HALL'S THEOREMS ON SOLVABLE GROUPS

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ABSTRACT. Representation theory is a way of studying the structure of groups using the tools and ideas of linear algebra. The goal of this paper is to demonstrate the power of representation theory to derive nontrivial properties about group structure. The culminating result of this paper will be a proof of all of Hall's theorems regarding solvable groups. These theorems serve as a natural extension to the Sylow theorems which apply in the case of solvable groups.

CONTENTS

1. INTRODUCTION

Representation theory was invented by Frobenius to understand a bizarre pattern. If one takes the Cayley table of a group and considers each element as a variable x_g, then the determinant is a product of irreducible polynomials with degree equal to their multiplicity. Representation theory seeks to use the tools and insights of linear algebra to study a wide range of diverse mathematical structures by "representing" them in a form that is easier to understand. This turns out to be an extremely fruitful endeavor, and today representation theory is extremely widespread. In this paper, I will define the basic notions and results of representation theory as well as character theory. Finally, I will use these results to provide a complete treatment of Hall's theorems regarding solvable groups. Along the way, I will use representation theory to prove Burnside's theorem, as well as a weakened version of Schur-Zassenhaus, both of which are extremely famous and powerful results.

2. Representation Theory

2.1. **Basic Representation Theory.**

Throughout this paper, we will assume the base field to be $\mathbb{C}$, all groups are finite, and all vector spaces are finite dimensional. Furthermore, if g is an element of some group which acts linearly on a vector space V via a group action, I will denote $g|_V \in \text{End}(V)$ as the linear transformation defined by g. We start by fixing a finite group G.

Definition 2.1. An *associative algebra* A over $\mathbb{C}$ is a ring with a ring homomorphism $\psi : \mathbb{C} \to Z(A)$.

Definition 2.2. The *group algebra* $\mathbb{C}[G]$ is the set of all linear combinations of elements in G with coefficients in $\mathbb{C}$. Multiplication and addition are defined in the natural way:

$$\sum_{g\in G} z_g g + \sum_{g\in G} z'_g g = \sum_{g\in G} (z_g + z'_g) g$$

$$\left(\sum_{g\in G} z_g g\right)\left(\sum_{g\in G} z'_g g\right) = \sum_{g,g'\in G} (z_g z'_{g'})(gg')$$

Note that $\mathbb{C}[G]$ is a associative algebra with $\psi : \mathbb{C} \to \mathbb{C}[G]$ as $\psi(z) = ze$.

Definition 2.3. A *(complex) representation* of G consists of the data of a pair (ρ, V), where V is a finite dimensional vector space over $\mathbb{C}$ and ρ is a group homomorphism $\rho : G \to GL(V)$ from G to the automorphism group of V. Using ρ, we give an action of $\mathbb{C}[G]$ over V as $\left(\sum_{g\in G} z_g g\right)|_V = \sum_{g\in G} z_g \rho(g)$

- A *subrepresentation* $(\rho|_W, W)$ of (ρ, V) consists of the data of a vector subspace W of V and group homomorphism $\rho|_W : G \to GL(W)$ given by restriction of ρ to W, i.e. for any $g \in G$, we have $\rho|_W(g) = \rho(g)|_W$.
- We say that the representation (ρ, V) is *irreducible* if it only has trivial subrepresentations, i.e. the zero representation $(\rho|_0, 0)$ and (ρ, V) itself.

One may think of a group representation more simply as an action of $\mathbb{C}[G]$ on V. For this reason, we sometimes only denote V for the representation (ρ, V). In this way, we can view a subrepresentation W of V as a vector subspace which is stable under the action of $\mathbb{C}[G]$. We often don't need to invoke the larger structure of the group algebra, and in many of our proofs we will only consider the actions of elements of G.

Example 2.4. Consider the Klein four group $\langle a, b | a^2 = b^2 = (ab)^2 = e\rangle$. It has this representation:

$$a \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This representation has a subrepresentation which consists of the vector subspace of $\mathbb{C}^2$ spanned by $(1, 0)$. In this subrepresentation our map looks like:

$$a \mapsto -1$$

$$b \mapsto 1$$

This representation is trivially irreducible because it is one dimensional and therefore has no nontrivial vector subspaces.

Lemma 2.5. *If V is a representation of G over $\mathbb{C}$ and $g \in G$, we have the following lemmas:*

(1) $e|_V = \mathrm{Id}$
(2) $g|_V$ *is diagonalizable.*
(3) The eigenvalues of $g|_V$ *are roots of unity.*

Proof. We use the spectral theorem

(1) $\rho : G \to GL(V)$ is a group homomorphism hence it must map the identity to the identity.
(2) Since G is a finite group, g has finite order and we must have $g|_V{}^{|g|} = \mathrm{Id}$. Then $g|_V$ satisfies the polynomial equation $X^{|g|} - 1 = 0$, which has no repeated roots. The minimal polynomial of $g|_V$ must divide this polynomial, and therefore it also has no repeated roots. Therefore by spectral theorem $g|_V$ is diagonalizable.
(3) Since the roots of $X^{|g|} - 1$ are the $|g|$-th roots of unity and the minimal polynomial of $g|_V$ divides $X^{|g|} - 1$, the eigenvalues of $g|_V$ must be $|g|$-th roots of unity.

□

Now that we have defined what a representation is, and identified a particularly useful type of representation (irreducible representations), we will study maps between representations.

Definition 2.6. Given two representations V and W, a *homomorphism* between them is a linear operator $\phi : V \to W$ which commutes with the action of G, i.e. for any $v \in V$ and $g \in G$, $\phi(gv) = g\phi(v)$. The set of homomorphisms from V to W is denoted $\mathrm{Hom}_G(V, W)$.

If ϕ is an isomorphism of vector spaces then it is an *isomorphism of representations*. Two representations are *isomorphic* if there is an isomorphism between them.

The following result, Schur's Lemma, is extremely powerful and fundamental to the study of representation theory.

Theorem 2.7 (Schur's Lemma)**.** *Suppose V and W are representations of G and $\phi : V \to W$ is a nonzero homomorphism between them. Then:*

(1) If V is irreducible then ϕ is injective.
(2) If W is irreducible then ϕ is surjective.
(3) If $W = V$ is irreducible then $\phi = \lambda Id$ for some $\lambda \in K$.

Proof. We use the defining property of irreducible representations.

(1) Consider the subspace $\ker(\phi)$ of V. Because ϕ commutes with the action of G, $\ker(\phi)$ is invariant under the action of G and hence is a subrepresentation of V. Therefore, since V is irreducible and ϕ is nonzero, we must have $ker(\phi) = 0$ and ϕ is injective.

(2) Similarly, consider the subspace $\mathrm{im}(\phi)$ of W. Because ϕ commutes with the action of G, $\mathrm{im}(\phi)$ is invariant under the action of G and is a subrepresentation of W. Therefore, since W is irreducible and ϕ is nonzero, we must have $\mathrm{im}(\phi) = W$ and ϕ is surjective.
(3) Choose an eigenvalue λ of ϕ and note that $\phi - \lambda\,\mathrm{Id}$ is singular. If $\phi - \lambda\,\mathrm{Id}$ is nonzero, then by parts (1) and (2), $\phi - \lambda\,\mathrm{Id}$ will be an isomorphism, which contradicts it being singular. Therefore $\phi - \lambda\,\mathrm{Id} = 0$, i.e. $\phi = \lambda\,\mathrm{Id}$.

□

Now we will generalize Schur's lemma to a more powerful form.

Lemma 2.8. *If $A = \bigoplus_i V_i^{n_i}$ and $B = \bigoplus_i V_i^{r_i}$ are two sums of distinct irreducible representations V_i with multiplicity then $\mathrm{Hom}_G(A, B) = \bigoplus_i M_{r_i \times n_i}(\mathbb{C})$.*

Suppose that $\phi : A \to B$ is a homomorphism of representations. We denote a_{ij} as the projection onto the jth copy of V_i in A and likewise b_{ij} as the projection onto the jth copy of V_i in B. For some i, j, k, l note that $b_{ij} \circ \phi \circ a_{kl}$ can be considered as a homomorphism of irreducible representations $V_{kl} \to V_{ij}$. Therefore by Schur's lemma:

$$b_{ij} \circ \phi \circ a_{kl} : V_{kl} \to V_{ij} = \begin{cases} \lambda\,\mathrm{Id} & i = k \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $b_{ij} \circ \phi \circ a_{il} = \lambda_{ijl}\,\mathrm{Id}_{V_{il} \to V_{ij}}$. Now note that $\phi = \sum_{i,j,k,l} b_{ij} \circ \phi \circ a_{kl}$ by linearity. Hence:

$$\begin{aligned} \phi &= \sum_{i,j,k,l} b_{ij} \circ \phi \circ a_{kl} \\ &= \sum_{i,j,l} b_{ij} \circ \phi \circ a_{il} \\ &= \sum_{i,j,l} \lambda_{ijl}\,\mathrm{Id}_{V_{il} \to V_{ij}} \\ &= \sum_{i} \sum_{j,l} \lambda_{ijl}\,\mathrm{Id}_{V_{il} \to V_{ij}} \\ &= \bigoplus_i M_i \end{aligned}$$

Where $M_i : V_i^{n_i} \to V_i^{r_i}$ is the $r_i \times n_i$ matrix with coefficients $(M_i)_{kl} = \lambda_{ikl}$.

2.2. **The Significance of Irreducible Representations.** Now we will prove some theorems which should establish the importance of the irreducible representations. In particular, we will eventually show that arbitrary representations factor uniquely (up to permutation) into a direct sum of irreducible representations (Krull-Schmidt theorem). In this way, irreducible representations are analogous to the prime numbers of representation theory.

Lemma 2.9. *If V is a representation of G, there exists a Hermitian inner product $\langle -, - \rangle$ on V which is invariant under the action of G. More specifically, for each $g \in G$ and $v_1, v_2 \in V$, $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$*

Proof. Take some arbitrary Hermitian innner product $[-, -]$ on V and let $\langle v_1, v_2 \rangle = \sum_{g \in G} [gv_1, gv_2]$. This inner product is invariant under the action of G because each element of G defines an automorphism on G by right multiplication. □

Theorem 2.10 (Maschke's Theorem + Krull-Schmidt Theorem). *Any representation decomposes uniquely into a direct sum of irreducible representations.*

Proof. Suppose V is a representation G. We first prove that V decomposes into a direct sum of irreducible representations (Maschke's theorem), and then prove any two such decompositions are unique (Krull-Schmidt theorem).

Maschke's theorem: By Lemma 2.9 we can fix a Hermitian inner product on V which is invariant under the action of G. If V is irreducible we are done. Otherwise, V must have a subrepresentation $W \subset V$. Since our inner product is invariant under the action of G, the vector subspace $W_\perp$ (the subspace of vectors orthogonal to all the vectors in W) is invariant under the action of G and is therefore a subrepresentation of V. By construction $V = W \oplus W_\perp$. We can continue decomposing V recursively until we are left with a direct sum of irreducible representations by finite dimensionality.

Krull-Schmidt theorem: Suppose that $V = \bigoplus_i V_i^{n_i} = \bigoplus_i V_i^{r_i}$ are two decompositions of V into distinct irreducible representations V_i with multiplicity. Consider the inclusion map $\bigoplus_i V_i^{n_i} \to \bigoplus_i V_i^{r_i}$. By Lemma 2.8 this map looks like $\bigoplus_i M_i$ where each $M_i : V_i^{n_i} \to V_i^{r_i}$ is a $r_i \times n_i$ matrix. This map must be invertable and therefore $r_i = n_i$ for each i and we are done. □

Lemma 2.11. *If $V = V_1^{p_1} \oplus \cdots \oplus V_n^{p_n}$ is a sum of distinct irreducible representations (with multiplicity) and $W \subseteq V$ is a subrepresentation of V then W is isomorphic to $V_1^{r_1} \oplus \cdots \oplus V_n^{r_n}$ where $r_i \leq p_i$.*

Proof. Consider $W_\perp$ under the inner product defined in Lemma 2.9. Note that W and $W_\perp$ are both subrepresentations of V with $V = W \oplus W_\perp$. The result follows from the Krull-Schmidt theorem. □

Hopefully, these theorems have provided some insight as to why we care so much about the irreducible representations. I will prove one more useful theorem which is closely related to these.

Theorem 2.12 (Density Theorem). *If V is an irreducible representation of G the map $\rho : \mathbb{C}[G] \to \mathrm{End}(V)$ is surjective.*

Proof. Choose an arbitrary basis $v_1, \ldots, v_n$ of V and consider the map $\alpha : \mathbb{C}[G] \to V^n$ as $\alpha(a) = (av_1, \ldots, av_n)$. It suffices to show that this map is surjective. Suppose the contrary. Then the image of α is a proper subrepresentation of V^n and by Lemma 2.11 we have $\mathrm{im}(\alpha) \cong V^r$ where $r < n$. Let $\phi : V^r \to V^n$ be the inclusion map. By Lemma 2.8 ϕ looks like an $r \times n$ matrix. Note that $\alpha(e) = (v_1, \ldots, v_n)$, therefore $(v_1, \ldots, v_n)$ is in the image of ϕ. However ϕ is not invertable since $r < n$, a contradiction with the fact that the v_is form a basis. □

The density theorem is a very useful result with many corollaries. For example, one can prove that $\mathbb{C}[G]$ is isomorphic to $\bigoplus_{V \in \mathrm{Irr}} \mathrm{End}(V)$ as an associative algebra, the direct sum of endomorphism rings of the irreducible representations. We will prove one corollary which will later be useful in proving Burnside's theorem.

Lemma 2.13. *If V is an irreducible representation of G and $\alpha \in Z(\mathbb{C}[G])$, then $\alpha|_V$ acts either by zero or by scaling on V.*

Proof. Note that the action of α on V must commute with the action of every element in $\mathbb{C}[V]$. Therefore by the density theorem $\alpha|_V$ commutes with every operator in $\operatorname{End}(V)$. Then by the spectral theorem $\alpha|_V = \lambda \operatorname{Id}$ and we are done. □

2.3. **Class Functions and Characters.** Now that we have introduced irreducible representations, a natural question that arises is how many irreducible representations can be found for a given group? In this section, we will prove that the number of irreducible representations a group has (up to isomorphism) is equal to the number of conjugacy classes it contains.

Definition 2.14. Given a group G, a *class function* $\alpha : G \to \mathbb{C}$ is a function which preserves conjugacy classes. Specifically for each $a, b \in G$, $\alpha(aba^{-1}) = \alpha(a)$.

Note that the space of class functions on G is a vector space over $\mathbb{C}$ whose dimension equals the number of conjugacy classes of G.

Definition 2.15. Given a representation V of a group G, we define the *character* $\chi_V : \mathbb{C}[G] \to \mathbb{C}$ of V as $\chi_V(a) = \operatorname{Tr} a|_V$. This is a class function because the trace is invariant under conjugation.

Definition 2.16. We define a Hermitian inner product on the vector space of class functions as $\langle \chi_W, \chi_V \rangle = \frac{1}{|G|} \sum_G \chi_W(g)\overline{\chi_V(g)}$.

As we will see, this is the most useful and natural inner product on the space of class functions. We will utilize its nice properties extensively throughout the rest of the paper. Remarkably, this inner product actually computes the dimension of $\operatorname{Hom}_G(V, W)$ as a vector space over $\mathbb{C}$. Recall that $\operatorname{Hom}_G(V, W)$ denotes the homomorphisms of representations between V and W. We will prove a weaker version of this result now.

Theorem 2.17. *If V, W are irreducible representations of G, then $V \cong W$ if and only if $\chi_V = \chi_W$.*

Proof. By Schur's lemma:

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & V \cong W \\ 0 & \text{otherwise} \end{cases}$$

Consider $\operatorname{Hom}(V, W)$, the vector space of linear transformations from V to W. We can define a group action of G on $\operatorname{Hom}(V, W)$ as $g\tau = g|_W \circ \tau \circ g^{-1}|_V$. G acts linearly on $\operatorname{Hom}(V, W)$, so we can consider the trace of $g|_{\operatorname{Hom}(V,W)}$. Fix an eigenbasis $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_r$ of W with respect to the $g|_V$ and $g|_W$ respectively. Let their eigenvalues be $\lambda_{v_1}, \ldots, \lambda_{v_n}$ and $\lambda_{w_1}, \ldots, \lambda_{w_r}$. We construct a basis of $\operatorname{Hom}(V, W)$ as the transformations which take an eigenvector of V and map it to an eigenvector of W and kill all others. Then $g|_{\operatorname{Hom}(V,W)}$ is a diagonal matrix whose trace we can calculate:

$$\begin{aligned} \operatorname{Tr} g|_{\operatorname{Hom}(V,W)} &= \sum \frac{\lambda_{w_i}}{\lambda_{v_j}} \\ &= \sum \lambda_{w_i} \overline{\lambda_{v_j}} && \textit{Lemma } 2.5 \\ &= \left(\sum \lambda_{w_i}\right)\left(\sum \overline{\lambda_{v_j}}\right) \\ &= \operatorname{Tr} g|_W \overline{\operatorname{Tr} g|_V} \end{aligned}$$

Now consider the linear transformation $\alpha : \mathrm{Hom}(V, W) \to \mathrm{Hom}_G(V, W)$ as $\alpha\tau = \frac{1}{|G|}\sum_{g\in G} g\tau$. Since $\tau \in \mathrm{Hom}_G(V, W)$ implies $\alpha\tau = \tau$, α is a surjection. α is then a projection from $\mathrm{Hom}(V, W)$ onto $\mathrm{Hom}_G(V, W)$ and therefore $\dim \mathrm{Hom}_G(V, W) = \mathrm{Tr}\,\alpha$. Now note that:

$$\begin{aligned}\mathrm{Tr}\,\alpha &= \frac{1}{|G|}\sum_G \mathrm{Tr}\, g|_W \,\mathrm{Tr}\, \overline{g|_V} \\ &= \frac{1}{|G|}\sum_G \chi_W(g)\overline{\chi_V(g)} \\ &= \langle \chi_W, \chi_V\rangle\end{aligned}$$

If $\chi_V = \chi_W$ then this expression must be nonzero and therefore $\dim \mathrm{Hom}_G(V, W) \neq 0$ and $V \cong W$ by Schur's lemma.

If $V \cong W$ then let ϕ be an isomorphism between them. Choose a basis $v_1, \ldots, v_n$ for V and let $\phi(v_1), \ldots, \phi(v_n)$ be our basis for W. By the condition that for each v_i, $\phi(gv_i) = g\phi(v_i)$, $g|_V$ and $g|_W$ have identical matrices in these bases. Therefore $\chi_V(g) = \chi_W(g)$ and V and W have the same character. □

This idea of averaging over the action of G to create a G invariant map is an extremely powerful problem solving technique. It is probably the single most useful idea in this whole paper.

Corollary 2.18 (First Orthagonality Relation)**.** *The characters of irreducible representations are orthonormal with respect to the outlined inner product.*

With this theorem, we have shown that the characters of the irreducible representations form a linearily independant orthonormal set. We have one more important orthagonality relation to show. However, we have some important work to do before we get there.

2.4. **The Regular Representation.** Now we wish to show that the characters of the irreducible representations form a basis for the vector space of class functions. For this, we introduce the regular representation. Just as examining action of G on itself is often useful, examining the action of $\mathbb{C}[G]$ on itself proves extremely useful in representation theory.

Note that $\mathbb{C}[G]$ is vector space over $\mathbb{C}$ spanned by the elements of G. Therefore, we may consider it as a representation of G, with the group action defined by left multiplication. Note that this representation is faithful on $\mathbb{C}[G]$ since we can consider how two elements $\alpha, \beta \in \mathbb{C}[G]$ act on the identity element $e \in \mathbb{C}[G]$.

This is a very important representation, called the regular representation.

Theorem 2.19. *The characters of the irreducible representations form a basis of the vector space of class functions.*

Proof. Suppose the contrary, then there is some nonzero class function ϕ with $\langle \phi, \chi_V\rangle = 0$ for all irreducible representations. Fix some irreducible representation V and define $\alpha = \frac{1}{|G|}\sum_G \phi(g)g$. Since ϕ is a class function $\alpha|_V : V \to V$ commutes with the action of G on V and defines a homomorphism of representations. Then by Schur's lemma $\alpha|_V = \lambda \operatorname{Id}$ for some λ. However note that:

$$\begin{aligned}\operatorname{Tr} \alpha|_V &= \frac{1}{|G|}\sum_G \phi(g)\chi_V(g)\\ &= \langle \phi, \overline{\chi_V}\rangle \\ &= \overline{\langle \phi, \chi_V\rangle} \\ &= 0\end{aligned}$$

Hence α acts by 0 on V. Therefore by Maschke's theorem α acts by 0 on any representation. Now consider the regular representation R. Since the elements of g are linearly independent in $\mathbb{C}[G]$, $\alpha|_R = 0$ implies $\phi(g) = 0$ for each $g \in G$. However we stated that ϕ was nonzero, a contradiction. □

Corollary 2.20. *The number of irreducible representations equals the number of conjugacy classes.*

This completes the brunt of the low level work. The rest is an application of these theorems.

2.5. **The Character Table.** Now that we have proven Corollary 2.20, an extremely important result, we have enough machinery to define a very important concept. That is the character table, a way to encode all the relevant information about the characters of a group's irreducible representations. Note that $C_G(g)$ denotes the centralizer of g.

Definition 2.21. The *character table* X of a group is the matrix whose rows correspond to irreducible representations and whose columns correspond to conjugacy classes. If the irreducible representations are $V_1, \dots, V_n$ and the conjugacy classes are $C_1, \dots, C_n$ then $X_{ij} = \chi_{V_i}(C_j)$. Note that the character table is only defined up to permutation of rows and columns.

Example 2.22. The character table of the Klein four group $\langle a, b | a^2 = b^2 = (ab)^2 = e\rangle$ is as follows:

	e	a	b	ab
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Theorem 2.23 (Second Orthagonality Relation)**.** *Suppose $h, g \in G$, then:*

$$\sum_{Irr} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)|, & \text{if } g \text{ and } h \text{ are conjugate} \\ 0, & \text{otherwise}\end{cases}$$

where the sum is over the characters of the irreducible representations.

Proof. Let $C_1, \dots, C_n$ be the conjugacy classes of our group. By the first orthagonality relation the character table X has orthonormal columns under our inner product. Let $\langle -, - \rangle$ be as in Lemma 2.9. Using Corollary 2.18 and the definition of our inner product gives:

$$\begin{aligned}
\mathrm{Id} &= \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \\
&= \begin{bmatrix} \langle \chi_{V_1}, \chi_{V_1} \rangle & \langle \chi_{V_1}, \chi_{V_2} \rangle & \cdots \\ \langle \chi_{V_2}, \chi_{V_1} \rangle & \langle \chi_{V_2}, \chi_{V_2} \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \\
&= X \begin{bmatrix} |C_1|/|G| & 0 & \cdots \\ 0 & |C_2|/|G| & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} X^\dagger \\
X^{-1} &= \begin{bmatrix} 1/|C_G(c_1)| & 0 & \cdots \\ 0 & 1/|C_G(c_2)| & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} X^\dagger \\
\mathrm{Id} &= \begin{bmatrix} 1/|C_G(c_1)| & 0 & \cdots \\ 0 & 1/|C_G(c_2)| & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} X^\dagger X
\end{aligned}$$

For any $c_i \in C_i$. But $(X^\dagger X)_{ij} = \sum_{Irr} \overline{\chi_i(g)} \chi_i(h)$. This proves the result. □

3. Burnside's Theorem

Now we will prove a big result, Burnside's theorem, which states than any group of order $p^\alpha q^\beta$ is solvable. This theorem is the premier demonstration of the power of representation theory to understand the structure of finite groups.

Theorem 3.1 (Burnside's Theorem)**.** *All groups of order $p^\alpha q^\beta$ for primes p, q are solvable.*

We proceed by contradiction, suppose that G is the smallest order not-solvable group of order $p^\alpha q^\beta$. Notice that G must be a simple group. Otherwise there is some normal subgroup N of G. Then G/N and N are solvable by minimality which implies G is solvable, a contradiction.

We claim that G cannot have any conjugacy class of order p^k or q^k. If this is true, then all the nontrivial conjugacy classes of G have order divisible by pq. Writing the class equation for G yields:

$$|G| = |Z(G)| + \sum_{C \text{ nontrivial}} |C|$$

Where the sum is over the nontrivial conjugacy classes. This will imply that:

$$|Z(G)| = 0 \pmod{pq}$$

However $|Z(G)| \geq 1$ because the identity lies in the center. This shows that the center of G is nontrivial, a contradiction with the simplicity of G.

Now we use the following steps to prove that G does not have conjugacy class of prime power order (Theorem 3.6).

Lemma 3.2. *Suppose that $\omega_1, \ldots, \omega_n$ are roots of unity. If $\frac{1}{n} \sum w_i$ is an algebraic integer either $w_1 = \cdots = w_n$ or $\sum w_i = 0$*

Proof. Let $\alpha = \frac{1}{n}\sum w_i$ be algebraic with $w_1, \ldots, w_n$ not satisfying $w_1 = \cdots = w_n$. We take the field norm of α as:

$$N_{\overline{\mathbb{Q}}/\mathbb{Q}}(\alpha) = \prod_{\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})} \sigma(\alpha)$$

Because α is an algebraic integer and $N_{\overline{\mathbb{Q}}/\mathbb{Q}}(\alpha) \in \mathbb{Q}$, $N_{\overline{\mathbb{Q}}/\mathbb{Q}}(\alpha)$ must be an integer. If the w_is are not all the same this implies $|\sigma(\alpha)| < 1$ for each σ by the triangle inequality. Therefore $|N_{\overline{\mathbb{Q}}/\mathbb{Q}}(\alpha)| < 1$ and $N_{\overline{\mathbb{Q}}/\mathbb{Q}}(\alpha) = 0$. This forces $\alpha = 0$. □

Lemma 3.3. *Suppose V is an irreducible representation of group G and let C be a conjugacy class of G. Then for $c \in C$, $\frac{|C|\chi_V(c)}{\dim V}$ is an algebraic integer.*

Proof. Consider $\alpha = \sum_{c \in C} c$. Note that α is in the center of $\mathbb{C}[G]$. Therefore by Lemma 2.13 $\alpha|_V = \lambda \operatorname{Id}$ for some λ. Because $\mathbb{Z}[G]$ is finitely generated over $\mathbb{Z}$, the center of $\mathbb{Z}[G]$ is integral over $\mathbb{Z}$. Therefore $\alpha \in Z(\mathbb{Z}[G])$ is integral over $\mathbb{Z}$ and λ is an algebraic integer. Taking the trace of α yields:

$$|C|\chi_V(c) = \lambda \dim V$$
$$\frac{|C|\chi_V(c)}{\dim V} = \lambda$$

□

Lemma 3.4. *Suppose V is an irreducible representation of group G and C is a conjugacy class of G with $gcd(|C|, \dim V) = 1$. Then for each $c \in C$ either $\chi_V(c) = 0$ or c acts on V by scaling.*

Proof. Choose some $c \in C$ and let $w_1, \ldots, w_n$ be the eigenvalues of $c|_V$. Note that the eigenvalues of $c|_V$ are roots of unity by Lemma 2.5. By Lemma 3.3, $\frac{|C|}{\dim V}\sum w_i$ is an algebraic integer. By the condition that $gcd(|C|, \dim V) = 1$, $\frac{1}{\dim V}\sum w_i$ must be algebraic. Then by Lemma 3.2 either $w_1 = \cdots = w_n$ and c acts by scaling, or $\sum w_i = 0$ and $\chi_V(c) = 0$. □

Lemma 3.5. *Suppose G is a group with a conjugacy class C of order p^k for some prime p and $k > 0$. Then for each $c \in C$, there exists a nontrivial irreducible representation whose dimension is not divisible by p with $\chi_V(c) \neq 0$.*

Proof. Denote S as the set of irreducible representations whose dimension is divisible by p. Denote T as the set of nontrivial irreducible representations whose dimension is not divisible by p.

Let c be an element in the conjugacy class C. Note that for each $V \in S$, $\frac{1}{p}\dim V \chi_V(c)$ is an algebraic integer. Then

$$\alpha = \frac{1}{p}\sum_S \dim V \chi_V(c)$$

is algebraic. Using the second orthogonality relation on the elements e and c gives:

$$\sum_{Irr} \dim V \chi_V(c) = 0$$

because they do not lie in the same conjugacy class. Splitting up the sum:

$$\begin{aligned} 0 &= \chi_{\mathbb{C}}(c) + \sum_S \dim V \chi_V(c) + \sum_T \dim V \chi_V(c) \\ &= 1 + p\alpha + \sum_T \dim V \chi_V(c) \end{aligned}$$

where $\chi_{\mathbb{C}}$ denotes the trivial representation. Since α is an algebraic integer it either has nonzero imaginary component or is an integer. Therefore $\sum_T \dim V \chi_V(c)$ must be nonzero. Hence there is at least one $V \in T$ with $\chi_V(c) \neq 0$. ☐

Theorem 3.6. *If a group G is simple, it has no conjugacy classes of prime power order.*

Proof. Suppose the contrary and let C be a conjugacy class of order p^k. By Lemma 3.5, pick a nontrivial irreducible representation V whose dimension is not divisible by p and with $\chi_V(c) \neq 0$. By Lemma 3.4, each $c \in C$ acts on V by scaling by some λ. Consider the following subgroup of G:

$$H := \{g \in G \mid g \text{ acts on } V \text{ as identity}\}.$$

We claim that H is a proper nontrivial normal subgroup of G.

(1) Since V is nontrivial H is a proper subgroup.
(2) For any $g \in G$ and $h \in H$ $(ghg^{-1})|_V = g|_V \operatorname{Id} g|_V^{-1} = \operatorname{Id}$, hence $ghg^{-1} \in H$. H is therefore normal.
(3) Let $K := \{ab^{-1} | a, b \in C\}$ and note that since each $c \in C$ scales by λ each ab^{-1} acts by identity. Therefore $K \subseteq H$. Note that K is nontrivial because C is nontrivial. Hence H is nontrivial.

This is a contradiction with the simplicity of G. ☐

4. Schur-Zassenhaus

The Schur-Zassenhaus theorem is an extremely deep result in finite group theory. It essentially allows us to determine in certain cases that a group can be constructed via a semidirect product. Most proofs use group cohomology, however it is possible to prove this theorem very nicely using representation theory.

Theorem 4.1 (Schur-Zassenhaus)**.** *If $|G| = ab$ where a and b are coprime, and N is a normal subgroup of G with order a, then G contains a subgroup of order b. Furthermore, all subgroups of order b are conjugate.*

For our purposes, we may assume that N is an elementary abelian p-group (a group in which every nonidentity element has order p). This simplifies the proof slightly. Nonetheless, one may find a proof that the general case reduces to our special case in Kargapolov and Merzljakov [6].

Definition 4.2. Let A, B be groups. We now define $W = A \wr B$ the *wreath product* of A and B. Let A^B={set-theoretic functions $B \to A$}. Note that A^B is a group under multiplication of functions. For each $f \in A^B$ and $b \in B$ we define f^b as:

$$f^b(x) = f(bx)$$

One can check that each $b \in B$ defines an automorphism on A^B. This gives us an action of B on A^B. The wreath product $A \wr B$ is the semidirect product $A^B \rtimes B$

with the action of B on A^B as defined. More explicitly, $A \wr B$ has the underlying set $B \times A^B$ with group operation given by:

$$bf \cdot b'f' = bb'f^{b'}f' \tag{4.3}$$

There is some distinction between the restricted and unrestricted wreath products in mathematical literature. However, since we are only dealing with finite groups we do not need to make this distinction.

Theorem 4.4. *If N is a normal subgroup of G, then G is a subgroup of $N \wr (G/N)$.*

Proof. Define a function $\tau : G/N \to G$ which chooses an element of G such that $\tau(k)N = k$. This is called a *coset representative function.* We define the map

$$\begin{aligned} \phi : G &\to N \wr (G/N) \\ g &\mapsto (gN)f_g \end{aligned} \tag{4.5}$$

where $f_g : G/N \to N$ is the function defined as $f_g(k) = \tau(gk)^{-1}g\tau(k)$. Since $\tau(kg)$ and $g\tau(k)$ land in the same coset of G/N this map is well defined. Furthermore, we can check that this is indeed a homomorphism.

$$\begin{aligned} (aN)f_a \cdot (bN)f_b &= (abN)f_a^{bN}f_b \\ &= (abN)\tau(ab-)^{-1}a\tau(b-)\tau(b-)^{-1}b\tau(-) \\ &= (abN)\tau(ab-)^{-1}ab\tau(-) \\ &= (abN)f_{ab} \end{aligned}$$

In this notation $-$ denotes a placeholder for some element of G/N. We can furthermore verify that ϕ is injective by computing f_a, f_b when a, b are members of the same coset. □

Lemma 4.6. *If $\tau_1, \tau_2 : G/N \to G$ are coset representation functions and $\phi_1, \phi_2 : G \to W = N \wr (G/N)$ are defined as in (4.5) using τ_1 and τ_2 respectively, then $\phi_1(G)$ and $\phi_2(G)$ are conjugate in W.*

Proof. We do some algebraic manipulations. Suppose $Nf \in W$, $g \in G$ and consider $\phi_2(g)$ conjugated by Nf:

$$\begin{aligned} (Nf)\phi_2(g)(Nf)^{-1} &= (Nf)((gN)f_{2g})(Nf)^{-1} && (4.5) \\ &= ((gN)f^{gN}f_{2g})(Nf)^{-1} && (4.3) \\ &= ((gN)f^{gN}f_{2g})(Nf^{-1}) \\ &= (gN)f^{gN}f_{2g}f^{-1} && (4.3) \end{aligned}$$

Now we expand $f^{gN}f_{2g}f^{-1}$ using the definition of f_{2g}:

$$\begin{aligned} f^g f_{2g} f^{-1} &= (f^g f_{2g} f^{-1})(-) \\ &= f(g-)\tau_2(g-)^{-1}g\tau_2(-)f(-)^{-1} \end{aligned}$$

We can similarily compute using (4.5) that $\phi_1(g) = (gN)f_{1g} = (gN)\tau_1(g-)^{-1}g\tau_1(-)$. From here we are motivated to fix $f = \tau_1^{-1}\tau_2$ in order to make these functions equal to eachother. Indeed:

$$\begin{aligned} f(g-)\tau_2(g-)^{-1}g\tau_2(-)f(-)^{-1} &= \tau_1^{-1}(g-)\tau_2(g-)\tau_2(g-)^{-1}g\tau_2(-)\tau_2^{-1}(-)\tau_1(-) \\ &= \tau_1^{-1}(g-)g\tau_1(-) \end{aligned}$$

Since our choice of f did not depend on g, $(Nf)\phi_2(g)(Nf)^{-1} = \phi_1(g)$ for any $g \in G$. Therefore $\phi_1(G)$ and $\phi_2(G)$ are conjugate in W. □

We have now proven all the necessary properties of the wreath product. Before we prove Schur-Zassenhaus, I will introduce a few miscellaneous results without proof.

Theorem 4.7 (Generalized Maschke's Theorem)**.** *If V is a representation of G over some field K whose characteristic does not divide $|G|$, and W is a subrepresentation of V, then V has a subrepresentation Z such that $V = W \oplus Z$.* [1]

Theorem 4.8 (Characterization of Finite Fields)**.** *If $\mathbb{F}$ is a finite field, then its order is p^k for some prime p which is equal to its characteristic. Furthermore, for every prime p and integer k there exists exactly one field of order p^k up to isomorphism.* [3]

Corollary 4.9. *Let M be an elementary abelian p-group of order p^k. Then there is a group isomorphism $M \to \mathbb{F}_{p^k}$ from M to the unique finite field of order p^k.* [3]

Now we have all the machinery we need to prove Theorem 4.10. One might be able to guess the overarching structure of this proof by the results we have introduced. Namely, we will consider $N^{G/N}$ as a vector space over the finite field $\mathbb{F}_{p^k}$ and then use representation theory to examine the structure of the wreath product.

Theorem 4.10 (Weak Schur-Zassenhaus)**.** *If $|G| = p^k b$ where b is coprime to p and N is an elementary abelian normal p-subgroup of G with order p^k, then G contains a subgroup of order b. Furthermore, all subgroups of order b are conjugate.*

Proof. Consider $W = N \wr (G/N) = N^{G/N} \rtimes (G/N)$ as above. We embed $N^{G/N}$ into W as the elements of the form $\{Nf | f \in N^{G/N}\}$ and we embed G/N into W as the elements of the form $\{k\bar{e} | k \in G/N\}$ where $\bar{e} : G/N \to N$ is the function that sends everything to the identity. Since N is an elementary abelian p-group, by Corollary 4.9 we can think of it as the field $\mathbb{F}_{p^k}$. Then $N^{G/N}$ is a vector space over $\mathbb{F}_{p^k}$ with multiplication by scalar defined as $(nf)(-) = n * f(-)$ (where $*$ denotes multiplication in $\mathbb{F}_{p^k}$) and addition defined as $(f_1+f_2)(-) = f_1(-)+f_2(-)$. Because $N^{G/N}$ is an abelian normal subgroup of W, we can consider an action of $G/N \cong W/(N^{G/N})$ on $N^{G/N}$ by conjugation in W, or equivalently as defined in the wreath product. $N^{G/N}$ is therefore a representation of G/N under this action. Consider the embedding $\phi : G \to W$ as defined in (4.5) and in this way we embed N into W. Note that G has elements in every coset of $W/(N^{G/N})$. Therefore, since N is normal in G it must be normal in W. By normality N is stable under the action of G/N and is therefore a subrepresentation of $N^{G/N}$.

Now, since p as the characteristic of $\mathbb{F}_{p^k}$ does not divide the order of our group $|G/N|$ by assumption, we can apply (generalized) Maschke's theorem to find a subrepresentation C of $N^{G/N}$ such that $N^{G/N} = N \oplus C$. Since C is a vector subspace of $N^{G/N}$ it is a subgroup of $N^{G/N}$ such that $N^{G/N} \cong N \times C$. Furthermore since C is stable under the action of G/N it is a normal subgroup of W.

Using this isomorphism we get $W \cong (N \times C) \rtimes (G/N)$, implying that $W/C \cong N \rtimes (G/N)$. We can see that $|W/C| = |G|$ so we may guess that these groups are isomorphic.

Let $\psi : W \to W/C$ be the quotient map and consider their composition $\psi \circ \phi : G \to W/C$. Since ϕ and ψ are group homomorphisms, so is $\psi \circ \phi$, and we claim that $\psi \circ \phi$ gives us the isomorphism between G and W/C. Since we already know that G and W/C have the same cardinality, it suffices to show that $\psi \circ \phi$ has trivial kernel. Suppose that $\psi \circ \phi(g) = e$ for some $g \in G$. We immediately have that $gN = N$ so $g \in N$. Finally, note that $g \in N$ does not map to identity in W/C unless $g = e$. Therefore $\psi \circ \phi$ is an isomorphism and $G \cong W/C$. Since we have previously identified $W/C \cong N \rtimes (G/N)$, we can identify G/N as a subgroup of order b in G. This completes the proof of the existence of such a subgroup.

Now we prove the conjugacy part. Suppose that $H_1, H_2 \in G$ are two complement subgroups to N. These both generate coset representation functions $\tau_1 : G/N \to H_1$ and $\tau_2 : G/N \to H_2$ which are also group homomorphisms. We can consider ϕ_1 and ϕ_2 as the maps $G \to W$ generated by τ_1, τ_2 as in Theorem 4.4. Because τ_1, τ_2 are group homomorphisms and N is abelian we can check that $\phi_1(H_1) = \phi_2(H_2) = G/N$ via direct computation. By Lemma 4.6, $\phi_1(G)$ and $\phi_2(G)$ are conjugate in W. By transitivity this gives that $\phi_1(H_1)$ and $\phi_1(H_2)$ are conjugate in W. Therefore $\psi \circ \phi_1(H_1)$ and $\psi \circ \phi_1(H_2)$ are conjugate in W/C. Since $W/C \cong G$, this means H_1 and H_2 are conjugate in G. □

5. Hall's Theorems

Hall's theorems provide a very nice generalization to the Sylow's theorems for the case of solvable groups. Similar to how the Sylow's theorems guarantee the existence of subgroups of prime power order, Hall's theorems guarantee the existence of subgroups of certain orders in the special case of solvable groups. Their proof makes use of Burnside's theorem as well as Schur-Zassenhaus, however there is a much more group theoretic flavor to these proofs. Most of the theorems are proven by supposing the existence of a "minimal criminal". This is a group of minimal order satisfying the conditions of the theorem but not satisfying the statement. The theorem is then proven by deriving a contradiction. This technique is essentially equivalent to induction on the order of G. For these sections I will use H^g to denote the subgroup gHg^{-1} and (a, b) to denote the greatest common divisor of a and b.

5.1. First Theorem.

We wish to establish a sufficient condition for a group to be solvable. We do this by combining Burnside's theorem with a group theoretic argument.

Definition 5.1. A *Hall subgroup* H of a group G is a subgroup with $(|H|, [G : H]) = 1$.

Definition 5.2. Given a set of primes π dividing the order of group G, a *π-subgroup* is a subgroup whose order is divisible by exactly the primes in π.

Definition 5.3. A *Hall π-subgroup* is a Hall subgroup which is also a π-subgroup.

Lemma 5.4. *Suppose H and K are subgroups of G and $([G : H], [G : K]) = 1$. Then $G = HK$ and $[G : H \cap K] = [G : K][G : H]$.*

Proof. Since $H \cap K$ is a subgroup of both H and K, $[G : H \cap K]$ is a multiple of $[G : H]$ and $[G : K]$. Therefore by the coprime condition $[G : H \cap K]$ is a multiple of $[G:H][G:K]$. Therefore:

$$\begin{aligned}\frac{|G|}{|H|}\frac{|G|}{|K|} &\leq \frac{|G|}{|H \cap K|} \\ |G| &\leq \frac{|H||K|}{|H \cap K|} \\ &= |HK|\end{aligned}$$

Therefore $G = HK$ and all our inequalities are equalities. By the first line we are done. □

Lemma 5.5. *If $H, K, L \subseteq G$ are solvable and $[G:H]$, $[G:K]$, and $[G:L]$ are coprime, then G is solvable.*

Proof. Suppose for the sake of contradiction that G is a not-solvable group of minimal order satisfying the conditions of the theorem. We claim that G must have a solvable normal subgroup N. Assuming the existence of N let $\phi : G \to G/N$ be the standard quotient map and let $\overline{G}, \overline{H}, \overline{K}, \overline{L}$ be the images of G, H, K, L under this map. Note that $\overline{G}, \overline{H}, \overline{K}, \overline{L}$ satisfy the conditions of the theorem and therefore by minimality of G, $\overline{G}$ is solvable. Then because G/N and N are solvable G must be solvable, a contradiction.

Now we prove the existence of N. Let M be a minimal normal subgroup of H and note that M is an elementary abelian p-group for some prime p. Note that p cannot divide both $[G:K]$ and $[G:L]$ so suppose without loss of generality that p does not divide $[G:K]$. Then by the first Sylow theorem K contains a Sylow p-subgroup P of G. By the second Sylow theorem we can choose some $g \in G$ so that P^g contains M. Therefore M is contained in K^g. Note that by Lemma 5.4 $G = K^g H$. Suppose that $m \in M$ and note for any $kh \in G$ with $k \in K^g$ and $h \in H$ we have $(kh)m(kh)^{-1} = k(hmh^{-1})k^{-1}$. By normality of M in H $(hmh^{-1}) \in M \subseteq K^g$ and therefore $k(hmh^{-1})k^{-1} \in K^g$. Therefore all the conjugates of elements in M are contained in K^g. Now let N be the normal subgroup generated by M. By the preceding argument $N \subseteq K^g$ and is therefore a proper normal subgroup. N is solvable since it is a subgroup of the solvable group K^g. □

Lemma 5.6. *If G is a group with the property that there exist subgroups of order a for each a such that $(a, |G|/a) = 1$, then all Hall subgroups of G have the same property.*

Proof. Suppose that $|G| = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ and H is a subgroup of G. Let $\pi_i = \{p_1, \dots, p_i\}$ and suppose without loss of generality that H is a Hall π_r subgroup for $r \leq n$. By (permutation) symmetry it suffices to show that for each $k \leq r$, H has a Hall π_k-subgroup. Let $\pi = \pi_k \cup \{p_{r+1}, \dots, p_n\}$ and let K be a Hall π-subgroup of G. Note that H and K have coprime indices in G so by Lemma 5.4:

$$\begin{aligned}[G : H \cap K] &= [G:H][G:K] \\ |H \cap K| &= |H||K|/|G| \\ &= p_1^{\alpha_1} \cdots p_k^{\alpha_k}\end{aligned}$$

Hence $H \cap K$ is a Hall π_k-subgroup and we are done. □

Note that using this lemma and an induction or "minimal criminal" approach, we can prove a stronger version of this lemma that applies to all subgroups. However this is not necessary as this lemma is sufficient to prove this next theorem.

Theorem 5.7. *If G is a group with subgroups of order a for each a such that $(a, |G|/a) = 1$, then G is solvable.*

Proof. Suppose that $|G| = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ is a not-solvable group of minimal order satisfying the conditions. If $n \leq 2$ then G is solvable by Burnside's theorem. Otherwise let H, K, L be Hall subgroups of G with indices $p_1^{\alpha_1}, p_2^{\alpha_2}, p_3^{\alpha_3}$ respectively. By Lemma 5.6 each of these subgroups satisfies the conditions of the theorem so by minimality of G they are all solvable. However note that they have coprime indices in G, so by Lemma 5.5 G is solvable, a contradiction. □

5.2. **Second Theorem.** Now that we have proven the first Hall theorem, we seek to prove the converse. This will establish remarkable necessary and sufficient conditions for a group to be solvable.

Theorem 5.8. *If G is solvable, then for any a, b such that $(a, b) = 1$ and $|G| = ab$, G contains a subgroup of order a.*

Proof. Suppose that G is a solvable group of minimal order which does not satisfy the theorem. Then let M be a minimal normal subgroup of G and let π be a set of primes dividing the order of G. Note that M is an elementary abelian p-group for some prime p. We have two cases:

(1) $p \in \pi$. Note that by minimality of G, G/M contains a Hall π-subgroup $\overline{H}$. The lift of $\overline{H}$ into G is a Hall π-subgroup of G and we are done with this case.
(2) $p \notin \pi$. Let π' be the set of primes dividing the order of G but not in π. Let $\overline{H}$ be a Hall π'-subgroup of G/M and let H be its lift into G. Note that $|M|$ and $[H : M]$ are coprime and therefore by Schur-Zassenhaus we can find a complement K of M in H. By construction K is a Hall π-subgroup and we are done.

□

5.3. **Third Theorem.** The final Hall theorem is analogous to the result that all Sylow p-subgroups are conjugate to each other. It states that Hall π-subgroups are conjugate and furthermore any π-subgroup is contained in a Hall π-subgroup. This result gives us a great level of understanding of the structure of solvable groups.

Theorem 5.9. *Suppose that G is a solvable group and π is a set of primes dividing $|G|$. If H is a π-subgroup and K is a Hall π-subgroup, then there is some $g \in G$ such that H is a subgroup of K^g.*

Proof. Suppose that G is a group of minimal order with subgroups H and K satisfying the conditions but not having any $g \in G$ such that $H \subseteq K^g$. Let M be a minimal normal subgroup of G and note it must be an elementary abelian p-group for some prime p. Then $\overline{K} = KM/M$ is a Hall π-subgroup of $\overline{G} = G/M$ and $\overline{H} = HM/M$ is a π-subgroup of $\overline{G}$. By minimality of G there is some $g \in G$ such that $\overline{H} \subseteq \overline{K}^{gM}$. Taking the lifts we have $HM \subseteq K^g M$. We have two cases:

(1) $p \in \pi$. $|K^g M|$ must equal K^g since otherwise $|K^g M|$ is divisible by a higher power of p than K. Therefore $HM \subseteq K^g$ and $H \subseteq K^g$.
(2) $p \notin \pi$. Then H and $K^g \cap (HM)$ are both complements to M in HM. Therefore by Schur-Zassenhaus they are conjugate in HM. Hence for $x \in HM$ we have $H = (K^g \cap (HM))^x \subseteq K^{gx}$.

□

Corollary 5.10. *For a given set of primes π dividing G. All Hall π-subgroups are conjugate.*

Acknowledgments

It is a pleasure to thank my mentor Yuqin Kewang for helping to guide me, helping me choose a topic, and giving feedback to my paper. I would not have been able to write this paper without her help. I would also like to thank Peter May for organizing the math REU and Daniil Rudenko for giving a fantastic series of lectures for the apprentice program. Finally, I would like to thank Justin Campbell for leading me to this topic with his incredible lectures on representation theory.

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