# HANDLE DECOMPOSITION AND WALL'S THEOREM ON $h$-COBORDISMS

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Abstract. We aim to present C.T.C. Wall's theorem on $h$-cobordisms, which states that any two simply-connected smooth 4-manifolds with isomorphic intersection forms must be $h$-cobordant. After briefly introducing the intersection form, which is a fundamental invariant for 4-manifolds, we develop the necessary tools to prove Wall's theorem, including basic Morse theory and handle decompositions. In the last section, we use these tools to sketch a proof of Wall's theorem.

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## 1. Introduction

In this paper we focus on the study of simply-connected 4-manifolds. All manifolds are assumed to be oriented. A key invariant for a 4-manifold is its intersection form. Intuitively, the intersection form of a 4-manifold $M$ keeps track of how the 2-homology classes of $M$, which may be represented by embedded surfaces, intersect with each other. Moreover, if $M$ is simply-connected, by Poincaré duality, we have $H_1(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0$, and thus the only nontrivial homological information is contained in $H_2(M;\mathbb{Z})$. It is then reasonable to ask to what extent the intersection form determines the topology of $M$. The notion of $h$-cobordism arises precisely in this context, serving as a bridge from the homological to the topological or differentiable. The theorem of C.T.C. Wall, which we aim to prove in this paper, states that two simply-connected smooth 4-manifolds with isomorphic intersection forms must be $h$-cobordant. Combining this with M. Freedom's remarkable

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topological $h$-cobordism theorem in dimension 4, we arrive at an affirmative answer to the aforementioned question: two smooth simply-connected 4-manifolds with isomorphic intersection forms must be homeomorphic. With these amazing results in mind, we now begin developing the necessary notions and tools towards understanding them. We start by rigorously defining the intersection form.

## 2. Intersection Forms

Let $M$ be a smooth simply-connected 4-manifold. As described in the introduction, we want to keep track of how 2-homology classes of $M$ intersect. It would be convenient to be able to represent the 2-homology classes by embedded surfaces, and then the intersection number would take on a more concrete geometric meaning. It turns out that we can always do this:

**Lemma 2.1.** *Every 2-homology class of a 4-manifold $M$ can be represented by an embedded surface.*

*Proof.* Since $\mathbb{CP}^\infty$ is an Eilenberg-Maclane space $K(\mathbb{Z}, 2)$, we have an isomorphism $[M, \mathbb{CP}^\infty] \simeq H^2(M;\mathbb{Z})$ via $f \mapsto f^*u$ where $u = [\mathbb{CP}^1]^*$ is the generator of $H^2(\mathbb{CP}^\infty;\mathbb{Z})$. Moreover, by cellular approximation, $f : M \to \mathbb{CP}^\infty$ can be homotoped to a map into $\mathbb{CP}^2$. Further perturbing $f$ to be smooth and transverse to $\mathbb{CP}^1 \subset \mathbb{CP}^2$ gives that $f^{-1}[\mathbb{CP}^1]$ is a surface in $M$ dual to $f^*u$. □

**Definition 2.2.** For a closed oriented 4-manifold $M$, define its **intersection form** to be the map

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}, \quad Q_M(\alpha, \beta) = S_\alpha \cdot S_\beta,$$

where $S_\alpha$ and $S_\beta$ are embedded surfaces representing $\alpha$ and $\beta$, and $S_\alpha \cdot S_\beta$ denotes the signed intersection number. Equivalently, by Poincaré duality, we may regard the intersection form as

$$Q_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}, \quad Q_M(\alpha^*, \beta^*) = (\alpha^* \cup \beta^*)[M],$$

where $[M]$ denotes the fundamental class.

Basic properties of the intersection form are:

**Lemma 2.3.** *$Q_M$ is $\mathbb{Z}$-bilinear, symmetric, and unimodular, i.e. the matrix representing it is invertible over $\mathbb{Z}$.*

*Proof.* The only nontrivial assertion is unimodularity. By linearity, $Q_M$ vanishes on torsion elements. Thus we may assume WLOG that $H_2(M;\mathbb{Z})$ is torsion-free. By Poincaré duality, the map

$$\hat{Q}_M : H_2(M;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(M;\mathbb{Z}), \mathbb{Z}), \quad \alpha \mapsto (x \mapsto \alpha \cdot x)$$

is an isomorphism. But this is equivalent to unimodularity by linear algebra. □

Intersection forms also behave nicely with respect to connected sum. This follows from the simple observation that performing connected sum on 4-manifolds does not affect 2-homology.

**Lemma 2.4.** *Let $M$ and $N$ be 4-manifolds. Then $Q_{M\#N} = Q_M \oplus Q_N$.*

From an algebraic perspective, by Lemma 2.3, the intersection form is a diagonalizable matrix in $GL_n(\mathbb{Z})$ where $n$ is the second Betti number $b_2(M)$. We may thus define a few algebraic invariants of the intersection form:

**Definition 2.5.** The **signature** of the intersection form $Q_M$, denoted by $\operatorname{sign} Q_M$, is defined to be its signature as a symmetric bilinear form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues. We say $Q_M$ is **even** if $Q_M(\alpha, \alpha)$ is an even number for all $\alpha \in H_2(M; \mathbb{Z})$, and we say $Q_M$ is **odd** if there exists $\alpha \in H_2(M; \mathbb{Z})$ for which $Q_M(\alpha, \alpha)$ is an odd number.

A remarkable feature of the signature is that it vanishes if and only if $M$ is a boundary. A proof can be found in [5].

**Theorem 2.6** (V. Rokhlin)**.** *A smooth 4-manifold $M$ has $\operatorname{sign} Q_M = 0$ if and only if $M = \partial W$ for some smooth 5-manifold $W$.*

## 3. Cobordisms and Statement of Wall's Theorem

Having defined the intersection form, which is a key homological invariant, we now turn to the notion of $h$-cobordisms, which serve as a bridge between the homological and the topological.

**Definition 3.1.** Two $m$-manifolds $M$ and $N$ are said to be **cobordant** through an $(m+1)$-manifold $W$ if $\partial W = \overline{M} \cup N$, where the overline denotes the same manifold endowed with the opposite orientation. We say $M$ and $N$ are ***h*-cobordant** through $W$ if in addition the inclusion of $M$ (equivalently, of $N$) into $W$ is a homotopy equivalence.

Intuitively, two $m$-manifolds $M$ and $N$ being $h$-cobordant means that they can be connected by an $(m+1)$-manifold in such a way that nothing happens homologically in between. Since as indicated above the intersection form captures the main homological information, it is reasonable to expect two 4-manifolds with isomorphic intersection forms to be $h$-cobordant. This is precisely the content of C.T.C. Wall's theorem:

**Theorem 3.2** (Wall's Theorem on $h$-Cobordisms)**.** *If two smooth simply-connected 4-manifolds $M$ and $N$ have isomorphic intersection forms, then they are $h$-cobordant.*

The rest of this paper will be devoted to sketching a proof of this theorem. But before that, we first derive a nice consequence of this theorem by combining it with a difficult result of M. Freedman, as mentioned in the introduction:

**Theorem 3.3** (Freedman's 4-Dimensional $h$-Cobordism Theorem)**.** *Suppose two simply-connected 4-manifolds $M$ and $N$ are $h$-cobordant through a simply-connected 5-manifold $W$. Then $W$ is homeomorphic to $M \times [0, 1]$. In particular, $M$ and $N$ are homeomorphic.*

Combining the preceding two theorems yields the following corollary, which represents the full passage from the homological to the topological: the intersection form completely determines the homeomorphism type.

**Corollary 3.4.** *If two smooth simply-connected 4-manifolds $M$ and $N$ have isomorphic intersection forms, then they are homeomorphic.*

In the following section, we develop the tools necessary to prove Wall's theorem.

## 4. Morse Theory and Handle Decompositions

To get started on proving Wall's theorem, a first observation is that by Rokhlin's theorem (Theorem 2.6), if the smooth simply-connected 4-manifolds $M$ and $N$ have isomorphic intersection forms, then in particular they have the same signature, and hence are cobordant through some 5-manifold $W$. The main task then is to "upgrade" this cobordism to an $h$-cobordism. To make this cobordism homologically trivial, it is convenient to have a "nice" decomposition of $W$ into simpler parts. This is precisely accomplished by Morse theory. The guiding philosophy is that the topology of $W$ can be completely determined by studying the singularities of certain smooth real-valued functions on $W$. In fact, these singularities give a recipe for reassembling $W$ from simple pieces known as handles.

**Definition 4.1.** Given a smooth real-valued function $f$ on a smooth manifold $W$, a critical point $p$ of $f$ is called **non-degenerate** if the differential $df$, as a section of the cotangent bundle $T^*W$, is transversal at $p$ to the zero section, or equivalently, the Hessian matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(p))_{i,j}$ of $f$ at $p$ is non-singular.
A smooth function $f : W \to \mathbb{R}$ is called a **Morse function** if all its critical points are non-degenerate and have distinct values.

**Remark 4.2.** In the definition of non-denegeracy, the explicit Hessian matrix depends on which chart we choose. However, if $p$ is a critical point, then whether the Hessian matrix at $p$ is singular is independent of the chart chosen. A quick way to see this is to think of the Hessian as a symmetric bilinear form on $T_pW$. More precisely, we define a map $H_p : T_pW \times T_pW \to \mathbb{R}$ as follows: given $u, v \in T_pW$, define $H_p(u, v) := U(V(f))(p)$, where $U$ (resp. $V$) is any extension of $u$ (resp. $v$) to a vector field in a neighborhood of $p$. $H_p$ is symmetric since

$$U(V(f))(p) - V(U(f))(p) = [U, V](f)(p) = 0$$

as $p$ is a critical point. From this we also see that $H_p$ is well-defined (independent of the vector field extensions) because on the one hand $H_p(u, v) = u(V(f))$ is independent of how we extend $u$ and on the other hand $H_p(u, v) = v(U(f))$ is independent of how we extend $v$. Finally, an explicit calculation shows that after choosing a chart, $H_p$ is precisely the Hessian of $f$ at $p$.

With this perspective, if $p$ is a critical point of a Morse function $f$, the Hessian $H_p$ is a non-degenerate symmetric bilinear form on $T_pW$ and hence is diagonalizable. More precisely, we have:

**Lemma 4.3** (Morse Lemma)**.** *Let $p$ be a non-degenerate critical point of $f$. Then there is a chart $x = (x_1, ..., x_n)$ around $p$ such that $x(p) = 0$ and in the chart domain we have*

$$f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

*for some integer $k \in [0, n]$. The integer $k$ is called the **Morse index** of $f$ at $p$.*

A first question regarding Morse functions is whether they exist. The answer is of course affirmative. For a proof, see [2].

**Lemma 4.4.** *Given a closed smooth manifold $M$, the collection of Morse functions on $M$ forms a dense subset of $C^\infty(M; \mathbb{R})$.*

We are now ready to see the remarkable interplay between the singularities of Morse functions on $W$ and the topology of $M$. Given a Morse function $f : W \to \mathbb{R}$, we write $W_{\leq a} := \{q \in W : f(q) \leq a\}$, $W_a := \{q \in W : f(q) = a\}$, and similarly for $W_{[a,b]}$. A first observation is that nothing happens topologically without encountering a singular point:

**Proposition 4.5.** *Suppose $[a, b]$ does not contain a critical value of $f$ and $f^{-1}([a, b])$ is compact. Then there exists a smooth retraction $r : W_{[a,b]} \to W_a$ such that $(f, r) : W_{[a,b]} \to [a, b] \times W_a$ is a diffeomorphism. In particular, $H_*(W_{a,b}, W_a) = 0$.*

*Proof.* Since $[a, b]$ does not contain a critical value of $f$, the restriction of $f$ to $W_{[a,b]}$ is a submersion. Since we also have $f^{-1}([a, b])$ compact, we may use a partition of unity to build a vector field $V$ on $W_{[a,b]}$ that lifts the vector field $\frac{d}{dt}$ on $\mathbb{R}$, i.e. $df(V) = \frac{d}{dt}$. Let $H_t$ be the local flow of this vector field $V$ on $W_{[a,b]}$. Note that $H_t$ is well-defined for all $x \in W_{[a,b]}$ with $t \in [a - f(x), b - f(x)]$. By construction, we have $f(H_t(x)) = f(x) + t$ since $V$ lifts $\frac{d}{dt}$. Thus $r(x) := H_{a-f(x)}(x)$ is the desired retraction. □

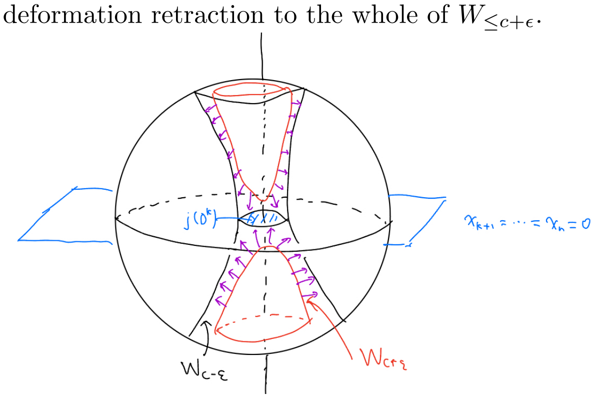
On the other hand, passing through a critical point of Morse index $k$ is homotopically the same as attaching a thickened $k$-cell:

**Proposition 4.6.** *Let $p \in W$ be a non-degenerate critical point of $f$ with index $k$. Write $c = f(p)$. Then there exists $\epsilon > 0$ and an embedding $j : D^k \hookrightarrow W_{[c-\epsilon,c+\epsilon]}$, where $D^k$ is the closed unit ball in $\mathbb{R}^k$, such that $c$ is the only critical value in $[c - \epsilon, c + \epsilon]$, $j^{-1}W_{c-\epsilon} = \partial D^k$, and $W_{\leq c-\epsilon} \cup j(D^k)$ is a deformation retract of $W_{\leq c+\epsilon}$.*

*Proof.* Let $x = (x_1, ..., x_n) : U \to \mathbb{R}^n$ be a chart as in Lemma 4.3, where $U$ is a neighborhood of $p$. From this lemma it is clear that the critical values of $f$ are isolated. Thus we may choose $\epsilon > 0$ such that $c$ is the only critical value in $[c - \epsilon, c + \epsilon]$, and $x(U) \subset \mathbb{R}^n$ contains the ball $B_{2\sqrt{\epsilon}}$ of radius $2\sqrt{\epsilon}$ centered at 0. Define $j : D^k \hookrightarrow U \subset W_{[c-\epsilon,c+\epsilon]}$ by

$$j(y) = x^{-1}(\sqrt{\epsilon}y_1, \cdots, \sqrt{\epsilon}y_k, 0, \cdots, 0).$$

As illustrated in the following figure, we obtain a deformation retraction of $W_{\leq c+\epsilon} \cap x^{-1}(B_{2\sqrt{\epsilon}})$ onto $(W_{\leq c-\epsilon} \cup j(D^k)) \cap x^{-1}(B_{2\sqrt{\epsilon}})$, which we may arrange to be given by the flow of a vector field that lifts $\frac{d}{dt}$ outside of $x^{-1}(B_{\sqrt{\epsilon}})$, as in the proof of Proposition 4.5 above. We may then use a partition of unity to further extend this deformation retraction to the whole of $W_{\leq c+\epsilon}$.




□

More precisely, if $\dim W = n$, then $W_{\leq c+\epsilon}$ is homeomorphic to the result of attaching to $W_{\leq c-\epsilon}$ a copy of $D^k \times D^{n-k}$ along the thickened boundary-sphere $S^{k-1} \times D^{n-k}$ according to the embedding $j$. We call this procedure attaching a ***k*-handle** to $W_{\leq c-\epsilon}$.

For a $k$-handle $D^k \times D^{n-k}$, we call $S^{k-1} \times 0$ the **attaching sphere** and call $0 \times S^{n-k-1}$ the **belt sphere**.

Given a smooth $n$-manifold $W$ with boundary, the homeomorphism type of the result of attaching a $k$-handle to $W$ is determined by precisely two ingredients: (1) the homotopy class of the attaching map from the attaching sphere $S^{k-1} \times 0$ to $\partial W$, and (2) the way in which we identify the thickened neighborhood of the attaching sphere, which can be specified by an automorphism of the trivial $D^{n-k}$-bundle over $S^{k-1}$.

Combining Propositions 4.5 and 4.6, we see that analogous to the CW structure on a cell complex, we can build $W$ by successively attaching handles of different orders. Moreover, analogous to the classical cellular approximation theorem, we can always assume that lower-order handles are attached before their higher-order counterparts. More precisely,

**Lemma 4.7.** *Let $M$ be a smooth $n$-manifold with boundary. Let $M_1$ be obtained from $M$ by attaching a $k$-handle and let $M_2$ be obtained from $M_1$ by attaching a $j$-handle. If $j \leq k$, then up to homeomorphism $M_2$ be also be obtained by first attaching a $j$-handle followed by a $k$-handle.*

*Proof.* To prove the lemma, it suffices to show that we can homotope the attaching map $f_j : S^{j-1} \to M_1$ to a map into $M$. We do this in two steps. First, we homotope $f_j$ to miss the belt sphere $0 \times S^{n-k-1}$ of the $k$-handle. This can be done by dimension counting: both the image of $S^{j-1}$ and the belt sphere $0 \times S^{n-k-1}$ live in the $(n-1)$-manifold $\partial M_1$. Thus their generic intersection will have dimension $(j-1)+(n-k-1)-(n-1) = j-k-1$, which is negative since $j \leq k$. Thus we can perturb $f_j$ to miss the belt sphere.
The next and last step is to push the image of $f_j$ into $\partial M$. Indeed, after step 1, the part of the image of $f_j$ outside of $\partial M$ can only be contained in $(D^k \backslash 0) \times S^{n-k-1}$. But this deformation retracts radially onto $S^{k-1} \times S^{n-k-1}$, which is contained in $\partial M$. This completes the proof. □

Propositions 4.5 and 4.6 suggest a strong connection between the CW structure of a manifold and its decomposition into handles. Indeed, cellular homology itself can be rephrased in terms of handles.

Let $M$ be a smooth $n$-manifold. We build a chain complex $\{C_k\}_{k\in\mathbb{N}}$ as follows. Fix a Morse function $f : M \to \mathbb{R}$ and consider the associated handle decomposition induced by $f$. We define $C_k$ to be the free abelian group on the set of $k$-handles in this decomposition. For the boundary maps $\partial_k : C_k \to C_{k-1}$, if $h^k_\alpha$ is a $k$-handle and $h^{k-1}_\beta$ is a $(k-1)$-handle, we define the **incidence number** $\langle h^k_\alpha, h^{k-1}_\beta\rangle$ to be the intersection number of the attaching sphere of $h^k_\alpha$ with the belt sphere of $h^{k-1}_\beta$. This allows us to define

$$\partial_k(h^k_\alpha) := \sum \langle h^k_\alpha, h^{k-1}_\beta\rangle h^{k-1}_\beta,$$

where the sum ranges over all $(k-1)$-handles.

We claim that homology groups $H_k(C_*) = \ker \partial_k / \operatorname{Im} \partial_{k+1}$ are naturally isomorphic to the cellular homology groups of $M$. In particular, the groups $H_k(C_*)$ are independent of the choice of the Morse function and the handle decomposition. Indeed, Proposition 4.6 shows that given a Morse function $f$ on $M$, there exists a CW structure on $M$ such that there is a bijection between the set of $k$-handles in the handle decomposition induced by $f$ and the set of $k$-cells in the CW structure. Moreover, the incidence number $\langle h^k_\alpha, h^{k-1}_\beta \rangle$ measures the number of times the attaching sphere of $h^k_\alpha$ wraps around the core of the $(k-1)$-handle $h^{k-1}_\beta$. Thus the boundary map $\partial_k$ defined above corresponds precisely to the cellular boundary formula for cellular homology.

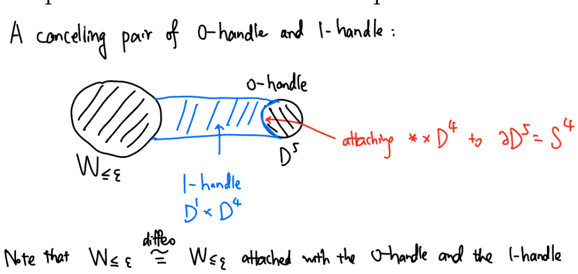
Now that we have developed the tools of Morse theory and the language of handle decompositions, we will proceed to sketch a proof of Wall's Theorem (Theorem 3.2).

## 5. Proof of Wall's Theorem

As explained at the beginning of the preceding section, since $M$ and $N$ have the same signature, they are cobordant through some 5-manifold $W$. The goal is to modify $W$ to be simply-connected and homologically-trivial. We do this in several steps:

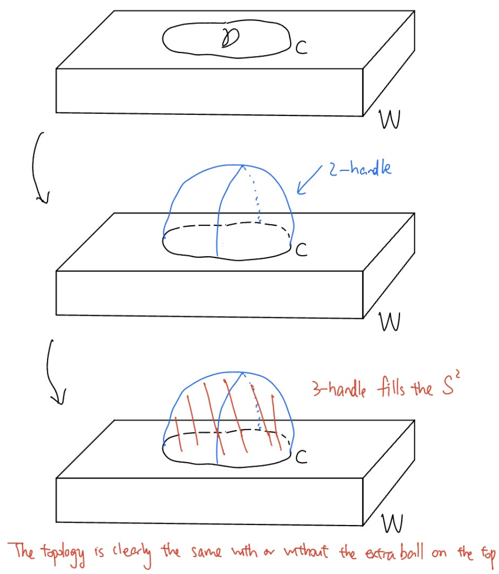
5.1. **Step 1: Kill the fundamental group.** Take generating loops $l_1, ..., l_n$ for $\pi_1(W)$. For dimension reasons, the loops can be realized as pairwise-disjoint embedded circles. For each $l_i$ (viewed as an embedded $S^1$), we cut out a tubular neighborhood $S^1 \times D^4$ in $W$ and glue in a copy of $D^2 \times S^3$ along the boundary. After performing this for each $i$, the modified $W$ is now simply-connected.

5.2. **Step 2: Simplify handles.** Fix a Morse function $f : W \to [0,1]$ with $f^{-1}(0) = M$ and $f^{-1}(1) = N$. As discussed in the preceding section, $W$ can be viewed as the result of successively attaching 0-, 1-, ..., 5-handles to $M$. We first claim that all 0-handles and 5-handles can be cancelled. Indeed, attaching a 0-handle is simply taking the disjoint union with a copy of $D^5$. Since $M$ and $W$ are connected, if a 0-handle were present in the handle decomposition, there must be a 1-handle that bridges the copy of $D^5$ with another component. However, the thickened attaching sphere of the 1-handle is merely two disjoint copies of $D^4$, one of which is embedded in the boundary $S^4$ of the 0-handle $D^5$. But as illustrated in the figure below, attaching such a pair of 0-handle and 1-handle has no affect on the topology. By a symmetric argument, we may also assume that no 5-handles are present in the handle decomposition.




We next claim that all 1-handles can also be eliminated at the price of adding extra 3-handles. Suppose there is a 1-handle $D^1 \times D^4$ in the handle decomposition of $W$. Since $W$ is connected, there must be a segment connecting the two endpoints of the core $D^1 \times 0$ of this 1-handle, and thus after attaching this 1-handle we get an embedded circle $C$ in $W$. Since we have made $W$ simply-connected and $\dim W = 5 \geq 2 \times 2 + 1$, $C$ must bound an embedded disk $\mathcal{D}$ in $W$. To cancel this 1-handle, our strategy will be first creating "out of thin air" a cancelling pair of a 2-handle and a 3-handle (by cancelling we mean attaching this pair does not affect the topology of $W$), and then using the newly created 2-handle to cancel with the 1-handle. We now carry out this strategy in more detail.

We attach a 2-handle $D^2 \times D^3$ by identifying the attaching sphere $S^1 \times 0$ with $C$ in such a way that the this two handle $D^2 \times D^3$ and a tubular neighborhood of $\mathcal{D}$ in $W$ (which is homeomorphic to $D^2 \times D^3$ glue together along $S^1 \times D^3$ to form a copy of $S^2 \times D^3$. We then attach a 3-handle $D^3 \times D^2$ by identifying its attaching sphere $S^2 \times 0$ with the $S^2$ factor above. (See the illustration below. The net effect of attaching this pair of a 2-handle and a 3-handle does not do anything to the topology of $W$.)




On the other hand, this newly added 2-handle cancels with the original 1-handle, since the attaching sphere of the 2-handle precisely wraps around the core of the 1-handle once. Thus, the net effect of this maneuver is that the 1-handle is now gone, and we get a new 3-handle in its place.

By symmetry, we may also eliminate all 4-handles at the price of adding 2-handles, without affecting the topology of $W$. This produces a handle decomposition of $W$ consisting only of 2- and 3-handles. Moreover, by Lemma 4.7, we may assume all 3-handles are attached after all 2-handles. This creates an interface, denoted by $M_{1/2}$, which is a 4-submanifold of $W$ obtained either by attaching all the 2-handles to $M$ or by attaching all the 3-handles to $N$ (recall that $N$, the other manifold whose intersection form is isomorphic to that of $M$ as given in the statement of Theorem 3.2, is the other "end" of the cobordism $W$) upside-down.

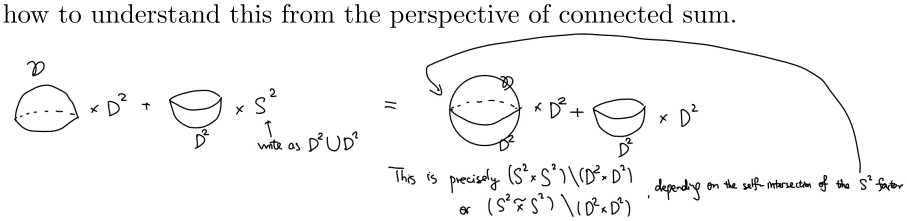
5.3. **Step 3: Analyzing the topology of $M_{1/2}$.** We claim that

$$M_{1/2} = M \# m(S^2 \times S^2) \# m'(S^2 \widetilde{\times} S^2)$$

for some $m, m' \in \mathbb{N}$. To see this, we analyze how a 2-handle can be attached to $M$: a 2-handle is a copy of $D^2 \times D^3$, and to attach it we need to specify (1) the homotopy class of the "un-thickened" attaching map $S^1 \times 0 \to M$ and (2) a self-diffeomorphism of the trivial $D^3$-bundle over $S^1$ (this determines how the thickening is performed). For (1), there is only one choice, since all embedded circles in a 4-manifold are isotopic. For (2), since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, we have two choices. More explicitly, the choices correspond to whether the $D^3$ factor rotates an even number of times as we travel around the base $S^1$ or an odd number of times.

To establish the claim made at the beginning of this subsection, it remains to show that attaching an even 2-handle (i.e. twisting an even number of times in the thickening) is equivalent to taking the connected sum with a copy of $S^2 \times S^2$, and analogously attaching an odd 2-handle is taking the connected sum with $S^2 \tilde{\times} S^2$ (this is the unique non-trivial $S^2$-bundle over $S^2$).

Indeed, since $\partial(D^2 \times D^3) = S^1 \times D^3 \cup D^2 \times S^2$, attaching a 2-handle deletes from $M$ a copy of $S^1 \times D^3$ and adds a copy of $D^2 \times S^2$. The attaching circle $S^1 \times 0$ bounds a disk $\mathcal{D}$ in $M$, which together with the attached $D^2$ factor forms a copy of $S^2 \times D^2$. The $S^2$ factor has self-intersection 0 if the 2-handle is even, and self-intersection 1 if the 2-handle is odd. See the illustration below, which explains how to understand this from the perspective of connected sum.




5.4. **Step 4: Getting rid of the twist.** We claim that

$$M_{1/2} = M \# m(S^2 \times S^2) \tag{5.1}$$

for some $m \in \mathbb{N}$, i.e. we do not need the twisted spheres $S^2 \tilde{\times} S^2$. We consider separately the cases where the intersection form of $M$ is odd or even.

If the intersection form $Q_M$ is odd, the claim follows from the following lemma. We omit the proof as it involves Kirby calculus, but it could be found in section 4.2 of [6].

**Lemma 5.2.** *If $M$ has odd intersection form, then there is a diffeomorphism*

$$M \# (S^2 \times S^2) \simeq M \# (S^2 \tilde{\times} S^2).$$

The case where $Q_M$ is even is more difficult. We do not explain the full details here, and simply take faith in it. The result is basically a combination of the fact that a simply-connected 4-manifold admits a spin structure if and only if its intersection form is even, and the following theorem of Rokhlin, which is a refinement of Theorem 2.6.

**Theorem 5.3** (Rokhlin)**.** *If two spin 4-manifolds $M$ and $N$ have the same signature, then they are cobordant through a spin 5-manifold $W$, and its spin structure induces on $M$ and $N$ their respective spin structures.*

With these results, we can start out with a cobordism without odd handles and thus are left with only the connect sum with copies of $S^2 \times S^2$.

5.5. **Step 5: Find a diffeomorphism of $M_{1/2}$ that kills the homology.** Let us first briefly summarize what we have achieved so far. By Step 2, we have simplified $W$ such that all the nontrivial homological information is contained in $H_2$ and $H_3$. By Steps 3 and 4, we have obtained a simple characterization of this nontrivial homological information: it is captured in the interface $M_{1/2}$. To make the cobordism homologically trivial, it remains to cut up $W$ along $M_{1/2}$ and then reglue along a properly chosen self-diffeomorphism of $M_{1/2}$ so that the 2- and 3-homologies cancel out. For this last step, we need to invoke another difficult theorem of Wall (see [7]):

**Theorem 5.4** (Wall's Theorem on Diffeomorphisms)**.** *Let $M$ be a smooth simply-connected 4-manifold with $Q_M$ indefinite. Then any automorphism of $Q_{M\#(S^2\times S^2)}$ can be realized by a self-diffeomorphism of $M\#(S^2\times S^2)$.*

By (5.1), we may write

$$H_2(M_{1/2};\mathbb{Z}) = H_2(M;\mathbb{Z})\oplus\mathbb{Z}\{\alpha_1,\overline{\alpha}_1,...,\alpha_m,\overline{\alpha}_m\} = H_2(N;\mathbb{Z})\oplus\mathbb{Z}\{\beta_1,\overline{\beta}_1,...,\beta_m,\overline{\beta}_m\}$$

where $\alpha_i,\overline{\alpha}_i$ represent the homology classes of the $i$-th copy of $S^2\times S^2$. By assumption, since $M$ and $N$ have isomorphic intersection forms, let $\phi : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ be such an intersections-preserving isomorphism. We may extend this to an isomorphism $\tilde{\phi} : H_2(M_{1/2};\mathbb{Z}) \to H_2(M_{1/2};\mathbb{Z})$ by setting $\tilde{\phi}(\alpha_i) = \beta_i$ and $\tilde{\phi}(\overline{\alpha}_i) = \overline{\beta}_i$. By Wall's Theorem on Diffeomorphisms, $\tilde{\phi}$ is induced by a diffeomorphism $\psi : M_{1/2} \to M_{1/2}$. Finally, we re-glue the upper and lower halves of $W$ along the interface $M_{1/2}$ by $\psi$. This kills the homology of $W$ and we obtain the desired $h$-cobordism.

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