AN INTRODUCTION TO LOEWNER ENERGY

DADU CHEN, ZHUOHAN GU

ABSTRACT. This paper provides an introduction to the study of Loewner Energy. We start by exploring the large deviation principle (LDP), which describes the exponential decay of the probability of certain rare events of families of probability measures. In large deviation theory, Schilder's theorem explains the LDP of Brownian motion. More specifically, it tells that the Dirichlet energy arises naturally as the good rate function of the scaled Brownian path, which almost surely has infinite Dirichlet energy. We then shift our focus to the formulation of Loewner theory as a preparation for the study of Schramm-Loewner-Evolution (SLE), which is a one-parameter family of random curves that describes the limiting behaviour of various statistical models. We introduce four fundamental terms in the Loewner theory (hulls, curves, conformal maps, and driving processes) and investigate their relationship between each other. Next, we discuss the LDP of chordal SLE analogy to Schilder's theorem. Loewner energy of a simple curve, which is defined as the Dirichlet energy of its driving function, arises as a good rate function. We finally discuss some properties of Loewner energy such as its reversibility.

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1. LARGE DEVIATION PRINCIPLE

We first start building an understanding of the Large Deviation Principle based on the following simple example.

Suppose that X is a random variable such that $X \sim \mathcal{N}(0, \sigma^2)$, the normal distribution, so that its probability density function is given by

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}.$$

For any $\epsilon > 0$, we have that $\sqrt{\epsilon}X \sim \mathcal{N}(0, \sigma^2\epsilon)$. Given $M > 0$, we can quantify how rare the event $\{\sqrt{\epsilon}X \geq M\}$ happens as $\epsilon \to 0+$. Since

$$\mathbb{P}(\sqrt{\epsilon}X \geq M) = \frac{1}{\sqrt{2\pi\sigma^2\epsilon}} \int_M^\infty e^{-\frac{x^2}{2\sigma^2\epsilon}} dx,$$

it follows that

$$\begin{aligned}\lim_{\epsilon\to 0+}\epsilon\log\mathbb{P}(\sqrt{\epsilon}X\geq M) &= \lim_{\epsilon\to 0+}\epsilon\log(\frac{1}{\sqrt{2\pi\sigma^2\epsilon}}\int_M^\infty e^{-\frac{x^2}{2\sigma^2\epsilon}}dx)\\ &= \lim_{\epsilon\to 0+}-\frac{1}{2}\epsilon\log(2\pi\sigma^2\epsilon)+\epsilon\log\int_M^\infty e^{-\frac{x^2}{2\sigma^2\epsilon}}dx\\ &= -\frac{M^2}{2\sigma^2}.\end{aligned}$$

Let $I_X(x) := \frac{x^2}{2\sigma^2}$. Note that

$$\lim_{\epsilon\to 0+}\epsilon\log\mathbb{P}(\sqrt{\epsilon}X\geq M) = -\inf_{x\in[M,\infty)}I_X(x), \tag{1.1}$$

and by similar deduction we have that

$$\lim_{\epsilon\to 0+}\epsilon\log\mathbb{P}(\sqrt{\epsilon}X\in[a,b]) = -\inf_{x\in[a,b]}I_X(x).$$

Intuitively, we can say that the probability of the event $\{\sqrt{\epsilon}X\in[c,d]\}$, where the interval $[c,d]$ is very small and contains x, decays exponentially fast in the order of $\frac{-I_X(x)}{\epsilon}$ as $\epsilon\to 0$. In fact, the main theory of the large deviation principle describes the exponential decay of the probability of certain rare events of families of probability measures.

We now give a precise definition of the large deviation principle. Let $\mathcal{X}$ be a Polish space [1], $\mathcal{B}$ the Borel $\sigma-$algebra on X, and$\{\mu_\epsilon\}_{\epsilon>0}$ a family of probability measures on $(\mathcal{X},\mathcal{B})$.

Definition 1.2. A **rate function** is a lower semicontinuous function $I:\mathcal{X}\to[0,\infty)$, i.e., for all $\alpha\geq 0$ the sub-level set $\{x : I(x)\leq\alpha\}$ is a closed subset of $\mathcal{X}$. A **good rate function** is a rate function for which all the sub-level sets are compact subsets of $\mathcal{X}$.

Definition 1.3. A family of probability measures $\{\mu_\epsilon\}_{\epsilon>0}$ on $(\mathcal{X},\mathcal{B})$ satisfies the **large deviation principle (LDP) of rate function** $I:\mathcal{X}\to[0,\infty)$ if for all open sets $O\in\mathcal{B}$ and closed sets $F\in\mathcal{B}$,

$$\limsup_{\epsilon\to 0+}\epsilon\log\mu_\epsilon(F)\leq -\inf_{x\in F}I(x);\ \liminf_{\epsilon\to 0+}\epsilon\log\mu_\epsilon(O)\geq -\inf_{x\in O}I(x).$$

Observe that by Equation (1.1), the distribution of $\{\sqrt{\epsilon}X\}_{\epsilon>0}$ satisfies the LDP with good rate function I_X.

We can show that if $\{\mu_\epsilon\}_{\epsilon>0}$ satisfies LDP of some rate function, i.e., the rate function exists, then it is unique.

Theorem 1.4. *If a family of probability measures satisfies the LDP of some rate function, then the rate function is unique.*

Proof. Let I, J be two rate functions for $\{\mu_\epsilon\}_{\epsilon>0}$. Suppose for contradiction that $I\neq J$. We can assume without loss of generality that there exists $x>0$ such that $J(x)<I(x)$.

Take $\alpha>0$ such that $J(x)<\alpha<I(x)$. Since I is lower semicontinuous, there exists $\epsilon>0$ such that if $x\in\{y : d(y,x)\leq\epsilon\}$, then $I(x)>\alpha$. Let $O=\{y : d(y,x)<$

[1] A Polish space is a separable, completely metrizable topological space.

$\epsilon\}$. By definition of LDP it follows that

$$\begin{aligned} -J(x) &\leq -J(O) \\ &\leq \liminf_{\epsilon \to 0+} \epsilon \log \mu_\epsilon(O) \\ &\leq \limsup_{\epsilon \to 0+} \epsilon \log \mu_\epsilon(\overline{O}) \\ &\leq -I(\overline{O}) \\ &\leq -\alpha \end{aligned}$$

which contradicts to $J(x) < \alpha$. □

A nice principle in the large deviation theory is that the LDP is preserved under any continuous map, after possibly changing the rate function. We summarize it as a theorem.

Theorem 1.5. (**Contraction principle**). *Let $\mathcal{X}, \mathcal{Y}$ be two Polish spaces, $f : \mathcal{X} \to \mathcal{Y}$ a continuous function, and $\{\mu_\epsilon\}_{\epsilon>0}$ a family of probability measures on $\mathcal{X}$ satisfying the LDP with a good rate function $I : \mathcal{X} \to [0, \infty)$. Let $I' : \mathcal{Y} \to [0, \infty)$ be defined by $I'(y) := \inf_{x \in f^{-1}\{y\}} I(x)$. Then, the family of pushforward probability measures $\{f_* \mu_\epsilon\}_{\epsilon>0}$[2] satisfies the LDP with good rate function I'.*

Proof. We first check that I' is a good rate function on $\mathcal{Y}$. For $\alpha > 0$, let $\Phi_{I'}(\alpha) := \{y : I'(y) \leq \alpha\}$. Since I is a good rate function, $\forall y \in f(\mathcal{X})$ we have that $I'(y) = I(x)$ for some $x \in f^{-1}\{y\}$, i.e., the infimum can be achieved. Hence, $\Phi_{I'}(\alpha) = \{f(x) : I(x) \leq \alpha\} = f(\Phi_I(\alpha))$. Since I is a good rate function, $\Phi_I(\alpha)$ is compact. Thus, $\Phi_{I'}(\alpha)$ is also compact as f is continuous.

Now, notice that for all $E \subset \mathcal{Y}$ we have that

$$\inf_{y \in E} I'(y) = \inf_{x \in f^{-1}(E)} I(x). \tag{1.6}$$

Since f is continuous, the preimage of f preserves open (closed) sets. Hence, by Equation (1.6) it follows that $\{f_* \mu_\epsilon\}_{\epsilon>0}$ satisfies the LDP with good rate function I'. □

We end this section with an important example in the large deviation theory, the scaled Brownian path. For $T \in (0, \infty)$, let

$$C^0[0,T] := \{W : [0,T] \to \mathbb{R} \mid t \mapsto W_t \text{ is continuous and } W_0 = 0\}.$$

Definition 1.7. The **Dirichlet energy** of $W \in C^0[0,T]$(resp. $W \in C^0[0,\infty)$) is given by

$$I_T(W) := \frac{1}{2} \int_0^T |\frac{dW_t}{dt}|^2 \, dt \left(\text{resp. } I_\infty(w)L = \frac{1}{2} \int_0^T |\frac{dW_t}{dt}|^2 \, dt \right)$$

if W is absolutely continuous, and set $I_T(W) = \infty$ otherwise.

It turns out that the Dirichlet energy $I_T(W)$ of $W \in C^0[0,T]$ arises naturally as the rate function of the scaled Brownian motion in the following way.

[2]This means that for all $E \subset \mathcal{Y}$. we have that $f_* \mu_\epsilon = \mu_\epsilon(f^{-1}(E))$

Theorem 1.8. (**Schilder**) [3]. *Fix $T \in (0, \infty)$. The family of processes $\{(\sqrt{\epsilon} B_t)_{t\in[0,T]}\}_{\epsilon>0}$, viewed as a family of random functions in $(C^0[0,T], \|\cdot\|)$, satisfies the LDP with a good rate function I_T, where $I_T(W) = \frac{1}{2}\int_0^T |\frac{dW_t}{dt}|^2 dt$ if W is absolutely continuous and ∞ otherwise.*

2. Chordal Loewner Chain

Throughout the paper, we let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the upper halfplane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk.

The definition of SLE relies on the Loewner transform, which is a deterministic way that encodes a simple curve on a simply connected domain into a driving function. The general idea is that given a simple curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$, we can parametrize it by a continuous time interval $[0, T)$ where $T \in (0, \infty)$ with $\gamma(0) = 0$ and $\gamma(t) \to \overline{\mathbb{H}}$ as $t \to T$. For each $t \in [0, T)$, we can associate it with a unique conformal map

$$g_t : \mathbb{H} \setminus \gamma_{[0,t]} \to \mathbb{H}$$

such that at near infinity the expansion of g_t satisfies

$$g_t(z) = z + \frac{2t}{z} + o(\frac{1}{z}).$$

We can extend g_t continuously to the prime ends and define $W_t := g_t(\gamma_t)$. We call W_t the driving function of γ.

Given a simple curve we can define a family of unique conformal mappings in the way above. Converesly, given a continuous function W_t we can find a family of conformal maps $(g_t)_{t\in[0,T)}$ by solving the Loewner equation for each $z \in \mathbb{H}$,

$$\begin{cases} g_0(z) = 0 \\ \partial_t g_t(z) = \dfrac{2}{g_t(z) - W_t} \end{cases}$$

and reconstruct a curve. Moreover, we shall see that if we choose $W_t = \sqrt{\kappa} B_t$ where κ is a parameter and B_t is the standard Brownian motion, then the random curve encoded by it is the chordal SLE_κ. The above arguments mostly rely on conformal theory in complex analysis, and we now give a brief overview.

Definition 2.1. A set $K \subset \overline{\mathbb{H}}$ is called a **hull** if K is compact and $\mathbb{H} \setminus K$ is simply connected.

Theorem 2.2. *For any hull K, there exists a unique conformal surjection $g_K : \mathbb{H} \setminus K \to \mathbb{H}$ such that*

$$\lim_{z\to\infty} (g_k(z) - z) = 0 \tag{2.3}$$

where the limit holds along any sequence $z_n \in \mathbb{H}$ such that $|z_n| \to \infty$. Such g_K is said to have hydrodynamic normalization. Near infinity, g_K has the expansion

$$g_K(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdot$$

where the coefficients $a_k, k \in \mathbb{N}$, are real.

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[3]For a detailed proof of this theorem, see [4].

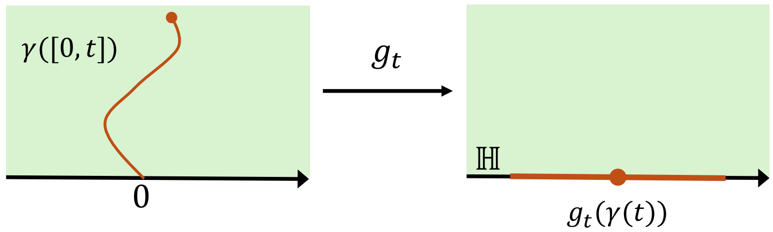

FIGURE 1. The hydrodynamical conformal map g_t.

Proof. By Riemann mapping theorem, there exists a conformal map $\tilde{g} : \mathbb{H} \setminus K \to \mathbb{D}$. Moreover, we can guarantee $\tilde{g}(\infty) \in \partial\mathbb{D}$ since there exists a holomorphic extension of $z \mapsto \tilde{g}(-1/z)$ to a neighborhood of 0 by Schwarz reflection principle. Then, by composing $\tilde{g}$ with a Mobius map $\phi : \mathbb{D} \to \mathbb{H}$ we can obtain a conformal map from $\mathbb{H} \setminus K$ to $\mathbb{H}$ mapping ∞ to ∞.

Pick one of them and call it $\hat{g}$. Let $H' = \{-1/z : z \in \mathbb{H} \setminus K\}$ and define

$$f(z) = -1/\tilde{g}(-1/g).$$

Notice that by Schwarz reflection again, f extends holomorphically and injectively to a neighborhood of 0. We can pick $\epsilon > 0$ such that $B(0,\epsilon) \cap \mathbb{H} \subset H'$, so f maps $B(0,\epsilon)$ to $\mathbb{R}$.

Note that if $f = u+iv$, then $f'(0) = \partial_x u(0) = \partial_y v(0) > 0$ since f maps $B(0,\epsilon)\cap\mathbb{H}$ to $\mathbb{H}$. Therefore, near 0 we have that

$$f(z) = b_1 z + b_2 z^2 + \cdots$$

where $b_1 > 0$ and $b_j \in \mathbb{R}$. This implies that for large $|z|$ we have

$$\hat{g}(z) = \hat{a}_{-1} z + \hat{a}_0 + \hat{a}_1 z^{-1} + \cdots$$

where $\hat{a}_{-1} > 0$ and $\hat{a}_j \in \mathbb{R}$. To normalize $\hat{g}$ as in the theorem, we require $\hat{a}_{-1} = 1$ and $\hat{a}_0 = 0$. Note that in fact there exists a unique choice of Mobius map $\phi : \mathbb{H} \to \mathbb{H}$ such that $g_k = \phi \circ \hat{g}$ has the expansion

$$g_K(z) = z + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

since $\hat{g}$ already fixes $\infty \mapsto \infty$. □

Definition 2.4. If K is a hull and g_K satisfies the hydrodynamic normalization, then the coefficient of $\frac{1}{z}$ in the expansion of g_K is denoted by $a_1(K)$. We call $a_1(K)$ the **half-plane capacity** of K.

Example 2.5. Suppose $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ is defined by $\gamma(t) = x_0 + it$. Then at any time T, we can associate γ with a hull by $\gamma([0,T]) = K = [x_0, x_0 + iT]$, a vertical line segment starting at $(x_0, 0)$. One can check that the corresponding conformal map satisfying hydrodynamic normalization is given by

$$\begin{aligned} g_K(z) &= x_0 + \sqrt{(z-x_0)^2 + T^2} \\ &= x_0 + z\sqrt{1 - \frac{2x_0}{z} + \frac{x_0^2 + T^2}{z^2}} \\ &= x_0 + z(1 - \frac{x_0}{z} + \frac{x_0^2+T^2}{2z^2} - \frac{1}{8}\frac{4x_0^2}{z^2} + \cdots) \\ &= z + \frac{T^2}{2z} + \cdots \end{aligned}$$

Hence, the half-plane capacity of the vertical line segment of length T is given by $a_1(K) = \frac{T^2}{2}$.

Proposition 2.6. *For any hull K, the half-plane capacity satisfies the following scaling rule and is translation invariance:*

$$\begin{aligned} a_1(\lambda K) &= \lambda^2 a_1(K) \\ a_1(K+x) &= a_1(K) \end{aligned}$$

Proof. Suppose that g_K satisfies the hydrodynamic normalization. Note that $\lambda g_K(\frac{z}{\lambda})$ is a conformal transformation from $\mathbb{H}\backslash\lambda K$ to $\mathbb{H}$ such that $|\lambda g_K \frac{z}{\lambda} - z| \to 0$ as $z \to \infty$. Hence, $\lambda g_K(\frac{z}{r}) = g_{\lambda K}$ by the uniqueness of $g_{\lambda K}$. Since

$$\lambda g_K(\frac{z}{\lambda}) = \lambda(\frac{z}{\lambda} + \frac{a_1(K)}{z/\lambda} + o(\frac{1}{z/\lambda})) = z + \frac{\lambda^2 a_1(K)}{z} + o(\frac{1}{z}),$$

the scaling property follows.

Similarly, $g_K(z-x)+x$ is a conformal surjection from $\mathbb{H}\backslash(K+x)$ to $\mathbb{H}$ such that $|g_K(z-x)+x-z| \to 0$ as $z \to \infty$. Hence, $g_{K+x}(z) = x + g_K(z-x)$ by uniqueness of g_{K+x} and the result follows from expanding $x + g_K(z-x)$. □

For any hull K, we can think of the half-plane capacity $a_1(K)$ as a notion that describes the size of K since it is monotone and non-negative.

It is not obvious that the half-plane capacity is non-negative. We can show this by introducing another way of defining the half-plane capacity. Before that, we need a classical result from conformal theory which states that Brownian motion is conformally invariant.

Theorem 2.7. (Conformal Invariance of Brownian Motion) [4]. *Let D and D' be domains in $\mathbb{C}$, and let $\phi : D \to D'$ be a conformal isomorphism. Fix $z \in D$, and let $z' = \phi(z)$. Let $(B_t)_{t\geq 0}$ and $(B'_t)_{t\geq 0}$ be complex Brownian motions starting at z and z', respectively. Define*

$$T = \inf\{t \geq 0 : B_t \notin D\}; T' = \inf\{t \geq 0 : B'_t \notin D'\}.$$

Let $\tilde{T} = \int_0^T |\phi'(B_t)|^2 dt$ and define for $t < \tilde{T}$

$$\tau(t) = \inf\{s \geq 0 : \int_0^s |\phi'(B_r)|^2 dr = t\}; \tilde{B}_t = \phi(B_{\tau(t)}).$$

Then, $(\tilde{T}, (\tilde{B}_t)_{t<\tilde{T}})$ and $(T', (B'_t)_{t<T'})$ have the same distribution.

Proposition 2.8. *If $J \subset K$ are hulls, then $a_1(J) \leq a_1(K)$. Also, $a_1(J) = a_1(K)$ only if $\mathbb{H} \cap (K \setminus J) = \emptyset$.*

[4]The proof requires Ito's formula and is given in [9].

Proof. Take any hull K and suppose g_K satisfies the hydrodynamic normalization.

Let τ denote the exit time of a complex Brownian motion from $\mathbb{H} \setminus K$, i.e., $\tau = \inf\{t \geq 0 : B_t \notin \mathbb{H} \setminus K\}$. We now define $\mathrm{hcap}(K) = \lim_{iy \in \infty} y\mathbb{E}[\mathrm{Im}(B_\tau)]$ and we will show that $a_1(K) = \mathrm{hcap}(K)$.

Let $G_t = g_K(B_t)$ and $M_t = G_t - B_t$. Since Brownian motion is conformally invariant, it follows that $(M_t)_{t<\tau}$ is a continuous local martingale. Note that since $G_t \to B_t$ as $|B_t| \to \infty$, M_t is bounded. Also, $M_t \to G_\tau - B_\tau$ as $t \to \tau$. Since τ is a stopping time and by optional stopping theorem we have that, for $B_0 = z$,

$$g_K(z) - z = G_0 - B_0 = \mathbb{E}_z[G_\tau - B_\tau].$$

Let $z = iy$. Then,

$$\begin{aligned} y\mathbb{E}_{iy}[\mathrm{Im}(B_\tau)] &= -y\mathrm{Im}\mathbb{E}_z[G_\tau - B_\tau] \\ &= \mathrm{Re}(z(g_K(z) - z)). \end{aligned}$$

Hence, $\mathrm{hcap}(K) = a_1(K)$. Monotonicity is easier. Suppose $J \subset K$ are hulls. Note that $g_{g_J(K \setminus J)}$ is a conformal surjection $\mathbb{H} \setminus g_{g_J(K\setminus J)} \to \mathbb{H}$, so $g_{g_J(K\setminus J)} \circ g_J$ is a conformal surjection $\mathbb{H} \setminus K \to \mathbb{H}$. Hence, by uniqueness of hydrodynamic nomalization we have that $g_K = g_{g_J(K\setminus J)} \circ g_J$. Thus,

$$a_1(K) = a_1(J) + a_1(g_{g_J(K\setminus J)}).$$

Therefore, monotonicity follows from non-negativity. $\square$

Before moving on to simple curve and hulls, we shall emphasize that the half-plane capacity is in fact a continuous function of the hull. The following lemma helps to prove the statement and is given in [5].

Lemma 2.9. *For a hull K and $\epsilon > 0$, let K^ϵ be the smallest hull containing the set $\mathbb{H} \cap \bigcup_{z\in K} \overline{B(z,\epsilon)}$. There exists $C, R, \alpha > 0$ such that if $K^\epsilon \subset B(z_0, R)$ for some $z_0 \in \mathbb{R}$, then*

$$|a_1(K) - a_1(K^\epsilon)| \leq C\epsilon^\alpha.$$

We can associate a curve with a family of hulls in the following way: Let I be an interval of the form $[0, \infty), [0, T]$, or $[0, T)$ where $T \in (0, \infty)$. For any curve $\gamma : I \to \overline{\mathbb{H}}$ that starts from the real line, define a family of hulls $(K_t)_{t\in I}$ associated to $\gamma(t), t \in I$ such that $K_t = \gamma([0, t])$ for any $t \in I$ if γ is simple. If γ is not simple, let $K_t = \overline{\mathbb{H} \setminus H_t}$ where H_t denotes the unbounded connected component of $\mathbb{H} \setminus \gamma([0, t])$.

With the constraints satisfied in the following proposition, we can reparametrize $(K_t)_{t\in I}$ so that $a_1(K_t) = 2t$ for any $t \in I$. We say that this family is parametrized with the half-plane capacity. A curve $\gamma : [0, T) \to \overline{\mathbb{H}}$ is said to be parametrized with the half-plane capacity if the associated hulls are parametrized with the half-plane capacity. We summarize the existence of reparametrization as a proposition.

Proposition 2.10. *Suppose that $(K_t)_{t\in I}$ is growing in the sense that $K_s \subset K_t$ for $s \leq t$ and that the growth is continuous in the sense that for any $\epsilon > 0$ and $S \in (0, \infty)$ such that $[0, S] \subset I$, $\exists \delta > 0$ such that $K_{t+\delta} \subset K_t^\epsilon$ for any $0 \leq t \leq S - \delta$. Then, we can reparametrize $(K_t)_{t\in I}$ with the half-plane capacity.*

Proof. We first note that since the half-plane capacity is continuous with respect to hulls and is monotone, the map $\phi : t \to a_1(K_t)$ is continuous and non-decreasing. By assuming $K_0 \in \mathbb{R}$ and $\mathbb{H} \cap (K_t \setminus K_s) \neq \emptyset$ for any $0 \leq s < t \leq T$, we have that $\phi(0) = 0$. Also, $\phi(t) > \phi(s)$ for any $0 \leq s < t \leq T$. Note that ϕ is an isomorphim and thus we can set $K'_t = K_{\phi^{-1}(2t)}$ so that $a_1(K') = \phi(\phi^{-1}(2t)) = 2t$. □

For a given family of hulls $(K_t)_{t\in I}$, we set $g_t = g_{K_t}$, where if the family is parametrized by the half-plane capacity then for each $t \in [0, T)$ we have that

$$g_t(z) = z + \frac{2t}{z} + \cdots$$

We will use g_t to denote the conformal map with this form from now.

For a simple curve $\gamma : [0, T] \to \mathbb{H}$ that starts from the real line, what we stated at the beginning of this section about the continuity of the driving function $W(t)$ can be summarized as follows.

Theorem 2.11. [5] *Suppose that γ is parametrized by the half-plane capacity. Then*

$$W(t) = \lim_{z\to\gamma(t)} g_t(z)$$

exists for any $t \in [0, T]$ and W is continuous. Here the limit is along any sequence $z_n \in \mathbb{H}\setminus\gamma(0, t]$. Moreover, the hydrodynamically normalized conformal maps $(g_t)_{t\in[0,T]}$ satisfy the Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W(t)} \tag{2.12}$$

with the initial value $g_0(z) = z$.

Given a continuous, real-valued function W_t, it turns out that we can find a family of conformal maps $(g_t)_{t\in[0,T]}$ that satisfies the Loewner equation (2.12) with $g_0(z) = z$ and we call $(g_t)_{t\in[0,T]}$ the Loewner chain. Moreover, there is a growing family of hulls parametrized with the half-plane capacity.

Definition 2.13. A **Loewner chain** is the solution g_t of the Loewner equation with a continuous driving term.

Note that if we fix $z \in \overline{\mathbb{H}}$, then the Loewner differential equation becomes an ordinary differential equation (ODE) of time t

$$\dot{z}_t = \frac{2}{z_t - W_t}$$

with initial condition $z_0 = z$.

Let $\tau(z) = \inf\{t \geq 0 : \liminf_{s\to t} |z_s - W_s| = 0\}$ which is the maximum survival time of the solution. Note that the map $\zeta \mapsto \frac{2}{\zeta - W_t}$ is continuous in t and Lipschitiz continuous in ζ in $\{(t, \zeta) \in [0, T] \times \overline{\mathbb{H}} : |\zeta - W_t| \geq \epsilon\}$ for $\epsilon > 0$. Hence, the solution to the ODE is unique and the solution at fixed time is a continuous function of the initial condition.

Let $g_t(z) = z_t$ for $t \in [0, T] \cap [0, \tau(z))$ and $z \in \overline{\mathbb{H}} \setminus \{W_0\}$. Let $H_t = \{z \in \mathbb{H} : \tau(z) > t\}$ and $K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$. By some effort [6] we can show that $g_t|_{H_t}$ is a conformal surjection and K_t is a hull. Moreover, there is a condition called the

[5] The full proof is rather technical and is in [5].

[6] See [5]

local growth that can give a sufficient and necessary condition to the fact that g_t has a continuous driving term.

Theorem 2.14. *Let $(K_t)_{t\in[0,T]}$ be a growing family of hulls and g_t be the associated conformal maps. Then the following statements are equivalent:*

- *(**Local growth**) For all $t \in [0, T], a_1(K_t) = 2t$ and for any $\epsilon > 0$ there is $\delta > 0$ such that for each $t \in [0, T - \delta]$, there exists a bounded connected set $C \subset \mathbb{H} \setminus K_t$ with diam$(g_t(C)) < \epsilon$ such that C separates $K_{t+\delta} \setminus K_t$ from infinity in $\mathbb{H} \setminus K_t$.*
- *There is a continuous $W(t), t \in [0, T]$ such that g_t is the solutioin of the Loewner differential equation.*

Intuitively, the local growth condition for a growing family of hulls is saying that at any small time step, the increment of the hull, i.e., $K_{t+\delta} \setminus K_t$, is contained in a small neighborhood C. In other words, the newly added portion of the hull over any small time step is confined to a small region rather than blowing up and spreading out of the place.

Remark 2.15. Note that by the previous Theorem, any one of the term $W(t), K_t, g_t$ can be obtained from each other as the concept of a Loewner chain relates all of them. In particular, if we wish to take a curve as the subject of study, we can simply replace K_t by the associated hulls generated by the curve. However, in order for $W(t)$ to be continuous, as a generalization of Theorem (2.11), although we do not require the curve to be simple, the associated hulls generated by the curve have to satisfy the local grwoth condition specified in Theorem (2.14).

We end this section by presenting an example of a curve that motivates the definition of SLE. Let $K_t = [0, 2\sqrt{t}i]$ and let $g_t : \mathbb{H} \setminus K_t$ be defined by $g_t(z) = \sqrt{z^2 + 4t}$.

Note that g_t is conformal and that $|g_t(z) - z| = |\sqrt{z^2 + 4t} - z| \to 0$ as $z \to \infty$. Hence, g_t satisfies the hydrodynamic normalization. Moreover, for each $z \in \mathbb{H}$ we have that $g_t(z)$ is a unique solution of the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z)}; g_0(z) = z.$$

Note that Theorem (2.11) states the existence of a continuous, real-valued function W that satisfies the chordal Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}; g_0(z) = z,$$

so the curve $\gamma(t) = 2\sqrt{t}i$ is the trivial case where $W = 0$, which actually corresponds to SLE_0.

3. Chordal Schramm-Loewner Evolution

The Schramm-Loewner Evolution (SLE) is a one-parameter family of random curves that plays an essential role in 2D random conformal geometry. One of the most exciting applications of SLE is that it becomes a good candidate for describing the scaling limits of many statistical models:

$$\begin{aligned}
SLE_2 &\iff \text{Loop-erased random walk}\\
SLE_3 &\iff \text{Critical Ising model interface}\\
SLE_6 &\iff \text{Critical independent percolation interface}\\
SLE_8 &\iff \text{Contour line of uniform spanning tree}
\end{aligned}$$

SLE are the only random curves that satisfy conformal invariance and the domain Markov property, which is why SLE curves play a perfect role in describing interfaces arising from conformally invariant systems. Moreover, SLE$_\kappa$ is almost surely generated, or traced out, by a continuous non-self-crossing curve γ^κ called its trace. We now give a brief introduction to chordal SLE$_\kappa$.

We first define SLE using the Loewner chain, which follows from the previous section. Then, we will define it as a random curve.

Previously, the Loewner chain obtained from family of hulls or driving process is deterministic and no random factor is involved. In this section, we show that given a continuous stochastic process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$, we can define a stochastic Loewner chain $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ corresponding to it, which makes the following definition of SLE well-defined. We call $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ a driving process.

Definition 3.1. Let $\kappa \geq 0$. A **chordal SLE$_\kappa$** is a stochastic Loewner chain with a driving process $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ equal to a Brownian motion with variance parameter κ, i.e., $W_t = \sqrt{\kappa}B_t$ where $(B_t)_{t\in\mathbb{R}_{\geq 0}}$ is a standard one-dimensional Brownian motion.

To see the correspondence between a stochastic Loewner chain and a driving process, we now show that under specific topological spaces, the mapping from the continuous functions $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ to Loewner chains $(g_t)_{t\in\mathbb{R}_{\geq 0}}$ is continuous.

Lemma 3.2. *For each $\delta > 0$ and $T > 0$ there exists a constant C such that the following holds. If $g_1(t,z)$ and $g_2(t,z)$ are solutions of the Loewner equation (2.11) with the continuous driving terms $(W_1(t))_{t\in[0,T]}$ and $(W_2(t))_{t\in[0,T]}$ respectively, then they satisfy*

$$|g_1(T,z_1) - g_2(T,z_2)| \leq C(||W_1 - W_2||_{\infty,[0,T]} + |z_1 - z_2])$$

for any z_1, z_2 such that $\mathrm{Im} g_k(T,z_k) > \delta > 0$.

Proof. Take $\delta > 0, T > 0$, and $z_1, z_2 \in \mathbb{H}$ such that $\mathrm{Im} z_1, z_2 > \delta$. Let $g_1(t,z)$ and $g_2(t,z)$ be solutions of the Loewner equation (2.11) with the continuous driving terms $(W_1(t))_{t\in[0,T]}$ and $(W_2(t))_{t\in[0,T]}$ respectively. Let $\psi(t) = g_1(t,z_1) - g_2(t,z_2)$. We have that

$$\partial_t\psi(t) = \zeta(t)(\psi(t) - D(t))$$

where $\zeta(t) = \frac{2}{(g_1(t,z_1)-W_1(t))(g_2(t,z_2)-W_2(t))}$ and $D(t) = W_1(t) - W_2(t)$.

Using an integrating factor, note that

$$\partial_t(e^{-\int_0^t \zeta(s)ds}\psi(t)) = -\zeta(t)e^{-\int_0^t \zeta(s)ds}D(t)$$

Then,

$$\psi(t) = e^{\int_0^t \zeta(s)ds}\psi(0) - \int_0^t \zeta(u)e^{\int_u^t \zeta(s)ds}D(u)du.$$

Since $|e^{\int_0^t \zeta(s)ds}| \leq e^{\int_0^t |\zeta(s)|ds}$, it follows that

$$
\begin{aligned}
|\int_0^t \zeta(u) e^{\int_u^t \zeta(s)ds} D(u)du| &\leq ||D||_{\infty,[0,T]} \int_0^t |\zeta(u)| e^{\int_0^u |\zeta(s)|ds} du \\
&= ||D||_{\infty,[0,T]} (e^{\int_0^t |\zeta(s)|ds} - 1).
\end{aligned}
$$

Notice that to prove the lemma it suffices to find an upper bound for $\int_0^t |\zeta(s)|ds$ for any z_1, z_2 such that $\text{Im} g_k(T, z_k) > \delta > 0$.

For $k \in \{1, 2\}$, let $I_k = \int_0^t \frac{2}{|g_k(t,z_k) - W_k(t)|^2} ds$. By the Cauchy-Schwarz inequality it follows that

$$
\int_0^t |\zeta(s)|ds \leq \sqrt{I_1 I_2}.
$$

Notice that for any $k \in \{1, 2\}$, we have that

$$
\begin{aligned}
I_k &= \int_0^t \frac{2}{|g_k(t, z_k) - W_k(t)|^2} ds \\
&\leq \log \frac{\text{Im} z_k}{\text{Im} g_k(t, z_k)} \\
&\leq \log \frac{\text{Im} z_k}{\max\{\delta, \sqrt{((\text{Im} z_k)^2 - 4t)^+}\}}
\end{aligned}
$$

where $a^+ = \max\{a, 0\}$ which completes the proof. □

We let the topology of the driving functions be given by the locally uniform convergence, and we say that the topology of the Loewner chains is given by the following form of Caratheodory convergence.

Definition 3.3. A sequence of Loewner chains $(g_n(t, \cdot), K_n(t))_{t \in \mathbb{R}_{\geq 0}})$ converges to $((g(t, \cdot), K(t))_{t \in \mathbb{R}_{\geq 0}})$ if for any $T > 0$ and any compact $J \subset \mathbb{H} \setminus K_T$, the sequence of functions $(t, z) \mapsto g_n(t, z)$ converges uniformly to $(t, z) \mapsto g(t, z)$ on $[0, T] \times J$.

A direct application of the above lemma is the following proposition, which justifies the definition of Chordal SLE_κ because the map $W_t \mapsto g_t$ is measurable.

Proposition 3.4. *Let K_0 be a hull and $G \subset \mathbb{H} \setminus K_0$ be a compact set. Then there exists a constant $C > 0$ such that if g_1, g_2 are two Loewner chains such that $K_k(T) \subset K_0$ for $k = 1, 2$, then*

$$
||g_1 - g_2||_{\infty,[0,T] \times G} \leq C ||W_1 - W_2||_{\infty,[0,T]}.
$$

We can also define SLE_κ using the uniqueness of those random curves satisfying conformal invariance and the domain Markov property.

Proof. Suppose that we have a arbitrary collection of probability measures $(\mu^{(U,a,b)})$ where U is a simply connected domain and $a \neq b$ are two bundary points of U. Assume that $(\mu^{(U,a,b)})$ is the law of a random curve $\gamma : [0, \infty) \to \mathbb{C}$ satisfying $\gamma([0, \infty)) \subset \overline{U}$ and $\gamma(0) = a, \gamma(\infty) = b$.

Definition 3.5. Let ϕ_* denote the pushforward defined by $\phi_* P = P \circ \phi^{-1}$. The family $(\mu^{(U,a,b)})$ satisfies **conformal invariance** if for all (U, a, b),

$$
\phi_* \mu^{(U,a,b)} = \mu^{(\phi(U), \phi(a), \phi(b))}.
$$

Definition 3.6. Let $(\mathcal{F}_t)_{t\in\mathbb{R}_{\geq 0}}$ be the filtration generated by $(\gamma(t))_{t\in\mathbb{R}_{\geq 0}}$. The family $(\mu^{(U,a,b)})$ satisfies **domain Markov property** if for all (U,a,b), for every $t\in\mathbb{R}_{\geq 0}$ and for any measurable set B in the space of curves,

$$\mu^{(U,a,b)}(\gamma|_{[t,\infty)}\in B|\mathcal{F}_t)=\mu^{(U\setminus\gamma([0,t]),\gamma(t),b)}.$$

Now, if the family $(\mu^{(U,a,b)})$ satisfies conformal invariance, domain Markov property, and that we can describe the curve γ by the Loewner equation in the sense that there is a $\mu^{(\mathbb{H},0,\infty)}$ almost sure event on which γ satisfies Theorem (2.14), then we can show that they are those where $\mu^{(\mathbb{H},0,\infty)}$ is the law of a random curve whose Loewner driving process is equal to a constant multiple of a one-dimensional Brownian motion. This is in fact the Schramm's principle.

To see this, notice that we only need to investigate one of the measures from the family since conformal invariance fixes the rest of them. We choose to work with $\mu^{(\mathbb{H},0,\infty)}$. By Theorem (2.14), for each realization of γ there is a continuous driving term W_t such that g_t is the solution of the corresponding Loewner equation. Note that the stochastic driving term W_t is the driving process of the random curve γ if we reparameterize with the half-plane capacity.

We now show that $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ has independent and stationary increments, since then by Probability knowledge we know that there eixsts a standard one-dimensional Brownian motion B_t and $\alpha\in\mathbb{R}_{\geq 0},\beta\in\mathbb{R}$ such that $W_t=\alpha B_t+\beta t$. In other words, $W_t=\sqrt{\kappa}B_t+\beta t$ for some $\kappa\geq 0$.

Let $\gamma'(s)=g_t(\gamma(t+s))-W_t$. Note that by Conformal invariance and domain Markov property, γ' is distributed as γ and independent of the realization of $\gamma|_{[0,t]}$. The conformal map associated with the hull $\gamma'([0,s])$ is given by

$$g'_s(z)=g_{t+s}\circ g_t^{-1}(z+W_t)-W_t.$$

Notice that $W'_s=W_{t+s}-W_t$ is the driving process of γ' since

$$\begin{aligned}\partial_s g'_s(z) &= (\partial_s g_{t+s})g_t^{-1}(z+W_t)\\ &= \frac{2}{g_{t+s}(g_t^{-1}(z+W_t))-W_{t+s}}\\ &= \frac{2}{g'_s(z)-(W_{t+s}-W_t)}.\end{aligned}$$

Then, it follows that (W'_s) is independent of $\mathcal{F}_t$ and is distributed as (W_t), which shows that $W_t=\sqrt{\kappa}B_t+\beta t$ for some $\kappa\geq 0$. Note that for $\gamma^{(\lambda)}(t)=\lambda\gamma(\frac{t}{\gamma^2})$, it has the same distribution as γ by conformal invariance of the measure. Its driving process is given by $W_t^{(\lambda)}=\lambda W_{\frac{t}{\lambda^2}}$. Notice that $(W_t^{(\lambda)})$ and (W_t) should have the same distribution, and this is only true when $\beta=0$.

Therefore $W_t=\sqrt{\kappa}B_t$.

□

There are several elementary properties of SLE_κ which we summarized as follows.

Theorem 3.7. *Let $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be SLE_κ, $\kappa>0$, and $(W_t)_{t\in\mathbb{R}_{\geq 0}}$ the corresponding driving process which is a Brownian motion with respect to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_{\geq 0}}$. Then, the SLE_κ satisfies the following properties.*

1. Scale invariance: For any $\lambda>0$, $(\lambda K_{t/\lambda^2})_{t\in\mathbb{R}_{\geq 0}}$ and $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ are equal in distribution.

2. Conformal Markov property: For any $s \in \mathbb{R}_{\geq 0}$, the family of hulls

$$(K'_{\tau,t})_{t\in\mathbb{R}_{\geq 0}} = (\overline{g_s(K_{s+t} \setminus K_s) - W_s})_{t\in\mathbb{R}_{\geq 0}}$$

is independent of $\mathcal{F}_s$, and $(K'_{s,t})_{t\in\mathbb{R}_{\geq 0}}$ and $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ are equal in distribution.

3. Strong conformal Markov property: For any almost surely finite stopping time τ with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_{\geq 0}}$, the family of hulls

$$(K'_{s,t})_{t\in\mathbb{R}_{\geq 0}} = (\overline{g_\tau(K_{\tau+t} \setminus K_\tau) - W_\tau})_{t\in\mathbb{R}_{\geq 0}}$$

is independent of $\mathcal{F}_\tau$, and $(K'_{\tau,t})_{t\in\mathbb{R}_{\geq 0}}$ and $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ are equal in distribution.

So far, we have defined SLE_κ as a stochastic Loewner chain and we can also derive SLE_κ from Schramm's principle. It turns out that SLE_κ can also be defined as a random curve by Theorem (3.10).

Definition 3.8. A growing family of hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ is generated by a curve γ if $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$ for all $t \in \mathbb{R}_{\geq 0}$.

Definition 3.9. For any Loewner chain g_t, we define its trace by the function $\gamma(t) = \lim_{\epsilon\to 0+} g_t^{-1}(W(t) + i\epsilon)$ if it exists.

Theorem 3.10. *For each κ, the trace γ exists and is a random curve such that the hulls $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ of SLE_κ are generated by γ almost surely.*

We end this section by presenting some phase transitions of SLE_κ. In other words, for different κ we can describe the behaviors of the random curves as simple, non-crossing, or space-filling.

Theorem 3.11. *Let the random curve $\gamma : [0,\infty) \to \mathbb{H}$ be SLE_κ. Then,*

1. For all $0 < \kappa \leq 4$, γ is simple and $\gamma(0,\infty) \cap \mathbb{R} = \emptyset$.

2. For all $4 < \kappa < 8, \gamma$ is self-intersecting. In fact, it is not simple on any interval: for any $0 \leq t_1 < t_2$ there exists $t_1 < s_1 < s_2 < t_2$ such that $\gamma(s_1) = \gamma(s_2)$. However, γ is not space-filling: for any $z \in \mathbb{H}, z \notin \gamma[0,\infty)$ almost surely.

3. For all $\kappa \geq 8, \gamma$ is not simple on any interval but is space-filling.

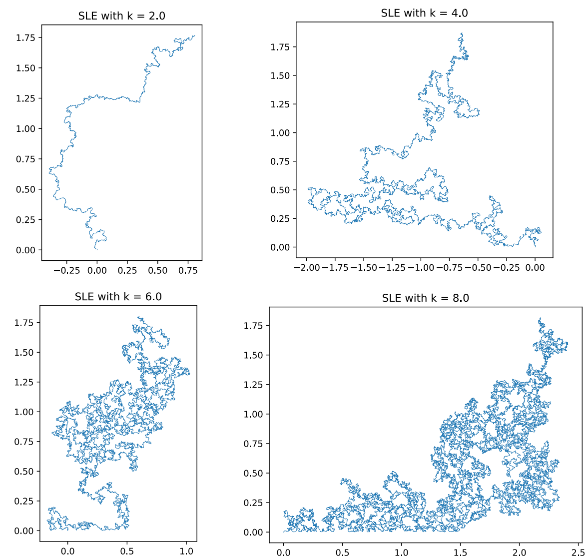

FIGURE 2. SLE with κ at different values.

4. LOEWNER ENERGY

Recall that in the first section we discussed the large deviation principle (LDP) of the scaled Brownian path. As we will see later in this section, the concept of Loewner energy arises when we investigate the LDP of Chordal SLE_κ.

Notice that so far, we've only defined SLE_κ on $(\mathbb{H}, 0, \infty)$. We now define SLE_κ for a general simply connected domain with two distinguished boundary points.

Definition 4.1. (Chordal SLE in a general simply connected domain). Let $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ be a chordal SLE_κ and let D be a simply connected domain and a, b be two boundary points of D with $a \neq b$. We define chordal SLE_κ in a domain D going from a to b to be the image of $(K_t)_{t\in\mathbb{R}_{\geq 0}}$ under any conformal onto map $\psi : \mathbb{H} \to D$ with $\psi(0) = a$ and $\psi(\infty) = b$.

Notice that extending the definition to a general simply connected domain is simple because any conformal map $\psi : \mathbb{H} \to D$ with the above properties only affects the time parametrization of the corresponding hulls: since SLE_κ is scale-invariant and the conformal onto maps can only be of the form $z \mapsto \psi(\lambda z)$, the definition is unique up to a linear time change.

Definition 4.2. The **Loewner energy** of a simple curve γ is defined as the Dirichlet energy of its driving function,

$$I_{D;a,b}(\gamma) := I_{\mathbb{H};0,\infty}(\psi(\gamma)) := I_T(W),$$

where ψ is any conformal map from D to $\mathbb{H}$ such that $\psi(a) = 0$, $\psi(b) = \infty$ and W is the driving function of $\psi(\gamma)$, $2T$ is the total capacity of $\psi(\gamma)$, and $I_T(W)$ is the Dirichlet energy.

Notice that the definition of Loewner energy of a simple curve also does not depend on the choice of ψ: since the driving function also has the scaling property, with different choices of ψ, W only changes to $t \mapsto \lambda W_{\frac{t}{\lambda^2}}$. This has the same Dirichlet energy as W.

Moreover, by definition, the Loewner energy is non-negative and $I_{D;a,b(\gamma)} = 0$ is achievable when $\gamma = \eta := \psi^{-1}(i\mathbb{R}_+)$: recall that at the end of section 2, we gave an example of SLE_0 in $(\mathbb{H}; 0, \infty)$, which is the Loewner chain driven by $W = 0$, i.e., $i\mathbb{R}_+$. Hence, the SLE_0 in $(D; a, b)$ is $\psi^{-1}(i\mathbb{R}_+)$. This is called the hyperbolic geodesic. Note that the driving function of $\psi(\eta)$ is $W = 0$ which makes $I_{D;a,b}(\eta) = 0$.

For any curve γ, the Loewner energy need not be finite. Conversely, any real-valued function W with finite energy does correspond to a simple curve γ.

Theorem 4.3. *If γ is simple and $I_{D;a,b}(\gamma) < \infty$, then γ has infinite total capacity, If a driving function W defined on $\mathbb{R}_+$ satisfies $I_\infty(W) < \infty$, then W generates a simple curve γ in $(\mathbb{H}; 0, \infty)$.*

One of the important motivations for the concept of Loewner energy comes from viewing the large deviations of chordal SLE_κ as $\kappa \to 0+$. Intuitively, this means that

$$\lim_{\kappa \to 0+} \mathbb{P}[SLE_\kappa \text{ in } (D; x, y) \text{ stays close to } \gamma] \approx \exp(-\frac{I_{D;x,y}(\gamma)}{\kappa}).$$

Recall that the large deviation principle depends on the space and topology that we are working in. Motivated by Theorem (4.3), we can consider the space $\mathcal{X}(D; a, b)$ of unparametrized simple curves with infinite total capacity in $(D; a, b)$. The topology we are working is induced by the Hausdorff metric.

Definition 4.4. The **Hausdorff distance** d_h of two compact subsets $F_1, F_2 \subset \overline{\mathbb{D}}$ is defined as

$$d_h(F_1, F_2) := \inf\left\{\epsilon \geq 0 \middle| F_1 \subset \bigcup_{x \in F_2} \overline{B}_\epsilon(x), F_2 \subset \bigcup_{x \in F_1} \overline{B}_\epsilon(x)\right\},$$

where $B\epsilon(x)$ is the Euclidean ball of radius ϵ centered at $x \in \overline{\mathbb{D}}$.

The Hausdorff metric on the set of closed subsets of a Jordan domain D is defined via the pullback by a conformal map $D \to \mathbb{D}$. For different conformal maps the Hausdorff distances are different, but the topology induced by them are since conformal automorphisms of $\mathbb{D}$ are Mobius transformations which are uniformly continuous.

In large deviation language, we make precise about being "staying close to" into the following theorem, which is analogous to Schilder's theorem.

Theorem 4.5. *The family of distributions $\{\mathbb{P}_\kappa\}_{\kappa > 0}$ on $\mathcal{X}(D; a, b)$ of the chordal SLE_κ curves satisfies the large deviation principle with good rate function $I_{D;a,b}$. That is, for any open set O and closed set F of $\mathcal{X}(D; a, b)$, we have*

$$\begin{aligned} \liminf_{\kappa \to 0+} \kappa \log \mathbb{P}_\kappa[\gamma_\kappa \in O] &\geq -\inf_{\gamma \in O} I_{D;a,b}(\gamma), \\ \limsup_{\kappa \to 0+} \kappa \log \mathbb{P}_\kappa[\gamma_\kappa \in F] &\leq -\inf_{\gamma \in F} I_{D;a,b}(\gamma) \end{aligned}$$

and the sub-level set $\{\gamma \in \mathcal{X}(D; a, b) | I_{D;a,b}(\gamma) \leq c\}$ is compact for any $c \geq 0$.

It seems like this result can be proved easily by Schilder's theorem and Theorem (1.5), but since we are working the Hausdorff metric and the map from continuous driving function to the hulls it generates is not continuous under this topology, this result requires more effort and is in [6].

It is natural to think that the Loewner energy of a curve from one boundary point a to another boundary point b is equal to the energy of the same curve going from b to a because it is proven that SLE_κ curves with $\kappa \leq 4$, have reversable properties.

Theorem 4.6. [7] *For $\kappa \in [0,4]$, the distribution of the trace γ of SLE_κ in $(\mathbb{H}; 0, \infty)$ coincides with that of its image under $\phi : z \mapsto -\frac{1}{z}$ upon forgetting the time parametrization.*

In fact, it is true that Loewner energy is reversible which reveals a deterministic nature of random curves.

Theorem 4.7. [8] *For any simple curve $\gamma \in \mathcal{X}(D; a, b)$, we have that $I_{D;a,b}(\gamma) = I_{D;b,a}(\gamma)$.*

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