 6.5, for all large N we have:

$$|\frac{1}{N}\sum_{n=1}^{N} P(n\theta) - \int_{S^1} P(x)\,dx| < \frac{\epsilon}{3}$$

Hence:

$$\begin{aligned}\left|\frac{1}{N}\sum_{n=1}^{N} f(n\theta) - \int_{S^1} f(x)\,dx\right| &\le \frac{1}{N}\sum_{n=1}^{N} |f(n\theta) - P(n\theta)| \\ &+ \frac{1}{N}\sum_{n=1}^{N} \left|P(n\theta) - \int_{S^1} P(x)\,dx\right| \\ &+ \int_{S^1} |P(x) - f(x)|\,dx < \epsilon.\end{aligned}$$

Thus, (1) holds for f. □

Proof of Theorem 6.3.

Proof. Consider a continuous function f. The sequence $S_N(f)$ converges to f by Corollary 6.1 as N approaches infinity. $S_N(f)$ satisfies (1) by Lemma 6.4 and Lemma 6.5. Therefore, f satisfies (1) by Lemma 6.6.

We estimate $\chi_{[a,b]}$ from above and below by continuous periodic functions $f_\epsilon^+, f_\epsilon^-$. These functions agree with $\chi_{[a,b]}$ except in two intervals of length 2ϵ as seen in Figure 7.

We know that (1) holds for these two functions and we know the following inequality holds:

$$f_\epsilon^- \le \chi_{[a,b]} \le f_\epsilon^+$$

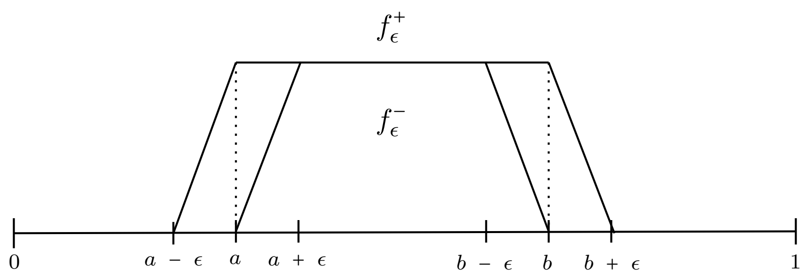

FIGURE 7. Approximation of $\chi_{[a,b]}$

For $S_N = \frac{1}{N}\sum_{n=1}^{N} \chi_{[a,b]}(n\theta)$, we obtain the following:

$$\frac{1}{N}\sum_{n=1}^{N} f_\epsilon^-(n\theta) \leq S_N \leq \frac{1}{N}\sum_{n=1}^{N} f_\epsilon^+(n\theta)$$

Then we get:

$$b - a - 2\epsilon \leq \lim_{N\to\infty} \inf S_N \qquad b - a + 2\epsilon \geq \lim_{N\to\infty} \sup S_N$$

$$b - a - 2\epsilon \leq \lim_{n\to\infty} S_N \leq b - a + 2\epsilon$$

Since this is true for any $\epsilon > 0$, $\lim_{N\to\infty} S_N$ exists and equals $b - a$.

□

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