PROGRAM NOTES AND ABSTRACTS FOR WEEK 1

Apprentice Program: Daniil Rudenko

We will cover a variety of topics in algebra, geometry and combinatorics. More details will be given Monday morning.

Probability: Ewain Gwynne

TITLE: Random trees

ABSTRACT: A discrete tree is a connected graph with no cycles. I will explain the bijection between discrete trees with n edges and $2n$-step walks in the non-negative integers which start and end at the origin. I will then discuss the construction of the continuum random tree, the random metric space which describes the large-scale behavior of uniform random trees with n edges as n goes to infinity. I will end by discussing the random graph obtained by gluing two discrete random trees together, which is connected to some (seemingly very difficult) open problems in the theory of Liouville quantum gravity. I will try to keep background knowledge to a minimum — a basic understanding of probability and metric spaces at the introductory undergraduate level should be sufficient.

Analysis: Elliot Cartee and Beniada Shabani

TITLE (First two talks: Elliot Cartee): A Brief History of Optimal Control Theory

ABSTRACT: In 1696, Johann Bernoulli challenged the leading mathematicians of the time to find a solution the Brachistochrone problem: "Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time/./" We will begin by discussing the history surrounding this problem, as well as Johann Bernoulli's own solution. Next we will trace out the history of optimal control theory by introducing and examining more modern tools for solving this problem, ranging from the Euler-Lagrange equations up to the Pontryagin Maximum Principle. A Calculus Sequence should suffice for understanding most of the concepts.

TITLE (Friday talk; Beniada Shabani): Hamilton-Jacobi equations

ABSTRACT: Hamilton-Jacobi equations are first order differential equations that arise naturally in optimal control theory, finance, game theory, physics, imaging, etc. In the first talk we will formally derive the equations from a deterministic control problem, using Bellman's dynamic programming principle. Next we describe a different approach to Hamilton-Jacobi equations in terms of calculus of variations and derive the Hopf-Lax formula for the solution of the problem. The final part will be dedicated to a weaker notion of solutions called the viscosity solutions. A Calculus sequence should suffice for understanding most of the concepts.

Number Theory: Matt Emerton and Zijian Yao

TITLE: Dirichlet's theorem on the infinitude of primes in arithmetic progression.

ABSTRACTS:

Tuesday (Emerton): We will explain Euler's analytic approach to the proof of infinitude of primes. We will then explain Dirichlet's approach to his theorem, which builds on Euler's ideas, but uses a key extra piece of theory: finite Fourier theory. This reduces Dirichlet's theorem to a key non-vanishing result: that the value of $s = 1$ of non-trivial L-functions is non-zero.

Wednesday (Emerton): After tying up whatever loose ends are left over from Tuesday :) , we will begin to explain the proof of the non-vanishing result. We will reduce to checking non-vanishing in certain key cases: L-functions of primitive quadratic characters.

Thursday (Yao): We discuss non-vanishing of L-functions for primitive quadratic characters. Although elementary proofs are possible, the non-vanishing holds for a deeper reason: it is a particular manifestation of the so-called class number formula. We will say something about this if time permits.

Special talk: Trevor Hyde

TITLE: Triangles, lattices, and elliptic curves

ABSTRACT: What could be simpler than a triangle? In this talk I'll explain how the space of all triangles (think: easy) is naturally equivalent to the space of all two dimensional lattices (think: hard) and to the space of all elliptic curves (think: harder!)

Geometry: Michael Klug

TITLE: Circle packings

ABSTRACT: What could be more central than a circle? Laying a bunch of nonoverlapping coins out on a table, we obtain a planar embedding of a graph with vertices at the centers and edges between tangent circles. We will prove the Koebe-Andreev-Thurston circle packing theorem which states that all such graphs can be realized through this construction and that for triangulations this realization is unique up to Möbius transformations.

Logic: Maryanthe Malliaris (Zoom, but in classroom)

TITLE: Complexity and randomness

ABSTRACT: The model-theoretic random graph is a very interesting object, both simple and complicated. Its appearance "in nature" in other mathematical objects can also be an indicator of simplicity or complexity. These three lectures will be broadly about such questions, starting with definitions and continuing to perhaps a few open questions. Interested people may also wish to consult the following recent paper: `https://nzjmath.org/index.php/NZJMATH/article/view/134`.

Algebraic Topology: Peter May and others

Work in progress: for now, here is an updated summary from last year. There will be talks and more informal get togethers in algebraic topology and related areas throughout the REU.

TITLE: Coambiguous concepts

ABSTRACT: We will start with some categorical linguistics, introducing the categorical language required for comparisons between subjects and comparisons between concepts. "Coambiguous concepts" are different definitions that non-obviously have the same or equivalent content. The idea leads to seriously interesting comparisons, either relating different areas of mathematics or giving alternative perspectives on a single area. The first week, we will introduce the categorical language that allows us to make such comparisons. Don't be put off by the undefined terms below!

Example: Finite T_0 topological spaces and finite posets are isomorphic concepts By contrast, what do we mean by equivalent concepts?

Example: Finite dimensional real vector spaces and their maps are equivalent to finite real matrices and matrix multiplication

Example: Finite sets are equivalent to the set of sets $\mathbf{n} = \{1, \cdots, n\}$

Example: Groups are equivalent to free groups modulo relations

Generalization next week: Monads and their algebras; the categorical "monadicity theorem".

TITLE: Homotopically coambiguous definitions

More deeply, what do we mean by concepts that are homotopically coambiguous, by which we mean that different definitions give homotopically equivalent concepts?

Example: Topological spaces, simplicial sets, small categories, and posets are homotopically equivalent concepts.

Example: Simplicial abelian groups and chain complexes are homotopically equivalent concepts.

Possible topics to be explored in more depth later on

Topic 1: Finite spaces and larger contexts (book in progress)

Topic 2: An introduction to stable homotopy theory and spectra

Slide talk: `https://www.youtube.com/watch?v=vRsrCNLkSA0`

Topic 3: An introduction to equivariant homotopy and cohomology theory

Slide talk: `https://www.cornell.edu/video/peter-may-equivariant-cohomology`

Topic 4: Operads and their algebras

Old new example: n-connective spaces are homotopically equivalent to "monadically coequalized" n-fold suspensions of E_n-spaces.

Old new example: Connective spectra are homotopically equivalent to "monadically coequalized" suspensions of E_∞-spaces.