SUM OF TWO SQUARES

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Abstract. We will first recall the work of B. Riemann on the meromorphic continuation of Riemann zeta function, which is ultimately related to the Poisson summation formula of Jacobi theta function.

It turns out that the Fourier expansion of the square of Jacobi theta function is related to the following classical problem in number theory:

- In how many different ways can one represent a positive integer n as the sum of two squares?

Let this number be $r(n)$. C. Jacobi first expressed $r(n)$ as generalized divisor sums which is easy to compute. We will establish the result of Jacobi via an identity between the square of Jacobi theta function and a suitable Eisenstein series.

The identity between theta function and Eisenstein series was later generalized by C. Siegel, polished by A. Weil, and refined by S. Kudla-S. Rallis and many others. We hope the talk can be a motivational introduction to the formula of Siegel-Weil and its variants.

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1. Meromorphic continuation of Riemann zeta function

One of the most notorious conjectures in number theory is the *Riemann hypothesis*, which conjectures that the nontrivial zeros of the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \tag{1}$$

all lie on the line $\mathrm{Re}(s)=\frac{1}{2}$.

It is known from L. Euler, which is a straightforward exercise in calculus that the power series in (1) is absolutely convergent only for $\mathrm{Re}(s)>1$ and defines a holomorphic function on the right half plane $\{s\in\mathbb{C}\mid \mathrm{Re}(s)>1\}$. Moreover, using the fact that any positive integer can be uniquely written as a product of prime numbers, $\zeta(s)$ admits the following local-global factorization where p runs over all the prime numbers

$$\zeta(s)=\prod_p(1-p^{-s})^{-1}. \tag{2}$$

In particular one can expect that the analytical properties of $\zeta(s)$ are ultimately related to the distribution of prime numbers.

To make sense of the value of $\zeta(s)$ at $\mathrm{Re}(s)=\frac{1}{2}$, we need to find a meromorphic continuation of $\zeta(s)$, which was first discovered by B. Riemann in [R].

In the following, we will introduce the following variant of $\zeta(s)$:

$$\xi(s):=\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot\zeta(s).$$

Here $\Gamma(s)$ is defined by the following absolutely convergent integral when $\mathrm{Re}(s)>0$

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t,\quad \mathrm{Re}(s)>0.$$

Using integration by parts, one can verify

$$s\cdot\Gamma(s)=\Gamma(s+1) \tag{3}$$

In particular (3) extends $\Gamma(s)$ to a meromorphic function on $s\in\mathbb{C}$ with simple poles at non-positive integers.

Remark 1.0.1. Following the factorization identity (2), $\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})$ can be viewed as the completion of $\zeta(s)$ by the archimedean local factor. For details see [T].

When $\mathrm{Re}(s)>0$, by a simple change of variable

$$\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot n^{-s}=\int_0^\infty e^{-\pi n^2t}t^{\frac{s}{2}-1}\,\mathrm{d}t.$$

Therefore for $\mathrm{Re}(s)>1$,

$$\xi(s)=\pi^{-\frac{s}{2}}\cdot\Gamma(\frac{s}{2})\cdot\zeta(s)=\sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2t}\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t. \tag{4}$$

Set

$$\theta(t)=\sum_{n\in\mathbb{Z}}e^{i\pi n^2t},\quad \mathrm{Im}(t)>0. \tag{5}$$

By direct calculation, the power series defining $\theta(t)$ is absolutely convergent for $\mathrm{Im}(t)>0$ and defines a holomorphic function on the upper-half plane. The function (5) is usually called the *Jacobi theta function*.

In particular, (4) can be transformed to be

$$\xi(s)=\int_0^\infty\left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t,\quad \mathrm{Re}(s)>1. \tag{6}$$

Theta function reminds us the *Poisson summation formula*, which claims that for any $\phi\in\mathcal{S}(\mathbb{R})$, the space of Schwartz functions on $\mathbb{R}$ consisting of rapidly decay smooth functions on $\mathbb{R}$,

$$\sum_{n\in\mathbb{Z}}\phi(n)=\sum_{n\in\mathbb{Z}}\mathcal{F}(\phi)(n).$$

Here we recall that the classical Fourier transform $\mathcal{F}$ gives a bijective map from $\mathcal{S}(\mathbb{R})$ to itself

$$\mathcal{F}(\phi)(x) = \int_{y\in\mathbb{R}} \phi(y)\cdot e^{2\pi i x y}\,\mathrm{d}y, \quad \phi\in\mathcal{S}(\mathbb{R}).$$

By direct calculation, for $\mathrm{Re}(t)>0$, the Fourier transform of $\mathfrak{G}_t(x) = e^{-\pi x^2 t}$ is equal to $t^{-\frac{1}{2}}\cdot\mathfrak{G}_{\frac{1}{t}}(x)$. Therefore

$$\theta(it) = t^{-\frac{1}{2}}\cdot\theta\left(\frac{i}{t}\right), \quad \mathrm{Re}(t)>0. \tag{7}$$

Plugging (7) into (6),

$$\begin{aligned}\xi(s) &= \int_0^\infty \left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t\\ &= \int_1^\infty \left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t + \int_0^1 \left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t.\end{aligned}$$

Notice that for $\mathrm{Re}(s)>1$,

$$\int_0^1 \left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t = \int_0^1 \frac{\theta(it)}{2}\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t - \frac{1}{s}.$$

Changing variable $t\mapsto t^{-1}$ and applying (7) to the identity, we get for $\mathrm{Re}(s)>1$

$$\begin{aligned}\int_0^1 \left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{s}{2}-1}\,\mathrm{d}t &= -\frac{1}{s} + \int_1^\infty \frac{\theta(it)}{2}\cdot t^{\frac{1-s}{2}-1}\,\mathrm{d}t\\ &= -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \left(\frac{\theta(it)-1}{2}\right)\cdot t^{\frac{1-s}{2}-1}\,\mathrm{d}t.\end{aligned}$$

It follows that we arrive at the following formula for $\mathrm{Re}(s)>1$,

$$\xi(s) + \frac{1}{s} + \frac{1}{1-s} = \int_1^\infty \left(\frac{\theta(it)-1}{2}\right)\cdot\left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right)\cdot\frac{\mathrm{d}t}{t}. \tag{8}$$

It turns out that the right hand side of (8) admits a holomorphic continuation to $s\in\mathbb{C}$. To make it more precise, for $t\geq 1$,

$$\left|\frac{\theta(it)-1}{2}\right| = \left|\sum_{n=1}^\infty e^{-\pi n^2 t}\right| \leq e^{-\pi t^2}\cdot\left|\sum_{n=1}^\infty e^{-\pi(n^2-1)t}\right| \leq C\cdot e^{-\pi t^2}$$

for some constant C independent of t. Moreover, (8) is invariant under $s\longleftrightarrow 1-s$. It follows that we arrive at the following theorem

Theorem 1.0.2. $\xi(s)$ admits a meromorphic continuation to $s\in\mathbb{C}$ with simple poles at $s=0$ and $s=1$. Moreover, the following functional equation holds

$$\xi(s) = \xi(1-s).$$

2. Theta function

Following the discussion as above section, the meromorphic continuation and functional equation of $\xi(s)$ is ultimately related to the symmetric property of theta function

$$\theta(t)=\sum_{n\in\mathbb{Z}}e^{i\pi n^2t},\quad t\in\mathcal{H}=\{z\in\mathbb{C}\mid \mathrm{Im}(z)>0\}.$$

We normalize $\theta(t)$ by

$$\Theta(t):=\theta(2t)=\sum_{n\in\mathbb{Z}}e^{2\pi in^2t}$$

The right hand side can be viewed as the Fourier series expansion of $\Theta(t)$. In particular (7) becomes

$$(-2it)^{\frac{1}{2}}\cdot\Theta(t)=\Theta\big(-\frac{1}{4t}\big),\quad t\in\mathcal{H}. \tag{9}$$

Set $q=e^{2\pi it}$. Then

$$\Theta(t)=\sum_{n\in\mathbb{Z}}q^{n^2}.$$

It turns out that

$$\mathfrak{R}(t)=\Theta(t)^2=\left(\sum_{n\in\mathbb{Z}}q^{n^2}\right)=\sum_{k\geq 0}r(k)\cdot q^k$$

where

$$r(k)=\#\{(x,y)\in\mathbb{Z}^2\mid x^2+y^2=k\}.$$

In particular, the Fourier coefficients of $\mathfrak{R}(t)$ carry arithmetic information.

Following (9), $\mathfrak{R}(t)$ admits the following transformation rules

$$(-2it)\cdot\mathfrak{R}(t)=\mathfrak{R}\big(-\frac{1}{4t}\big),\quad t\in\mathcal{H}. \tag{10}$$

Exercise 2.0.1. Prove the following facts:

(1) The special linear group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ via

$$g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})\curvearrowright z\in\mathcal{H}\quad g\cdot z=\frac{az+b}{cz+d}.$$

It enjoys the following property

$$g_1\big((g_2(z))\big)=(g_1\cdot g_2)(z)\quad g_1,g_2\in\mathrm{SL}_2(\mathbb{R}),z\in\mathcal{H};$$

(2) The stabilizer of $i\in\mathcal{H}$ is given by

$$\mathrm{SO}_2(\mathbb{R})=\left\{\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}\mid\theta\in[0,2\pi)\right\}$$

In particular, $\mathcal{H}$ can be identified as the quotient manifold $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$;

By periodicity of $q=e^{2\pi it}$,

$$\mathfrak{R}(t)=\mathfrak{R}(t+1),\quad t\in\mathcal{H}. \tag{11}$$

Using (10),

$$\big(-2i\cdot\frac{t}{4t+1}\big)\cdot\mathfrak{R}\big(\frac{t}{4t+1}\big)=\mathfrak{R}\big(-1-\frac{1}{4t}\big)=(-2it)\cdot\mathfrak{R}(t)$$

and hence

$$\mathfrak{R}(\frac{t}{4t+1}) = (4t+1)\cdot \mathfrak{R}(t). \tag{12}$$

In particular, (11) and (12) can be reformulated as

$$\mathfrak{R}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot t) = \mathfrak{R}(t), \quad \mathfrak{R}(\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \cdot t) = (4t+1)\cdot \mathfrak{R}(t). \tag{13}$$

Let $\Gamma_1(4)$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the following two matrices

$$\Gamma_1(4) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \rangle.$$

Exercise 2.0.2. Establish the following facts:

(1) Show that

$$\Gamma_1(4) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 4 \right\};$$

(2) Show that

$$\mathfrak{R}(\gamma \cdot t) = (ct+d)\cdot \mathfrak{R}(t), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4).$$

A holomorphic function $\mathfrak{F}$ on upper-half plane with Fourier expansion

$$\mathfrak{F}(t) = \sum_{n\geq 0} a_n \cdot q^n, \quad t \in \mathcal{H}, q = e^{2\pi i t}$$

satisfying (13) is called a *modular form* of weight 1 and level $\Gamma_1(4)$. For a detailed introduction to modular forms, see [DS].

3. Eisenstein series and its meromorphic continuation

In this section, we will construct another modular form of weight 1 and level $\Gamma_1(4)$ using the so called *Eisenstein series*.

For any positive integer k, set

$$\mathrm{G}_k^{\chi}(t) = \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2, 4|c} \frac{\chi(d)}{(ct+d)^k}$$

where $\chi : (\mathbb{Z}/4\mathbb{Z})^{\times} \simeq \{\pm 1\}$ the unique nontrivial character and $\chi(d)$ is defined to be zero if $(4,d) = 0$.

For $t \in \mathcal{H}$, the series defining $\mathrm{G}_k^{\chi}(t)$ is absolutely convergent for $k \geq 3$ and hence defines a holomorphic function on $\mathcal{H}$. Moreover $\mathrm{G}_k^{\chi}(t)$ is nontrivial only when k is odd since χ is nontrivial.

By direct calculation, for any $\gamma \in \Gamma_1(4)$,

$$\mathrm{G}_k^{\chi}(\gamma \cdot t) = (ct+d)^k \cdot \mathrm{G}_k^{\chi}(t), \quad t \in \mathcal{H}.$$

Therefore $\mathrm{G}_k^{\chi}(t)$ is a modular form of weight k and level $\Gamma_1(4)$.

In the following, we are going to calculate the Fourier expansion of $\mathrm{G}_k^{\chi}(t)$.

Lemma 3.0.1. Fix $k \geq 3$ an odd integer. For n an integer,

$$\int_0^1 \mathrm{G}_k^{\chi}(t) \cdot e^{-2\pi i n t} \,\mathrm{dRe}(t) = \begin{cases} 2 \cdot L(k,\chi) & n = 0 \\ 0 & n < 0 \\ \frac{4 \cdot L(k,\chi)}{L(1-k,\chi)} \cdot \sum_{d|n} \chi(d) \cdot d^{k-1} & n \geq 1 \end{cases}$$

Here for $t = x + iy \in \mathcal{H}$,

$$\int_0^1 \mathrm{G}_k^{\chi}(t) \cdot e^{-2\pi i n t} \,\mathrm{dRe}(t) = \int_0^1 \mathrm{G}_k^{\chi}(x+iy) \cdot e^{-2\pi i n(x+iy)} \,\mathrm{d}x.$$

Proof. In the absolute convergence region, the integral is equal to

$$\begin{aligned}
&= \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2, 4|c} \chi(d) \cdot \int_0^1 \frac{e^{-2\pi i n t}}{(ct+d)^k} \,\mathrm{dRe}(t) \\
&= \delta_{n,0} \cdot \sum_{0\neq d\in\mathbb{Z}} \frac{\chi(d)}{d^k} + \sum_{0\neq c\in\mathbb{Z}, 4|c} \frac{1}{c^k} \sum_{d\in\mathbb{Z}} \chi(d) \cdot \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})^k} \,\mathrm{dRe}(t) \\
&= \delta_{n,0} \cdot \sum_{0\neq d\in\mathbb{Z}} \frac{\chi(d)}{d^k} + \sum_{0\neq c\in\mathbb{Z}, 4|c} \frac{1}{c^k} \cdot \left(\sum_{d\equiv 1 \mod 4} \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})^k} \,\mathrm{dRe}(t) - \sum_{d\equiv 3 \mod 4} \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})^k} \,\mathrm{dRe}(t) \right).
\end{aligned}$$

Here $\delta_{n,0}$ is the Kronecker delta. By definition

$$\sum_{0\neq d\in\mathbb{Z}} \frac{\chi(d)}{d^k} = 2 \cdot \sum_{n\geq 1} \frac{\chi(d)}{d^k} = 2 \cdot L(k,\chi)$$

where $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ is the Dirichlet L-function. Following the same discussion as Section 1, one can show that $L(s,\chi)$ admits a holomorphic continuation to $s \in \mathbb{C}$.

Now

$$\sum_{d\equiv 1 \mod 4} \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})^k} \,\mathrm{dRe}(t) = \sum_{d \mod \frac{c}{4}} e^{2\pi i n(\frac{4d+1}{c})} \cdot \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t^k} \,\mathrm{dRe}(t)$$

and

$$\sum_{d\equiv 3 \mod 4} \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})^k} \,\mathrm{dRe}(t) = \sum_{d \mod \frac{c}{4}} e^{2\pi i n(\frac{4d+3}{c})} \cdot \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t^k} \,\mathrm{dRe}(t)$$

Exercise 3.0.2. Use residual calculus to show that for

$$\int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t^k} \,\mathrm{dRe}(t) = \begin{cases} \frac{(-2\pi i)^k}{(k-1)!} \cdot n^{k-1} & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

Since

$$\begin{aligned}
\sum_{d \mod \frac{c}{4}} e^{2\pi i n(\frac{4d+1}{c})} - \sum_{d \mod \frac{c}{4}} e^{2\pi i n(\frac{4d+3}{c})} &= \begin{cases} |\frac{c}{4}| \cdot \left(e^{2\pi i \frac{n}{c}} - e^{2\pi i \frac{3n}{c}}\right) & \frac{c}{4} \mid n \\ 0 & \frac{c}{4} \nmid n \end{cases} \\
&= \begin{cases} |\frac{c}{4}| \cdot (2i) & \frac{n}{c} \equiv \frac{1}{4} \mod \mathbb{Z} \\ |\frac{c}{4}| \cdot (-2i) & \frac{n}{c} \equiv \frac{3}{4} \mod \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

it follows that when $n \leq 0$, the resulting integral is equal to

$$\int_0^1 \mathrm{G}_k^{\chi}(t) \cdot e^{-2\pi i n t} \, \mathrm{dRe}(t) = \begin{cases} 2 \cdot L(k,\chi) & n = 0 \\ 0 & n < 0 \end{cases}$$

and when $n \geq 1$,

$$\int_0^1 \mathrm{G}_k^{\chi}(t) \cdot e^{-2\pi i n t} \, \mathrm{dRe}(t) = \sum_{0 \neq c \in \mathbb{Z}, 4|c} \frac{1}{c^k} \cdot \frac{(-2\pi i)^k}{(k-1)!} \cdot n^{k-1} \cdot \begin{cases} |\frac{c}{4}| \cdot (2i) & \frac{n}{c} \equiv \frac{1}{4} \mod \mathbb{Z} \\ |\frac{c}{4}| \cdot (-2i) & \frac{n}{c} \equiv \frac{3}{4} \mod \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{(-1)^{\frac{k+1}{2}+1} \cdot \pi^k}{2^{k-1} \cdot (k-1)!} \cdot \sum_{0 \neq c \in \mathbb{Z}, 4|c} \left(\frac{4n}{c}\right)^{k-1} \cdot \frac{c}{|c|} \cdot \begin{cases} 1 & \frac{4n}{c} \equiv 1 \mod 4 \\ -1 & \frac{4n}{c} \equiv 3 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{(-1)^{\frac{k+1}{2}+1} \cdot \pi^k}{2^{k-2} \cdot (k-1)!} \cdot \sum_{d|n} \chi(d) \cdot d^{k-1}. \tag{14}$$

Exercise 3.0.3. Establish the functional equation for $L(s,\chi)$. Deduce from the functional equation of $L(s,\chi)$ that for a positive integer k,

$$L(k,\chi) = \frac{(-1)^{\frac{k-1}{2}} \cdot \pi^k}{(k-1)! \cdot 2^k} \cdot L(1-k,\chi).$$

Plugging Exercise 3.0.3 into (14), it follows that the lemma has been proved. □

Based on Lemma 3.0.1, the following Fourier expansion holds

$$\frac{\mathrm{G}_k^{\chi}(t)}{2 \cdot L(k,\chi)} = 1 + \frac{2}{L(1-k,\chi)} \cdot \sum_{n \geq 1} \left(\sum_{d|n} \chi(d) \cdot d^{k-1} \right) \cdot q^n, \quad q = e^{2\pi i t}. \tag{15}$$

Although $\mathrm{G}_k^{\chi}(t)$ is no longer absolutely convergent when $k = 1$, the right hand side of (15) is still absolutely convergent when $k = 1$ and defines a holomorphic function on $\mathcal{H}$. A natural question is whether the right hand side of (15) defines a modular form of weight 1 and level $\Gamma_1(4)$ when $k = 1$. This is indeed the case. There are several ways to justify this fact, one way is to consider the following continuous family of Eisenstein series

$$\mathrm{G}_1^{\chi}(t,s) = \sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2, 4|c} \frac{\chi(d)}{(ct+d) \cdot |ct+d|^{2s}}, \quad t \in \mathcal{H} \tag{16}$$

At prior $\mathrm{G}_1^{\chi}(t,s)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{1}{2}$ and satisfies

$$\mathrm{G}_1^{\chi}(\gamma \cdot t, s) = (ct+d) \cdot |ct+d|^{2s} \cdot \mathrm{G}_1^{\chi}(t,s), \quad \gamma \in \Gamma_1(4). \tag{17}$$

We will calculate the Fourier expansion of $\mathrm{G}_1^{\chi}(t,s)$ and show that it admits a holomorphic continuation to $s \in \mathbb{C}$. In particular the value at $s = 0$ is exactly given by the right hand side of (15).

Lemma 3.0.4. For $\mathrm{Re}(s) > \frac{1}{2}$ and an integer n,

$$\int_0^1 \mathrm{G}_1^{\chi}(t,s) \cdot e^{-2\pi i n t} \, \mathrm{dRe}(t) \tag{18}$$

admits a holomorphic continuation to $s \in \mathbb{C}$, with

$$\int_0^1 \mathrm{G}_1^{\chi}(t,s)\cdot e^{-2\pi i n t}\,\mathrm{dRe}(t)\Big|_{s=0} = \begin{cases} 2\cdot L(1,\chi) & n=0 \\ 0 & n<0 \\ \frac{4\cdot L(1,\chi)}{L(0,\chi)}\cdot \sum_{d|n}\chi(d) & n\geq 1 \end{cases}$$

Proof. In the absolute convergence region, the integral (18) is equal to

$$=2\cdot L(1+2s,\chi)\cdot \delta_{n,0} + \sum_{0\neq c\in\mathbb{Z},4|c} \frac{1}{c\cdot |c|^{2s}}\cdot \sum_{d\in\mathbb{Z}}\chi(d)\cdot \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})\cdot |t+\frac{d}{c}|^{2s}}\,\mathrm{dRe}(t).$$

For the second term above, following the same idea as Lemma 3.0.1,

$$\begin{aligned} &=\sum_{0\neq c\in\mathbb{Z},4|c} \frac{1}{c\cdot |c|^{2s}}\cdot \sum_{d\in\mathbb{Z}}\chi(d)\cdot \int_0^1 \frac{e^{-2\pi i n t}}{(t+\frac{d}{c})\cdot |t+\frac{d}{c}|^{2s}}\,\mathrm{dRe}(t) \\ &=\frac{i}{2}\cdot \sum_{0\neq c\in\mathbb{Z},4|c} \frac{|c|^{1-2s}}{c}\cdot \begin{cases} 1 & \frac{4n}{c}\equiv 1 \mod 4 \\ -1 & \frac{4n}{c}\equiv 3 \mod 4 \\ 0 & \text{otherwise} \end{cases} \cdot \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t\cdot |t|^{2s}}\,\mathrm{dRe}(t) \\ &=i\cdot \sum_{d|n}\chi(d)\cdot \left(\frac{4n}{d}\right)^{-2s}\cdot \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t\cdot |t|^{2s}}\,\mathrm{dRe}(t). \end{aligned}$$

It remains to estimate the function

$$s\in\{s\in\mathbb{C}\mid \mathrm{Re}(s)>\frac{1}{2}\}\mapsto \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t\cdot |t|^{2s}}\,\mathrm{dRe}(t). \tag{19}$$

It is clear that when $n=0$ the above function is identically zero. Hence admits a holomorphic continuation to $s\in\mathbb{C}$. When $n\neq 0$, using integration by parts, the following identity holds when $\mathrm{Re}(s_1)+\mathrm{Re}(s_2)$ is sufficiently large

$$\int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t^{s_1}\cdot \bar{t}^{s_2}}\,\mathrm{dRe}(t) = -\frac{1}{2\pi i n}\cdot\left(s_1\cdot \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t^{s_1+1}\cdot \bar{t}^{s_2}}\,\mathrm{dRe}(t) + s_2\cdot \int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t^{s_1}\cdot \bar{t}^{s_2+1}}\,\mathrm{dRe}(t)\right) \tag{20}$$

In particular (20) provides a holomorphic continuation of (19) to $s\in\mathbb{C}$.

Exercise 3.0.5. After holomorphic continuation via (20), show the following identity

$$\int_{\mathbb{R}} \frac{e^{-2\pi i n t}}{t\cdot |t|^{2s}}\,\mathrm{dRe}(t)\Big|_{s=0} = \begin{cases} 0 & n\leq 0 \\ -2\pi i & n\geq 1 \end{cases}$$

The rest of the calculation is the same as Lemma 3.0.1 and we omit. □

Based on Lemma 3.0.4 and (20), the Fourier coefficients

$$n\in\mathbb{Z}\mapsto \int_0^1 \mathrm{G}_1^{\chi}(t,s)\cdot e^{-2\pi i n t}\,\mathrm{dRe}(t)$$

are of moderate growth in variable n. Therefore the Fourier expansion

$$\mathrm{G}_1^{\chi}(t,s) = \sum_{n\in\mathbb{Z}}\left(\int_0^1 \mathrm{G}_1^{\chi}(t,s)\cdot e^{-2\pi i n t}\,\mathrm{dRe}(t)\right)\cdot q^n \quad q=e^{2\pi i t}$$

admits a holomorphic continuation to $s \in \mathbb{C}$ and its value at $s = 0$ is exactly given by the right hand side of (15) when $k = 1$. In particular (17) shows that indeed $\mathrm{G}_1^\chi(t) := \mathrm{G}_1^\chi(t, 0)$ is a modular form of weight 1 and level $\Gamma_1(4)$. After normalization, let us set

$$\mathfrak{E}_k^\chi(t) = \frac{\mathrm{G}_k^\chi(t)}{2 \cdot L(k, \chi)}$$

4. Sum of two squares

The discussion from Section 2 and Section 3 provides two different modular forms of weight 1 and level $\Gamma_1(4)$:

$$\mathfrak{R}(t) = 1 + \sum_{n \geq 1} r(n) \cdot q^n$$

$$\mathfrak{E}_1^\chi(t) = 1 + 4 \cdot \sum_{n \geq 1} \left(\sum_{d|n} \chi(d) \right) \cdot q^n$$

here we use the fact that $L(0, \chi) = \frac{1}{2}$.

It turns out that the space of modular forms of weight 1 and level $\Gamma_1(4)$ is actually of dimension 1 ([DS, §4]). In particular we arrive at the identity

$$\mathfrak{R}(t) = \mathfrak{E}_1^\chi(t).$$

Comparing the Fourier coefficients we arrive at the following theorem, which was first established by C. Jacobi [J] using the theory of elliptic functions.

Theorem 4.0.1. Let n be a positive integer, then

$$r(n) = 4 \cdot \sum_{d|n} \chi(d).$$

As a corollary, let $n = p$ be an odd prime number. Then

$$r(p) = 8 \text{ resp. } 0 \iff p \equiv 1 \text{ resp. } 3 \mod 4. \tag{21}$$

This matches with what we learned from algebraic number theory. One can show that an odd prime is split (resp. inert) in $\mathbb{Z}[i]$ if and only if $p \equiv 1$ (resp. 3) $\mod 4$. This is essentially equivalent to (21).

5. Further reading

As one may already observe, the Fourier expansion of k-th power of theta function is ultimately related to the arithmetic problem of sum of k-squares. Following similar discussion as above, One can try to find a similar expression for

$$r(n, k) = \#\{(x_1, ... x_k) \in \mathbb{Z}^k \mid \sum x_i^2 = k\}.$$

using Eisenstein series. For details see [DS].

C. Siegel generalized Theorem 4.0.1 from higher dimensional aspect [S1][S2][S3][S4][S5], comparing *Siegel theta function* on *Siegel upper-half plane* and the special values of *Siegel Eisenstein series*. Later the work of Siegel was polished by A. Weil using the so called *Weil representations* [A] on metaplectic groups, and extended by S. Kudla-S. Rallis in a series of works [KR1][KR2][KR3]. The identity between theta function and Eisenstein series is usually called the *Sigel-Weil formula*.

The meromorphic continuation of Eisenstein series is the fulcrum in the theory of automorphic forms. It is really unfortunate that I do not have enough time to share my thoughts on this extremely important topic. I will only refer the readers who have interst to the work of R. Langlands [R], the recent work of J. Bernstein and E. Lapid [BL], and many related works.

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