

Finite Spaces and Larger Contexts

J. P. May and Elle Pishavar

Contents

Introduction	viii
Part 1. Alexandroff spaces, posets, and simplicial complexes	1
Chapter 1. Alexandroff spaces and posets	3
1.1. The basic definitions of point set topology	3
1.2. Alexandroff and finite spaces	4
1.3. Bases and subbases for topologies	6
1.4. Operations on spaces	8
1.5. Continuous functions and homeomorphisms	8
1.6. Alexandroff spaces, preorders, and partial orders	10
1.7. Finite spaces and homeomorphisms	11
1.8. Spaces with at most four points	14
Chapter 2. Homotopy equivalences of Alexandroff and finite spaces	17
2.1. Connectivity and path connectivity	17
2.2. Function spaces and homotopies	19
2.3. Homotopy equivalences	21
2.4. Cores of finite spaces	22
2.5. Hasse diagrams and homotopy equivalence	24
Chapter 3. Homotopy groups and weak homotopy equivalences	29
3.1. Homotopy groups	29
3.2. Weak homotopy equivalences	29
3.3. A local characterization of weak equivalences	30
3.4. The non-Hausdorff suspension	30
3.5. 6-point spaces and height	36
Chapter 4. Elementary properties of finite spaces	39
4.1. The Euler characteristic of a finite space	39
4.2. The Möbius function	40
4.3. The fundamental group of a finite space	41
4.4. Covers of finite spaces	45
Chapter 5. Minimal models	51
5.1. the Euler characteristic of finite representations of homology manifolds	55
Chapter 6. Simplicial complexes	57
6.1. Abstract and ordered simplicial complexes	57
6.2. Geometric simplicial complexes	58
6.3. Comparison of abstract and geometric simplicial complexes	61

6.4. Cones and subdivisions of simplicial complexes	61
6.5. The simplicial approximation theorem	64
Chapter 7. A concise introduction to categories	67
7.1. Categories	67
7.2. Functors	69
7.3. Natural transformations	71
7.4. The Yoneda lemma	73
7.5. The fundamental groupoid of a space	74
Chapter 8. The relation between A-spaces and simplicial complexes	75
8.1. The construction of simplicial complexes from A-spaces	75
8.2. The construction of A-spaces from simplicial complexes	76
8.3. Mapping spaces	77
8.4. Simplicial approximation and A-spaces	78
8.5. Contiguity of maps between A-spaces	79
Chapter 9. Homotopy Theory for Subdivision and A-Space Models	83
9.1. Modeling Topological Spaces	84
9.2. Motivation and Colimit Setup	86
9.3. Revisiting contiguity	87
9.4. Modeling Homotopy Classes of Maps	88
9.5. Suspensions and the Group Structure on $\langle X, Y \rangle$	89
9.6. Applications to Homotopy Groups	93
Chapter 10. Characterization of finite manifolds	97
10.1. Pure complexes	98
Chapter 11. Group actions and finite groups	105
11.1. Equivariance and finite spaces	105
11.2. The basic posets and Quillen's conjecture	107
11.3. Some exploration of the posets $\mathcal{A}_p(G)$	110
11.4. The components of $\mathcal{S}_p(G)$	112
Chapter 12. Really finite H -spaces	115
Part 2. Topological spaces, simplicial sets, and categories	121
Chapter 13. Simplicial sets	123
13.1. Motivation for the introduction of simplicial sets	123
13.2. The definition of simplicial sets	125
13.3. Standard simplices and their role	126
13.4. The total singular complex SX and the nerve $N\mathcal{C}$	128
13.5. The geometric realization of simplicial sets	130
13.6. The adjoint relationship between S and T	132
13.7. CW complexes	132
13.8. The fundamental category functor Π	135
13.9. Tensor products of functors?	136
Chapter 14. The big picture: a schematic diagram and the role of subdivision	137

Chapter 15. Subdivision and Properties A , B and C in the Category of Sets	139
15.1. Properties A , B and C of simplicial sets	139
15.2. The definition of the subdivision of a simplicial set	141
15.3. Combinatorial properties of subdivision	144
15.4. Subdivision and Properties A , B , and C of simplicial sets	145
15.5. The proof of Theorem 15.4.1	146
15.6. Isomorphisms of subdivisions	147
15.7. Regularity and non-singularity of simplicial sets and CW complexes	147
15.8. Characterizing nerves of posets	149
Chapter 16. Subdivision and Properties A , B , and C in \mathcal{Cat}	155
16.1. Properties A , B , and C of categories	155
16.2. The definition of the subdivision of a category	155
16.3. Subdivision and Properties A , B , and C of categories	156
16.4. The proof of Theorem 16.3.1	157
16.5. Relations among Sd^* , Sd^c , N , and Π	158
16.6. Horn-filling conditions and nerves of categories	160
16.7. Quasicategories, subdivision, and posets	163
Part 3. Appendices	165
Chapter 17. Cores of Alexandroff spaces	167
Chapter 18. The enumeration of homotopy classes of F -spaces	173
Chapter 19. An outline summary of point set topology	181
19.1. Metric spaces	181
19.2. Compact and locally compact spaces	182
19.3. Further separation properties	184
19.4. Metrization theorems and paracompactness	186
19.5. Finite metric spaces and their embedding into Lebesgue spaces	188
19.6. Finite Metric Spaces	188
19.7. The Problem with Finite Metric Spaces	190
19.8. Embedding Finite Metric Spaces	192
Chapter 20. Reduction methods of finite spaces	205
Chapter 21. Epilogue	213
Bibliography	215

Introduction

A finite space is a topological space that has only finitely many points. At first glance, it seems ludicrous to think that such spaces can be of any interest. In fact, from the point of view of homotopy theory, they are equivalent to finite simplicial complexes. Therefore they support the entire range of invariants to be found in classical algebraic topology. For a striking example that sounds like nonsense, there is a space with six points and infinitely many non-zero homotopy groups. That is like magic: it sounds impossible until you know the trick, when it becomes obvious. We usually restrict attention to finite T_0 -spaces¹, and those are precisely equivalent to finite posets (partially ordered sets). Therefore finite spaces are also of interest in combinatorics. In fact, there is a large and growing literature about finite spaces and their role in other areas of mathematics and science.

My own interest in the subject was aroused by 1966 papers by McCord [50] and Stong [65] that are the starting point of this book. However, I should admit that I came upon these papers while casting about for material to teach in Chicago's large scale REU, which I organize and run. I wanted something genuinely fascinating, genuinely deep, and genuinely accessible, with lots of open problems. Finite spaces provide a perfect REU topic for an algebraic topologist. Most experts in my field know nothing at all about finite spaces, so the material is new even to the experts, and yet it really is accessible to smart undergraduates. This book will feature several contributions made by undergraduates, some from Chicago's REU and some not.

When I first started talking about finite spaces, in the summer of 2003, my interest had nothing at all to do with my own areas of research, which seemed entirely disjoint. However, it has gradually become apparent that finite spaces can be integrated seamlessly into a global picture of how posets, simplicial complexes, simplicial sets, topological spaces, small categories, and groups are interrelated by a web of adjoint pairs of functors with homotopical meaning. The undergraduate may shudder at the stream of undefined terms!

The intention of this book is to introduce the algebraic topology of finite topological spaces and to integrate that topic into an exposition of a global view of a large swathe of modern algebraic topology that is accessible to undergraduates and yet has something new for the experts. A slogan of our REU is that "all concepts will be carefully defined", and we will follow that here. However, proofs will be selective. We aim to convey ideas, not all of the details. When the results are part of the mainstream of other subjects (group theory, combinatorics, point-set topology, and algebraic topology) we generally quote them. When they are particular to our main topics and not to be found on the textbook level, we give complete details.

These notes started out entirely concretely, without even a mention of things like categories or simplicial sets. Chicago students won't stand for oversimplification, and their questions always led me into deeper waters than I intended. They were also impatient with the restriction to finite spaces and finite simplicial complexes, one reason being that as soon as their questions forced me to raise the level of discourse, the restriction to finite things seemed entirely unnatural to them.

The infinite version of finite topological spaces is readily defined and goes back to a 1937 paper of Alexandroff [2]. We call these spaces Alexandroff spaces, and we use the abbreviation A -space for Alexandroff T_0 -space. To go along with this, we

¹The T_0 property means that the topology distinguishes points.

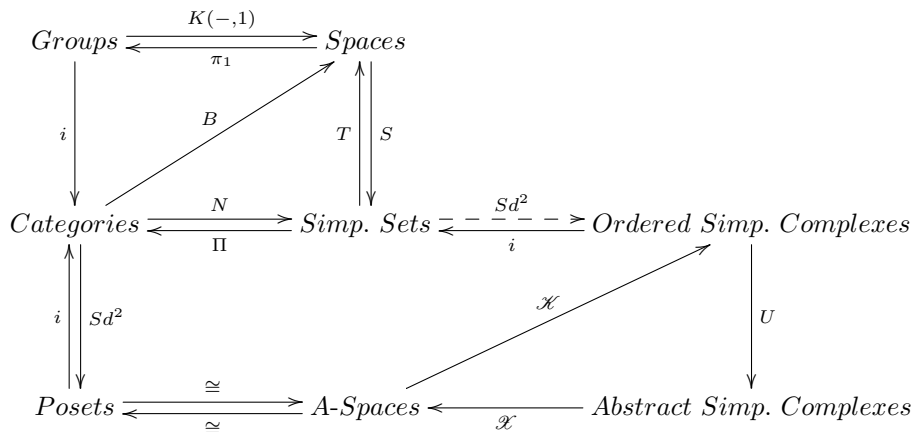
also use the abbreviation F -space for finite T_0 -space. Just as F -spaces are equivalent to finite posets, so A -spaces are equivalent to general posets. Similarly, from the point of view of homotopy theory, F -spaces are equivalent to finite simplicial complexes and A -spaces are equivalent to general simplicial complexes.

Roughly speaking, the first part of the book focuses on the homotopy theory of F -spaces and A -spaces. A central theme is the difference between weak homotopy equivalences and homotopy equivalences. A continuous map $f: X \rightarrow Y$ is a homotopy equivalence if there is a map $g: Y \rightarrow X$ such that the composite $g \circ f$ is homotopic to the identity map of X and the composite $f \circ g$ is homotopic to the identity map of Y . The map f is a weak homotopy equivalence (usually abbreviated to weak equivalence) if for every choice of basepoint $x \in X$ and every $n \geq 0$, the induced map $f_*: \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ is an isomorphism (of sets if $n = 0$, of groups if $n = 1$, and of abelian groups if $n \geq 2$).

Every homotopy equivalence is a weak homotopy equivalence. A map between nice spaces, namely CW complexes, is a homotopy equivalence if it is a weak homotopy equivalence. All of the spaces that one encounters in standard introductions to algebraic topology are nice, so that the distinction seems parenthetical and of minor interest. It is by now very well understood by algebraic topologists that the definitively “right” notion of equivalence is weak equivalence, not homotopy equivalence. However, to get a feel for the strength of the distinction, one needs to see serious examples where the two notions are genuinely different.

The first half of the book offers just such a perspective. The work of Stong makes it very easy to understand homotopy equivalences of finite spaces. The work of McCord relates weak equivalences of Alexandroff spaces to weak equivalences, and therefore homotopy equivalences, of simplicial complexes. As we shall explain, a reinterpretation in terms of finite spaces of a conjecture of Quillen about the poset of non-trivial elementary subgroups of a finite group illuminates precisely this distinction between weak homotopy equivalences and actual homotopy equivalences. Another open problem also illuminates the distinction. The problem of enumerating homotopy equivalences of finite spaces combinatorially has been solved by a pair of Chicago undergraduates, Alex Fix and Stephan Patrias. The problem of enumerating weak homotopy equivalences combinatorially is still open.

The second half of the book guides the reader through the following oversimplified diagram of categories and functors between them ([Chapter 14](#)).



The connections among these categories are remarkably close. It has been understood since the 1950's that topological spaces and simplicial sets can in principle be used interchangeably in the study of homotopy theory. In fact, except that groups only model very special spaces, called $K(\pi, 1)$'s, all of these categories can in principle be used interchangeably in the study of homotopy theory. We'd like people outside algebraic topology to become more aware of these interconnections.

One thing that is largely new is a careful combinatorial analysis of exactly how subdivision ties together the categories of simplicial sets, (small) categories, and posets, alias A -spaces. This is due in large part to Rina Foygel, a recent Chicago PhD and now faculty member in Statistics, and her work is included with her permission. In particular, we give a careful explanation of the classical result that the second subdivision of a suitably well-behaved simplicial set is a simplicial complex and the folklore result that the second subdivision of any (small) category is a poset. One striking result is that, when regarded as a simplicial set, any classical (ordered) simplicial complex is the nerve of a category. As far as I know, that has never before been noticed. We ask the novice not to be intimidated. We will go slow! We ask the expert to be patient. There will be new things along the way.

There are all sorts of possible choices of material and presentation for a book on this general topic, and I'll explain, but not justify, my choices rather flippantly. The main justification is that the REU is supposed to be fun, and so is this book.

It is a standard saying that one picture is worth a thousand words. It is a defect of the (senior) author that he is not good at drawing pictures and is too lazy to learn. That is one among many reasons that this book, although started by the senior author, the one who is writing this introduction, has been joined by his friend and student Elle Pischevar as a coauthor. She has drawn all of the pictures, edited all of the contributions by REU participants, and helped in countless other ways. Mistakes that remain are due to the senior author.

In mathematics, it is perhaps fair to also say that one good definition is worth a thousand calculations. The author likes to make up definitions and to see relations between seemingly unrelated concepts, so we will do lots of that. However, to quote a slogan from a T-shirt worn by one of the author's students, "calculation is the way to the truth". There is a need for more calculational understanding of the subject

here, and the author, being too lazy to compute himself, hopes that readers will be inspired.

In fact, the author's notes on this subject have been online since 2003, and a number of people have been inspired by them. In particular, Gabriel Minian, in Buenos Aires, and his students have followed up problems in my notes. His student Jonathan Barmak wrote a 2009 thesis, now a book [7], that has a good deal of overlap with the first half of this book.² I'll content myself with the basic theory and refer to Barmak's book for more recent advances made in Argentina.

Pedagogically, I've been using this material as a device to offer beginning undergraduates capsule introductions to point-set topology, algebraic topology, and category theory. I've also used the evolution of concepts as a means to help students gain an intuition for abstraction and conceptualization in modern mathematics.³ These twin purposes pervade and guide the exposition.

Elle and I have drawn inspiration from a number of REU papers over the years. With the permission of their authors, we have included several, sometimes edited so as to fit into the context of the book. The topics were often suggested during REUs, and several are original related research. We will highlight contributions as they appear, but here is a list of contributors, with the title of their REU paper, a link to it on the senior author's web page, and a reference to where their contribution appears in the book.

Adam Black.

The Euler Characteristic of Finite Topological Spaces.

<http://math.uchicago.edu/~may/REU2015/REUPapers/Black.pdf> (2015), [Chapter 4].

Cathy (Xi) Chen.

Cores of Alexandroff Spaces.

[http://math.uchicago.edu/~may/REU2015/REUPapers/Chen,Xi\(Cathy\).pdf](http://math.uchicago.edu/~may/REU2015/REUPapers/Chen,Xi(Cathy).pdf) (2015), [Chapter 17].

Alex Fix and Stephen Patrias.

Enumeration of Homotopy Classes of Finite T_0 Topological Spaces.

<https://math.uchicago.edu/~may/VIGRE/VIGRE2008/REUPapers/Fix.pdf> (2008), [Section 2.5, Chapter 18].

Isaac Friend.

Finite Connected H -Spaces are Contractible.

²I'll quote from his introduction. "In 2003, Peter May writes a series of unpublished notes in which he synthesizes the most important ideas on finite spaces until that time. In these articles, May also formulates some natural and interesting questions and conjectures which arise from his own research. May was one of the first to note that Stong's combinatorial point of view and the bridge constructed by McCord could be used together to attack algebraic topology problems using finite spaces. Those notes came to the hands of my PhD advisor Gabriel Minian, who proposed me to work on this subject. May's notes and problems, jointly with Stong's and McCord's papers, were the starting point of our research on the Algebraic Topology of Finite Topological Spaces and Applications."

³Entirely independent of this book, an advertisement for just such a use of the subject of finite spaces as a pedagogical tool has been published by two students of a student of mine [32].

<http://math.uchicago.edu/may/REU2015/REUPapers/Friend.pdf> (2015), [Chapter 12].

Doron L. Grossman-Naples.

Finite Manifolds and Minimal Finite Models of Closed Surfaces.

<https://math.uchicago.edu/may/REU2018/REUPapers/Grossman-Naples.pdf> (2018), [??].

Ryan Hopkins.

Finite Metric Spaces and their Embedding into Lebesgue Spaces.

<http://math.uchicago.edu/may/REU2015/REUPapers/Hopkins.pdf> (2015), [Chapter 19].

Eleanor McSpirit.

Homotopy Theory from Subdivision and A-Space Models.

<http://math.uchicago.edu/may/REU2018/REUPapers/McSpirit.pdf> (2018), [Definition 6.4.5].

Adele Padgett.

An Investigation into Covers of Some Finite Spaces.

<https://math.uchicago.edu/may/REU2015/REUPapers/Padgett.pdf> (2015), [Section 4.4].

David Ramsey.

A Result on Representations of Homology Manifolds by Finite Spaces.

<http://www.math.uchicago.edu/may/VIGRE/VIGRE2010/REUPapers/RamseyD.pdf> (2010), [??].

Jasha Sommer-Simpson.

Barycentric Subdivision and Isomorphisms of Groupoids.

<http://math.uchicago.edu/may/REU2015/REUPapers/Sommer-Simpson.pdf> (2015), [to be added].

Daping Weng.

On Minimal Finite Models.

<http://www.math.uchicago.edu/may/VIGRE/VIGRE2010/REUPapers/Weng.pdf> (2010), [??].

Zhenghui (Sunny) Zhang.

Characterization of Nerves of Posets.

<http://math.uchicago.edu/may/REU2022/REUPapers/Zhang,Zhenghui.pdf> (2022), [Section 15.7].

Part 1

Alexandroff spaces, posets, and simplicial complexes

Alexandroff spaces and posets

1.1. The basic definitions of point set topology

The intuitive notion of a set in which there is a prescribed description of nearness of points is obvious. So is the intuitive notion of a function that takes nearby points to nearby points. However, formulating the “right” general abstract notion of what a “topology” on a set should be and what a “continuous map” between topological spaces should be is not so obvious. Since, intuitively, nearness is thought of in terms of distance, the most immediate way to make the intuition precise is to use distance functions. That leads to metric spaces and the ε - δ description of continuity, which is how we usually think of spaces and maps. Hausdorff came up with a much more abstract and general notion that is now universally accepted.

DEFINITION 1.1.1. A *topology* on a set X consists of a set \mathcal{U} of subsets of X , called the “open sets of X in the topology \mathcal{U} ”, with the following properties.

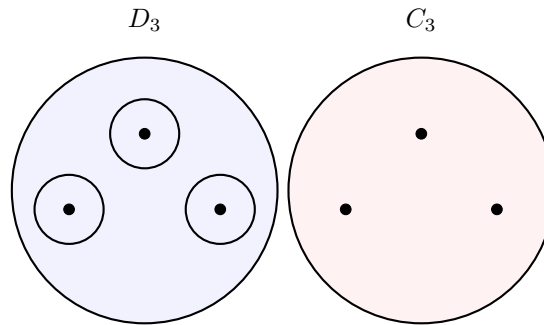
- (i) The empty set \emptyset and the set X are in \mathcal{U} .
- (ii) A finite intersection of sets in \mathcal{U} is in \mathcal{U} .
- (iii) An arbitrary union of sets in \mathcal{U} is in \mathcal{U} .

A *neighborhood* of a point $x \in X$ is an open set U such that $x \in U$.

We write (X, \mathcal{U}) for the set X with the topology \mathcal{U} . More usually, when the topology \mathcal{U} is understood, we just say that X is a topological space. We say that a topology \mathcal{U} is *finer* than a topology \mathcal{V} if every set in \mathcal{V} is also in \mathcal{U} (\mathcal{U} has more open sets). We then say that \mathcal{V} is *coarser* than \mathcal{U} . We have two obvious and uninteresting topologies on any set X .

DEFINITION 1.1.2. The *discrete topology* on X is the topology in which all sets are open. It is the finest topology on X . The *trivial* or *coarse* or *indiscrete* topology on X is the topology in which \emptyset and X are the only open sets. It is the coarsest topology on X . We write D_n and C_n for the discrete and coarse topologies on a set with n elements. These are the largest and the smallest possible topologies (in terms of the number of open subsets).

EXAMPLE 1.1.3. In pictures, we shall display non-empty open sets as the set of points interior to circles drawn on a space. We only draw circles around the smallest open sets, remembering that the union of open sets is open. The following figure depicts D_3 and C_3 , each contained within a large circle for ease of presentation.



DEFINITION 1.1.4. Let X be a topological space. A subset of X is *closed* if its complement is open. The closed sets satisfy the following conditions.

- (i) The empty set \emptyset and the set X are closed.
- (ii) An arbitrary intersection of closed sets is closed.
- (iii) A finite union of closed sets is closed.

We shall make little or no use of the following definition, but it may help make clear how the abstract definitions correspond to common notions in calculus.

DEFINITION 1.1.5. Let A be a subset of a topological space X . The *interior* $\overset{\circ}{A}$ of A is the union of the open subsets of X contained in A . The *closure* \bar{A} of A is the intersection of the closed sets containing A . A point $x \in X$ is a *limit point* of A if every neighborhood of x contains a point $a \neq x$ of A . A is *dense* in X if $\bar{A} = X$.

We shall omit proofs of many standard results that are part of basic point-set topology, such as the next one. While this result is not too hard and can safely be left as an exercise, other omitted proofs will be more substantial.

PROPOSITION 1.1.6. *A point $x \in X$ is in \bar{A} if and only if every neighborhood of x contains a point of A , and \bar{A} is the union of A and the set of limit points of A . The set A is closed if and only if it contains all of its limit points.*

1.2. Alexandroff and finite spaces

It is very often interesting to see what happens when one takes a standard definition and tweaks it a bit. The following tweaking of the notion of a topology is due to Alexandroff [2], except that he used a different name for the notion¹.

DEFINITION 1.2.1. A topological space X is an *Alexandroff space* if the set \mathcal{U} is closed under arbitrary intersections, not just finite ones.

REMARK 1.2.2. The notion of an Alexandroff space has a pleasing complementarity. If X is an Alexandroff space, then the closed subsets of X give it a new topology in which it is again an Alexandroff space. We write X^{op} for X with this opposite topology. Then $(X^{op})^{op}$ is the space X back again.

A space is *finite* if the set X is finite. Since any intersection in a finite space is finite, the following observation is immediate.

LEMMA 1.2.3. *A finite space is an Alexandroff space.*

¹His name was Diskrete Räume, which translates as discrete spaces.

It turns out that a great deal of what can be proven for finite spaces applies equally well more generally to Alexandroff spaces, with exactly the same proofs. When that is the case, we will prove the more general version. However, finite spaces have recently captured people's attention. Since digital processing and image processing start from finite sets of observations and seek to understand pictures that emerge from a notion of nearness of points, finite topological spaces seem a natural tool in many such scientific applications. There are quite a few papers on the subject, although few of much mathematical depth, starting from the 1980's.

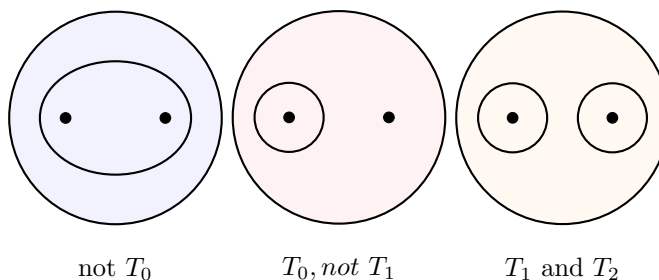
There was a brief early flurry of beautiful mathematical work on this subject. Two independent papers, by McCord and Stong [50, 65], both published in 1966, are especially interesting. We will work through them. We are especially interested in questions that are raised by the union of these papers but are answered in neither. These questions have only recently been pursued. We are also interested in calculational questions about the enumeration of finite topologies.

There is a hierarchy of "separation properties" on spaces, and intuition about finite spaces is impeded by too much habituation to the stronger of them.

DEFINITION 1.2.4. Let (X, \mathcal{U}) be a topological space.

- (i) X is a T_0 -space if for any two points of X , there is an open neighborhood of one that does not contain the other. That is, the topology distinguishes points.
- (ii) X is a T_1 -space if each point of X is a closed subset.
- (iii) X is a T_2 -space, or *Hausdorff space*, if any two points of X have disjoint open neighborhoods.²

EXAMPLE 1.2.5. The following are examples of 2 point spaces with or without the aforementioned separation properties; keep in mind that the smallest open sets are pictured in the interiors of the drawn circles.



LEMMA 1.2.6. *If X is a T_2 -space, then it is a T_1 -space. If X is a T_1 -space, then it is a T_0 -space.*

There are still stronger separation properties. In most of topology, the spaces considered are at least Hausdorff. For example, metric spaces are Hausdorff. We discuss them briefly in the final section. Observe that discrete spaces are clearly Hausdorff. Conversely, the following commonly used property implies that a finite Hausdorff space is discrete.

²The terminology is due to a 1935 paper of Alexandroff and Hopf [3]. The German word for separation is "Trennung", hence the letter T for the hierarchy of separation properties.

PROPOSITION 1.2.7. *Let A be a subset of a Hausdorff space X and let $x \in X$ but $x \notin A$. Then x is a limit point of A if and only if every neighborhood of x contains infinitely many points in A .*

Obviously, intuition gained from thinking about Hausdorff spaces is likely to be misleading when thinking about finite spaces! In fact, the following result shows more generally that Hausdorff Alexandroff spaces are discrete. In fact, there are no interesting spaces that are both Alexandroff and T_1 , let alone T_2 .

LEMMA 1.2.8. *If an Alexandroff space is T_1 , then it is discrete. Therefore an Alexandroff T_1 -space is a T_2 -space.*

PROOF. Every subset of any set is the union of its subsets with a single element. In an Alexandroff space, all unions of closed subsets are closed. In a T_1 -space, all singleton subsets are closed. If both of these conditions hold, every subset is closed. Therefore every subset is open. \square

Thus Alexandroff spaces are Hausdorff, or equivalently T_1 , if and only if they are discrete. They are of no topological interest. In contrast, Alexandroff T_0 -spaces are very interesting. The following warm-up problem might seem a bit difficult right now, but its solution will shortly become apparent.

EXERCISE 1.2.9. Show that a finite T_0 -space has at least one point which is a closed subset.

NOTATION 1.2.10. As in the introduction, we define an F -space to be a finite T_0 -space and an A -space to be an Alexandroff T_0 -space.

1.3. Bases and subbases for topologies

Alexandroff spaces have canonical minimal bases, which we describe in this section. We first recall the notions of a basis and a subbasis for a topology. The idea is that one often has a preferred collection of “small” or canonical open sets, a “basis” from which all other open sets are generated.

DEFINITION 1.3.1. A *basis* for a topology on a set X is a set \mathcal{B} of subsets of X such that

- (i) For each $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$.
- (ii) If $x \in B' \cap B''$ where $B', B'' \in \mathcal{B}$, then there is at least one $B \in \mathcal{B}$ such that $x \in B \subset B' \cap B''$.

The topology \mathcal{U} generated by the basis \mathcal{B} is the set of subsets U such that, for every point $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$. Equivalently, a set U is in \mathcal{U} if and only if it is a union of sets in \mathcal{B} .

In the definition, we did not assume that we started with a topology on X . If we do start with a given topology \mathcal{U} , then it usually admits many different bases. We can easily characterize which subsets of \mathcal{U} give bases.

LEMMA 1.3.2. *Let (X, \mathcal{U}) be a topological space. A subset \mathcal{B} of \mathcal{U} is a basis that generates \mathcal{U} if and only if for every $U \in \mathcal{U}$ and every $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$.*

We can generate bases for topologies from subbases.

DEFINITION 1.3.3. A *subbasis* for a topology on a set X is a set \mathcal{S} of open subsets of X whose union is X ; that is, \mathcal{S} is a *open cover* of X . The set of finite intersections of sets in \mathcal{S} is the basis generated by \mathcal{S} . If (X, \mathcal{U}) is a topological space, a subbasis \mathcal{S} for the topology \mathcal{U} is a subset of \mathcal{U} such that every set in \mathcal{U} is a union of finite intersections of sets in \mathcal{S} .

EXAMPLE 1.3.4. The set of singleton sets $\{x\}$ is a basis for the discrete topology on X . The set of open balls $B(x, r) = \{y \mid d(x, y) < r\}$ is a basis for the topology on a metric space X .

Returning to Alexandroff spaces, we find that such a space has a canonical basis which is minimal in the strong sense that the open sets in the canonical basis are open sets in *any* basis for the topology on X .

DEFINITION 1.3.5. Let X be an Alexandroff space. For $x \in X$, define U_x to be the intersection of the open sets that contain x . Define a relation \leq on the set X by $x \leq y$ if $x \in U_y$ or, equivalently, $U_x \subset U_y$. Write $x < y$ if the inclusion is proper.

LEMMA 1.3.6. *The set of open sets U_x is a basis \mathcal{B} for X . If \mathcal{C} is any other basis, then $\mathcal{B} \subset \mathcal{C}$. Therefore \mathcal{B} is the unique minimal basis for X .*

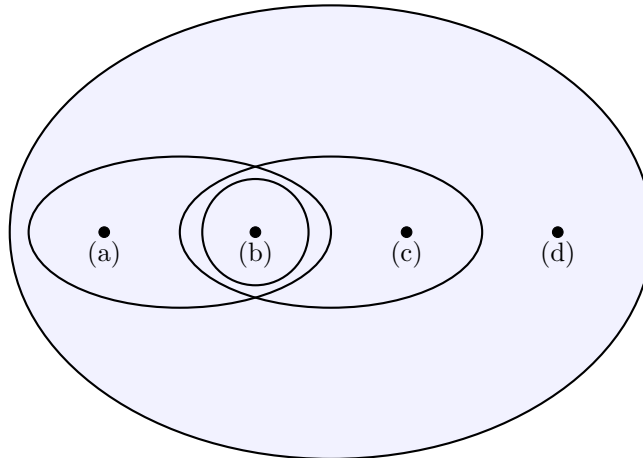
PROOF. The first statement is clear from the definitions. If \mathcal{C} is another basis and $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subset U_x$. This implies that $C = U_x$, so that $U_x \in \mathcal{C}$. \square

As you may have guessed, we can detect whether or not an Alexandroff space is T_0 in terms of its minimal basis. This is formalized as follows.

LEMMA 1.3.7. *Two points x and y in X have the same neighborhoods if and only if $U_x = U_y$. Therefore X is T_0 if and only if $U_x = U_y$ implies $x = y$.*

PROOF. If x and y have the same neighborhoods, then obviously $U_x = U_y$. Conversely, suppose that $U_x = U_y$. If $x \in U$ where U is open, then $U_y = U_x \subset U$ and therefore $y \in U$. Similarly if $y \in U$, then $x \in U$. Thus x and y have the same neighborhoods. \square

EXERCISE 1.3.8. Identify the inclusion relations among U_a , U_b , U_c , and U_d in the following pictured topology.



1.4. Operations on spaces

There are many standard operations on spaces that we shall have occasion to use. We record four of them now and will come back to others later.

DEFINITION 1.4.1. The *subspace topology* on $A \subset X$ is the set of all intersections $A \cap U$ for open sets U of X .

Subspace topologies are defined for injective functions. There is a perhaps less intuitive analogue for surjective functions.

DEFINITION 1.4.2. Let X be a topological space and $q : X \rightarrow Y$ be a surjective function. The *quotient topology* on Y is the set of subsets U such that $q^{-1}(U)$ is open in X .

DEFINITION 1.4.3. The *topology of the union* on the disjoint union $X \amalg Y$ has as open sets the unions of an open set of X and an open set of Y . More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the topology of the union on the disjoint union $\coprod_{i \in I} X_i$ has as open sets the unions of open sets $U_i \subset X_i$. Note that a subset is closed if and only if it intersects each X_i in a closed subset.

DEFINITION 1.4.4. The *product topology* on the cartesian product $X \times Y$ is the topology with basis the products $U \times V$ of an open set U in X and an open set V in Y . More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the product topology on the product set $\prod_{i \in I} X_i$ is the topology generated by the basis consisting of all products $\prod_{i \in I} U_i$ where U_i is open in X_i and $U_i = X_i$ for all but finitely many i .

There is a consistency observation relating the subspace and product topologies.

PROPOSITION 1.4.5. *If $A \subset X$ and $B \subset Y$, then the subspace and product topologies on $A \times B \subset X \times Y$ coincide.*

EXERCISE 1.4.6. Verify the following properties of Alexandroff spaces.

PROPOSITION 1.4.7. *A subspace of an Alexandroff space is an Alexandroff space. A quotient of an Alexandroff space is an Alexandroff space. A disjoint union of Alexandroff spaces is an Alexandroff space. A product of finitely many Alexandroff spaces is an Alexandroff space.*

Here is a thought exercise.

PROBLEM 1.4.8. *Is the product of infinitely many Alexandroff spaces an Alexandroff space?*

1.5. Continuous functions and homeomorphisms

DEFINITION 1.5.1. Let X and Y be spaces. A function $f : X \rightarrow Y$ is *continuous* if $f^{-1}(U)$ is open in X for all open subsets U of Y . A continuous function is often called a *map*.

It suffices that $f^{-1}(U)$ be open for each U in a basis for the topology on Y , or even for each U in a subbasis. The reader is encouraged to use that to verify that the abstract definition of continuity just given coincides with the usual ε - δ definition of continuity on metric spaces (see [Section 19.1](#)). By passage to complements, a function f is continuous if and only if $f^{-1}(C)$ is closed in X for all closed subsets C of Y . This can be reinterpreted in terms of closures (and thus in terms of limit points).

LEMMA 1.5.2. *A function $f : X \rightarrow Y$ is continuous if and only if, for all $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*

LEMMA 1.5.3. *Let A be a subspace of a space X . A continuous function from A to a Hausdorff space Y admits at most one extension to a continuous map $\overline{A} \rightarrow Y$.*

Identity functions and composites of continuous functions are continuous.

LEMMA 1.5.4. *Let X be a space, let $A \subset X$, and give A the subspace topology. Then the inclusion $i : A \rightarrow X$ is a continuous function. If B is a space and $j : B \rightarrow A$ is a function such that $i \circ j$ is continuous, then j is continuous.*

LEMMA 1.5.5. *Let X be a space, let $q : X \rightarrow Y$ be a surjective function, and give Y the quotient topology. Then q is a continuous function. If Z is a space and $r : Y \rightarrow Z$ is a function such that $r \circ q$ is continuous, then r is continuous.*

LEMMA 1.5.6. *Let X_i be spaces and let $\iota_i : X_i \rightarrow \coprod X_i$ be the inclusion. Then ι_i is a continuous function. If Z is a space and $\eta_i : X_i \rightarrow Z$ are continuous functions, then the unique function $\coprod X_i \rightarrow Z$ that restricts to η_i on X_i is continuous.*

LEMMA 1.5.7. *Let X_i be spaces and let $\pi_i : \prod X_i \rightarrow X_i$ be the projection. Then π_i is a continuous function. If Y is a space and $\rho_i : Y \rightarrow X_i$ are continuous functions, then the unique function $Y \rightarrow \prod X_i$ with i^{th} coordinate ρ_i is continuous.*

The four previous propositions state that the subspace, quotient, union, and product topologies satisfy certain “universal properties”. In each of these results, the specified topology is the only topology for which the last statement is true.

Continuity is a local condition on a function.

LEMMA 1.5.8. *A function $f : X \rightarrow Y$ is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

LEMMA 1.5.9. *A function $f : X \rightarrow Y$ is continuous if and only if its restriction to each set in an open cover of X is continuous.*

There is an analogue for finite closed covers.

LEMMA 1.5.10. *A function $f : X \rightarrow Y$ is continuous if and only if its restriction to each set in a finite closed cover of X is continuous.*

In particular, if $X = A \cup B$ where A and B are closed subsets of X , then continuous functions $A \rightarrow Y$ and $B \rightarrow Y$ that agree on $A \cap B$ induce a continuous function $X \rightarrow Y$.

DEFINITION 1.5.11. A continuous bijection $f : X \rightarrow Y$ is a *homeomorphism* if its inverse f^{-1} is also continuous. That is, a homeomorphism is a continuous bijection with a continuous inverse. Equivalently, a map $f : X \rightarrow Y$ is a homeomorphism if there is a map $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$. An *inclusion* or *embedding* is a continuous injection that is a homeomorphism onto its image. We write $X \cong Y$ to indicate that X is homeomorphic to Y .

Intuitively, homeomorphism is the topological counterpart of the algebraic notion of isomorphism. Topologists are interested in properties of spaces that are invariant under homeomorphism. We shall later (Lemma 1.7.1, Theorem 19.2.7) give conditions on X and Y that ensure that a continuous bijection is a homeomorphism.

1.6. Alexandroff spaces, preorders, and partial orders

Here we relate Alexandroff spaces to the combinatorial notions of preorder and partial order.

DEFINITION 1.6.1. A *preorder* on a set X is a reflexive and transitive relation, denoted \leq . This means that $x \leq x$ and that $x \leq y$ and $y \leq z$ imply $x \leq z$. A preorder is a *partial order* if it is antisymmetric, which means that $x \leq y$ and $y \leq x$ imply $x = y$. Then (X, \leq) is called a *poset*. A poset is totally ordered if for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Recall from Definition 1.3.5 that, in an Alexandroff space X , $x \leq y$ means that $U_x \subset U_y$.

LEMMA 1.6.2. *The relation \leq on an Alexandroff space X is reflexive and transitive, so that the relation \leq is a preorder. The relation is also antisymmetric, so that (X, \leq) is a poset, if and only if the space X is T_0 .*

PROOF. The first statement is clear and the second holds by Lemma 1.3.7. \square

LEMMA 1.6.3. *A preorder (X, \leq) determines a topology \mathcal{U} on X with basis the set of all sets $U_x = \{y \mid y \leq x\}$. It is called the *order topology* on X . The space (X, \mathcal{U}) is an Alexandroff space. It is a T_0 -space if and only if (X, \leq) is a poset.*

PROOF. If $x \in U_y$ and $x \in U_z$, then $x \leq y$ and $x \leq z$, hence $x \in U_x \subset U_y \cap U_z$. Therefore $\{U_x\}$ is a basis for a topology. The intersection U of a set $\{U_i\}$ of open subsets is open since if $x \in U$, then $U_x \subset U_i$ for each i and therefore U is the union of these U_x . Therefore (X, \mathcal{U}) is an Alexandroff space with minimal basis $\{U_x\}$. Since $U_x = U_y$ if and only if $x \leq y$ and $y \leq x$, Lemma 1.3.7 implies that (X, \mathcal{U}) is T_0 if and only if (X, \leq) is a poset. \square

We put things together to obtain the following conclusion.

PROPOSITION 1.6.4. *For a set X , the Alexandroff space topologies on X are in bijective correspondence with the preorders on X . The topology \mathcal{U} corresponding to \leq is T_0 if and only if the relation \leq is a partial order.*

REMARK 1.6.5. If \leq is a preorder on X , the opposite preorder is given by $x \leq^{op} y$ if and only if $y \leq x$. The corresponding Alexandroff space is X^{op} .

The real force of the comparison between Alexandroff spaces and preorders comes from the fact that continuous maps correspond precisely to order-preserving functions.

DEFINITION 1.6.6. Let X and Y be preorders. A function $f: X \rightarrow Y$ is order-preserving if $w \leq x$ in X implies $f(w) \leq f(x)$ in Y .

LEMMA 1.6.7. *A function $f: X \rightarrow Y$ between Alexandroff spaces is continuous if and only if it is order preserving.*

PROOF. Let f be continuous and suppose $w \leq x$. Then $w \in U_x \subset f^{-1}U_{f(x)}$ and thus $f(w) \in U_{f(x)}$. This means that $f(w) \leq f(x)$. For the converse, let f be order preserving and let V be open in Y . If $f(x) \in V$, then $U_{f(x)} \subset V$. If $w \in U_x$, then $w \leq x$ and thus $f(w) \leq f(x)$ and $f(w) \in U_{f(x)} \subset V$, so that $w \in f^{-1}(V)$. Thus $f^{-1}(V)$ is the union of these U_x and is therefore open. \square

1.7. Finite spaces and homeomorphisms

In this section we specialize the theory above to finite spaces. Thus let X be a finite space and write $|X|$ for the number of points in X . One might think that finite spaces are uninteresting since they are just finite preorders in disguise, but that turns out to be far from the case.

Topologists are only interested in spaces up to homeomorphism, and we proceed to classify finite spaces up to homeomorphism.

LEMMA 1.7.1. *A map $f: X \rightarrow X$ is a homeomorphism if and only if f is either one-to-one or onto.*

PROOF. By finiteness, one-to-one and onto are equivalent; either means that f is a bijection, as holds for a homeomorphism. If f is a bijection, then f induces a bijection 2^f from the set 2^X of subsets of X to itself. Since f is continuous, if $f(U)$ is open, then so is U . Therefore the bijection 2^f must restrict to a bijection from the topology \mathcal{U} to itself. Alternatively, observe that the function f is a permutation of the set X and the set of permutations of X is a finite group. Therefore f^n is the identity for some n , and the continuous function f^{n-1} is f^{-1} . \square

The previous lemma fails if we allow different topologies on X : there are continuous bijections between different topologies. We proceed to describe how to enumerate the distinct topologies up to homeomorphism. We say that two topologies \mathcal{U} and \mathcal{V} on X are equivalent if there is a homeomorphism $(X, \mathcal{U}) \rightarrow (X, \mathcal{V})$. There are quite a few papers on this enumeration problem in the literature, although some of them focus on enumeration of all topologies, rather than homeomorphism classes of topologies [14, 20, 24, 24, 37, 38, 39, 41, 59, 60]. The difference already appears for two point spaces, where there are four distinct topologies but three inequivalent topologies, that is three non-homeomorphic two point spaces. Here is a table lifted from Wikipedia that gives an idea of the enumeration.

n	Distinct topologies	Distinct T_0 -topologies	Inequivalent topologies	Inequivalent T_0 -topologies
1	1	1	1	1
2	4	3	3	2
3	29	19	9	5
4	355	219	33	16
5	6942	4231	139	63
6	209,527	130,023	718	318
7	9,535,241	6,129,859	4,535	2,045
8	642,779,354	431,723,379	35,979	16,999
9	63,260,289,423	44,511,042,511	363,083	183,231
10	8,977,053,873,043	6,611,065,248,783	4,717,687	2,567,284

Through $n = 9$, a published source for the fourth column is [39]. However, this is not the kind of enumeration problem for which one expects to obtain a precise answer for all n . Rather, one expects bounds and asymptotics. There is a precise formula relating the second column to the first column, but we are really only interested in the last column. In fact, we are far more interested in refinements of the last column that shrink its still inordinately large numbers to smaller numbers of far greater interest to an algebraic topologist.

We shall explain how to reduce the determination of the third and fourth columns to a matrix computation, using minimal bases. For this purpose, it is convenient to describe minimal bases for a topology on X without reference to their enumeration by the elements $x \in X$, since the latter can give redundant information when the space is not T_0 . The following sequence of lemmas applies to the study of general Alexandroff spaces, not necessarily finite.

LEMMA 1.7.2. *A set \mathcal{B} of nonempty subsets of X is the unique minimal basis for an Alexandroff topology \mathcal{U} if and only if the following conditions hold.*

- (i) *Every point of X is in some set B in \mathcal{B} .*
- (ii) *The intersection of two sets in \mathcal{B} is a union of sets in \mathcal{B} .*
- (iii) *If a union of sets B_i in \mathcal{B} is again in \mathcal{B} and if $x \in B \subset \cup B_i$ with $B \in \mathcal{B}$, then $B = B_i$ for some i .*

PROOF. Conditions (i) and (ii) are equivalent to saying that \mathcal{B} is a basis for a topology, which we call \mathcal{U} . We suppose this topology is Alexandroff. Then each B in \mathcal{B} must be a union of sets of the form U_x and each U_x must be in \mathcal{B} by Lemma 1.3.6. If \mathcal{B} is the minimal basis $\{U_y\}$, then each given set B_i in (iii) must be U_y for some $y \in X$. If the union of these U_y is also in \mathcal{B} , then the union must be U_x for some $x \in X$. But then x is in U_y for some y and thus $U_x = U_y$, so that (iii) holds. If \mathcal{B} is a possibly larger basis, we still have that any open set B is a union of sets U_y . If that union is in \mathcal{B} and not of the form U_x for any x , then $\mathcal{B} \setminus \{B\}$ is still a basis, so that \mathcal{B} is not minimal. \square

This result implies the following relationships between minimal bases and subspaces, quotients, disjoint unions, and products of Alexandroff spaces.

LEMMA 1.7.3. *If A is a subspace of X , the minimal basis of A consists of the intersections $A \cap U$, where U is in the minimal basis of X .*

LEMMA 1.7.4. *If Y is a quotient space of X with quotient map $q: X \rightarrow Y$, the minimal basis of Y consists of the subsets U of Y such that $q^{-1}(U)$ is in the minimal basis of X .*

LEMMA 1.7.5. *The minimal basis of $X \amalg Y$ is the union of the minimal basis of X and the minimal basis of Y .*

LEMMA 1.7.6. *The minimal basis of $X \times Y$ is the set of products $U \times V$, where U and V are in the minimal bases of X and Y .*

Returning to finite spaces X , we shall show how to enumerate the homeomorphism classes of spaces with finitely many elements. This is meant only to illustrate how such an enumeration problem can be reduced to computationally accessible form. To allow spaces that are not T_0 , the finite number to focus on is not the number of elements in X but rather the number of elements in the minimal basis for the topology on X . These numbers are equal if and only if X is a T_0 -space.

DEFINITION 1.7.7. Consider square matrixes $M = (a_{i,j})$ with integer entries that satisfy the following properties.

- (i) $a_{i,i} \geq 1$.
- (ii) $a_{i,j}$ is $-1, 0$, or 1 if $i \neq j$.
- (iii) $a_{i,j} = -a_{j,i}$ if $i \neq j$.

- (iv) $a_{i_1, i_s} = 0$ if there is a sequence of distinct indices $\{i_1, \dots, i_s\}$ such that $s > 2$ and $a_{i_k, i_{k+1}} = 1$ for $1 \leq k \leq s - 1$.

Say that two such matrices M and N are equivalent if there is a permutation matrix T such that $T^{-1}MT = N$ and let \mathcal{M} denote the set of equivalence classes of such matrices.

THEOREM 1.7.8. *The homeomorphism classes of finite spaces are in bijective correspondence with \mathcal{M} . If the homeomorphism class of X corresponds to the equivalence class of an $r \times r$ matrix M , then r is the number of sets in a minimal basis for X , and the trace of M is the number of elements of X . Moreover, X is a T_0 -space if and only if the diagonal entries of M are all one.*

PROOF. We work with minimal bases for the topologies rather than with elements of the set. For a minimal basis U_1, \dots, U_r of a topology \mathcal{U} on a finite set X , define an $r \times r$ matrix $M = (a_{i,j})$ as follows. If $i = j$, let $a_{i,i}$ be the number of elements $x \in X$ such that $U_x = U_i$. Define $a_{i,j} = 1$ and $a_{j,i} = -1$ if $U_i \subset U_j$ and there is no k (other than i or j) such that $U_i \subset U_k \subset U_j$. Define $a_{i,j} = 0$ otherwise. Clearly (i)–(iv) hold, and a reordering of the basis results in a permutation matrix that conjugates M into the matrix determined by the reordered basis. Thus X determines an element of \mathcal{M} .

If $f: X \rightarrow Y$ is a homeomorphism, then f determines a bijection from the basis for X to the basis for Y . This bijection preserves inclusions and the number of elements that determine corresponding basic sets, hence X and Y determine the same element of \mathcal{M} . Conversely, suppose that X and Y have minimal bases $\{U_1, \dots, U_r\}$ and $\{V_1, \dots, V_r\}$ that give rise to the same element of \mathcal{M} . Reordering bases if necessary, we can assume that they give rise to the same matrix. For each i , choose a bijection f_i from the set of elements $x \in X$ such that $U_x = U_i$ and the set of elements $y \in Y$ such that $V_y = V_i$. We read off from the matrix that the f_i together specify a homeomorphism $f: X \rightarrow Y$. Therefore our mapping from homeomorphism classes to \mathcal{M} is one-to-one.

To see that our mapping is onto, consider an $r \times r$ -matrix M of the sort under consideration and let X be the set of pairs of integers (u, v) with $1 \leq u \leq r$ and $1 \leq v \leq a_{u,u}$. Define subsets U_i of X by letting U_i have elements those $(u, v) \in X$ such that either $u = i$ or $u \neq i$ but $u = i_1$ for some sequence of distinct indices $\{i_1, \dots, i_s\}$ such that $s \geq 2$, $a_{i_k, i_{k+1}} = 1$ for $1 \leq k \leq s - 1$, and $i_s = i$. We see that the U_i give a minimal basis for a topology on X by verifying the conditions specified in [Lemma 1.3.6](#).

Condition (i) is clear since $(u, v) \in U_u$. To verify (ii) and (iii), we observe that if $(u, v) \in U_i$ and $u \neq i$, then $U_u \subset U_i$. Indeed, we certainly have $(u, v) \in U_i$ for all v , and if $(k, v) \in U_u$ with $k \neq u$, then we must have a sequence connecting k to u and a sequence connecting u to i . These can be concatenated to give a sequence connecting k to i , which shows that (k, v) is in U_i . To see (ii), if $(u, v) \in U_i \cap U_j$, then $U_u \subset U_i \cap U_j$, which implies that $U_i \cap U_j$ is a union of sets U_u . To see (iii), if a union of sets U_i is a set U_j , there is an element of U_j in some U_i and then $U_j \subset U_i$, so that $U_j = U_i$. A counting argument for the diagonal entries and consideration of chains of inclusions show that the matrix associated to the topology whose minimal basis is $\{U_i\}$ is the matrix M that we started with. \square

1.8. Spaces with at most four points

We describe the homeomorphism classes of spaces with at most four points, with just a start on taxonomy. Recall from [Definition 1.1.2](#) that D_n and C_n denote the discrete and coarse topologies on an n -element set.

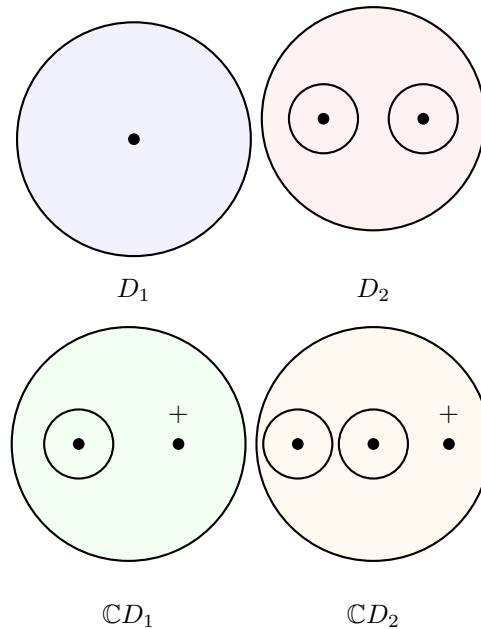
- There is a unique space with one point, namely $C_1 = D_1$.
- There are three spaces with two points, namely C_2 , $P_2 = \mathbb{C}D_1$, and D_2 .

Proper subsets of X are those not of the form \emptyset or X . Since \emptyset and X are in any topology, we often restrict to proper subsets when specifying topologies. The following definitions prescribe the two names for the second space in the short list just given.

DEFINITION 1.8.1. We define certain topologies on a set S_n with n elements. Let $P_n = P_{1,n}$ be the space (unique up to homeomorphism) which has only one proper open set, containing only one point $s \in S_n$; for $1 < m < n$, let $P_{m,n}$ be the space whose proper open subsets are all of the non-empty subsets of a given subset S_m of S_n with m elements.

DEFINITION 1.8.2. For a space X define the *non-Hausdorff cone* by $\mathbb{C}X := X \coprod \{*\}$, where $\{*\}$ is a disjoint added basepoint. We let the open subsets of $\mathbb{C}X$ be the open subsets of X along with the set $X \cup \{*\}$.

EXAMPLE 1.8.3. We observed earlier that $P_{1,2} = \mathbb{C}D_1$. That is the start of a pattern. We claim that $\mathbb{C}D_{n-1}$ is homeomorphic to $P_{n-1,n}$ for any n . We see that by identifying D_{n-1} with $S_{n-1} \subset S_n$ and identifying the cone point $+$ with the point of S_n not in S_{n-1} .



We shall see that $\mathbb{C}X$ is contractible in [Lemma 2.3.2](#) below. This means that it is a point to the eyes of homotopy theory or algebraic topology.

Here is a table of the nine homeomorphism classes of topologies on a three point set $X = \{a, b, c\}$. All of these spaces are disjoint unions of contractible spaces. A space that is not the disjoint union of proper open and closed subspaces is *connected*.

Proper open sets	Name	T_0 ?	connected?
all	D_3	yes	no
a, b, (a,b), (b,c)	$D_1 \amalg P_2$	yes	no
a, b, (a,b)	$P_{2,3} \cong \mathbb{C}D_2$	yes	yes
a	P_3	no	yes
a, (a,b)	$\mathbb{C}P_2 \cong (\mathbb{C}P_2)^{op}$	yes	yes
a, (b,c)	$D_1 \amalg C_2$	no	no
a, (a,b), (a,c)	$(\mathbb{C}D_2)^{op}$	yes	yes
(a,b)	$\mathbb{C}C_2 \cong P_3^{op}$	no	yes
none	$C_3 = D_3^{op}$	no	yes

EXERCISE 1.8.4. Check that the spaces said to be homeomorphic in the above list are in fact homeomorphic.

We tabulate the proper open subsets of the thirty-three homeomorphism classes of topologies on a four point space $X = \{a, b, c, d\}$. That is, these topologies are obtained by adding in the empty set and the whole set. The list is ordered by decreasing number of singleton sets in the topology, and, when that is fixed, by decreasing number of two-point subsets and then by decreasing number of three-point subsets.³

³I thank Mark Bowron for sending me a correction and suggesting a reordering.

1	all
2	a, b, c, (a,b), (a,c), (b,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
3	a, b, c, (a,b), (a,c), (b,c), (a,b,c), (a,b,d)
4	a, b, c, (a,b), (a,c), (b,c), (a,b,c)
5	a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
6	a, b, (a,b), (a,c), (b,d), (a,b,c), (a,b,d)
7	a, b, (a,b), (a,c), (a,b,c), (a,b,d)
8	a, b, (a,b), (a,c), (a,b,c), (a,c,d)
9	a, b, (a,b), (c,d), (a,c,d), (b,c,d)
10	a, b, (a,b), (a,c), (a,b,c)
11	a, b, (a,b), (a,b,c), (a,b,d)
12	a, b, (a,b), (a,b,c)
13	a, b, (a,b), (a,c,d)
14	a, b, (a,b)
15	a, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d)
16	a, (a,b), (a,c), (a,b,c), (a,b,d)
17	a, (a,b), (a,c), (a,b,c)
18	a, (a,b), (c,d), (a,c,d)
19	a, (a,b), (a,b,c), (a,b,d)
20	a, (b,c), (a,b,c), (b,c,d)
21	a, (a,b), (a,b,c)
22	a, (a,b), (a,c,d)
23	a, (b,c), (a,b,c)
24	a, (a,b)
25	a, (a,b,c)
26	a, (b,c,d)
27	a
28	(a,b), (c,d)
29	(a,b), (a,b,c), (a,b,d)
30	(a,b), (a,b,c)
31	(a,b)
32	(a,b,c)
33	none

PROBLEM 1.8.5.

- (1) Determine which of these spaces are T_0 and which are connected.
- (2) Give a taxonomy in terms of explicit general constructions that accounts for all of these topologies. That is, determine appropriate “names” for all of these spaces.
- (3) How many are not contractible spaces or disjoint unions of contractible spaces? (Hint: there is one connected 4-point space that is not contractible; which one of the 33 is it?)

Homotopy equivalences of Alexandroff and finite spaces

2.1. Connectivity and path connectivity

We begin the exploration of homotopy properties of Alexandroff spaces by discussing connectivity and path connectivity. We recall the general definitions. We let $I = [0, 1]$ denote the unit interval with its usual metric topology as a subspace of \mathbb{R} . A *path* in a space X is a map $f : I \rightarrow X$; it is said to connect the points $f(0)$ and $f(1)$.

DEFINITION 2.1.1. Let X be a space.

- (i) X is *connected* if the only subspaces of X that are both open and closed are \emptyset and X .
- (ii) X is *path connected* if any two points of X can be connected by a path.

A path connected space is connected, but not conversely. The following results can be found in any text in point-set topology, such as [54]. They also make good exercises.

LEMMA 2.1.2. *Let Y be a subspace of a space X and let $Y = A \cup B$. Then A and B are both open and closed in Y if and only if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty or, equivalently, A contains no limit point of B and B contains no limit point of A . We then say that $Y = A \cup B$ is a separation of Y . Thus Y is connected if and only if it has no separation.*

The following consequence is used very frequently.

PROPOSITION 2.1.3. *Let $X = A \cup B$ be a separation. If $Y \subset X$ is connected, then Y is contained in either A or B .*

PROPOSITION 2.1.4. *A union of connected or path connected spaces that have a point in common is connected or path connected.*

PROPOSITION 2.1.5. *If $f : X \rightarrow Y$ is a continuous map and X is connected or path connected, then the image of f is connected or path connected.*

For example, I is a connected space, hence the image of a path in X is a connected subspace of X .

PROPOSITION 2.1.6. *Any product of connected or path connected spaces is connected or path connected.*

DEFINITION 2.1.7. Define two equivalence relations \sim and \approx on X .

- (i) $x \sim y$ if x and y are both in some connected subspace of X . A *component* of X is an equivalence class of points under \sim . Let $\pi'_0(X)$ denote the set of components of X .

- (ii) $x \approx y$ if there is a path connecting x and y . A *path component* of X is an equivalence class of points under \approx . Let $\pi_0(X)$ denote the set of path components of X .

If $x \approx y$, then $x \sim y$ since the image of a path connecting x and y is a connected subspace. Therefore each component of X is the union of some of its path components. For nice spaces, components and path components are the same.

DEFINITION 2.1.8. Let X be a space.

- (i) X is *locally connected* if for each $x \in X$ and each neighborhood U of x , there is a connected neighborhood V of x contained in U .
(ii) X is *locally path connected* if for each $x \in X$ and each neighborhood U of x , there is a path connected neighborhood V of x contained in U .

PROPOSITION 2.1.9. Let X be a space.

- (i) X is locally connected if and only if every component of an open subset U is open in X .
(ii) X is locally path connected if and only if every path component of an open subset U is open in X .
(iii) If X is locally path connected, then the components and path components of X coincide.

Now return to a finite or, more generally, Alexandroff space X . At first sight, one might imagine that there are no continuous maps from I to a finite space, but that is far from the case. The most important feature of finite spaces is that they are surprisingly richly related to the “real” spaces that algebraic topologists care about.

LEMMA 2.1.10. Let X be an Alexandroff space. Then each U_x is connected. If X is connected and $x, y \in X$, there is a finite sequence of points z_i , $1 \leq i \leq q$, such that $z_1 = x$, $z_q = y$ and either $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for $i < q$.

PROOF. Suppose that $U_x = A \amalg B$, where A and B are open and disjoint. We may as well assume that x is in A . Then $U_x \subset A$ and therefore $B = \emptyset$ and $U_x = A$. Therefore U_x is connected. Now assume that X is connected. Fix x and consider the set A of points y that are connected to x by some sequence of points z_i , as in the statement. We see that A is open since if z is in A then the open set U_z of points $w \leq z$ is contained in A . We see that A is closed since if y is not connected to x by a finite sequence of points, then neither is any point of U_y , so that the complement of A is open. Since X is connected, it follows that $A = X$. \square

LEMMA 2.1.11. If $x \leq y$ in an Alexandroff space X , then there is a path $p: I \rightarrow X$ connecting x and y .

PROOF. Define $p(t) = x$ if $t < 1$ and $p(1) = y$. We claim that p is continuous. Let V be an open set of X . If neither x nor y is in V , then $p^{-1}(V) = \emptyset$. If x is in V and y is not in V , then $p^{-1}(V) = [0, 1)$. If y is in V , then x is in $U_y \subset V$ since $x \leq y$, hence $p^{-1}(V) = I$. In all cases, $p^{-1}(V)$ is open. \square

PROPOSITION 2.1.12. An Alexandroff space is connected if and only if it is path connected.

PROOF. The previous two lemmas, the second generalized by concatenation of paths to finite sequences as in the first, imply that $x \sim y$ if and only if $x \approx y$. \square

2.2. Function spaces and homotopies

An open cover of a space X is any set of open subsets whose union is all of X . The following notion is fundamental to point-set topology. It is discussed in more detail in [Chapter 19](#).

DEFINITION 2.2.1. A space is *compact* if every open cover has a finite subcover.

For example, a classical result called the Heine-Borel theorem says that a subspace of \mathbb{R}^n is compact if and only if it is closed and bounded.

DEFINITION 2.2.2. Let X and Y be spaces and consider the set Y^X of maps $X \rightarrow Y$. The *compact-open topology* on Y^X is the topology in which a subset is open if and only if it is a union of finite intersections of sets

$$W(C, U) = \{f \mid f(C) \subset U\},$$

where C is compact in X and U is open in Y . This means that the set of all $W(C, U)$ is a subbasis for the topology.

Ignoring topology, for sets X , Y , and Z , functions $f: X \times Y \rightarrow Z$ are in bijective correspondence with functions $\hat{f}: X \rightarrow Z^Y$ via the relation

$$f(x, y) = \hat{f}(x)(y).$$

Returning to topology, and so restricting Z^Y to consist only of the continuous functions $Y \rightarrow Z$, one would like to have that f is continuous if and only if \hat{f} is continuous. The compact-open topology, which at first sight seems to be unmotivated, is designed to minimize conditions on X , Y , and Z which force this conclusion. In fact, there are several different criteria which guarantee the conclusion. We recall one due to Fox [25] which applies to both Alexandroff spaces and metric spaces.

DEFINITION 2.2.3. A space is *first countable* if every point x has a countable neighborhood basis \mathcal{B}_x . This means that if U is a neighborhood of x , then there is a $B \in \mathcal{B}_x$ such that $x \in B \subset U$.

EXAMPLE 2.2.4. An Alexandroff space X is first countable since the singleton set $\{U_x\}$ is a neighborhood basis for x . A metric space is first countable since the ε -neighborhoods $B(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ for positive rational numbers ε form a countable neighborhood basis.

PROPOSITION 2.2.5. *Let X and Y be first countable spaces. Then a function $f: X \times Y \rightarrow Z$ is continuous if and only if $\hat{f}: X \rightarrow Z^Y$ is continuous.*

We shall use function spaces to study the notion of homotopy.

DEFINITION 2.2.6. A *homotopy* $h: f \simeq g$ is a map $h: X \times I \rightarrow Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. Two maps are homotopic, written $f \simeq g$, if there is a homotopy between them.

It is impossible to overstate the importance of this notion. We will be studying the homotopy theory of finite topological spaces. For finite spaces, the use of function spaces allows us to recognize homotopic maps in a very simple way. The first statement of the following result is clear, and the reader should check the second statement from the definitions. The conclusion reduces the determination of whether or not two maps are homotopic to the determination of whether or not they are in the same path component of Y^X .

COROLLARY 2.2.7. *If X is first countable, then homotopies $h: X \times I \rightarrow Y$ correspond bijectively to paths $j: I \rightarrow Y^X$ via $h \leftrightarrow j$ if $h(x, t) = j(t)(x)$. Therefore the homotopy classes of maps $X \rightarrow Y$ are in canonical bijective correspondence with the path components of Y^X .*

When Y is Alexandroff, we can use its preorder to compare maps $X \rightarrow Y$ for any space X .

DEFINITION 2.2.8. If Y is Alexandroff, define the pointwise ordering of maps $X \rightarrow Y$ by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

PROPOSITION 2.2.9. *If Y is Alexandroff, then the intersection V_g of the open sets in Y^X that contain a map g is $\{f \mid f \leq g\}$.*

PROOF. Let $f \in V_g$ and $x \in X$. Since $g \in W(\{x\}, U_{g(x)})$ and $\{x\}$ is compact, $f \in W(\{x\}, U_{g(x)})$, so $f(x) \in U_{g(x)}$ and $f(x) \leq g(x)$. Since x was arbitrary, $f \leq g$. Conversely, let $f \leq g$. Consider any $W(C, U)$ that contains g and let $x \in C$. Then $g(x) \in U$ and, since $f(x) \leq g(x)$, $f(x) \in U_{g(x)} \subset U$. Therefore $f \in W(C, U)$ and f is in all open subsets of Y^X that contain g . \square

Unfortunately, however, V_g need not be open in Y^X in general. This problem is addressed in work of Kukiela [42]. Since our primary interest is in finite spaces, we shall not go into detail, but the following remarks indicate the subtleties here.

REMARK 2.2.10. Michał Kukiela [42] studied the behavior of the compact open topology on Y^X when X and Y are possibly infinite Alexandroff spaces.¹ He showed that Y^X is rarely an Alexandroff space. In particular X^X is never an Alexandroff space if X is infinite, which contradicts an assumption made by Arenas [4]. However, Kukiela proved that Y^X is Alexandroff if X is finite. For any X we have an ordering on the set Y^X , hence we have the Alexandroff topology on Y^X that it determines. However the Alexandroff topology is generally finer (has more open sets) than the compact open topology.

When X and Y are both finite, so is Y^X , and then **Proposition 2.2.9** has the following interpretation.

COROLLARY 2.2.11. *If X and Y are finite, then the pointwise ordering on Y^X coincides with the preordering associated to its compact open topology.*

Here, finally, is our easy way to recognize homotopic maps between finite spaces. Part of the result holds for all Alexandroff spaces.

PROPOSITION 2.2.12. *If X and Y are Alexandroff spaces and $f \leq g$, then $f \simeq g$ by a homotopy h such that $h(x, t) = f(x)$ for all t and all points $x \in X$ such that $f(x) = g(x)$. Conversely, if X and Y are finite and $f \simeq g$, then there is a sequence of maps $\{f = f_1, f_2, \dots, f_q = g\}$ such that either $f_i \leq f_{i+1}$ or $f_{i+1} \leq f_i$ for $i < q$.*

PROOF. For the first statement, we have the path p connecting f to g in Y^X that is specified by $p(t) = f$ if $t < 1$ and $p(1) = g$. By **Lemma 2.1.11**, it is continuous if we give Y^X the Alexandroff topology associated to \leq . Since that topology has more open sets than the compact open topology, by Kukiela's result

¹Kukiela made his contribution as an undergraduate at Nicolaus Copernicus University, in Toruń, Poland. Quoting from an email from him, "my study of Alexandroff spaces was in a great degree inspired by your notes on finite spaces".

just mentioned, it is also continuous if we give Y^X the compact open topology. By [Proposition 2.2.9](#), the corresponding function $X \times I \rightarrow Y$ is also continuous, giving us the claimed homotopy. For the second statement, [Corollary 2.2.7](#) shows that homotopies between maps $X \rightarrow Y$ are paths in Y^X , hence two maps are homotopic if and only if they are in the same path component. Now [Lemma 2.1.10](#) and [Corollary 2.2.11](#) give the conclusion. \square

2.3. Homotopy equivalences

We have seen that enumeration of finite sets with reflexive and transitive relations \leq amounts to enumeration of the topologies on finite sets. We have refined this to consideration of homeomorphism classes of finite spaces. We are much more interested in the enumeration of the homotopy types of finite spaces. We will come to a still weaker and even more interesting enumeration problem later, one which is still unsolved. [unsolvable?!](#)

DEFINITION 2.3.1. Two spaces X and Y are *homotopy equivalent* if there are maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. A space is *contractible* if it is homotopy equivalent to a point.

This relationship can change the number of points. We have a first example.

LEMMA 2.3.2. *If X is a space containing a point y such that the only open (or only closed) subset of X containing y is X itself, then X is contractible. In particular, the non-Hausdorff cone $\mathbb{C}X$ is contractible for any X .*

PROOF. This is a variation on a theme we have already seen twice. Let $*$ denote a space with a single point, also denoted $*$. Define $r: X \rightarrow *$ by $r(x) = *$ for all x and define $i: * \rightarrow X$ by $i(*) = y$. Clearly $r \circ i = \text{id}$. Define $h: X \times I \rightarrow X$ by $h(x, t) = x$ if $t < 1$ and $h(x, 1) = y$. Then h is continuous. Indeed, let U be open in X . If $y \in U$, then $U = X$ and $h^{-1}(U) = X \times I$, while if $y \notin U$, then $h^{-1}(U) = U \times [0, 1)$. The argument when X is the only closed subset containing y is the same. Clearly h is a homotopy $\text{id} \simeq i \circ r$. \square

DEFINITION 2.3.3. A point x of an Alexandroff space X is *maximal* if there is no $y > x$ in X ; minimal points are defined similarly.

COROLLARY 2.3.4. *If X is an Alexandroff space and $x \in X$, then U_x is contractible. If X is finite and has a unique maximal point or a unique minimal point, then X is contractible.*

PROOF. The only open subset of U_x that contains x is U_x itself. If X is finite and x is the unique maximal point in X , then $X = U_x$. If x is the unique minimal point in X , then the only closed set containing x is X . \square

A result of McCord [[50](#), Thm. 4] says that, when studying finite or, more generally, Alexandroff spaces up to homotopy type, there is no loss of generality if we restrict attention to T_0 -spaces, that is, to posets. The proof is based on use of the Kolmogorov quotient of a space.

DEFINITION 2.3.5. Let X be any space. Define an equivalence relation \sim on X by $x \sim y$ if x and y have the same open neighborhoods. The Kolmogorov quotient X_0 of X is the quotient space $X/(\sim)$ obtained by identifying equivalent points. It is a T_0 space. Let $q_X: X \rightarrow X_0$ be the quotient map.

The Kolmogorov quotient satisfies a universal property.

LEMMA 2.3.6. *Let Z be a T_0 -space and $f: X \rightarrow Z$ be a map. Then there is a unique map $f_0: X_0 \rightarrow Z$ such that $f_0 \circ q_X = f$. Therefore, if $f: X \rightarrow Y$ is any map, there is a unique map $f_0: X_0 \rightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$.*

PROOF. Since the topology on Z separates points, f must take equivalent points to the same point. Therefore f factors through a function $f_0: X_0 \rightarrow Y_0$, and f_0 is continuous by the universal property of the quotient topology. \square

THEOREM 2.3.7. *For an Alexandroff space X , the quotient map $q_X: X \rightarrow X_0$ is a homotopy equivalence.*

PROOF. The equivalence relation \sim on X is given by $x \sim y$ if $U_x = U_y$, or, equivalently, if $x \leq y$ and $y \leq x$. The relation \leq on X induces a relation \leq on X_0 . We claim that $q(U_x) = U_{q(x)}$ for all $x \in X$. To see this, observe first that $q^{-1}q(U_x) = U_x$ since if $q(y) = q(z)$ where $z \in U_x$, then $y \in U_y = U_z \subset U_x$. Therefore $q(U_x)$ is open, hence it contains $U_{q(x)}$. Conversely, $U_x \subset q^{-1}(U_{q(x)})$ by continuity and thus $q(U_x) \subset U_{q(x)}$.

We conclude that the quotient topology on X_0 agrees with the topology determined by \leq . It follows that $q(x) \leq q(y)$ if and only if $x \leq y$. Indeed, $q(x) \leq q(y)$ implies $q(x) \in U_{q(y)} = q(U_y)$. Thus $q(x) = q(z)$ for some $z \in U_y$ and $U_x = U_z \subset U_y$, so that $x \leq y$. Conversely, if $x \leq y$, then $U_x \subset U_y$ and therefore $U_{q(x)} \subset U_{q(y)}$, so that $q(x) \leq q(y)$.

To prove that q is a homotopy equivalence, let $f: X_0 \rightarrow X$ be any function such that $q \circ f = \text{id}$. That is, we choose a point from each equivalence class. By what we have just proven, f preserves \leq and is therefore continuous.² Let $g = f \circ q$. We must show that g is homotopic to the identity. We see that g is obtained by first choosing one x_u with $U_{x_u} = U$ for each U in the minimal basis for X and then letting $g(x) = x_u$ if $U_x = U$. Thus $U_{g(x)} = U_x$ and $g(x) \in U_x$, which means that $g \leq \text{id}$. Now Proposition 2.2.12 gives the required homotopy $h: \text{id} \simeq g$. Note that $h(g(x), t) = g(x)$ for all t . \square

We conclude that to classify Alexandroff spaces up to homotopy equivalence, it suffices to classify A -spaces up to homotopy equivalence.

2.4. Cores of finite spaces

Stong [65, §4] has given an interesting way of studying homotopy types of finite spaces. An attempt to extend his results to Alexandroff spaces was made by Arenas [4], but his work had a mistake that was noticed and corrected by Kukiela [42]; see Remark 2.2.10. Since the generalization is not an immediate one, we give proofs for the finite space case in this section, turning to Alexandroff spaces in Chapter 17. However, we give the basic definitions in full generality. We change Stong's language a bit in the following exposition. We first single out an especially nice class of homotopy equivalences.

DEFINITION 2.4.1. Let Y be a subspace of a space X , with inclusion denoted by $i: Y \rightarrow X$. We say that Y is a *deformation retract* of X if there is map $r: X \rightarrow Y$

²I have seen it claimed in an undergraduate thesis (not at the University of Chicago, which does not have undergraduate theses) that Theorem 2.3.7 holds for any space X , not necessarily Alexandroff. However, there need not be a continuous function $f: X_0 \rightarrow X$ such that $q \circ f = \text{id}$.

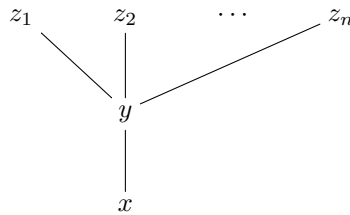
such that $r \circ i$ is the identity map of Y and there is a homotopy $h: X \times I \rightarrow X$ from the identity map of X to $i \circ r$ such that $h(y, t) = y$ for all $y \in Y$ and $t \in I$.

DEFINITION 2.4.2. Let X be a finite space.

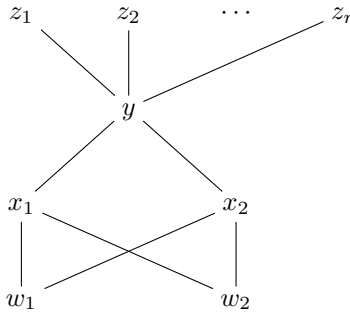
- (a) A point $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$. Note that y is unique if X is T_0 .
- (b) A point $x \in X$ is *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$.
- (c) A point $x \in X$ is a *beat point* if it is either an upbeat point or a downbeat point.

X is a *minimal finite space* if it is a T_0 -space and has no beat points. A *core* of a finite space X is a subspace Y that is a minimal finite space and a deformation retract of X .

REMARK 2.4.3. If we draw a graph of a poset by drawing a line downwards from y to x if $x < y$, we see that, above an upbeat point x , the graph of those edges with y as a vertex looks like



For a more complicated example, both x_1 and x_2 are upbeat points in the poset



Turning the pictures upside down, we see what the graphs below downbeat points look like. The essential point is that a beat point has either exactly one edge connecting to it from above or exactly one edge connecting to it from below.

Intuitively, identifying x and y and erasing the line between them should not change the homotopy type. We say this another way in the proof of the following result, looking at inclusions rather than quotients in accordance with our definition of a core.

THEOREM 2.4.4. *Any finite space X has a core.*

PROOF. With the notations of the proof of [Theorem 2.3.7](#), identify X_0 with its image $f(X_0) \subset X$. The proof of [Theorem 2.3.7](#) shows that X_0 , so interpreted, is a deformation retract of X . Thus we may as well assume that X is T_0 . Suppose that

X has an upbeat point x . We claim that the subspace $X - \{x\}$ is a deformation retract of X . To see this define $f: X \rightarrow X - \{x\} \subset X$ by $f(z) = z$ if $z \neq x$ and $f(x) = y$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \text{id}$. We claim that f preserves order and is therefore continuous. Thus suppose that $u \leq v$. We must show that $f(u) \leq f(v)$. If $u = v = x$ or if neither u nor v is x , there is nothing to prove. When $u = x < v$, $f(u) = y$ and $f(v) = v \geq y$. When $u < x = v$, $f(u) = u < x < y = f(v)$. Now [Proposition 2.2.12](#) gives the required deformation. A similar argument applies to show that $X - \{x\}$ is a deformation retract of X if x is a downbeat point. Starting with X_0 , define X_i from X_{i-1} by deleting one upbeat or downbeat point. After finitely many stages, there are no more upbeat or downbeat points left, and we arrive at the required core. \square

THEOREM 2.4.5. *If X is a minimal finite space and $f: X \rightarrow X$ is homotopic to the identity, then f is the identity.*

PROOF. First suppose that $f \geq \text{id}$. For all x , $f(x) \geq x$. If x is a maximal point, then $f(x) = x$. Let x be any point of X and suppose inductively that $f(z) = z$ for all $z > x$. Then, by continuity, $z > x$ implies $z = f(z) \geq f(x) \geq x$. If $f(x) \neq x$, this implies that x is an upbeat point, contradicting the minimality of X . Therefore $f(x) = x$. By induction, $f(x) = x$ for all x . A similar argument shows that $f \leq \text{id}$ implies $f = \text{id}$. By [Proposition 2.2.12](#), it now follows that the component of the identity map in the finite space X^X consists only of the identity map. That is, any map homotopic to the identity is the identity. \square

COROLLARY 2.4.6. *If $f: X \rightarrow Y$ is a homotopy equivalence of minimal finite spaces, then f is a homeomorphism.*

PROOF. If $g: Y \rightarrow X$ is a homotopy inverse, then $g \circ f \simeq \text{id}$ and $f \circ g \simeq \text{id}$. By the theorem, $g \circ f = \text{id}$ and $f \circ g = \text{id}$. \square

COROLLARY 2.4.7. *Finite spaces X and Y are homotopy equivalent if and only if they have homeomorphic cores. In particular, the core of X is unique up to homeomorphism.*

PROOF. This is immediate since the cores of X and Y are minimal finite spaces that are homotopy equivalent to X and Y . \square

REMARK 2.4.8. In any homotopy class of finite spaces, there is a representative with the least possible number of points. This representative must be a minimal finite space, since its core is a homotopy equivalent subspace. The minimal representative is homeomorphic to a core of any finite space in the given homotopy class.

In an appendix ([Chapter 17](#)) is an exposition on cores of Alexandroff Spaces, included from an REU paper written by Xi (Cathy) Chen in 2015.

2.5. Hasse diagrams and homotopy equivalence

This section is taken from an REU paper written by Alex Fix and Stephen Patrias in 2008. While this portion of their paper is expository, they went on to do original research on the enumeration of homotopy types of finite spaces. That portion of their work will appear in the appendix [Chapter 18](#). Their remarkable conclusion is that, as n grows large, the number of homotopy classes of F -spaces

is asymptotically equivalent to the number of homeomorphism classes of F -spaces. We urge the skeptical reader to read the final paragraph of [Chapter 18](#).

Conceptually, the reason for this is that homotopy equivalence of finite spaces, in contrast to homotopy equivalence between the usual spaces of algebraic topology, is far too strict. The notion of weak homotopy equivalence, studied in the following chapter is the right one.

2.5.1. Hasse diagrams. The correspondence between F -Spaces and partial orders leads to a graphical visualization of F -spaces.

DEFINITION 2.5.1. For a partial order P , we define its associated *Hasse diagram* H , a directed graph which captures all of the relevant order information of P . Let the vertices of H be the points of P and let there be a directed edge from y to x whenever $x < y$ but there is no other vertex z such that $x < z < y$. We then say that y is a predecessor of x and x is a successor of y .

If there is a path $y \rightarrow x_1 \rightarrow \cdots \rightarrow x_k \rightarrow x$ then $y > x_1 > x_2 > \cdots > x_k > x$ so $y > x$. Conversely, if $y > x$ and x is not a successor of y , then we can find z so that $y > z > x$ and by doing this recursively (since the graph is finite), we can find $y > x_1 > \cdots > x_k > x$ so that each step is to a successor, and thus there is a path $y \rightarrow x_1 \rightarrow \cdots \rightarrow x_k \rightarrow x$ in H . From this, we also see that the Hasse diagram is necessarily acyclic, that is, there are no directed cycles $x \rightarrow x_1 \rightarrow \cdots \rightarrow x_k \rightarrow x$ or else we would have $x > x$.

We can also go the other way, from a directed acyclic graph G back to a partial order P , by saying $y \geq x$ in P whenever there is a path (including trivial paths) from y to x in G . However, to do this uniquely, we need the following definitions.

DEFINITION 2.5.2. We say that an edge $y \rightarrow x$ is a *shortcut* in a directed graph G if there is also a path $y \rightarrow x_1 \rightarrow \cdots \rightarrow x_k \rightarrow x$ with at least two edges between x and y . We say that a directed acyclic graph is a *partial order diagram* if it has no shortcuts.

THEOREM 2.5.3. *The above construction of the Hasse diagram gives a bijection from partial orders to partial order diagrams. Furthermore, there is an isomorphism of partial orders between two posets P and Q if and only if there is a graph isomorphism between the associated diagrams H_P and H_Q .*

PROOF. It is easy to check that these two constructions are in fact inverses of each other, so that we have a bijection on objects. Then, a bijection $\sigma : P \rightarrow Q$ is order-preserving if and only if it preserves successors and predecessors, that is it preserves edges in the graph. Therefore σ is an isomorphism of posets if and only if it is also a graph isomorphism of the associated Hasse diagrams. \square

COROLLARY 2.5.4. *We have a bijection between F -space topologies and Hasse diagrams, so that homeomorphism of F -spaces is equivalent to graph isomorphism of their diagrams.*

It is also useful to have a convention for drawing these diagrams, as having an orderly presentation allows both a consistent visual understanding of their structure and an additional handle for computation with these graphs.

DEFINITION 2.5.5. The height $h(X)$ of a poset X is the maximal length k of a chain $x_1 < \cdots < x_k$ in X . If we fix a vertex v , we define the *level* ℓ of v as the maximal length of the chain that ends at v .

We have the following important facts about levels:

- (1) The level of a vertex v is the length of the longest downward path beginning at v .
- (2) There is always an edge from a point v with level ℓ to some v' with level $\ell - 1$.
- (3) There is never an edge from a point v with level ℓ to any v' in level $\ell' \geq \ell$.
- (4) Level 1 consists of precisely the minimal points of the graph.

REMARK 2.5.6. When drawing the Hasse diagram of a poset, we always draw level 1 at the bottom, and each subsequent level ℓ immediately above its predecessor, level $\ell - 1$. Thus, all edges in the graph point downwards in the graph, allowing us to omit specifying the directions of edges.

Our theorems about cores and minimal finite spaces have the following immediate corollary:

COROLLARY 2.5.7. *In order to enumerate all the finite spaces with n points up to homotopy equivalence, it suffices to enumerate the minimal spaces with at most n points up to homeomorphism.*

PROOF. Since any finite space X on n points has a core, and this core is a deformation retract of the original space, X is homotopy equivalent to a minimal space on no more than n points. Thus, there is at least one minimal space in every homotopy equivalence class. Additionally, if there are two minimal spaces X and Y in the same class, then there is a homotopy equivalence $f : X \rightarrow Y$. But then f is a homeomorphism. So if we enumerate the minimal spaces up to homeomorphism, we pick exactly one representative from each homotopy class. \square

2.5.2. Minimal Spaces as Graphs. We now begin the process of converting these topological notions into graph theory, from which actual computations can be made. Primarily, we wish to categorize minimal spaces via a property of the associated Hasse diagram. We start first with a description of upbeat and downbeat points as they appear in the graph.

THEOREM 2.5.8. *A point x in a finite space X is an upbeat point if and only if it has in-degree one in the associated Hasse diagram (that is, it has only one incoming edge). Similarly, x is downbeat if and only if it has out-degree one (it has only one outgoing edge).*

PROOF. Assume that x is upbeat. Then there exists $y > x$ such that for all $z > x$, $z \geq y$. First, we have that y is a successor of x , since there cannot be any z with $y > z > x$. Thus, there is an edge $y \rightarrow x$ in the Hasse diagram. We claim that there is no other edge $y' \rightarrow x$ with $y' \neq y$. If there were, then $y' > x$ so since x is upbeat, $y' > y$. But since $>$ is equivalent to the existence of a path, we have that there exists a path $y' \rightarrow \cdots \rightarrow y$. Hence there is both a path $y' \rightarrow \cdots \rightarrow y \rightarrow x$ and an edge $y' \rightarrow x$ which violates the requirement that the Hasse diagram have no shortcuts. Thus, x has exactly one incoming edge.

Conversely, assume there is exactly one y such that $y \rightarrow x$. Then for any $z > x$ we have that there is a path $z \rightarrow \cdots \rightarrow x$. But since there is only one vertex y such that $y \rightarrow x$, this path must actually be $z \rightarrow \cdots \rightarrow y \rightarrow x$ so there is also a path from z to y so $z \geq y$. Thus x is upbeat.

The proof for the second claim is exactly symmetric. \square

COROLLARY 2.5.9. *A space is minimal if and only if for every vertex in its associated Hasse diagram, the in-degree and out-degree are both not equal to one.*

DEFINITION 2.5.10. Henceforth, we will refer to such a graph as a *minimal graph* for brevity.

We can derive several useful consequences from this classification. For starters, we can begin enumerating the minimal spaces by explicitly constructing graphs which satisfy the above condition (which we will do in [Chapter 18](#)). However, we can also use this theorem to derive additional facts about the structure of minimal graphs which might otherwise be difficult to derive using only topological arguments.

PROPOSITION 2.5.11. *Let G be a minimal graph with at least two vertices. Then each level of G contains at least two vertices.*

PROOF. Assume first that level 1 has exactly one vertex v . Then, since G has at least two vertices, there is some vertex v' in level 2. But every vertex in level 2 has an edge to a vertex in level 1, so $v' \rightarrow v$ is an edge in the graph. But then v' has exactly one downwards edge, contradicting the minimality of G .

Now, assume that some level $\ell > 1$ has exactly one vertex v . This vertex has a neighbor v' on level $\ell - 1$, so $v \rightarrow v'$ is an edge in the graph. Now, assume there is some other $w \neq v$ such that $w \rightarrow v'$ is also an edge in the graph. Since all edges proceed downwards, we have that w is on some level $k > \ell - 1$. Level ℓ has exactly one vertex and w is not it, so $k > \ell$. We claim that this implies that there is in fact a path $w \rightarrow \cdots \rightarrow v$ in the graph, so that the edge $w \rightarrow v'$ is a shortcut of the path $w \rightarrow \cdots \rightarrow v \rightarrow v'$, which is not allowed.

To prove this claim, we induct on k : for a vertex w on level $k = \ell + 1$, w must have a neighbor on level ℓ , so $w \rightarrow v$ is an edge in the graph, and hence also a path. Then, for w on level $k > \ell + 1$, we again have that w has a neighbor on the next lowest level, so there is some w' on level $k - 1$ such that $w \rightarrow w'$ is an edge. By induction, there is a path $w' \rightarrow \cdots \rightarrow v$ in G , so $w \rightarrow w' \rightarrow \cdots \rightarrow v$ is also a path in G . \square

Homotopy groups and weak homotopy equivalences

3.1. Homotopy groups

We recall the definition of the homotopy groups $\pi_n(X, x)$ of a space X at $x \in X$. We shall not give adequate motivation here. This is the first of several places where the first author will advertise his book [48] as a source for a more complete treatment, but in fact all standard textbooks in algebraic topology treat these definitions. For $n = 0$, we define $\pi_0(X)$ to be the set of path components of X , with the component of x taken as a basepoint (and there is no group structure). When $n = 1$, we define $\pi_1(X, x)$, or $\pi_1(X)$ when the basepoint is assumed, to be the fundamental group of X at the point x .

For all $n \geq 0$, $\pi_n(X)$ can be described most simply by considering the standard sphere S^n with a chosen basepoint $*$. One considers all maps $\alpha: S^n \rightarrow X$ such that $\alpha(*) = x$. One says that two such maps α and β are based homotopic if there is a based homotopy $h: \alpha \simeq \beta$. Here a homotopy h is based if $h(*, t) = x$ for all $t \in I$. If $n = 1$, the map α is a loop at x , and we can compose loops to obtain a product which makes $\pi_1(X, x)$ a group. The homotopy class of the constant loop at x gives the identity element, and the loop $\alpha^{-1}(t) = \alpha(1 - t)$ represents the inverse of the homotopy class of α . There is a similar product on the higher homotopy groups, but, in contrast to the fundamental group, the higher homotopy groups are abelian.

A path p from x to x' induces an isomorphism $\pi_n(X, x) \rightarrow \pi_n(X, x')$. On the fundamental group, it maps a loop α to the composite $p \circ \alpha \circ p^{-1}$, where p^{-1} is the reverse path $p^{-1}(t) = p(1 - t)$ from x' to x .

A map $f: X \rightarrow Y$ induces a function $f_*: \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$. One just composes maps α and homotopies h as above with the map f . If $n \geq 1$, f_* is a homomorphism.

3.2. Weak homotopy equivalences

DEFINITION 3.2.1. A map $f: X \rightarrow Y$ is a *weak homotopy equivalence* if

$$f_*: \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $x \in X$ and all $n \geq 0$. If $n = 0$, this means that components are mapped bijectively. Two spaces X and Y are weakly homotopy equivalent if there is a finite chain of weak homotopy equivalences $Z_i \rightarrow Z_{i+1}$ or $Z_{i+1} \rightarrow Z_i$ starting at $X = Z_1$ and ending at $Z_q = Y$.

The definition may seem strange at first sight, but it has gradually become apparent that the notion of a weak homotopy equivalence is even more important in algebraic topology than the notion of a homotopy equivalence. The notions

are related. We state some theorems that the reader can take as reference points. Proofs can be found in [48]. We mention CW complexes in the following result because they give the appropriate level of generality. They will be defined later, in Definition 6.1.1. However, all the reader needs to know here is that the geometric realizations of simplicial complexes, which will be defined in Definition 6.2.6, are special cases of CW complexes.

THEOREM 3.2.2. *A homotopy equivalence is a weak homotopy equivalence. Conversely, a weak homotopy equivalence between CW complexes (for example, between simplicial complexes) is a homotopy equivalence.*

THEOREM 3.2.3. *Spaces X and Y are weakly homotopy equivalent if and only if there is a space Z and weak homotopy equivalences $Z \rightarrow X$ and $Z \rightarrow Y$. Moreover, there is such a Z which is a CW complex.*

That is, the chains that appear in the definition need only have length two. For those who know about homology and cohomology, we record the following result.

THEOREM 3.2.4. *A weak homotopy equivalence induces isomorphisms of all singular homology and cohomology groups.*

3.3. A local characterization of weak equivalences

An essential point in our work, which we will take for granted, is that weak homotopy equivalence is a local notion in the sense of the following theorem. McCord [50] relies on point-by-point comparison with arguments in the early paper [22] which proves the result using quasifibrations. More modern references are [47, 69].

THEOREM 3.3.1. *Let $f: A \rightarrow B$ be a continuous map. Suppose that B has a basis \mathcal{O} such that for each $U \in \mathcal{O}$, the restriction $f: f^{-1}(U) \rightarrow U$ is a weak homotopy equivalence. Then f is a weak homotopy equivalence.*

proof?

3.4. The non-Hausdorff suspension

The suspension is one of the most basic constructions in all of topology. Following McCord [50], we show that it comes in two weakly equivalent versions, the classical one and a non-Hausdorff analogue that preserves finite spaces. For the purposes of this book, we shall use the following unbased variant of the classical suspension.

DEFINITION 3.4.1. Define the *cone* CX of a topological space X to be the quotient space $X \times I / X \times \{1\}$ obtained by identifying $X \times \{1\}$ to a single point, denoted $+$. Define the *suspension* SX of X to be the quotient space obtained from $X \times [-1, 1]$ by identifying $X \times \{1\}$ to a single point $+$ and identifying $X \times \{-1\}$ to another single point, denoted $-$. Thus SX can be thought of as obtained by gluing together the bases of two cones on X . For a map $f: X \rightarrow Y$, define $Sf: SX \rightarrow SY$ by $(Sf)(x, t) = (f(x), t)$.

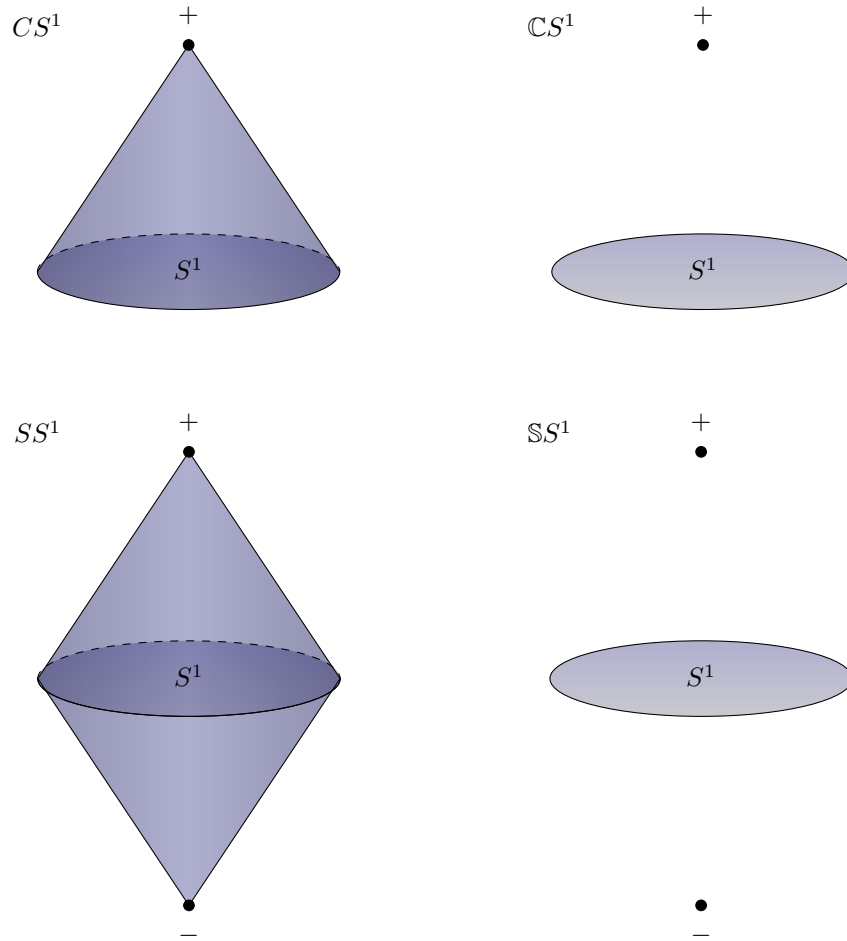
It should be clear that CX is contractible to its cone point $+$. We defined the non-Hausdorff cone $\mathbb{C}X$ by adjoining a new cone point $*$ and letting the proper open subsets of $\mathbb{C}X$ be all of the open subsets of X , and we saw that $\mathbb{C}X$ is contractible. We now change the notation for the cone point $*$ and call it $+$.

DEFINITION 3.4.2. The *non-Hausdorff suspension* of X is defined to be

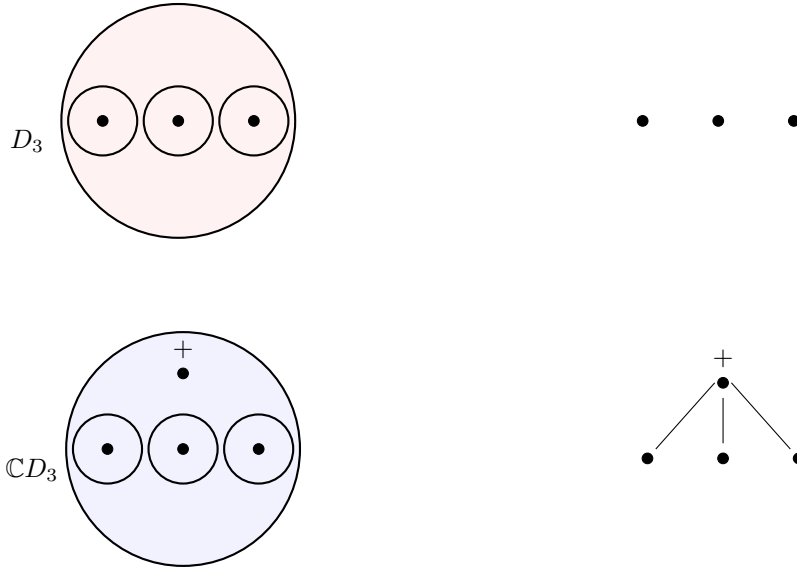
$$\mathbb{S}X := X \amalg \{+\} \amalg \{-\}.$$

Here $\{+\}$ and $\{-\}$ are new points disjoint from X . We let all of the proper open sets of $\mathbb{S}X$ be the open subsets of X along with the sets $X \cup \{+\}$ and $X \cup \{-\}$.

REMARK 3.4.3. A visualization of CS^1 gives geometric meaning to the description of the cone construction. In fact, if we let $X := S^1$, the unit circle, we can visualize and compare all of the constructions just defined.



EXAMPLE 3.4.4. Observe that if X is a T_0 -space, then so are $\mathbb{C}X$ and $\mathbb{S}X$. If, for example, $X = D_3$, the finite space of 3 points endowed with the discrete topology, then the above constructions produce the following T_0 spaces (pictured left), while the associated Hasse diagram is shown on the right.



REMARK 3.4.5. When X is an A -space, $x < +$ and $x < -$ for all $x \in X$. Notice that the only open set containing both of the points $\{+\}$ and $\{-\}$ in $\mathbb{S}X$ is the entire space. In the language of posets, the non-Hausdorff suspension just adds two elements on top of the Hasse diagram of X .

Given a map $f : X \rightarrow Y$, we construct a map $\mathbb{S}f : \mathbb{S}X \rightarrow \mathbb{S}Y$ by defining

$$\mathbb{S}f(x) = \begin{cases} f(x) & \text{if } x \in X \\ + & \text{if } x = + \\ - & \text{if } x = - \end{cases}$$

EXERCISE 3.4.6. Check that $\mathbb{S}f$ is continuous since f is continuous.

With these definitions in hand, we can relate the classical suspension of a space with the non-Hausdorff suspension by defining the following comparison map.

DEFINITION 3.4.7. Define $\gamma_X : \mathbb{S}X \rightarrow \mathbb{S}X$ by

$$\gamma(x, t) = \begin{cases} x & \text{if } -1 < t < 1 \\ + & \text{if } t = 1 \\ - & \text{if } t = -1 \end{cases}$$

EXERCISE 3.4.8. We've defined γ so that it is continuous and $\gamma_Y \circ \mathbb{S}f = \mathbb{S}f \circ \gamma_X$. Check that these statements are true.

LEMMA 3.4.9. *The map $\gamma_X : \mathbb{S}X \rightarrow \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f : X \rightarrow Y$, the maps $\mathbb{S}f : \mathbb{S}X \rightarrow \mathbb{S}Y$ and $\mathbb{S}f : \mathbb{S}X \rightarrow \mathbb{S}Y$ are weak homotopy equivalences.*

PROOF. This is an application of [Theorem 3.3.1](#). Take the three subspaces X , $X \cup \{+\}$, and $X \cup \{-\}$ as our open cover of $\mathbb{S}X$. This is a basis since the intersection of any two of these open sets is X . The inverse images under γ of these open subsets

are $X \times (-1, 1)$, $X \times [-1, 1)$, and $X \times (-1, 1]$. We claim that if we restrict γ to each of these subspaces, γ becomes a homotopy equivalence and, therefore, γ is a weak homotopy equivalence. The proof of this claim is clear: $\gamma(X \times (-1, 1)) = X$ and the domain and target are both contractible in the other two cases.

Similarly, taking the three subspaces Y , $Y \cup \{+\}$, and $Y \cup \{-\}$ as our basis of $\mathbb{S}Y$, their inverse images under $\mathbb{S}f$ are X , $X \cup \{+\}$, and $X \cup \{-\}$, and the restrictions of $\mathbb{S}f$ on these three subspaces are weak homotopy equivalences. Finally, take the images in $\mathbb{S}Y$ of $Y \times (-1/2, 1/2)$, $Y \times [-1, 1/2)$, and $Y \times (-1/2, 1]$ as our basis of $\mathbb{S}Y$. Their inverse images under Sf are the corresponding subspaces of SX , and the restrictions of Sf to these subspaces are weak homotopy equivalences. \square

DEFINITION 3.4.10. The *n*th non-Hausdorff suspension of X is $\mathbb{S}^n X := \mathbb{S}(\mathbb{S}^{n-1} X)$. The *n*th classical suspension of X is $S^n X := S(\mathbb{S}^{n-1} X)$. Inductively, we have a map $\gamma^n : S^n X \rightarrow \mathbb{S}^n X$.

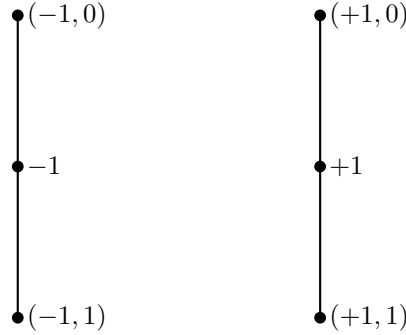
THEOREM 3.4.11. For a space X , the map $\gamma^n : S^n X \rightarrow \mathbb{S}^n X$ is a weak homotopy equivalence.

PROOF. We appeal to the diagram

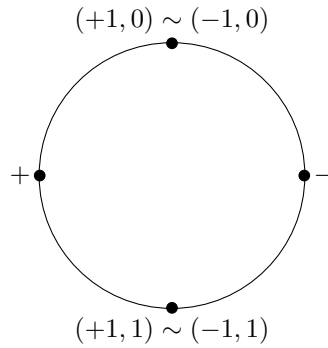
$$\begin{array}{ccc}
 S^n X & \xrightarrow{S\gamma^{n-1}} & \mathbb{S}\mathbb{S}^{n-1} X \\
 \gamma \downarrow & \searrow \gamma^n & \downarrow \gamma \\
 \mathbb{S}\mathbb{S}^{n-1} X & \xrightarrow{\mathbb{S}\gamma^{n-1}} & \mathbb{S}^n X
 \end{array}$$

The commutativity of the diagram follows from [Exercise 3.4.8](#). We may assume inductively that γ^{n-1} is a homotopy equivalence. It follows that $S\gamma^{n-1}$ and $\mathbb{S}\gamma^{n-1}$ are also weak homotopy equivalences by the preceding lemma. By the commutativity of the diagram, we have that γ^n is also a weak homotopy equivalence. \square

We apply the previous theorem to a simple example. Consider S^0 , which is just a two-point space. Building $\mathbb{S}S^0$ is a process that's pictured below. Namely, we first cross the two points with the unit interval, obtaining:

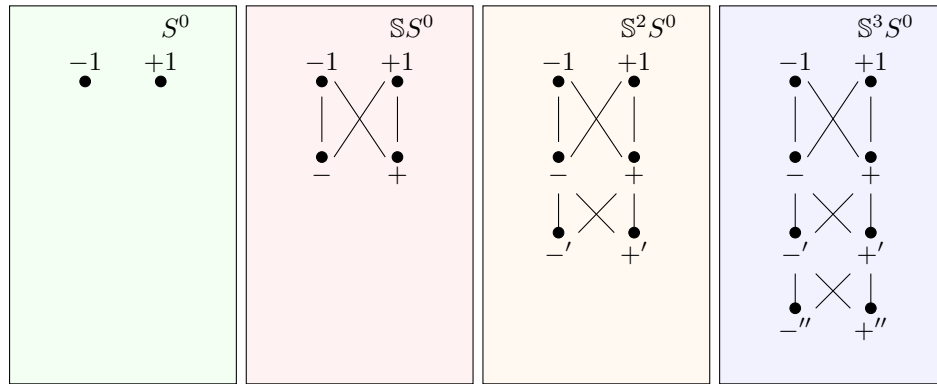


We then identify $(+1, 1) \sim (-1, 1)$ and $(+1, 0) \sim (-1, 0)$, which produces what's pictured in the following diagram.



It's then pictorially clear that $SS^0 \simeq S^1$. A little visualization, gluing two hollow cones onto a circle (one upwards, one downwards), will convince the reader that $SS^1 \cong S^2$. This then gives that $SS^1 = S^2S^0 \simeq S^2$. This result holds in fact for all n : the n -fold classical suspension of S^0 is homeomorphic to S^n .

The non-Hausdorff suspension of a finite space is easy to visualize, and we can draw the iteration of this process as follows in the case $X = S^0$.



Thus, n iterations of the non-Hausdorff suspension of S^0 yields a finite space with $2n$ new points, in addition to the 2 that we started with. The clear implication is stated as follows.

THEOREM 3.4.12. *Each $\mathbb{S}^n S^0$ is a finite minimal space with $2n + 2$ points.*

The minimality holds since $\mathbb{S}^n S^0$ has no upbeat or downbeat points. Instead, each point has incomparable points above and below it in the partial ordering.

We then have the following result.

THEOREM 3.4.13. *The n -sphere S^n is weak homotopy equivalent to the finite minimal space $\mathbb{S}^n S^0$ with $2n + 2$ points.*

PROOF. By [Theorem 3.4.11](#), we have that $\gamma_n : S^n S^0 \rightarrow \mathbb{S}^n S^0$ is a weak homotopy equivalence. As mentioned before, $S^n S^0 \cong S^n$. □

It is classical that infinitely many of the homotopy groups of S^2 are non-zero. Thus we have a six point space with infinitely many non-zero homotopy groups!

The following example gives two weakly equivalent five point spaces that are not homotopy equivalent.

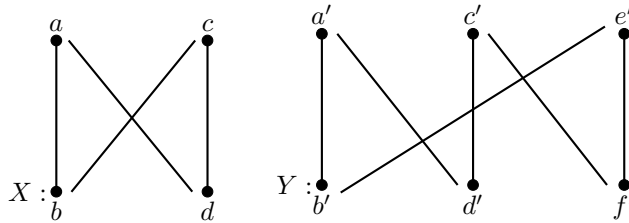
EXAMPLE 3.4.14. Consider the Hasse diagram of $\mathbb{S}D_3$, as shown in Example 3.4.4, and the opposite space $\mathbb{S}D_3$ (pictured below), which has two minimal points.



We can check that $\mathbb{S}D_3$ is homotopy equivalent to the wedge, or one-point union, of two circles. We know that $\gamma_{D_3} : \mathbb{S}D_3 \rightarrow \mathbb{S}D_3$ is a weak homotopy equivalence, and we have a very similar weak equivalence $\mathbb{S}D_3 \rightarrow (\mathbb{S}D_3)^{op}$. It will later become clear that X and X^{op} have the same weak homotopy type for any finite space X . However, a comparison of the minimal Hasse diagrams of $(\mathbb{S}D_3)^{op}$ and $\mathbb{S}D_3$ shows that the two spaces are not homotopy equivalent.

Thus, our five point example gives two weakly homotopy equivalent minimal finite spaces with the same number of points that are not homotopy equivalent. Moreover, there is no direct weak homotopy equivalence from one to the other: one needs a chain, like $\mathbb{S}D_3 \leftarrow \mathbb{S}D_3 \rightarrow (\mathbb{S}D_3)^{op}$.

EXAMPLE 3.4.15. There are minimal finite spaces with more than $2n+2$ points that are also weakly homotopy equivalent to S^n . For example, consider the four-point circle and the six point space pictured below. Both X and Y can be seen to be minimal, and they are clearly not homeomorphic and therefore not homotopy equivalent. However, these spaces are weak homotopy equivalent.



Take the unit circle in the complex plane. Let $f : S^1 \rightarrow X$ and $g : S^1 \rightarrow Y$ be given by

$$f(x) = \begin{cases} a & \text{if } x = 1 \\ b & \text{if } x = e^{i\theta}, 0 < \theta < \pi \\ c & \text{if } x = -1 \\ d & \text{if } x = e^{i\theta}, \pi < \theta < 2\pi \end{cases}$$

and

$$g(x) = \begin{cases} a' & \text{if } x = 1 \\ b' & \text{if } x = e^{i\theta}, 0 < \theta < 2\pi/3 \\ c' & \text{if } x = e^{2\pi i\theta/3} \\ d' & \text{if } x = e^{i\theta}, 2\pi/3 < \theta < 4\pi/3 \\ e' & \text{if } x = e^{4\pi i\theta/3} \\ f' & \text{if } x = e^{i\theta}, 4\pi/3 < \theta < 2\pi \end{cases}$$

One can verify both f and g are weak homotopy equivalences, and thus X and Y are both finite models of the circle. It should be clear that X is the unique minimal finite model of the circle.

3.5. 6-point spaces and height

Up to homeomorphism, the only minimal connected spaces with at most five points are the one point space, the 4-point circle, and the two 5-point minimal spaces described in [Example 3.4.14](#).

PROPOSITION 3.5.1. *Up to homeomorphism, there are seven connected minimal 6-point spaces X , and none of them are weakly contractible. One is the six point two sphere \mathbb{S}^2S^0 , two are $\mathbb{S}D_4$ and its opposite. The remaining four have three maximal and three minimal points.*

PROOF. We must have at least two minimal and at least two maximal points. Indeed, if we have just one intermediate point y , any point greater or less than it is upbeat or downbeat. If we have two intermediate points, they cannot be comparable without again contradicting minimality, and if they are incomparable we arrive by minimality at \mathbb{S}^2S^0 , which is homeomorphic to its opposite. The only remaining cases have all points either minimal or maximal. By the minimality of X , each minimal point must be less than at least two maximal points and each maximal point must be greater than at least two minimal points. There is only one example with two minimal points, and its opposite is the only example with four minimal points. We are left with the case when there are three minimal and three maximal points. Here each minimal point must be less than at least two maximal points and zero, one, two, or all three of them can be less than all three maximal points. In all four cases, the resulting space is homeomorphic to its opposite. \square

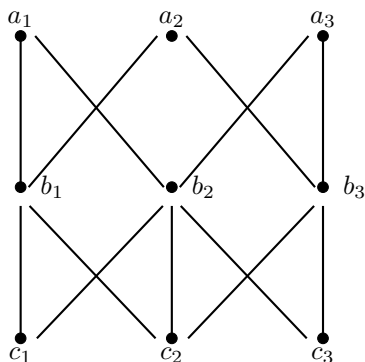
Recall that, by definition, minimal finite spaces can contain neither upbeat nor downbeat points. As said before, any non-maximal point in the Hasse diagrams of such models must have at least two points below, and similarly any non-minimal point must contain two points above. When the senior author first taught finite spaces in the REU, in 2003, he asked if $2n + 2$ was the least number of points in a finite space of the weak homotopy type of S^n . Barmak and Minian [\[8\]](#) proved that using homology, but we shall give a direct elementary proof. Recall the definition of the height $h(X)$ of a poset X (definition [Definition 2.5.5](#)).

PROPOSITION 3.5.2. *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)$ points. It has exactly $2h(X)$ points if and only if it is homeomorphic to $\mathbb{S}^{h(X)-1}S^0$ and therefore weakly homotopy equivalent to $S^{h(X)-1}$.*

PROOF. Let $x_1 < \dots < x_h$ be a maximal chain in X . Since X cannot have a minimum point, there is a y_1 which is not greater than x_1 . Since no x_i is an upbeat point, $1 \leq i < h$, there must be some $y_{i+1} > x_i$ such that y_{i+1} is not greater than x_{i+1} . The points y_i are easily checked to be distinct from each other and from the x_j . Now suppose that X has exactly these $2h$ points. By the maximality of our chain, the x_i and y_j are incomparable. For $i < j$, we started with $x_i < x_j$, and we check by cases from the absence of upbeat and downbeat points that $y_i < x_j$, $y_i < y_j$, and $x_i < y_j$. Comparing with the iterated suspension, we see that this implies that X is homeomorphic to $\mathbb{S}^{h-1}S^0$. \square

Drawing posets, and thinking about them, leads to lots of eliminations from the list of F -spaces that might not be contractible or weakly contractible (weakly homotopy equivalent to a point).

It presents a rather challenging exercise to determine the least number of points n giving an n -point weakly contractible space that is not contractible. The reference ([17]) gives the answer as $n = 9$ with the following example.



Elementary properties of finite spaces

4.1. The Euler characteristic of a finite space

The notion of the Euler characteristic is one that exists for any arbitrary topological space, not necessarily finite. For classical treatment of the Euler characteristic of a simplicial complex, a notion which will be introduced in [Chapter 6](#), section 4.4 of *A Concise Course in Algebraic Topology* gives a short explanation. For now, we define the equivalent notion for finite spaces. It will soon become clear that the notions of the Euler characteristic of a finite space and the Euler characteristic of a simplicial complex are equivalent in a formal way.

DEFINITION 4.1.1. The *Euler characteristic of a finite T_0 -space* is given by

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\sharp C+1}$$

where $\mathcal{C}(X)$ is the set of non-empty chains of X and $\sharp C$ is the cardinality of some element of that set.

For ordinary topological spaces, the Euler characteristic is a homotopy invariant. Using this last definition we can prove for finite spaces that the Euler characteristic is also homotopy invariant.

THEOREM 4.1.2. *Let X and Y be finite T_0 -spaces that are homotopy equivalent. Then $\chi(X) = \chi(Y)$.*

PROOF. Let X_c and Y_c be the cores of X and Y respectively, which must exist by 1.9. 1.10 implies that X_c and Y_c are homeomorphic and thus $\chi(X_c) = \chi(Y_c)$. As per 1.9, we may think of X_c as part of a sequence of subspaces of X , where each successive element in the sequence is generated by removing a beat point. Thus, it suffices to show that removing a beat point does not affect the Euler characteristic. Let P be a finite poset with beat point p , where there must exist some q such that if r is comparable with p then r is also comparable with q . We can then construct a bijection

$$\begin{aligned} \varphi : \{C \in \mathcal{C}(P) \mid p \in C, q \notin C\} &\rightarrow \{C \in \mathcal{C}(P) \mid p \in C, q \notin C\} \\ C &\mapsto C \cup \{q\} \end{aligned}$$

We may thus write:

$$\begin{aligned} \chi(P) - \chi(P - \{p\}) &= \sum_{p \in C \in \mathcal{C}(P)} (-1)^{\sharp C+1} = \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \in C \ni p} (-1)^{\sharp C+1} = \\ &= \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \notin C \ni p} (-1)^{\sharp \varphi(C)+1} = \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \notin C \ni p} (-1)^{\sharp C} = 0 \end{aligned}$$

□

4.2. The Möbius function

The Euler characteristic of finite T_0 -spaces is particularly interesting because of its relationship to the Möbius function of posets, a combinatorial object. To define the Möbius function we first define an *incidence algebra* \mathfrak{A} on P . $\mathfrak{A}(P)$ is the set of functions $P \times P \rightarrow \mathbb{R}$ such that for $f \in \mathfrak{A}(P)$, $f(x, y) = 0$ if $x \not\leq y$. This forms a vector space over \mathbb{R} where we have a product defined as

$$fg(x, y) = \sum_{z \in P} f(x, z)g(z, y)$$

We let $\xi_p \in \mathfrak{A}$ be the function such that $\xi_p(x, y) = 1$ whenever $x \leq y$. This function has an inverse in \mathfrak{A} which we call the *Möbius function* and denote μ_p . Note that $\xi_p(x, y)$ is invertible according to [7] page 26. The identity of \mathfrak{A} is

$$\delta(s, t) = \begin{cases} 1 & : s = t \\ 0 & : s \neq t \end{cases}$$

Note that in equations where elements of \mathfrak{A} are added to some integer that integer simply denotes a multiple of δ .

It follows directly from the definition of the multiplication that:

$$\xi^2(s, u) = \sum_{s \leq t \leq u} 1$$

so we may deduce that $\xi^2(s, u)$ is the number of chains of length 2 between s and u (note that the length is given as the number of elements minus 1). Similarly

$$\xi^k(s, u) = \sum_{s=s_0 \leq s_1 \leq \dots \leq s_k=u} 1$$

which is the number of chains of length k . Observing that

$$(\xi - 1)(s, u) = \begin{cases} 1 & : s < u \\ 0 & : s = u \end{cases}$$

we can use $(\xi - 1)^k$ to count the number of strictly-increasing chains. Note furthermore that

$$(2 - \xi)(s, t) = \begin{cases} 1 & : s = t \\ -1 & : s < t \end{cases}$$

PROPOSITION 4.2.1. $(2 - \xi)^{-1}(s, t)$ gives the total number of strictly increasing chains from s to t .

PROOF. Let ℓ be the length of the longest chain between s and t so that $(\xi - 1)^{\ell+1}(u, v) = 0$ for $s \leq u \leq v \leq t$. For such u and v

$$\begin{aligned} (2 - \xi)[1 + (\xi - 1) + (\xi - 1)^2 + \dots + (\xi - 1)^\ell](u, v) &= \\ [1 - (\xi - 1)][1 + (\xi - 1) + (\xi - 1)^2 + \dots + (\xi - 1)^\ell](u, v) &= \\ [1 - (\xi - 1)^{\ell+1}](u, v) &= \delta(u, v) \end{aligned}$$

The equality from the second line to the third comes from multiplying out so that all of the central terms cancel. Because δ is the identity, $(2 - \xi)^{-1} = 1 + (\xi - 1) + (\xi - 1)^2 + \dots + (\xi - 1)^\ell$ when restricted to the elements between some s and t . But as explained above, $(\xi - 1)^k$ are just the chains of length k between s and t so it follows that $(2 - \xi)^{-1}$ is the total number of chains from s to t . \square

The following theorem connects the combinatorial notion of the Möbius function to the topological notion of the Euler characteristic:

THEOREM 4.2.2 (Hall's Theorem). *Let P be a finite poset and let \hat{P} be $P \cup \{\hat{0}, \hat{1}\}$ where $\hat{0}$ and $\hat{1}$ are minimum and maximum elements. Let c_i be the number of strictly increasing chains between $\hat{0}$ and $\hat{1}$ of length i . Then*

$$(4.2.3) \quad \mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \dots$$

PROOF.

$$\begin{aligned} \mu_{\hat{P}}(\hat{0}, \hat{1}) &= (1 + (\xi - 1))^{-1}(\hat{0}, \hat{1}) \\ &= (1 - (\xi - 1) + (\xi - 1)^2 - \dots)(\hat{0}, \hat{1}) \\ &= 1(\hat{0}, \hat{1}) - (\xi - 1)(\hat{0}, \hat{1}) + (\xi - 1)^2(\hat{0}, \hat{1}) - \dots \\ &= c_0 - c_1 + c_2 - \dots \end{aligned}$$

□

This expression is very close to the expression developed for the Euler characteristic. Indeed the only difference is that when computing the Euler characteristic, the empty-set is not regarded as a face of the simplicial complex whereas in this expression it is, entering the sum as -1 . Thus by defining the reduced Euler characteristic, $\tilde{\chi}(X) = \chi(X) - 1$ we have the following remarkable fact:

PROPOSITION 4.2.4. *Let P be a finite poset.*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\mathcal{H}(P)).$$

For more information on Hall's theorem ([Theorem 4.2.2](#)) see [\[64\]](#)[p.307-8].

Rephrase without reference to simplicial complexes

4.3. The fundamental group of a finite space

4.3.1. H-loop groups. The following section outlines a method of computing the fundamental group of a finite space. A second process is outlined in ?? after the introduction of simplicial complexes, and we shall soon prove that the notions are in fact equivalent.

The following definitions are due to Barmak and Minian [\[8\]](#).

DEFINITION 4.3.1. For any poset X with a base point x_0 , let $H(X)$ be the associated Hasse diagram. We call an ordered pair $e = (x, y)$ an *H-edge* if $(x, y) \in E(H(X))$ or $(y, x) \in E(H(X))$. The point x is called the *origin* of e , denoted by $o(e)$ and the point y is called the *end* of e , denoted by $e(e)$. The *inverse* of an *H-edge* $e = (x, y)$ is the *H-edge* $e^{-1} = (y, x)$.

DEFINITION 4.3.2. If we have a sequence of *H-edges* e_1, e_2, \dots, e_n with $e(e_i) = o(e_{i+1})$ for all $1 \leq i \leq n - 1$, we can connect them together to get an *H-path* $\xi = e_1 e_2 \dots e_n$. Typically we say the *origin* of this *H-path* is $o(\xi) = o(e_1)$ and the *end* of this *H-path* is $e(\xi) = e(e_n)$. The *inverse* of an *H-path* $\xi = e_1 e_2 \dots e_n$ is the *H-path* $\xi^{-1} = e_n^{-1} e_{n-1}^{-1} \dots e_1^{-1}$.

DEFINITION 4.3.3. An *H-path* $\xi = e_1 e_2 \dots e_n$ is said to be *monotonic* if either $e_i \in E(\mathcal{H}(X))$ for all $1 \leq i \leq n$ or $e^{-1} \in E(\mathcal{H}(X))$ for all $1 \leq i \leq n$.

DEFINITION 4.3.4. For two *H-paths* $\xi_1 = e_1 e_2 \dots e_n$ and $\xi_2 = f_1 f_2 \dots f_m$ with $e(\xi_1) = o(\xi_2)$, it makes sense to define a *composition* of ξ_1 and ξ_2 :

$$\xi_1 \xi_2 = e_1 e_2 \dots e_n f_1 f_2 \dots f_m.$$

Broken reference

DEFINITION 4.3.5. An H -loop at x_0 is an H -path ξ such that $o(\xi) = e(\xi) = x_0$.

DEFINITION 4.3.6. Two H -loops ξ and ξ' at x_0 are said to be *close* if there exist four H -paths ξ_1, ξ_2, ξ_3 , and ξ_4 with ξ_2 and ξ_3 being monotonic, such that $\xi = \xi_1\xi_4$ and $\xi' = \xi_1\xi_2\xi_3\xi_4$. Denote this close relation by $\xi \simeq \xi'$.

Two H -loops ξ and ξ' at x_0 are said to be H -equivalent if there exists a sequence of loops at x_0 , $\xi = \xi_0, \xi_1, \xi_2, \dots, \xi_n = \xi'$ such that $\xi_{i-1} \simeq \xi_i$ for each $1 \leq i \leq n$.

It is not hard to verify that H -equivalence is actually an equivalence relation. Therefore, we obtain the equivalence classes for H -loops at x_0 . Let us denote the equivalence class of the H -loop ξ by $\langle \xi \rangle$ and collect all the equivalence classes into the set $\mathcal{H}(X, x_0)$. Similar to the way we handle the idea of fundamental group, we can define a product on these equivalence classes by taking $\langle \xi_1 \rangle \langle \xi_2 \rangle = \langle \xi_1\xi_2 \rangle$. It is not hard to show that this product is well defined. This gives a group structure on the set $\mathcal{H}(X, x_0)$, which is called the H -loop group.

REMARK 4.3.7. The H -loop group in the Hasse diagram provides a way to compute the fundamental group of a topological space by just looking at its minimal finite model. As we know, a minimal finite model is weak homotopy equivalent to the original space, and hence all the information of every homotopy group is carried by its minimal finite model. However, it is not known yet whether there is an efficient way to extract the information of higher homotopy groups just from a minimal finite model.

Reference to minimal model chapter later in the book

Finite graphs are a class of geometric objects whose minimal finite models have been completely understood. One important fact about finite graphs that makes it easier to find their minimal models is that a finite graph is a 1-dimensional CW complex, i.e. a wedge sum of circles. Therefore, the weak homotopy type of a finite graph is determined by its Euler characteristic, and from this we can work out a way to compute minimal finite models of finite graphs.

Before we go into the actual argument, we would like to study the Hasse diagram a little further. As we know from the previous section, the H -loop group $\mathcal{H}(x, x_0)$ is isomorphic to the fundamental group of (X, x_0) . But what is more about the Hasse diagram is that it can provide another way of looking at the fundamental group in terms of generators and relations. Here we first want to show how to get the generators.

PROPOSITION 4.3.8. Let (X, x_0) be a poset. If $x \in X$ is neither maximal nor minimal and $x \neq x_0$, then the inclusion $i : X - \{x\} \rightarrow X$ induces an epimorphism

$$i_* : \mathcal{H}((X - \{x\}), x_0) \rightarrow \mathcal{H}(X, x_0).$$

PROOF. Since every H -loop at x_0 in $X - \{x\}$ has a natural image as an H -loop in X under inclusion, i_* is naturally a homomorphism. Therefore, to show i_* is an epimorphism, it is sufficient to check that every H -loop in X that goes through x is equivalent to an H -loop that does not go through x .

Suppose $\epsilon_1\epsilon_2 \dots (y, x)(x, z) \dots \epsilon_n$ is an H -loop. Then without loss of generality, we can assume either $y \leq x \leq z$ or $y \leq x$ and $z \leq x$.

If $y \leq x \leq z$, then $\{x, y, z\}$ is within a triangle and therefore we can deduce that

$$\epsilon_1\epsilon_2 \dots (y, x)(x, z) \dots \epsilon_n \approx \epsilon_1\epsilon_2 \dots (y, z) \dots \epsilon_n.$$

If $y \leq x$ and $z \leq x$, then since x is not maximal, we can find $w \in X - \{x\}$ such that $w > x$. Then

$$\begin{aligned} \epsilon_1 \epsilon_2 \dots (y, x)(x, z) \dots \epsilon_n &\approx \epsilon_1 \epsilon_2 \dots (y, w)(w, x)(x, w)(w, z) \dots \epsilon_n \\ &\approx \epsilon_1 \epsilon_2 \dots (y, w)(w, z) \dots \epsilon_n. \end{aligned}$$

□

We know that for a path connected space, the fundamental group does not depend on the choice of the base point. Thus without loss of generality, we can always choose the base point x_0 to be one of the minimal points. Now imagine that if we eliminate all the points that are neither maximal nor minimal in X , then we will be left only with all the maximals and minimals. Call this subspace with only maximals and minimals Y . Then we have the following corollary:

COROLLARY 4.3.9. *For any finite poset (X, x_0) , let (Y, x_0) be the subspace that consists of only maximals and minimals in (X, x_0) . Then the inclusion induces an epimorphism $i_* : \pi_1(Y, x_0) \rightarrow \pi_1(X, x_0)$.*

REMARK 4.3.10. Note that since there are only maximals and minimals in Y , $h(Y) \leq 2$. Typically, for a non-contractible space X , $h(Y) = 2$. Also, if X is connected, then removing middle points will not disconnect the space, i.e. Y remains connected.

standardize notation
for height

REMARK 4.3.11. When $h(Y) = 2$, we know that Y can be represented as a finite 1-dimensional simplicial complex, i.e. a finite graph. Since a finite graph is always homotopy equivalent to a wedge sum of circles, we can assume that the graph is homotopy equivalent to $\bigvee_{i=1}^m S^1$. Therefore, we have

$$\pi_1(Y, x_0) \cong \mathcal{H}(Y, x_0) \cong \pi_1\left(\bigvee_{i=1}^m S^1, s_0\right) \cong \mathbb{Z}^{*m}.$$

Now we can go into the search for minimal finite models of finite graphs:

THEOREM 4.3.12. *If X is a minimal finite model of $\bigvee_{i=1}^n S^1$, then $h(X) = 2$.*

PROOF. Take the subspace of maximals and minimals Y . Since X is a minimal finite model of a noncontractible space, we know that $h(Y) = 2$. By [Remark 4.3.11](#), $\pi_1(Y, x_0) = \mathbb{Z}^{*m}$.

By [Proposition 4.3.8](#), there is an epimorphism $i_* : \pi_1(Y, x_0) \rightarrow \pi_1(X, x_0)$. Note that since $\pi_1(X, x_0) = \mathbb{Z}^{*n}$, thus we must have $m \geq n$.

Consider the finite graph of a space Y . Since it is a wedge sum of m circles, there are m edges that are not contained in any maximal tree of the graph. If we remove $m - n$ of these edges by forgetting the relations between the vertices, we obtain a new finite space Z and the graph in question is homotopy equivalent to $\bigvee_i^n S^1$.

standardize notation
for cardinality

Note that $\#Z = \#Y \leq \#X$. But since X is a minimal finite model of $\bigvee_i^n S^1$, we also have $\#X \leq \#Z$. Therefore, $\#Z = \#Y = \#X$, which implies $X = Y$. □

The following theorem will conclude our search:

THEOREM 4.3.13. *Let j be the number of maximal points in X and k be the number of minimal points in X . Then X is a minimal finite model of $\bigvee_i^n S^1$ if and only if $h(X) = 2$, $\#X = \min\{j + k | (j - 1)(k - 1) \geq n\}$ and the number of edges in the graph of X is $\#X + n - 1$.*

PROOF. We have shown that if X is a minimal finite model of $\vee_i^n S^1$, then $h(X) = 2$. Since j is the number of maximal points and k is the number of minimal points in X , we know that there can be at most jk many edges in the graph of X . Let E be the number of edges in and V be the number of vertices, then the Euler characteristic formula tells us that

$$1 - n = V - E \geq j + k - jk.$$

Therefore, we must have $(j - 1)(k - 1) = jk - j - k + 1 \geq n$, and hence $\#X = j + k \geq \min\{j + k | (j - 1)(k - 1) \geq n\}$.

Now suppose we have j and k such that $(j - 1)(k - 1) \geq n$. Then consider the finite poset $W = \{x_1, x_2, \dots, x_j, y_1, y_2, \dots, y_k\}$ with $xs > yt$ for any $1 \leq s \leq j$ and $1 \leq t \leq k$. As we can see, W is a finite model of $\vee_{i=1}^{(j-1)(k-1)} S^1$. But then we $i = 1$ can remove $(j - 1)(k - 1) - n$ edges from the graph of W by forgetting the corresponding relations, and the resulting finite poset would be a finite model of $\vee_i^n S^1$.

Now since for any j and k with $(j - 1)(k - 1) \geq n$ we can find a finite model with $j + k$ points, we conclude that $\#X = \min\{j + k | (j - 1)(k - 1) \geq n\}$, and the number of edges just follows from the Euler characteristic formula.

Conversely, suppose we have a finite poset X with $h(X) = 2$, $\#X = \min\{j + k | (j - 1)(k - 1) \geq n\}$ and the number of edges in the associated graph being $\#X + n - 1$. Note that if X is connected, then we are done, for the reason that the graph of X will also be connected, and the three conditions will determine that it is a finite graph with the Euler characteristic $1 - n$. Therefore, the only thing we need to show here is connectedness.

Suppose X is disconnected. Let X_1, X_2, \dots, X_l be distinct connected components in X . Let M_i be the set of maximal points in X_i and m_i be the set of minimal points in X_i . Then $j = \sum_{i=1}^l \#M_i$ and $k = \sum_{i=1}^l \#m_i$. Since $\#X = \min\{j + k | (j - 1)(k - 1) \geq n\}$, we must have $(j - 2)(k - 1) < n$. But at the same time, $n = E - j - k + 1$ by the Euler characteristic formula. Therefore,

$$(j - 2)(k - 1) < E - j - k + 1$$

$$jk < E + (k - 1).$$

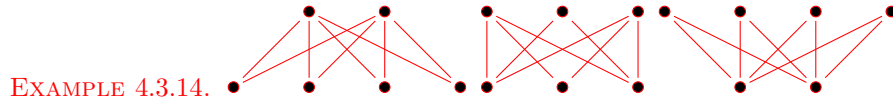
Note that jk is in fact the number of edges in the complete bipartite graph $(\bigcup_{i=1}^l m_i, \bigcup_{i=1}^l M_i)$. The inequality above shows that the graph of X differs from the complete bipartite graph in less than $k - 1$ edges.

Since there are no edges between M_i and m_r for $i \neq r$, we have

$$k = 1 > \sum_{i=1}^l \#M_i(k - \#m_i) \geq \sum_{i=1}^l (k - \#m_i) = (l - 1)k.$$

This forces $l = 1$ and hence X is connected. \square

This theorem gives a method to compute minimal finite models of all finite graphs. Unlike the n -spheres, some finite graphs have more than one minimal finite model, with the same number of points but different arrangements. For example, the following three are minimal finite models of $\vee_{i=1}^3 S^1$:



Up to this point are the minimal finite models that have been completely understood. But we want to push the frontier a little bit further to some slightly more complicated spaces and investigate the possible size of their minimal finite models.

Hotlink to minimal models chapter

4.4. Covers of finite spaces

One of the problems mathematicians face as they explore the new territory of finite spaces is how to classify them. A useful way to differentiate between some spaces is to calculate their fundamental groups. However, it is difficult to intuitively understand what a loop in a finite space would look like, and it can be difficult to calculate the fundamental groups of finite spaces.

Information about covers of finite spaces can be obtained through observing similarity with covers of spaces we are accustomed to working with. For instance, we investigate the relationship between the wedge of two circles and the 5-point space weakly homotopy equivalent to it. Similarly, we suggest an intuitive way to find covers for any height-2 poset by looking at other wedges of circles.

Omit mention of Galois correspondence?

Covering spaces share a deep connection to the fundamental group. If a space X is path-connected, locally path-connected, and semi-locally simply connected, there is a *Galois correspondence* between the covers of X and the subgroups of its fundamental group. We will show that any connected finite space is also path-connected and locally contractible. It follows that any connected finite space is path-connected, locally path-connected, and semi-locally simply connected. Thus, there exists a Galois correspondence for finite spaces.

Given a space X , a cover is intuitively a larger space \tilde{X} which can be projected neatly into X in such a way that locally \tilde{X} can be regarded as a stack of pancakes.

DEFINITION 4.4.1. A *cover* of a space X is a space \tilde{X} and a map $p : \tilde{X} \rightarrow X$ such that for each point x in X , there is an open neighborhood U of x where $p^{-1}(U)$ is the union of disjoint open sets in \tilde{X} , and p maps each of these sets homeomorphically onto U .

DEFINITION 4.4.2. A *lift* of $f : X \rightarrow Y$ along $g : Z \rightarrow Y$ is a map $\tilde{f} : X \rightarrow Z$ such that $g \circ \tilde{f} = f$.

$$\begin{array}{ccc} & & Z \\ & \nearrow \tilde{f} & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

THEOREM 4.4.3. Given a cover $p : \tilde{X} \rightarrow X$, a homotopy $f_t : Y \rightarrow X$, and a map $\tilde{f}_0 : Y \rightarrow \tilde{X}$ lifting f_0 , there exists a unique homotopy $\tilde{f}_t : Y \rightarrow \tilde{X}$ of \tilde{f}_0 that lifts f_t .

Proofs of this theorem and the ones below can be found in Allen Hatcher's Algebraic Topology [31] (p.60, 84-85). We will only need this theorem to lift paths, not any larger spaces, because none of the objects we will be working with in this paper will have dimension greater than 1.

Now, the Galois correspondence between the covers of a space X and the subgroups of its fundamental group gives us the following:

THEOREM 4.4.4. *Suppose X is path-connected, locally path-connected, and semi-locally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. This bijection is obtained by associating each subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the cover (\tilde{X}, \tilde{x}_0) . If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \rightarrow X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Notice that if K is a subgroup of H , then the space corresponding to K will cover the space corresponding to H . This means that the bijection in the previous theorem is order reversing. Also note that the fundamental group of a space X must have a trivial subgroup, which is a subgroup of every other subgroup, so there must be a cover of X that covers every other cover. This is called the *universal cover of X* , and it is unique up to isomorphism. In fact, a space need only be locally path-connected and semi-locally simply connected to have a universal cover, and the proof of the previous theorem actually uses the existence of a universal cover.

This theorem is also stated as an *equivalence of categories*, which will give us a general relationship between the covers of any two weakly homotopy equivalent spaces. However, the aim of this paper is to suggest an explicit geometric relationship between covers of weakly equivalent spaces. We will describe this explicit relationship for the wedge of circles and the 5-point space weakly equivalent to it.

The following theorem will provide some intuition about the relationship between posets (which we will see are equivalent to finite spaces) and wedges of circles, if both are considered as graphs.

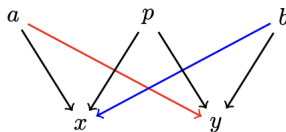
THEOREM 4.4.5. *For a connected graph X with maximal tree T , $\pi_1(X)$ is a free group with basis the classes of loops $[f_\alpha]$ corresponding to the edges e_α of $X - T$.*

This makes sense intuitively because a tree has no non-trivial loops and can be retracted to a single vertex. Collapsing a maximal tree in a connected graph leaves one vertex with a bouquet of edges, forming a wedge of circles, and the fundamental group of a wedge of κ circles is the free group on κ generators. Because trees are contractible, collapsing a maximal tree is a homotopy equivalence. This means that the fundamental group of X is the same as the fundamental group of the wedge of circles made up of the edges left over when the maximal tree is collapsed. These edges are exactly the edges e_α of $X - T$.

PROPOSITION 4.4.6. *For any connected finite space X , there is a Galois correspondence between the covers of X and the subgroups of its fundamental group.*

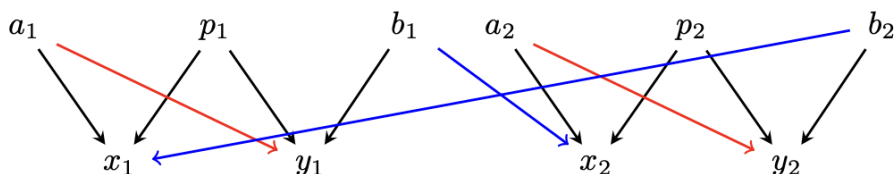
PROOF. Since X is connected, it is path-connected, and X is locally path-connected because its connected components and path components coincide. By the previous lemma, X is locally contractible and hence semi-locally simply connected, so [Theorem 4.4.4](#) applies. \square

4.4.1. Covers of the 5-point space. Consider the following 5-point space, W , which is weakly homotopy equivalent to the wedge of two circles.



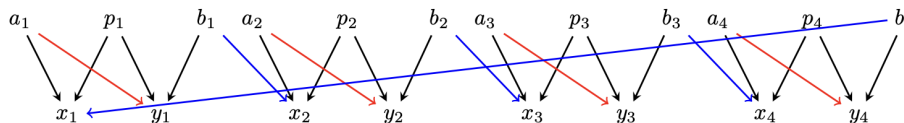
If we consider the two zigzags $p < x > a < y > p$ and $p < x > b < y > p$, then the first is a loop containing the red edge, and the second is a loop containing the blue edge.

We can construct a 2-fold cover of W by connecting two copies of the space as follows.

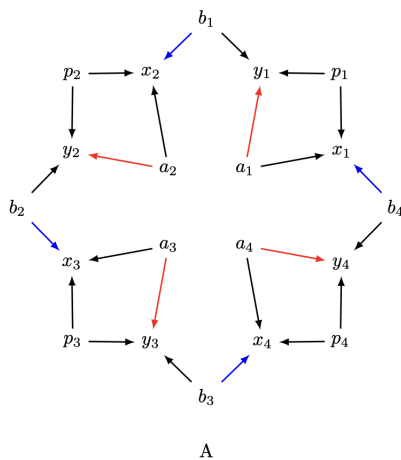


It is easy to imagine simply picking up one of the two W shapes, shifting it over so that its points match up with the points in the other W shape, and flattening it to get the original space.

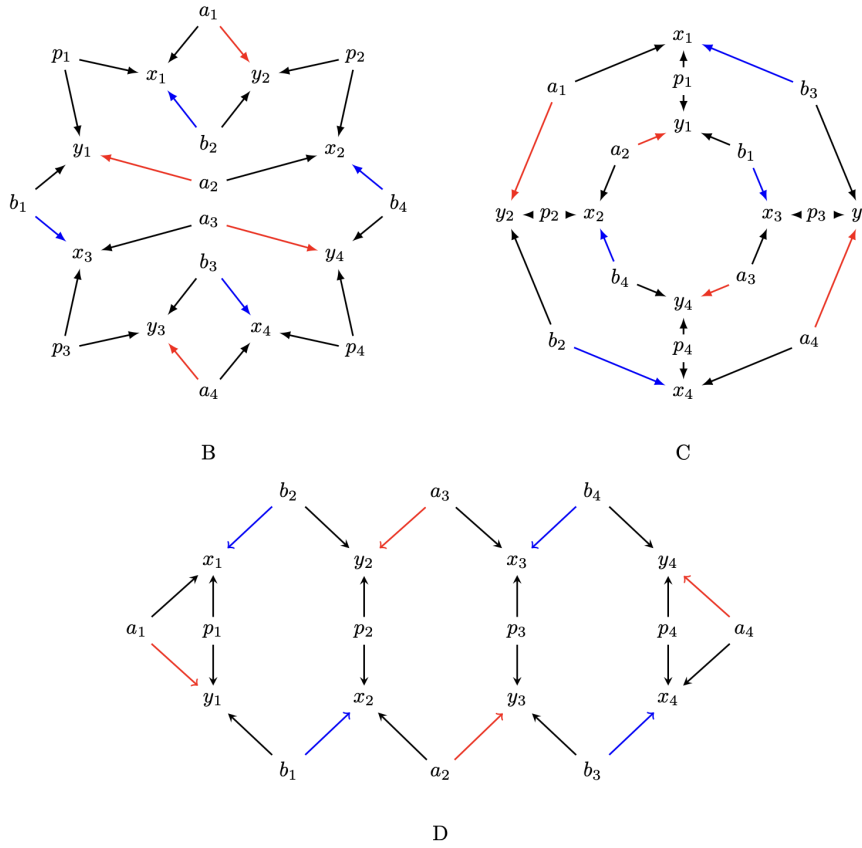
Similarly, we can construct a 3-fold cover by taking three copies of W without the blue zigzag $b < x$, and then connecting b_i to x_{i+1} for $i = 1, 2$ and b_3 to x_1 . A 4-fold cover, shown below, can be constructed in the same way by taking four copies of W without the blue zigzag and connecting them with the blue zigzags $b_i < x_{i+1}$ for $i = 1, \dots, 3$ and $b_4 < x_1$. It is easy to see that an n -fold cover of this space may be constructed by stringing together n of these W -shaped “beads” using the blue edges.



The symmetry of this 4-fold cover is clearer if it is drawn planar:

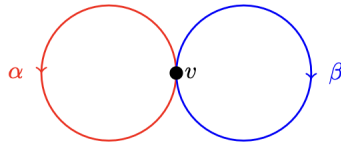


Three more 4-fold covers follow:

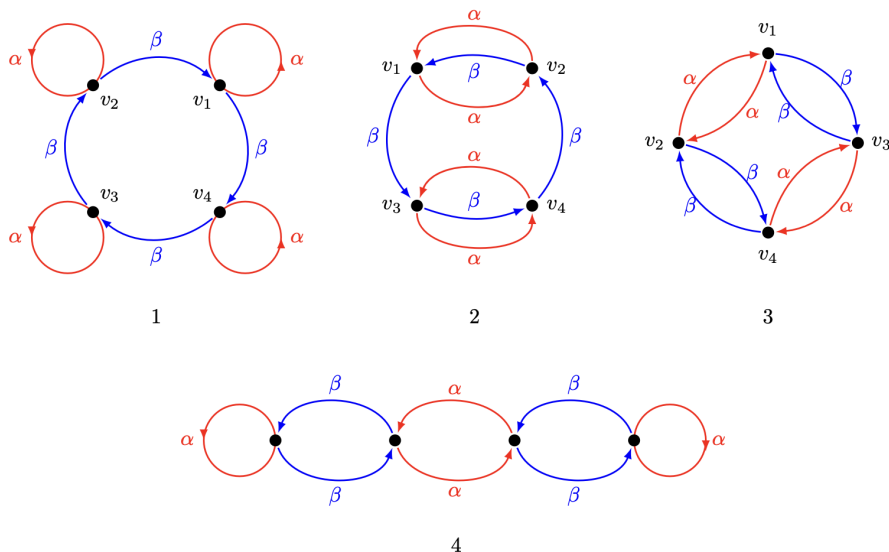


Now that we have found several 4-fold covers and a way to construct an n -fold cover of W for any n , it makes sense to ask whether there is a simple way to construct all covers of this space. To address this question, we will look at the wedge of two circles.

4.4.2. Finding a relationship to the wedge of circles. Depicted below is the wedge of two circles. Note that we have colored the edges to distinguish the two generators α and β and have given them orientations.



Four 4-fold covers of the wedge of two circles are shown below.



There are some immediate visual parallels between these covers and those of W in the previous section. On one hand, looking at the covers of the wedge of circles as graphs, each vertex has one red edge and one blue edge going in, and one red edge and one blue edge going out. On the other hand, each zigzag $a_i < x_i > p_i < y_i > b_i$ is connected to one red edge and one blue edge pointing in, and one red edge and one blue edge pointing out.

If we think of collapsing the points in W along $a < x > p < y > b$, then $a < y$ and $b < x$ would correspond to the red and blue generators of the wedge of two circles. This fits with [Theorem 4.4.5](#) because $a < x > p < y > b$ is a maximal tree in W , and W has the same fundamental group as the wedge of two circles. Similarly, we can collapse the black edges in covers A, B, C , and D above to get 1, 2, 3, and 4, respectively.

Now, if we think of the wedge of two circles and its covers as graphs, it becomes clear that for every node, we will have a zigzag $a_i < x_i > p_i < y_i > b_i$ in the corresponding poset cover of W . We already know that generator α corresponds to the zigzag containing $a < y$ in W , β corresponds to the zigzag containing $b < x$, and the direction of each generator is preserved by which point is reached first in the zigzag. Therefore, given a cover of the wedge of two circles, we need only turn each point in the cover into five points, endow these points with the appropriate ordering, and connect the colored edges to the correct points to create an analogous cover of W . We shall turn to the formalism of category theory to show how this correspondence between the covers of W and the covers of $S^1 \vee S^1$ can be made more precise.

CHAPTER 5

Minimal models

The following two chapters cover minimal models of various surfaces.

In characterizing finite manifolds, a comparatively well-understood class of finite spaces, the following definitions are useful.

DEFINITION 5.0.1. If X is a topological space, a *finite model* of X is a finite T_0 space which is weak homotopy equivalent to X .

Since every space is weak homotopy equivalent to a regular CW complex¹, this implies that every space has a T_0 Alexandroff model, and if the regular CW complex is finite, so is the model.

We describe Stong minimality and absolute minimality of finite models of topological spaces, exhibit Stong minimal models of all closed surfaces, and derive several elementary lower bounds for the size of absolutely minimal models. We define the notion of a finite manifold and characterize finite surfaces, then use this characterization to show that a finite model of a closed surface is a finite surface if and only if it is induced by a regular CW structure on the surface. Finally, we use this result to deduce a better lower bound for the size of models which are finite surfaces and construct minimal finite surface models of orientable surfaces whose genera satisfy nice number-theoretic properties.

One basic example which illustrates the relationship between ordinary spaces and finite models is that of the non-Hausdorff suspension. Recall that the non-Hausdorff suspension of S^n produces a weakly equivalent version with $2n+2$ points for each n .

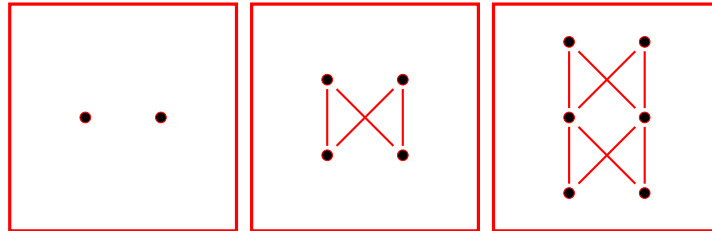


Figure 1: Hasse diagrams for finite models of S^0 , S^1 and S^2 .

Notice that a given space will have many finite models. For example, every finite simplicial structure gives rise to a finite model, and we can always enlarge a model by adding beat points. To avoid superfluous information, reduce complexity, and gain a better understanding of these models, it is desirable to find finite models which are minimal in one of two senses.

¹Every space is weak homotopy equivalent to a CW complex, while every CW complex is homotopy equivalent to a simplicial complex of the same dimension; see, for example, Theorem 2C.5 in [31].

DEFINITION 5.0.2. We say a finite T_0 space is *Stong minimal* if its cardinality is minimal in its homotopy class. We say a finite T_0 space is *absolutely minimal* if its cardinality is minimal in its weak homotopy class.²

Note that the second notion of minimality is stronger than the first; the sphere is a rare case where the evident Stong minimal model is absolutely minimal. It is also perhaps a more natural notion of minimality when it comes to the study of finite models, since it is equivalent to being the smallest finite model of a space. However, Stong minimality is easier to check, and Stong minimal models are easier to find: two finite T_0 spaces are homotopy equivalent if and only if their cores are homeomorphic, so a space is Stong minimal if and only if it has no beat points, and a Stong minimal model can be obtained from any finite model simply by removing beat points. In contrast, at the time of the writing of this paper, there is no known algorithm for reducing an arbitrary finite T_0 space to an absolutely minimal space, or even for determining whether a space is absolutely minimal.

For this reason, results regarding absolutely minimal models have typically involved exhibiting a particular model for a space and showing that no smaller space can have the same homotopy or homology groups. Barmak and Minian take this approach in [8] in which they show that the finite models described above are the unique absolutely minimal models of S^n for each n . Having found such models for the most basic topological spaces, we turn next to another well-known countable collection of spaces with simple homology: closed surfaces. Cianci and Ottina exhibit absolutely minimal models of the torus, the projective plane, and the Klein bottle in [16], but their methods for bounding model size below are not related to the genus of these surfaces, and hence do not generalize directly to surfaces of higher genus. In this paper, we begin by describing Stong minimal models for all closed surfaces.³ We then find some lower bounds for the size of arbitrary finite models of closed surfaces using results from [16] together with some elementary combinatorial facts. Having derived some minor results for the general case, we specialize to a particularly well-behaved class of finite T_0 spaces called finite manifolds, characterize them in dimension 2, and conclude by deriving a much stronger bound for finite models of this type, which we use to find some finite models which are minimal among finite surfaces.

5.0.1. Stong minimal models of closed surfaces. In this section, we construct regular CW models for all closed surfaces and show that the associated posets have no beat points, making them Stong minimal. These models are generalizations of absolutely minimal models presented in [16].

Given an orientable surface S of genus g , the usual CW structure for S is a regular $4g$ -gon with edge identifications represented by the word $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$.

However, this structure is not regular. To fix this issue, we add in the perpendicular bisectors of each edge with no new identifications, splitting each external edge into two 1-cells attached by a 0-cell. We also gain one new vertex in the center at the intersection of all the bisectors. This gives a regular CW structure for S

²These definitions are standard, but the terminology is not: the first is typically called a “minimal finite space” or simply “minimal”, and the second a “minimal finite model”. This nomenclature allows for such peculiar entities as spaces which are both minimal and finite models, but are not minimal finite models. We use different terminology to avoid confusion.

³We assume throughout that our surfaces are connected, as all our results can be immediately generalized to disconnected closed surfaces by taking coproducts.

and thus a finite model (see Figure 2). It is easy to check that this model has $14g + 2$ points. Note that every 0-cell is contained in at least two 1-cells, every 1-cell contains exactly two 0-cells and is contained in at least two 2-cells, and every 2-cell contains exactly two 1-cells. Consequently, no vertex in the Hasse diagram of the model has in-degree or out-degree 1, so there are no beat points. Thus, this model is Stong minimal.

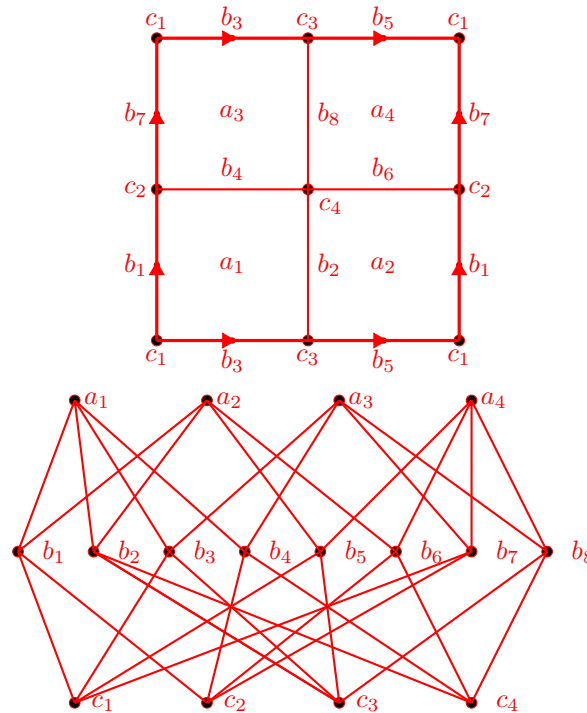


Figure 2: The regular CW structure and associated finite model for the orientable surface of genus 1. Taken from [16].

The construction of the models for nonorientable surfaces is similar. Given a nonorientable surface S of genus g , we begin with the usual CW model: the regular $2g$ -gon with edge identifications given by the word $a_1^2 \dots a_g^2$. To make this regular, we add in both the perpendicular bisectors of the edges and the line segments between opposing vertices. This yields a regular CW structure with $11g + 2$ points, and the face poset is Stong minimal for the same reason as in the orientable case.

5.0.2. Elementary bounds. In this section, we use the weak homotopy invariance of Euler characteristic and several results from [16] to derive lower bounds for the size of arbitrary finite models of closed surfaces other than S^2 and $\mathbb{R}P^2$, whose absolutely minimal finite models are already known. We denote the Euler characteristic of a space X by $\chi(X)$ and the cardinality of X by $\#X$.

In [16], Cianci and Ottina define what they call a *splitting property* (S2) for finite posets. The details are not relevant, but the following result they derive is.

PROPOSITION 5.0.3. *Let X be a finite T_0 space which is Eilenberg-MacLane of type $(G, 1)$. If X satisfies (S2) then $H_1(X)$ is free abelian and $H_n(X) = 0$ for $n > 1$.*

We obtain the following corollary.

PROPOSITION 5.0.4. *No closed surface other than S^2 or $\mathbb{R}P^2$ can have a model satisfying (S2).*

PROOF. Let S be a closed surface that is not S^2 or $\mathbb{R}P^2$. Then S is covered by \mathbb{R}^2 , so it is Eilenberg-MacLane of type $(\pi_1(S), 1)$. If S is nonorientable, then $H_1(S)$ is not free abelian, and if S is orientable, $H_2(S)$ is nontrivial. Since homology is a weak homotopy invariant, the same is true of any finite model of S , so the result follows by the previous proposition. \square

There are two relevant consequences derived in [16] of not satisfying (S2).

PROPOSITION 5.0.5. *If X is a finite T_0 space not satisfying (S2) that is connected and Stong minimal, then $\#X \geq 16$.*

PROPOSITION 5.0.6. *If X is a finite T_0 space with at most two maximal points or at most two minimal points, X satisfies (S2).*

These give us our lower bounds.

THEOREM 5.0.7. *Let X be a finite model of a surface S other than S^2 or $\mathbb{R}P^2$. Then $\#X \geq \max(16, \log_2(|\chi(S)|))$.*

PROOF. Since we can reduce any model to a Stong minimal model by removing beat points, we may assume without loss of generality that X is already Stong minimal. Furthermore, because path-connectedness is detected by homology, X must be path-connected and thus connected. It follows that $\#X \geq 16$.

To obtain the other bound, note that since Euler characteristic is weak homotopy invariant, $\chi(S) = \chi(\mathcal{K}(X))$, which is the alternating sum of the number of chains in X of various lengths, $\sum_{k \text{ a chain}} (-1)^{\#k+1}$. By the triangle inequality, the absolute value of the Euler characteristic must be less than or equal to the total number of chains in X , $\sum_{k \text{ a chain}} 1$. Since chains are subsets of X , this is less than or equal to the number of subsets of X , $2^{\#X}$. The result follows. \square

We can improve our logarithmic bound to a square root bound in the case where X has height 3.

PROPOSITION 5.0.8. *Let X be a height-3 finite model of a surface other than S^2 or $\mathbb{R}P^2$. Then $\#X \geq \sqrt{2|\chi(S) - 7|}$.*

PROOF. Let $n = \#X$. The only negative contribution to $\chi(\mathcal{K}(X))$ is from the edges, of which there are at most $\binom{n}{2}$ since they are 2-chains in X . We know that there are at least 6 vertices since X does not satisfy (S2): there are at least three maximal points and three minimal points and no point can be both maximal and minimal since X is connected. We also know there must be at least one face because X is of height 3. Thus, there must be at least $|\chi(S) - 7|$ edges, so $n^2 \geq n^2 - n = 2\binom{n}{2} \geq 2|\chi(S) - 7|$, from which the result follows. \square

It is conceivable that this method could be extended to posets of greater height. (It is trivial from the simplicial homology of $\mathcal{K}(X)$ that any finite model must have height at least 3.)

5.1. the Euler characteristic of finite representations of homology manifolds

Define closed homology manifold.

Though the formula of the Euler characteristic of a finite T_0 space X has already been derived, there is more that can be said when the geometric realization of the associated simplicial complex $|K(X)|$ is a closed homology manifold.

Given a topological space X , let $\chi(X)$ be the Euler characteristic of X . If K is a finite simplicial complex, it is clear that

$$\chi(|K|) = \sum_{\sigma \in K} (-1)^{\dim(\sigma)}.$$

Let X be a finite T_0 space. Since $|K(X)|$ and X are weak homotopy equivalent, their homology groups are isomorphic, and hence they have the same Euler characteristic. Let $\mathcal{C}(X)$ be the set of non-empty chains of X . The definition of $|K(X)|$ allows us to conclude

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{ht(C)}$$

[7].

We can relate the Euler characteristic of a finite T_0 space X to the Euler characteristics of lower links in X with the following proposition.

PROPOSITION 5.1.1. *Let X be a finite T_0 space. Then*

$$\chi(X) = \sum_{x \in X} (1 - \chi(\hat{U}_x)).$$

PROOF. Proof by induction on the cardinality $\#X$ of X . The case $\#X = 0$ is trivial. Assume our hypothesis is true for $\#X = k$. Let $\#X = k + 1$, and let $x_0 \in X$ be a maximal point. Since $x_0 \notin \hat{U}_y$ for all $y \neq x_0$, we have

$$\chi(X \setminus \{x_0\}) = \sum_{y \in X \setminus \{x_0\}} (1 - \chi(\hat{U}_y^X))$$

by our hypothesis. Furthermore,

$$\begin{aligned} \chi(X) &= \sum_{C \in \mathcal{C}(X)} (-1)^{ht(C)} \\ &= \sum_{C \in \mathcal{C}(X), x_0 \in C} (-1)^{ht(C)} + \sum_{D \in \mathcal{C}(X), x_0 \notin D} (-1)^{ht(D)} \\ &= \sum_{C \in \mathcal{C}(X), x_0 \in C} (-1)^{ht(C)} + \chi(X \setminus \{x_0\}). \end{aligned}$$

Clearly, if $x_0 \in C \subset X$, then $C \in \mathcal{C}(X)$ if and only if $C = \{x_0\}$ or $C \setminus \{x_0\} \in \mathcal{C}(\hat{U}_{x_0})$. Hence,

$$\begin{aligned} \sum_{C \in \mathcal{C}(X), x_0 \in C} (-1)^{ht(C)} &= 1 - \sum_{C \in \mathcal{C}(\hat{U}_{x_0})} (-1)^{ht(C)} \\ &= 1 - \chi(\hat{U}_{x_0}). \end{aligned}$$

Our induction immediately follows. \square

Of course, this proof can be altered slightly to provide an analogous result for upper links.

We now reach the main result.

THEOREM 5.1.2. *Let X be a finite T_0 space. If $|(K)(X)|$ is a closed homology manifold, then*

$$\chi(X) = \sum_{x \in X} (-1)^{\ell_X(x)}$$

PROOF. Recall that a compact polyhedron M is a closed homology manifold if its underlying simplicial complex K is such that for any simplex σ of K , the homology groups of $|lk_K(\sigma)|$ are isomorphic to the homology groups of $S^{\dim(M) - \dim(\sigma) - 1}$.⁴ Note that the polyhedron condition implies that K is pure.

For $x \in X$, let C be a maximal chain in X containing x , and let $C_{\geq} = \{y \in C \mid y \geq x\}$. Since $(K)(X)$ is pure, X is pure, so $ht(C_{\geq}) = ht(\hat{F}_x^X) + 1$. Furthermore, $lk_X(C_{\geq}) = \hat{U}_x^X$. Hence,

$$\begin{aligned} \chi(\hat{U}_x^X) &= \chi(S^{ht(X) - ht(C_{\geq}) - 1}) \\ &= 1 + (-1)^{ht(X) - ht(\hat{F}_x^X)} \\ &= 1 + (-1)^{ht(X) - ht(\hat{U}_x^{X^{OP}})} \\ &= 1 + (-1)^{ht(X) - \ell_{X^{OP}}(x) + 1} \\ &= 1 + (-1)^{\ell_X(x) + 1} \end{aligned}$$

Our result follows from the above proposition □

With this result, we can now provide another proof of a well-known fact.

COROLLARY 5.1.3. *All odd-dimensional polyhedral closed homology manifolds have Euler characteristic 0.*

PROOF. Let M be an odd-dimensional polyhedral homology manifold with underlying complex K . Then K_{Δ} is a finite T_0 space such that $(K)(K_{\Delta})$ is a triangulation of M . Thus,

$$\chi(X) = \sum_{x \in K_{\Delta}} (-1)^{\ell_{K_{\Delta}}(x)}.$$

But $(K_{\Delta})^{OP}$ is also a finite T_0 space such that $(K)((K_{\Delta})^{OP})$ is a triangulation of M , so

$$\chi(X) = \sum_{x \in (K_{\Delta})^{OP}} (-1)^{\ell_{(K_{\Delta})^{OP}}(x)}.$$

Since $\ell_{K_{\Delta}}(x) = ht_{K_{\Delta}}(x) - \ell_{(K_{\Delta})^{OP}}(x)$, and since $ht(K_{\Delta})$ is odd, $\ell_{K_{\Delta}}(x)$ and $\ell_{(K_{\Delta})^{OP}}(x)$ have different parities. Hence we conclude that $\chi(X) = -\chi(X) = 0$, and thus that $\chi(M)$. □

⁴Note the similarity between this definition and piecewise-linear triangulations of a manifold, in which the link of a simplex is homeomorphic to a sphere of appropriate dimension.

Simplicial complexes

6.1. Abstract and ordered simplicial complexes

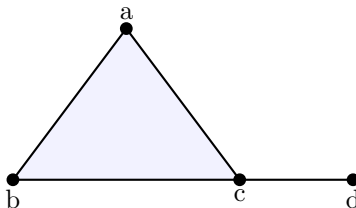
Simplicial complexes provide a general class of spaces that is sufficient for most purposes of basic algebraic topology. There are more general classes of spaces, in particular the CW complexes, that are more central to the modern development of the subject, but they give exactly the same collection of homotopy types, as we shall recall. We shall give a quick introduction to simplicial complexes here, largely restricting ourselves to what we shall use later. More detail can be found in many textbooks in algebraic topology (although not in my own book [48]). However, it is hard to find as precise a demarkation between simplicial complexes and ordered simplicial complexes as is needed for conceptual understanding, and this will become increasingly important as we go on. We implicitly focus on finite simplicial complexes, waiting for simplicial sets for full rigor in the infinite case.

DEFINITION 6.1.1. An *abstract simplicial complex* K is a set $V = V(K)$, whose elements are called *vertices*, together with a set K of (non-empty) finite subsets of V , whose elements are called *simplices*, such that every vertex is an element of some simplex and every subset of a simplex is a simplex; such a subset is called a *face* of the given simplex. We say that K is finite if V is a finite set. The *dimension* of a simplex is one less than the number of vertices in it.

EXAMPLE 6.1.2. Abstract complexes can be understood in a diagrammatic way. Consider for example, the abstract simplicial complex whose vertex set and simplices are given by

$$V(K) = \{a, b, c, d\}, K = \{a, b, c; ab, bc, ac, cd; abc\}.$$

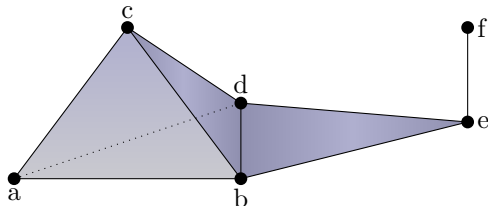
The vertices contain one simplex, and thus have dimension zero. They can be drawn simply as points. The simplices composed of two vertices can be drawn as lines as they are one-dimensional objects. Finally, faces (simplices containing three vertices) are shaded to indicate solidity. This produces the diagram pictured:



EXAMPLE 6.1.3. Notice that K must contain all points in the vertex set, as well as all subsets of the vertices included in higher dimensional simplices. Thus, we note the following examples that are NOT abstract simplicial complexes.

- (1) $V = \{a, b, c\}$, $K = \{a, b; ab\}$
- (2) $V = \{a, b, c\}$, $K = \{a, b, c; ab, bc; abc\}$

EXERCISE 6.1.4. Write down the set of simplices of the simplicial complex X given below:



DEFINITION 6.1.5. A map $g: K \rightarrow L$ of abstract simplicial complexes is a function $g: V(K) \rightarrow V(L)$ that takes simplices to simplices. That is, for each set S in K , the set $g(S)$ is in L . We say that K is a *subcomplex* of L if the vertices and simplices of K are some of the vertices and simplices of L . We say that K is a *full subcomplex* of L if, further, every simplex of L whose vertices are in K is a simplex of K .

As already said, there is a very important distinction to be made between simplicial complexes as we have just defined them and *ordered* simplicial complexes.

DEFINITION 6.1.6. An *ordering* of an abstract simplicial complex K is a partial order on the vertices of K that restricts to a total order on the vertices of each simplex of K . A map of ordered simplicial complexes is a map of simplicial complexes that is given by an order preserving map on its poset of vertices.

While imposition of an ordering may seem artificial, since we have no canonical choice, it is essential to a serious calculational theory. We shall later introduce simplicial sets, which generalize simplicial complexes and elegantly systematize orderings. Many of the definitions below have evident ordered variants. We shall not belabor the point. However, orderings will be essential to understanding the relationship between simplicial complexes and finite spaces. Of course, this is not surprising since finite spaces are essentially the same as finite posets.

Unless otherwise stated, simplicial complexes without an adjective (such as ordered or geometric) means abstract simplicial complexes henceforward.

6.2. Geometric simplicial complexes

Following geometric intuition, we must first define geometric simplices.

DEFINITION 6.2.1. Let $\{v_0, \dots, v_n\}$ be a set of points in some \mathbb{R}^N such that the vectors

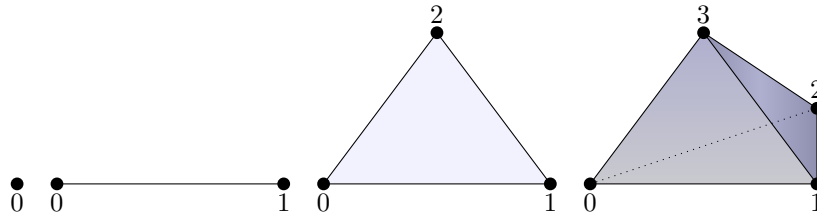
$$\{(v_1 - v_0), (v_2 - v_0) \dots (v_n - v_0)\}$$

are linearly independent. The (geometric) *n-simplex* σ spanned by $\{v_0, \dots, v_n\}$ is the set of all points $\sum_{i=0}^n t_i v_i$, where $0 \leq t_i \leq 1$ and $\sum t_i = 1$. The t_i are called the *barycentric coordinates* of the point x . When each $t_i = 1/(n+1)$, the point x is called the *barycenter* of σ . The points v_i are the *vertices* of σ . A simplex spanned by a subset of the vertices is a *face* of σ ; it is a *proper face* if the subset is proper.

DEFINITION 6.2.2. The *standard* n -simplex $\Delta[n]$ is the n -simplex spanned by the standard basis of \mathbb{R}^{n+1} . Thus the standard 0-simplex is the point $1 \in \mathbb{R}$, the standard 1-simplex is the line $\{t, 1-t\} \subset \mathbb{R}^2$, and so forth. Later, when necessary for clarity, we will sometimes denote these topological n -simplices by $\Delta[n]^t$ to distinguish them from other kinds of n -simplices that will appear.

As we noticed before, n -simplices are easy to visualize for small n .

EXAMPLE 6.2.3. Below are depictions of the standard ordered 0-, 1-, 2-, and 3-simplices $\Delta[0]$, $\Delta[1]$, $\Delta[2]$ and $\Delta[3]$.



EXERCISE 6.2.4. How many 0,1 and 2-dimensional simplices does the 3-dimensional simplex shown above contain?

DEFINITION 6.2.5. A *geometric simplicial complex* K is a set of simplices in some \mathbb{R}^N such that every face of a simplex in K is a simplex in K and the intersection of two simplices in K is a simplex in K . The set of vertices of K is the union of the sets of vertices of its simplices. Note that although we require all vertices to lie in some \mathbb{R}^N and we require each set of vertices that spans a simplex of K to be geometrically independent, we do not require the entire set of vertices to be geometrically independent. For example, we can have three vertices on a single line in \mathbb{R}^N , as long as the two vertices furthest apart do not span a 1-simplex of K . A subcomplex L of a simplicial complex K is a simplicial complex whose simplices are some of the simplices of K . It is a full subcomplex if every simplex of K with vertices in L is in L .

The simplices of a geometric simplicial complex are the building blocks of a subspace of \mathbb{R}^N .

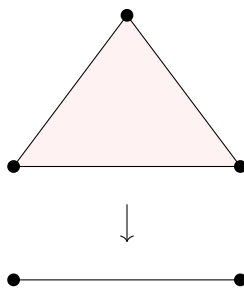
DEFINITION 6.2.6. The *geometric realization* $|K|$ of a geometric simplicial complex K is the union of the simplices of K , each regarded as a subspace of \mathbb{R}^N , with the topology whose closed sets are the sets that intersect each simplex in a closed subset. If K is finite, but not in general otherwise, this is the same as the topology of $|K|$ as a subspace of \mathbb{R}^N . The open simplices of $|K|$ are the interiors of its simplices (where a vertex is an interior point of its 0-simplex), and every point of $|K|$ is an interior point of a unique simplex. The *boundary* $\partial\sigma$ of a simplex σ is the subcomplex given by the union of its proper faces. A space homeomorphic to $|K|$ for some K is called a *polytope*.

The *dimension* of a simplicial complex is the maximal dimension of its simplices, and that of course corresponds to our geometric intuition.

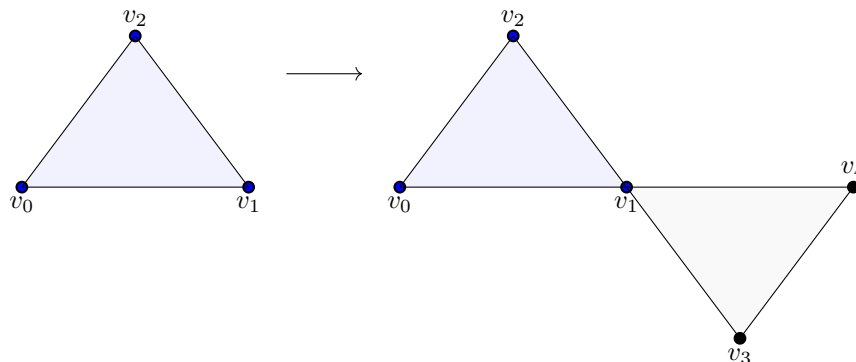
DEFINITION 6.2.7. Maps $g: K \rightarrow L$ of geometric simplicial complexes are defined in the same way as maps of abstract simplicial complexes: g is a function from the vertex set $V(K)$ to the vertex set $V(L)$ such that, for each subset S of

$V(K)$ that spans a simplex of K , the set $g(S)$ is the set of vertices of a simplex of L . Then g determines the continuous map $|g|: |K| \rightarrow |L|$ that sends $\sum t_i v_i$ to $\sum t_i g(v_i)$. Note that although we do not require g to be one-to-one on vertices, $|g|$ is nevertheless well-defined and continuous. If g is a bijection on vertices and simplices, we say that it is an isomorphism, and then $|g|$ is a homeomorphism.

EXAMPLE 6.2.8. An important example of a simplicial map is one that collapses simplices. The introduction of simplicial sets later in the book preserves the information given by all possible collapses of faces, such as the one pictured below.



EXAMPLE 6.2.9. Inclusions of geometric simplicial complexes are easily visualized since the image must be a subcomplex, as in the following picture.



It is usual to abbreviate $|g|$ to g and to refer to it as a simplicial map, but for now we prefer to keep the distinction between g and $|g|$ clear.

REMARK 6.2.10. The reader can and should object to our insistence that all of the vertices of K are in some \mathbb{R}^N . Why not allow an infinite set of vertices with no bound on the allowed size of the simplices? The idea is to take the topological space given by the disjoint union of the simplices of a geometric simplicial complex, ignoring their embeddings in Euclidean space, and to then form a quotient space by gluing them together along their common faces. We might instead think of sets of standard n -simplices $\Delta[n]$, and we might think of taking their disjoint union and then gluing together along prescribed faces to construct the geometric realization more abstractly. We shall allow ourselves to think of such infinite dimensional simplicial complexes, but it is best not to take them too seriously for now. We shall come back to them under the guise of simplicial sets, which are best treated later. In that context, we will make the intuition precise and show how best to define geometric realization in general.

6.3. Comparison of abstract and geometric simplicial complexes

DEFINITION 6.3.1. The abstract simplicial complex aK determined by a geometric simplicial complex K has vertex set the union of the vertex sets of the simplices of K . Its simplices are the subsets that span a simplex of K . An abstract finite simplicial complex K determines a geometric finite simplicial complex gK by choosing any bijection between the vertices of K and a geometrically independent subset of some \mathbb{R}^N . For specificity, we can take the standard basis elements of \mathbb{R}^N where N is the number of points in the vertex set $V(K)$. The geometric simplices are spanned by the images of the simplices of K under this bijection. For an abstract simplicial complex K , agK is isomorphic to K , the isomorphism being given by the chosen bijection. Similarly, for a finite geometric simplicial complex K , gaK is isomorphic to K .

We could remove the word finite from the previous definition by defining geometric simplicial complexes more generally, without reference to a finite dimensional ambient space \mathbb{R}^N , as in Remark 6.2.10. We also note that we do not have to realize in such a high dimensional Euclidean space as a count of vertices would dictate. The following result holds no matter how many vertices there are. It is rarely used, but it is conceptually attractive. A proof can be found in [34, 1.9.6].

THEOREM 6.3.2. *Any finite simplicial complex K of dimension n can be geometrically realized in \mathbb{R}^{2n+1} .*

REMARK 6.3.3. Abstract and geometric simplicial complexes contain the same combinatorial information and an abstract simplicial complex can be constructed from a geometric one without ambiguity. However, with our present definitions a finite geometric simplicial complex can only be constructed up to homeomorphism from an abstract one. Use of ordered simplicial complexes and simplicial sets will later give an elegant way around this and will do so without the finiteness restriction.

In view of the discussion above, abstract and geometric finite simplicial complexes can be used interchangeably. In particular, the geometric realization of a finite abstract simplicial complex K is understood to mean the geometric realization of any gK .

We need a criterion for when the geometric realizations of two simplicial maps are homotopic.

DEFINITION 6.3.4. Continuous maps f and g from a topological space X to the geometric realization $|K|$ of a simplicial complex are *simplicially close* if, for each $x \in X$, both $f(x)$ and $g(x)$ are in a single simplex $\sigma(x)$ of K .

PROPOSITION 6.3.5. *If f and g are simplicially close continuous maps from a topological space X to some $|K| \subset \mathbb{R}^N$, then f and g are homotopic.*

PROOF. Define $h: X \times I \rightarrow \mathbb{R}^N$ by

$$h(x, t) = (1 - t)f(x) + tg(x).$$

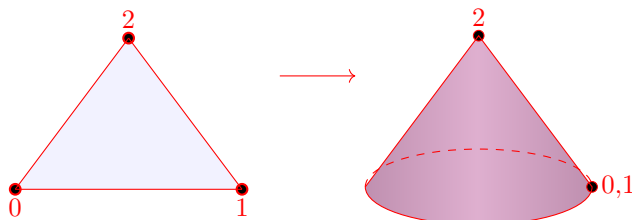
Since simplices are convex, $h(x, t)$ is in some $\sigma(x) \subset |K|$. From this, we see that h is continuous and specifies a homotopy as required. \square

6.4. Cones and subdivisions of simplicial complexes

We've so far seen cones on topological spaces, as well as cones on F -spaces. The notion of a cone also exists for abstract and finite geometric simplicial complexes.

DEFINITION 6.4.1. The *cone* $K * x$ on an abstract simplicial complex K is constructed by adding a new vertex x and taking the simplices to be all subsets of all unions of x with a simplex in K .

EXAMPLE 6.4.2. The geometric motivation behind the cone can be easily seen in the case of $|\Delta[2]| * x$. Notice that this does not again yield a simplicial complex, as the faces of the simplex are no longer unique.



If K is instead a finite geometric simplicial complex in \mathbb{R}^n , consider x as a point of $\mathbb{R}^n - K$ such that each ray starting at x intersects $|K|$ in at most one point. Observe that the union of $\{x\}$ and the set of vertices of a simplex of K is a geometrically independent set. Define the cone $K * x$ on K with vertex x to be the geometric simplicial complex whose simplices are all of the faces of the simplices spanned by such unions.

REMARK 6.4.3. Notice that K is a subcomplex of $K * x$, x is the only vertex not in K , and $|K * x|$ is homeomorphic to $C|K|$.

EXAMPLE 6.4.4. A simplex is the cone of any one of its vertices with the subcomplex spanned by the remaining vertices (the opposite face).

Subdivisions of simplicial complexes will play a central role in our work.

DEFINITION 6.4.5. A simplicial complex L is a *subdivision* of a simplicial complex K if $V(K) \subset V(L)$, each simplex of K is contained in a simplex of L , and each simplex of K is the union of finitely many simplices of L . If L is ordered, then the partial order on $V(L)$ restricts to a partial order on $V(K)$ that gives K an ordering.

DEFINITION 6.4.6. The *canonical subdivision* K' of an abstract simplicial complex K is the ordered simplicial complex whose vertices are the simplices of K , partially ordered by inclusion, and whose simplices are the totally ordered finite subsets $\{\sigma_0, \dots, \sigma_n\}$ of simplices of K . With $\sigma_0 < \dots < \sigma_n$ we call σ_n the barycenter of the simplex $\{\sigma_0, \dots, \sigma_n\}$.

DEFINITION 6.4.7. A subdivision L of a finite geometric simplicial complex K is a geometric simplicial complex such that each simplex of L is contained in a simplex of K and each simplex of K is the union of finitely many simplices of L .

The following observation should be clear.

LEMMA 6.4.8. *If L is a subdivision of K , then $|L| = |K|$ (as spaces).*

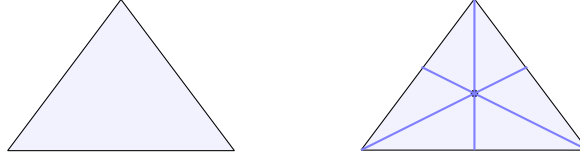
The n -skeleton K^n of K is the union of the simplices of K of dimension at most n . It is a subcomplex. There are many ways to subdivide both abstract and geometric simplicial complexes, and in applications there can be advantages to one or another of them. However, we will focus on the standard canonical choices. We have defined that already for abstract simplicial complexes. We give a somewhat

pedantic inductive geometric construction for geometric simplicial complexes that should make the idea clear and then reexpress the answer combinatorially, proving that the canonical choices agree under the passages a and g back and forth.

CONSTRUCTION 6.4.9. We construct the *barycentric subdivision* K' of a geometric simplicial complex K . We subdivide the skeleta of K inductively. Let $L_0 = K^0$. Suppose that a subdivision L_{n-1} of K^{n-1} has been constructed. Let b_σ be the barycenter of an n -simplex σ of K . The space $|\partial\sigma|$ coincides with $|L_\sigma|$ for a subcomplex L_σ of L_{n-1} , and we can define the cone $L_\sigma * b_\sigma$. Clearly $|L_\sigma * b_\sigma| = |\sigma|$ and $|L_\sigma * b_\sigma| \cap |L_{n-1}| = |L_\sigma| = |\partial\sigma|$.

If τ is another n -simplex, then $|L_\sigma * b_\sigma| \cap |L_\tau * b_\tau| = |\sigma \cap \tau|$, which is the realization of a subcomplex of L_{n-1} and therefore of both L_σ and L_τ . Define L_n to be the union of L_{n-1} and the complexes $L_\sigma * b_\sigma$, where σ runs over all n -simplices of K . Our observations about intersections show that L_n is a simplicial complex which contains L_{n-1} as a subcomplex. The union of the L_n is denoted K' and called the barycentric subdivision of K .

EXAMPLE 6.4.10. The barycentric subdivision of a 2-simplex is easily visualized pictorially.



The second barycentric subdivision of K is the barycentric subdivision of the first barycentric subdivision, and so on inductively.

We can enumerate the simplices of K' explicitly rather than inductively.

PROPOSITION 6.4.11. *Define $\sigma < \tau$ if σ is a proper face of τ . Then K' is the geometric simplicial complex whose vertices are the barycenters of simplices of K and whose n -simplices σ' are the spans of the geometrically independent sets $\{b_{\sigma_0}, \dots, b_{\sigma_n}\}$, where $\sigma_0 > \dots > \sigma_n$. The vertex b_{σ_0} is called the leading vertex of the simplex σ' .*

PROOF. We show this inductively for the subcomplexes L_n . Since $L_0 = K^0$, this is clear for L_0 . Assume that it holds for L_{n-1} . If τ is a simplex of L_n such that $|\tau|$ is contained in $|K^n|$ but not contained in K^{n-1} , then τ is a simplex in the cone $L_\sigma * b_\sigma$ for some n -simplex σ . By the induction hypothesis and the definition of L_σ , each simplex of L_σ is the span of a set $\{b_{\sigma_0}, \dots, b_{\sigma_m}\}$, where $\sigma > \sigma_0 > \dots > \sigma_m$. Therefore τ is the span of a set $\{b_\sigma, b_{\sigma_0}, \dots, b_{\sigma_m}\}$. \square

PROPOSITION 6.4.12. *There is a simplicial map $\xi = \xi_K: K' \rightarrow K$ whose realization is simplicially close to the identity map and therefore homotopic to the identity map.*

PROOF. Let ξ map each vertex b_σ of K' to any chosen vertex of σ . If σ' is a simplex of K' with leading vertex b_{σ_0} , then all other vertices of σ' are barycenters of faces of σ_0 , hence are mapped under ξ to vertices of σ_0 . Therefore the images under ξ of the vertices of σ' span a face of σ_0 , so that ξ is a simplicial map. With these notations, if $x \in |K'|$ is an interior point of the simplex σ' , then it is mapped under $|\xi|$ to a point of $\sigma_0 \supset \sigma'$, and we let $\sigma(x) = \sigma_0$. Since ξ maps every vertex of σ' to a vertex of σ_0 , x and $\xi(x)$ are both in the closure of σ_0 . \square

DEFINITION 6.4.13. Just as for abstract simplicial complexes, we say that a geometric simplicial complex is ordered if its vertices are partially ordered and the partial order restricts to a total order of the vertices of each simplex. For an ordered geometric simplicial complex K , define the *standard simplicial map* $\xi: K' \rightarrow K$ by letting $\xi(b_\sigma)$ be the maximal vertex x_n of the simplex $\sigma = \{x_0, \dots, x_n\}$.

REMARK 6.4.14. Observe that K' has a canonical ordering even when K does not. Explicitly, the partial ordering of the set of vertices $\{b_\sigma\}$ of K' is given by $b_\sigma \leq b_\tau$ if σ is a face of τ . Notice for this that a vertex of K , regarded as a simplex, is its own barycenter. This partial order clearly restricts to a total order on the vertices of each simplex.

PROPOSITION 6.4.15. *If K is a geometric simplicial complex with canonical subdivision K' , then aK' is isomorphic to the canonical subdivision of aK .*

PROOF. Letting the vertex b_σ in K' correspond to the vertex σ in aK , this is an immediate comparison of definitions. \square

REMARK 6.4.16. The barycenters of the simplices of K that are not vertices correspond to the vertices of aK' that are not vertices of aK . All simplices of aK' with more than one vertex have at least one vertex that is not in aK . Thus the only simplices in aK' that are also simplices in aK are the vertices of aK . However, if we think geometrically, then every simplex τ of K' is contained in a unique simplex σ of K , as must be so since K' is a subdivision and is also clear from a picture of the barycentric subdivision. The simplex σ is called the *carrier* of τ .

PROPOSITION 6.4.17. *A simplicial map $g: K \rightarrow L$ induces a subdivided simplicial map $g': K' \rightarrow L'$ whose realization is simplicially close to $|g|$ and hence homotopic to $|g|$. Moreover, g' is order-preserving.*

PROOF. The images under g of the vertices of a simplex σ of K span a simplex $g(\sigma)$, of possibly lower dimension than σ , and we define $g'(b_\sigma) = b_{g(\sigma)}$ on vertices. If b_{σ_0} is the leading vertex of a simplex σ' of K' , then all other vertices of σ' are barycenters of faces of σ_0 . Their images under g' are barycenters of faces of $g(\sigma_0)$. If x is an interior point of σ' , then both $g(x)$ and $g'(x)$ are in the closure of $g(\sigma_0)$. \square

REMARK 6.4.18. When K and L are ordered and g is an order-preserving simplicial map, the following “naturality” diagram commutes if we use the standard simplicial maps ξ for K and L .

$$\begin{array}{ccc} K' & \xrightarrow{g'} & L' \\ \xi \downarrow & & \downarrow \xi \\ K & \xrightarrow{g} & L \end{array}$$

6.5. The simplicial approximation theorem

The classical point of barycentric subdivision is its use in the simplicial approximation theorem, which in its simplest form reads as follows. Starting with $K^{(0)} = K$, let $K^{(n)} = K^{(n-1)'}$ be the n th barycentric subdivision of a simplicial complex K . By iteration of $\xi: K' \rightarrow K$, we obtain a simplicial map $\xi^{(n)}: K^{(n)} \rightarrow K$ whose geometric realization is homotopic to the identity map.

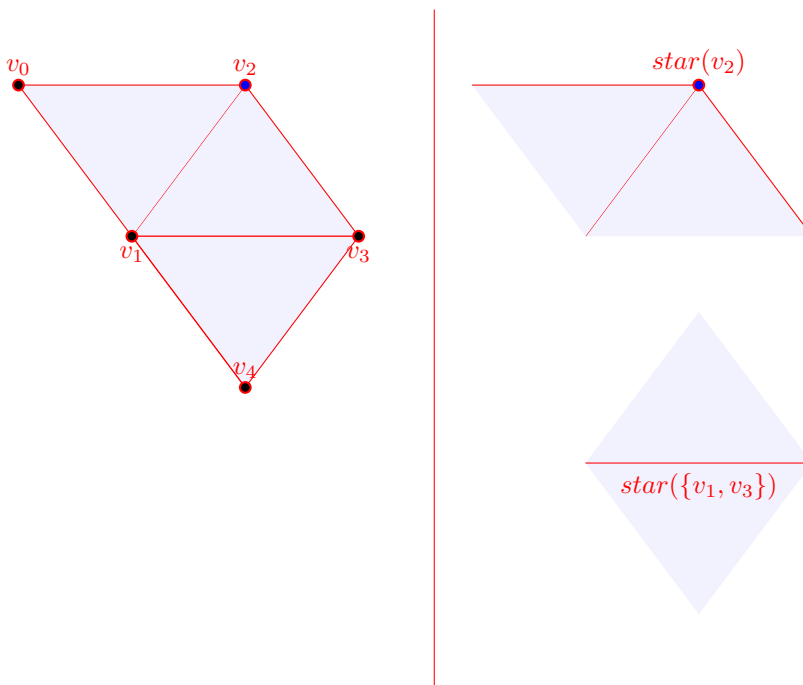
THEOREM 6.5.1. *Let K be a finite simplicial complex and L be any simplicial complex. Let $f: |K| \rightarrow |L|$ be any continuous map. Then, for some sufficiently large n , there is a simplicial map $g: K^{(n)} \rightarrow L$ such that f is homotopic to $|g|$.*

This means that, for the purposes of homotopy theory, general continuous maps may be replaced by simplicial maps. Since this is proved in so many places, we shall content ourselves with a slightly sketchy proof. It relies on the classical Lebesgue lemma, whose proof is not hard but just a little far afield.

LEMMA 6.5.2 (Lebesgue lemma). *Let (X, d) be a compact metric space with a given open cover \mathcal{U} . Then there exists a number $\lambda > 0$ such that every subset of X with diameter less than λ is contained in some set $U \in \mathcal{U}$. The smallest such λ is called the Lebesgue number of the cover.*

DEFINITION 6.5.3. For a vertex v of a simplicial complex K , define $star(v)$ to be the union of the interiors of all simplices of $|K|$ that contain v as a vertex. For a subcomplex L of K , define $star(L) \subset |K|$ to be the union over $v \in L$ of the open spaces $star(v)$.

EXAMPLE 6.5.4. Consider the following simplicial complex. Below illustrates examples of both the star of a vertex, as well as the star of a simplex.



PROOF OF THE SIMPLICIAL APPROXIMATION THEOREM. We are given a map $f: |K| \rightarrow |L|$. Give $|K|$ the open cover by the sets $f^{-1}(star(w))$, where w runs over the vertices of L . Since $|K|$ is a compact subspace of a metric space, the Lebesgue lemma ensures that there is a number λ such that any subset of $|K|$ of diameter less than λ is contained in one of the open sets $f^{-1}(star(w))$. The diameter of a (closed) simplex is easily seen to be the maximal length of a one-dimensional face. Each

barycentric subdivision therefore has the effect of decreasing the maximal diameter of a simplex. Precisely, the maximal diameter of the subdivision of a q -simplex turns out to be $q/q + 1$ times the maximal diameter of the given simplex (e.g. [63, p.124], [34, p.24], [31, p. 120]), but the precise estimate is not important.

What is important is that, since K is finite, for any $\delta > 0$ there is a large enough n such that every simplex of $K^{(n)}$ has diameter less than $\delta/2$. Then each $star(v)$ for a vertex v of $K^{(n)}$ has diameter less than δ , and we conclude that $f(star(v)) \subset star(w)$ for some vertex w of L . Define $g: V(K^{(n)}) \rightarrow V(L)$ by letting $g(v) = w$ for some w such that $f(star(v)) \subset star(w)$. One checks that g maps simplices to simplices and so specifies a map of simplicial complexes. If u is an interior point of a simplex σ of K , then $f(x)$ is an interior point of some simplex τ of L . One can check that g maps each vertex of σ to a vertex of τ . This implies that $|g|$ is simplicially close to f and therefore homotopic to f . \square

A concise introduction to categories

7.1. Categories

In order to properly understand how Alexandroff Spaces interact with the world of classical algebraic topology, we must introduce the language of category theory. The language of categories is utilized in many fields of mathematics, but in this part of the book, we will primarily explore how category theory is used in the study of finite spaces.

DEFINITION 7.1.1. A *category* \mathcal{C} is a collection of objects (X, Y, Z, \dots) , denoted $Ob(\mathcal{C})$, together with, for each pair (X, Y) of objects of \mathcal{C} , a set of *morphisms* (alias maps) $f : X \rightarrow Y$, denoted $\mathcal{C}(X, Y)$, satisfying the following: For each object X of \mathcal{C} there is a given identity morphism $1_X : X \rightarrow X$ and for each triple (X, Y, Z) of objects of \mathcal{C} and pair of morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ there is given a morphism $g \circ f : X \rightarrow Z$. This is viewed as a composition law:

$$\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z).$$

We require $1_Y \circ f = f = f \circ 1_X$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphism h with domain Z .

REMARK 7.1.2. We do not require that $Ob(\mathcal{C})$ be a set; it may be a proper class. If it is a set, we say that the category is small.

EXAMPLE 7.1.3. The collection of all sets is a category denoted $\mathcal{S}et$. Its morphisms are functions. For example, $\mathbb{R} \in Ob(\mathcal{S}et)$ is just the set of all real numbers, without any addition operation, multiplication operation or any topology. A morphism in $\mathcal{S}et$ does not have to be continuous at all, it simply must be well-defined for every point in its domain.

EXAMPLE 7.1.4. The collection of all groups is a category denoted $\mathcal{G}rp$. Its morphisms are group homomorphisms. Given a group G , we can construct its categorical version \mathcal{G} by letting $Ob(\mathcal{G})$ be one object, namely $\{x\}$, and each element $g \in G$ correspond to a unique morphism $g : x \rightarrow x$ such that if $gh = k$ for $g, h, k \in G$, then $gh = k$ as compositions of morphisms. Observe that for any $g \in G$, its corresponding morphism in \mathcal{G} has an inverse, namely the morphism corresponding to g^{-1} . This is another example of a small category.

EXAMPLE 7.1.5. The collection of all topological spaces is a category denoted $\mathcal{T}op$. Its morphisms are continuous functions. Since the number of maps between two space X and Y cannot exceed $|X|^{|Y|}$, we see that $\mathcal{T}op(X, Y)$ is a set, implying that $\mathcal{T}op$ is locally small.

EXAMPLE 7.1.6. A poset P can also be understood as a category such that each object in the category corresponds to a unique element of P , and a morphism

$f : x \rightarrow y$ exists, where $x, y \in P$ iff $x \leq y$. In other words, the morphisms encode all the information about the order relation on the poset. Since the number of morphisms between two objects can be at most 2 and a poset is a set, we see that P is a small category.

EXAMPLE 7.1.7. We can also define the category of posets, denoted \mathcal{P} . $Ob(\mathcal{P})$ is the collection of all posets. This is a large category. A morphism $f : P \rightarrow Q$ for any two posets $P, Q \in Ob(\mathcal{P})$ is an order preserving function. In other words a morphism $f : P \rightarrow Q$ corresponds to a function such that for any $x, y \in P$ with $x \leq y$, $f(x) \leq f(y)$. Since the number of morphisms between P and Q cannot exceed $|P|^{|Q|}$, we see that \mathcal{P} is locally small.

EXAMPLE 7.1.8. Moving in to the world of classical algebraic topology, we can turn the collection of simplicial complexes into a category, denoted \mathcal{SC} . Its objects are simplicial complexes, and its maps are simplicial maps. By an argument made in the previous example, this is also a locally small category.

EXAMPLE 7.1.9. Similarly, we can define the locally small category of ordered simplicial complexes, denoted \mathcal{OSC} , where the objects are ordered simplicial complexes, and the morphisms are simplicial maps that preserve the ordering on the vertices. For a such a map $f : K \rightarrow L$, where K and L are ordered simplicial complexes, $x \leq y$ with $x, y \in V(K)$ implies that $f(x) \leq f(y)$.

EXAMPLE 7.1.10. \mathcal{F} -spaces is the category whose objects are finite T_0 topological spaces and morphisms are continuous maps between them. Observe that \mathcal{F} -spaces is clearly included in \mathcal{Top} , in the sense that every object and morphism that is in \mathcal{F} -spaces is also in \mathcal{Top} . In such a case, we see that \mathcal{F} -spaces is a subcategory of \mathcal{Top} .

EXAMPLE 7.1.11. Another notable subcategory of \mathcal{Top} is that of \mathcal{A} -Spaces, where we restrict our objects to just T_0 Alexandroff spaces, and the morphisms are continuous maps between them. Since every finite space is an Alexandroff Space, we can see that \mathcal{F} -Spaces is a subcategory of \mathcal{A} -Spaces.

EXAMPLE 7.1.12. A *monoid* is a set \mathcal{M} with an associative binary operation and an identity element. Note that in a category \mathcal{C} the composition law \circ on the set $\mathcal{C}(X, X)$ is just such a binary operation with identity element 1_X . Therefore a monoid is a category with one object. A category can be thought of as a “monoid with many objects”.

In any category, there is a notion of isomorphism. It answers the sensible version of the question “when are two things the same”. The nonsensical version would have the answer “when they are equal”. The sensible version interprets “things” to mean objects of a category” and the sensible answer is that we think of two objects as essentially the same when they are isomorphic.

DEFINITION 7.1.13. A morphism $f : X \rightarrow Y$ in a category \mathcal{C} is called an *isomorphism* if there is a morphism $g : Y \rightarrow X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.

EXERCISE 7.1.14. If a morphism f has a left inverse and a right inverse then it is an isomorphism and the left and right inverses coincide.

DEFINITION 7.1.15. A *groupoid* is a category in which every morphism is an isomorphism. Just as a monoid can be defined to be a category with just one object, a *group* can be defined to be a groupoid with just one object. Similarly, a *groupoid* can be thought of as a “group with many objects”.

Finally, in many instances, it becomes necessary to determine whether certain objects in a category were constructed from other objects. In particular, it is valuable to realize when constructions exist in a particular category can be generalized to an arbitrary category. For example, in $\mathcal{S}et$, the Cartesian product of two sets is defined in the familiar fashion:

$$S_1 \times S_2 = \{(s_1, s_2) | s_1 \in S_1, s_2 \in S_2\}$$

. This notion is generalized in the following definition of a product.

DEFINITION 7.1.16. Given any category \mathcal{C} , the *product* $X \times Y$ of two objects $X, Y \in \mathcal{C}$ such that for any $A \in \mathcal{C}$ there exists a unique map $h : A \rightarrow X \times Y$ making the following diagram commute.

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & \downarrow \exists! h & \searrow & \\ X & \xleftarrow{f} & X \times Y & \xrightarrow{\pi_2} & Y \end{array}$$

The following exercises test comprehension:

- EXERCISE 7.1.17. (1) In $\mathcal{T}op$, the product of two topological spaces is nothing but the Cartesian product equipped with the product topology. (Lemma 1.5.7)
- (2) In \mathcal{P} , the product of two posets P, Q exists and is the same product that is defined in ??.
- (3) In the category $\mathcal{S}\mathcal{C}$, the product exists and is the same as in ??.

EXERCISE 7.1.18. For a poset P regarded as a category, what is the product of two elements $x, y \in P$?

7.2. Functors

In the previous chapters it was shown that A -spaces are in bijective correspondence with posets. We can make a similar and formalized statement about their respective categories. Thus, we introduce the following definitions. A morphism of categories is called a functor.

DEFINITION 7.2.1. Let \mathcal{C}, \mathcal{D} be categories. A *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of a rule that assigns to each object X of \mathcal{C} an object FX of \mathcal{D} , together with, for each pair (X, Y) of objects of \mathcal{C} , a function

$$F : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY),$$

written $f \mapsto Ff$, such that $F(1_X) = 1_{FX}$ and $F(g \circ f) = Fg \circ Ff$.

EXERCISE 7.2.2. If f is an isomorphism in \mathcal{C} , then Ff is an isomorphism in \mathcal{D} .

EXAMPLE 7.2.3. The *abelianization of a group* G is the group $G/[G, G]$ where $[G, G]$ is the commutator subgroup, that is, the subgroup generated by the set $\{ghg^{-1}h^{-1} | g, h \in G\}$. Abelianization defines a functor $A : \mathcal{G}rp \rightarrow \mathcal{A}b$ where $\mathcal{A}b$ is the category of abelian groups.

EXAMPLE 7.2.4. The collection of all small categories is a category denoted $\mathcal{C}at$. Its morphisms $F : \mathcal{C} \rightarrow \mathcal{D}$ are the functors.

REMARK 7.2.5. We insist that categories be small for the purposes of this definition to ensure that we have a well-defined set and not just a proper class of functors between any two categories.

EXAMPLE 7.2.6. In $\mathcal{C}at$, the product of two categories $\mathcal{C} \times \mathcal{D}$ is defined such that $Ob(\mathcal{C} \times \mathcal{D}) = \{(c, d) | c \in Ob(\mathcal{C}), d \in Ob(\mathcal{D})\}$ and $Mor(\mathcal{C} \times \mathcal{D}) = \{(f, g) | f \in Mor(\mathcal{C}), g \in Mor(\mathcal{D})\}$.

EXERCISE 7.2.7. Let \cong denote an isomorphism in $\mathcal{C}at$. Prove the following:

- (1) $\mathcal{D} \cong \mathcal{A} - Spaces$
- (2) $\mathcal{F} - Spaces \cong \mathcal{F} \mathcal{D}$ ($\mathcal{F} \mathcal{D}$ is the category of finite posets)

EXERCISE 7.2.8. Given two posets P and Q regarded as categories, describe a functor $F : P \rightarrow Q$. What properties must it satisfy?

DEFINITION 7.2.9. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be *faithful* if the function

$$F : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$$

is injective for every pair (X, Y) of objects of \mathcal{C} .

DEFINITION 7.2.10. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be *full* if the function

$$F : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$$

is surjective for every pair (X, Y) of objects of \mathcal{C} .

DEFINITION 7.2.11. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be an *isomorphism of categories* if there is a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that FG is the identity functor on \mathcal{D} and GF is the identity functor on \mathcal{C} .

DEFINITION 7.2.12. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be *essentially surjective* if, for every object Y of \mathcal{D} , there is an object X of \mathcal{C} and an isomorphism $FX \cong Y$.

DEFINITION 7.2.13. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be an *equivalence of categories* if it is full, faithful, and essentially surjective.

DEFINITION 7.2.14. A *subcategory* of a category \mathcal{C} is a category that consists of some of the objects and some of the morphisms of \mathcal{C} ; it is a full subcategory if it contains all of the morphisms in \mathcal{C} between any two of its objects. The *skeleton* of a category \mathcal{C} is a full subcategory which contains exactly one object from each isomorphism class of objects of \mathcal{C} .

PROPOSITION 7.2.15. *The inclusion of a skeleton of \mathcal{C} in \mathcal{C} is an equivalence of categories.*

PROOF. We understand a skeleton to be a full subcategory, so the inclusion is full and faithful, and it is essentially surjective by definition. \square

7.3. Natural transformations

Naturally, there are also morphisms of functors. Let $F, F' : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A natural transformation $\eta : F \rightarrow F'$ is a collection of maps $\eta_X : FX \rightarrow F'X$, one for each object X of \mathcal{C} , such that the following diagram commutes for each map $f : X \rightarrow Y$ in \mathcal{C} :

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow \eta_X & & \downarrow \eta_Y \\ F'X & \xrightarrow{F'f} & F'Y. \end{array}$$

An example of a natural transformation arises when considering functors between posets when viewed as categories.

DEFINITION 7.3.1. Two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ are *adjoint* if there exists an isomorphism

$$\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd)$$

for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$ that is natural in both variables. Precisely, for a map [add commutative diagram]. In this case, we say that F is *left-adjoint* to G , and that G is *right-adjoint* to F .

PROPOSITION 7.3.2. Given two posets P and Q regarded as categories, and two functors $F, G : P \rightarrow Q$ (order preserving functions from P to Q), there exists a natural transformation $\eta : F \rightarrow G$ iff $F \leq G$, i.e. for all $x \in P$, $F(x) \leq G(x)$.

PROOF. Suppose there exists a natural transformation $\eta : F \rightarrow G$. Then for $x \in Ob(P)$, there exists $\eta_x : F(x) \rightarrow G(x)$, which implies that $F(x) \leq G(x)$. Conversely, suppose that for all $x \in P$, $F(x) \leq G(x)$, where F and G are now viewed as order preserving functions. As functors, this implies that for all $x \in Ob(\mathcal{C})$, there always exists a morphism (let us call it η_x) from $F(x) \rightarrow G(x)$. Now, we can define $\eta : F \rightarrow G$ using η_x as coordinate maps. Further, if $x \leq y$ with $x, y \in Ob(\mathcal{C})$, then we combine the fact that $F(x) \leq F(y)$ and $G(x) \leq G(y)$ with $F(x) \leq G(x)$ and $F(y) \leq G(y)$ via η_x and η_y to get the commutative diagram:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \downarrow \eta_x & & \downarrow \eta_y \\ G(x) & \xrightarrow{G(f)} & G(y). \end{array}$$

□

An analogous notion of similar categories can be defined for functors as well.

DEFINITION 7.3.3. A natural transformation η is said to be a *natural isomorphism* if each of the maps η_X is an isomorphism.

Simply by interpreting the results for [Theorem 8.1.2](#) we observe another example of a natural transformation.

PROPOSITION 7.3.4. (Via [Theorem 8.1.2](#)) There exists a natural transformation $\eta : |K(-)| \rightarrow id_{\mathcal{A}\text{-Spaces}}$, where $|K(-)|, id_{\mathcal{A}\text{-Spaces}} : \mathcal{A}\text{-Spaces} \rightarrow \mathcal{T}op$.

By [Proposition 2.2.12](#), we observe the following:

PROPOSITION 7.3.5. *Given two finite posets P, Q regarded as categories, there exists a natural transformation from F to G , where $F, G : P \rightarrow Q$ iff $F \leq G$.*

This proposition suggests that the concept of a natural transformation “generalizes” the concept of a homotopy between two maps between spaces. To make this more precise, we introduce an equivalent definition of natural transformations.

DEFINITION 7.3.6. Let \mathcal{J} denote the interval category $\{0 \rightarrow 1\}$; or the category with the objects 0 and 1, along with only one non-trivial morphism $0 \rightarrow 1$. A natural transformation between two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ is a functor $H : \mathcal{C} \times \mathcal{J} \rightarrow \mathcal{D}$ such that, on objects $H(x, 1) = F(x)$ and $H(x, 0) = G(x)$, where $x \in \text{Ob}(\mathcal{C})$, and on morphisms $H(f, 1) = F(f)$ and $H(f, 0) = G(f)$, where $f \in \text{Mor}(\mathcal{C})$.

EXERCISE 7.3.7. Check that the previous definitions of natural transformations are equivalent.

DEFINITION 7.3.8. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be an *equivalence of categories* if there is a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and there are natural isomorphisms $FG \rightarrow \text{Id}_{\mathcal{D}}$ and $GF \rightarrow \text{Id}_{\mathcal{C}}$. Note that an isomorphism of categories is an equivalence, but not conversely.

The senior author has a liking for theorems of the following form.

PROPOSITION 7.3.9. *An equivalence of categories is an equivalence of categories.*

That is, the two definitions of what it means for a functor to be an equivalence of categories are equivalent. It is easy to show that if F is an equivalence of categories in our second sense, then F is certainly full, faithful, and essentially surjective. The converse requires a little work and a use of the axiom of choice that the fastidious set-theoretically minded reader may find distasteful: the first step is to choose an object $G(D)$ in \mathcal{C} such that $FG(D)$ is isomorphic to D for each object D of \mathcal{D} . The second is to choose an isomorphism $\eta : FG(D) \rightarrow D$ for each D . We then define G on morphisms so as to make η a natural isomorphism by definition, using that

$$F : \mathcal{C}(G(D), G(D')) \rightarrow \mathcal{D}(FG(D), FG(D'))$$

is a bijection. For a morphism $g : D \rightarrow D'$ in \mathcal{D} , we define $Gg : G(D) \rightarrow G(D')$ to be F^{-1} of the composite

$$FG(D) \xrightarrow{\eta} D \xrightarrow{g} D' \xrightarrow{\eta^{-1}} FG(D').$$

The reader can see how composition must be defined in order to complete the proof.

Note that the proof of [Proposition 7.2.15](#) is easy using our first definition of an equivalence of categories, but not so easy using the second. [Proposition 7.3.9](#) has real force: it makes it easy to recognize equivalences of categories (in the second sense) when we see them.

EXERCISE 7.3.10. Use [Definition 7.2.13](#) to prove the natural equivalence $\mathcal{O}\mathcal{I}\mathcal{C} \simeq \mathcal{I}\mathcal{C}$ via the inclusion functor $i : \mathcal{O}\mathcal{I}\mathcal{C} \rightarrow \mathcal{I}\mathcal{C}$.

7.4. The Yoneda lemma

DEFINITION 7.4.1. Given an element $C \in \mathcal{C}$ for a locally small category \mathcal{C} , we can define a functor $h_C : \mathcal{C} \rightarrow \mathcal{S}et$ such that for any object $D \in \mathcal{C}$, $h_C(D) = Hom(C, D)$. For a morphism $f : D \rightarrow E$ with $E \in \mathcal{C}$, we can define a set map $h_C(f) : Hom(C, D) \rightarrow Hom(C, E)$ by post-composition i.e. $h_C(f)$ sends a morphism $g : C \rightarrow D$ to $f \circ g : C \rightarrow E$.

EXERCISE 7.4.2. Show that for all all objects $C \in \mathcal{C}$, the functor h_C is well defined.

DEFINITION 7.4.3. A functor $F : \mathcal{C} \rightarrow \mathcal{S}et$ is *representable* if there exists an object $C \in \mathcal{C}$, such that F is naturally isomorphic to h_C .

The Yoneda Lemma gives an "upper bound" on the number of representable functors can be assigned to a given h_C .

THEOREM 7.4.4. For any object $C \in \mathcal{C}$, and any functor $F : \mathcal{C} \rightarrow \mathcal{S}et$, the set of natural transformations from h_C to F , denoted $Nat(h_C, F)$, is in bijective correspondence with $F(C) \in \mathcal{S}et$. In other words, $Nat(h_C, C) \simeq F(C)$ in $\mathcal{S}et$.

PROOF. Consider a natural transformation in $Nat(h_C, C)$ defined by the maps $\eta_A : Hom(C, A) \rightarrow F(A)$.

Define the set map $\Theta : Nat(h_C, C) \rightarrow F(C)$ such that

$$\Theta(\eta) = \eta_C(id_C).$$

Further define the set map $\Psi : F(C) \rightarrow Nat(h_C, C)$ such that every $d \in F(C)$ gives the natural transformation $\Psi(d) := \eta^d : h_C \rightarrow C$ where

$$\eta_X^d : (f : C \rightarrow X) \mapsto F(f)(d)$$

with $F(f) : F(C) \rightarrow F(X)$.

We will prove that $\Psi = \Theta^{-1}$. For $\eta \in Nat(h_C, C)$, we have that

$$\Psi(\Theta(\eta)) = \Psi(\eta_C(id_C)) = \eta^{h_C(id_C)}.$$

Thus, any given morphism $f : C \rightarrow X$ is sent to $F(f)(\eta_C(id_C))$ via $\eta_X^{h_C(id_C)}$. In addition, since η is a natural transformation from h_C to F , the following diagram commutes.

$$\begin{array}{ccc} Hom(C, C) & \xrightarrow{\eta_C} & F(C) \\ \downarrow h_C(f) & & \downarrow F(f) \\ Hom(C, X) & \xrightarrow{\eta_X} & F(X). \end{array}$$

Thus, we have that $\eta_X(h_C(f)(id_C)) = F(f)(\eta_C(id_C))$. By definition of h_C ,

$$\eta_X(h_C(f)(id_C)) = \eta_X(f \circ id_C) = \eta_X(f),$$

and therefore $\eta_X^{h_C(id_C)} = \eta_X$. Thus Ψ is a left inverse of Θ . Now for any $d \in F(C)$:

$$\begin{aligned} \Theta(\Psi(d)) &= \Theta(\eta^d) \\ &= \eta_C^d(id_C) \\ &= F(id_C)(d) = d. \end{aligned}$$

Therefore, Ψ is also a right inverse of Θ . We can therefore conclude the set isomorphism $Nat(h_C, C) \simeq F(C)$. \square

A similar statement can be made for contravariant functors.

7.5. The fundamental groupoid of a space

We illustrate the idea of translating topology into algebra by explaining the fundamental groupoid. This brief section will leave the drawing of relevant diagrams to the reader.

We construct a functor $\Pi : \mathcal{T}op \rightarrow \mathcal{G}pd$, where $\mathcal{G}pd$ is the full subcategory of $\mathcal{C}at$ whose objects are groupoids. For a topological space X , the objects of the category ΠX are the points of the space X . Let $I = [0, 1]$ be the unit interval. A path $p : x \rightarrow y$ is a continuous map $p : I \rightarrow X$ such that $p(0) = x$ and $p(1) = y$. Two paths p and p' from x to y are said to be equivalent if there is a map $h : I \times I \rightarrow X$ such that, for all $t \in I$,

$$h(t, 0) = x, \quad h(t, 1) = y, \quad h(0, t) = p(t), \quad \text{and} \quad h(1, t) = p'(t).$$

h is said to be a homotopy from p to p' through paths from x to y . The set of morphisms $x \rightarrow y$ in ΠX is the set of equivalence classes of paths $x \rightarrow y$. For a path $q : y \rightarrow z$, the composite $q \circ p$ is defined by

$$(q \circ p)(t) = \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Define id_x to be the constant path at x , $\text{id}_x(t) = x$. Define $p^{-1}(t) = p(1 - t)$. Composition is not associative or unital, but it becomes so after passage to equivalence classes. Verifications that we leave to the reader (or the first chapter of [48]) show that ΦX is a well-defined groupoid. For a map $f : X \rightarrow Y$, we define Πf on objects by sending x to $f(x)$ and on morphisms by sending the equivalence class $[p]$ to the equivalence class $[f \circ p]$. Then Π is a well-defined functor.

If we fix basepoints, we get a functor that is perhaps more familiar. The fundamental group of X at the basepoint x is the group $\pi_1(X, x)$ given by the morphisms $x \rightarrow x$ in the groupoid ΠX . If we define $\mathcal{T}op_*$ to be the category of spaces X with a chosen basepoint x and maps $f : X \rightarrow Y$ that preserve basepoints, $f(x) = y$, then π_1 gives a functor from based spaces to groups, called the fundamental group functor. Its construction is the first step towards algebraic topology.

EXERCISE 7.5.1. By definition, $\pi_1(X, x)$, regarded as a category with a single object x , is a full subcategory of ΠX . Show that if X is path connected, then $\pi_1(X, x)$ is a skeleton of ΠX . Thus the essential information in ΦX is captured by the fundamental group.

The relation between A-spaces and simplicial complexes

Following McCord [50], we are going to relate A -spaces, and in particular F -spaces, with simplicial complexes, explaining how to go back and forth between them. Since any Alexandroff space is homotopy equivalent to a T_0 -space, there is no loss of generality if we restrict attention to A -spaces. As usual, the reader may prefer to think only in terms of F -spaces.

8.1. The construction of simplicial complexes from A-spaces

DEFINITION 8.1.1. Let X be an A -space. Define $\mathcal{K}(X)$ to be the abstract simplicial complex whose vertex set is X and whose simplices are the finite totally ordered subsets of the poset X ; $\mathcal{K}(X)$ is often called the *order complex* of A . Observe that the partial order of X gives an ordering of $\mathcal{K}(X)$, since it restricts to a total order on each simplex. Observe too that if V is a subspace of X , then $\mathcal{K}(V)$ is a full subcomplex of $\mathcal{K}(X)$ since any totally ordered subset of X whose points are in V is a totally ordered subset of V . Since a map $f: X \rightarrow Y$ is an order-preserving function, it may be regarded as a simplicial map $\mathcal{K}(f): \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$.

THEOREM 8.1.2. *For an A-space X , there is a weak homotopy equivalence*

$$\psi = \psi_X: |\mathcal{K}(X)| \rightarrow X$$

such that the following diagram commutes for each map $f: X \rightarrow Y$.

$$\begin{array}{ccc} |\mathcal{K}(X)| & \xrightarrow{|\mathcal{K}(f)|} & |\mathcal{K}(Y)| \\ \psi_X \downarrow & & \downarrow \psi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

PROOF. Each point $u \in |\mathcal{K}(X)|$ is an interior point of a simplex σ spanned by some strictly increasing sequence $x_0 < x_1 < \dots < x_n$ of points of X . We define $\psi(u) = x_0$. For $f: X \rightarrow Y$, $\mathcal{K}(f)(u)$ is in the simplex spanned by the $f(x_i)$ and $f(x_0) \leq f(x_1) \leq \dots \leq f(x_n)$. Omitting repetitions, we see that $f(x_0)$ is the minimal vertex of this simplex, so that $\psi(\mathcal{K}(f)(u)) = f(x_0) = f(\psi(u))$, which proves that the diagram commutes. We must still prove that ψ is continuous and that it is a weak homotopy equivalence.

For $x \in X$, let $star(x)$ denote the union of the interiors of the simplices of $\mathcal{K}(X)$ that have x as a vertex; it is an open neighborhood of x in $|\mathcal{K}(X)|$. For an open subset V of X , define the open star, $star(V)$, to be the union over the vertices $v \in V$ of the open subspaces $star(v)$. It is the complement of $|\mathcal{K}(X - V)|$ in $|\mathcal{K}(X)|$. To see that ψ is continuous, we show that $\psi^{-1}(V) = star(V)$. If

$\psi(u) = v \in V$, then v is the initial vertex x_0 of a simplex σ . Since a vertex v is the unique interior point of the simplex $\{v\}$, $u \in \text{star}(V)$. Conversely, suppose that $u \in \text{star}(v)$, where $v \in V$. Then u is an interior point of a simplex σ determined by an increasing sequence $x_0 < x_1 < \cdots < x_n$ such that some $x_i = v \in V$. Since $x_0 \leq v$, $x_0 \in U_v$. Since V is open, $U_v \subset V$. Thus $\psi(u) = x_0$ is in V .

It remains to prove that ψ is a weak homotopy equivalence. We shall do so by applying [Theorem 3.3.1](#) to the minimal open cover $\{U_x\}$ of X . If x is in $U_y \cap U_z$, then x is in both U_y and U_z , so that U_x is contained in both U_y and U_z . This verifies the first hypothesis of the cited theorem. For the second hypothesis, we know that each U_x is a contractible subspace of V . We also know that each $|\mathcal{K}(U_x)|$ is a contractible space. In fact, $\mathcal{K}(U_x)$ is a simplicial cone, in the sense that for every simplex σ of $\mathcal{K}(U_x)$ which does not contain x , $\sigma \cup \{x\}$ is a simplex of $\mathcal{K}(U_x)$. The realization of such a simplicial cone is contractible to the cone vertex x since $h(y, t) = (1 - t)y + tx$ gives a well-defined contracting homotopy. Specializing the following general result to $L = \mathcal{K}(U_x)$, we see that $\text{star}(U_x)$ is also contractible. Therefore each restriction $\psi: \psi^{-1}(U_x) \rightarrow U_x$ is a weak homotopy equivalence and [Theorem 3.3.1](#) applies to show that ψ is a weak equivalence. \square

PROPOSITION 8.1.3. *Let L be a full subcomplex of a simplicial complex K . Then $|L|$ is a deformation retract of its open star, $\text{star}L$, in $|K|$.*

PROOF. Again, $\text{star}L$, is defined to be the union of the open stars of the vertices of L . This result is a standard fact in the theory of simplicial complexes, and a more detailed proof than we shall give can be found in [\[62, 70.1 and p. 427\]](#). Consider a simplex σ that is in the closure of $\text{star}(L)$. Then σ has vertex set the disjoint union of a set of vertices in L and a set of vertices in $K - L$. Each point u of σ that is neither in the span s of the vertices in L nor in the span t of the vertices not in L is on a unique line segment joining a point in t to a point in s . Define the required retraction r by sending u to the end point in $s \subset L$ of this line segment, letting r be the identity map on L and thus on s . Deformation along such line segments gives the required homotopy showing that $i \circ r$ is homotopic to the identity, where i is the inclusion of $|L|$ in its open star. \square

EXAMPLE 8.1.4. Suppose that $|\mathcal{K}(X)|$ is homotopy equivalent to a sphere S^n . Then the dimension of $|\mathcal{K}(X)|$, which is $h(X) - 1$, must be at least n . Thus $h(X) \geq n + 1$. Therefore, by [Proposition 3.5.2](#), X has at least $2n + 2$ points and, if X has exactly $2n + 2$ points, then it is homeomorphic to $\mathbb{S}^n S^0$.

8.2. The construction of A-spaces from simplicial complexes

Now let K be a finite geometric simplicial complex with first barycentric subdivision K' . Remember that $|K| = |K'|$.

DEFINITION 8.2.1. Define an A-space $\mathcal{X}(K)$ as follows. The points of $\mathcal{X}(K)$ are the barycenters b_σ of the simplices of K , that is, the vertices of K' . The required partial order \leq is defined by $b_\sigma \leq b_\tau$ if $\sigma \subset \tau$. The open subspace U_{b_σ} coincides with $\mathcal{X}(\sigma)$, where σ (together with its faces) is regarded as a subcomplex of K . For a simplicial map $g: K \rightarrow L$, define $\mathcal{X}(g): \mathcal{X}(K) \rightarrow \mathcal{X}(L)$ by $\mathcal{X}(g)(b_\sigma) = b_{g(\sigma)}$, and note that this function is order-preserving and therefore continuous. Using the barycenters themselves to realize the vertices geometrically, we see from the description of K' in [Proposition 6.4.11](#) that $\mathcal{X}\mathcal{X}(K) = K'$ and $\mathcal{X}\mathcal{X}(g) = g'$.

We use [Theorem 8.1.2](#) to obtain the following complementary result.

THEOREM 8.2.2. *For a simplicial complex K , there is a weak homotopy equivalence*

$$\phi = \phi_K: |K| \longrightarrow \mathcal{X}(K)$$

such that the following diagram is commutative

$$\begin{array}{ccc} |K'| & \xrightarrow{|g'|} & |L'| \\ \phi_K \downarrow & & \downarrow \phi_L \\ \mathcal{X}(K) & \xrightarrow{\mathcal{X}(g)} & \mathcal{X}(L) \end{array}$$

PROOF. Define

$$\phi_K = \psi_{\mathcal{X}(K)}: |K'| = |\mathcal{X} \mathcal{X}(K)| \longrightarrow \mathcal{X}(K).$$

Then ϕ_K is a weak homotopy equivalence and the diagram commutes by [Theorem 8.1.2](#). Since $|K| = |K'|$ and $|L| = |L'|$, we can replace $|g'|$ by $|g|$ in the diagram. By [Proposition 6.4.17](#), $|g'|$ is simplicially close to $|g|$ and hence homotopic to $|g|$. Therefore, after the replacement, the diagram would only be homotopy commutative, in the sense that the two composite maps in the diagram would be homotopic. \square

8.3. Mapping spaces

For completeness, we record results of Stong [[65](#), §6] that were obtained about the same time as the results of McCord recorded above and that give a quite different approach to the relationship between finite simplicial complexes and finite spaces. Since the proofs are fairly long and combinatorial in flavor, and since the statements do not have quite the same immediate impact as those in McCord's work, we shall not work through the details here.

Rather than constructing finite models for finite simplicial complexes, Stong studies all maps from the geometric realizations of simplicial complexes K into finite spaces X by studying the properties of the function space $X^K \equiv X^{|K|}$. More generally, he fixes a subcomplex L of K and a basepoint $* \in X$ and studies the subspace $(X, *)^{(K,L)}$ of maps $f: |K| \longrightarrow X$ such that $f(|L|) = *$. Homotopies relative to $|L|$ between such maps are homotopies h such that $h(p, t) = *$ for $p \in |L|$.

THEOREM 8.3.1. *Let L be a subcomplex of a finite simplicial complex K , let X be a finite space with basepoint $*$, and let $F = (X, *)^{(K,L)}$ denote the subspace of X^K consisting of those maps $f: |K| \longrightarrow X$ such that $f(|L|) = *$.*

- (i) *For any $f \in F$, there is a map $g \in F$ such that the set $V = \{h | h \leq g\} \subset F$ is a neighborhood of f in F ; that is, there is an open subset U such that $f \in U \subset V$.*
- (ii) *If $f \simeq f'$ relative to L , then there is a sequence of elements $\{g_1, \dots, g_s\}$ in F such that $g_1 = f$, $g_s = f'$, and either $g_i \leq g_{i+1}$ or $g_{i+1} \leq g_i$ for $1 \leq i < s$.*

The essential point of this analysis is the following consequence.

COROLLARY 8.3.2. *The path components and components of F coincide. That is, the homotopy classes of maps $f: (K, L) \longrightarrow (X, *)$ are in bijective correspondence with the components of F .*

Recheck: add?
Expository paper
topic?

8.4. Simplicial approximation and A-spaces

There are two papers, [29, 30], that start with the simplicial approximation theorem and take up where McCord and Stong leave off. In view of the explicit constructions of $\mathcal{K}(X)$ and $\mathcal{X}(K)$, the following definition is reasonable.

DEFINITION 8.4.1. Define the *barycentric subdivision* of an A-space X to be $X' = \mathcal{X}\mathcal{K}(X)$. For a map $f: X \rightarrow Y$, define $f': X' \rightarrow Y'$ to be $\mathcal{X}\mathcal{K}(f)$. Iterating the construction, define $X^{(n)} = (X^{(n-1)})'$, where $X^{(0)} = X$. Observe inductively that $\mathcal{K}(X^{(n)}) = \mathcal{K}(X)^{(n)}$ since $\mathcal{K}\mathcal{X}(K) = K'$.

PROPOSITION 8.4.2. *There is a map $\zeta = \zeta_X: X' \rightarrow X$ that makes the following diagram commute, and ζ is a weak homotopy equivalence.*

$$\begin{array}{ccc} |\mathcal{K}\mathcal{X}\mathcal{K}(X)| & \xlongequal{\quad} & |\mathcal{K}(X')| \xrightarrow{|\xi_{\mathcal{K}(X)}|} |\mathcal{K}(X)| \\ \psi_{\mathcal{X}\mathcal{K}(X)} \downarrow & & \downarrow \psi_X \\ X' = \mathcal{X}\mathcal{K}(X) & \xrightarrow{\quad \zeta_X \quad} & X. \end{array}$$

The simplicial map $\xi_{\mathcal{K}(X)}$ coincides with $\mathcal{K}(\zeta_X): \mathcal{K}(X') \rightarrow \mathcal{K}(X)$. The following diagram commutes for a map $f: X \rightarrow Y$.

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \zeta_X \downarrow & & \downarrow \zeta_Y \\ X & \xrightarrow{f} & Y \end{array}$$

PROOF. The points of $\mathcal{X}\mathcal{K}(X)$ are the barycenters of the simplices of $\mathcal{K}(X)$. These simplices σ are spanned by increasing sequences $x_0 < \dots < x_n$ of elements of X . Let $\zeta(b_\sigma) = x_n$. Since $b_\sigma \leq b_\tau$ implies $\sigma \subset \tau$ and thus $\zeta(b_\sigma) \leq \zeta(b_\tau)$, ζ is continuous. We understand $\xi_{\mathcal{K}(X)}$ to be the standard choice specified in [Definition 6.4.13](#). Inspection of definitions shows that $\xi_{\mathcal{K}(X)} = \mathcal{K}(\zeta_X)$. The commutativity of the first diagram follows from the “naturality” of ψ with respect to the map ζ_X . That is, this diagram is a specialization of the commutative diagram of [Theorem 8.1.2](#), with f there taken to be ζ_X here. That ζ_X is a weak homotopy equivalence follows from the diagram, since all other maps in it are weak homotopy equivalences. The last statement is clear by inspection of definitions. \square

THEOREM 8.4.3. *Let X be an F-space and Y be an A-space, and let $f: |\mathcal{K}(X)| \rightarrow |\mathcal{K}(Y)|$ be any map. Then for some sufficiently large n there is a map $g: X^{(n)} \rightarrow Y$ such that f is homotopic to $|\mathcal{K}(g)|$. We call g a finite approximation to f .*

PROOF. By the classical simplicial approximation theorem for simplicial complexes, for a sufficiently large n there is a simplicial approximation

$$j: \mathcal{K}(X^{(n-1)}) = \mathcal{K}(X)^{(n-1)} \rightarrow \mathcal{K}(Y)$$

to f . Let g be the composite

$$X^{(n)} = \mathcal{X}\mathcal{K}(X^{(n-1)}) \xrightarrow{\mathcal{X}(j)} \mathcal{X}\mathcal{K}(Y) = Y' \xrightarrow{\zeta_Y} Y.$$

Then

$$\mathcal{K}(g) = \mathcal{K}(\zeta_Y) \circ \mathcal{K}\mathcal{X}(j) = \mathcal{K}(\zeta_Y) \circ j'.$$

We have $|j'| \simeq |j|$ by [Proposition 6.4.17](#) and $|j| \simeq f$ by assumption. Since we also have $|\mathcal{K}(\zeta_Y)| = |\xi_{\mathcal{K}(Y)}| \simeq \text{id}$, we have $|\mathcal{K}(g)| \simeq f$, as required. \square

The point to emphasize here is that finite models for spaces have far too few maps between them. For example, $\pi_n(S^n, *) = \mathbb{Z}$, but there are only finitely many distinct maps from any finite model for S^n to itself. The theorem says that, after subdividing the domain sufficiently, we can realize any of these homotopy classes in terms of maps between (different) finite models for S^n .

8.5. Contiguity of maps between A-spaces

Remembering the definition of $\mathcal{K}(X)$, we may as well refer to points of an A-space X as vertices and to finite ordered subsets of X as simplices. Thus “simplex” is just a convenient abbreviation of “finite totally ordered subset”. We use that language in translating the notion of contiguity from simplicial complexes to finite spaces. If $q > n$, we agree to write ζ for the composite $X^{(q)} \rightarrow X^{(n)}$ determined by iteration of maps ζ .

DEFINITION 8.5.1. Let $f, g: X \rightarrow Y$ be continuous maps between A-spaces. We say that f is *contiguous* to g if for every simplex σ of X , there is a simplex of Y that contains both $f(\sigma)$ and $g(\sigma)$. More generally, let $f: X \rightarrow Y$ and $g: X^{(n)} \rightarrow Y$ be continuous maps. We say that f is contiguous to g if for each simplex σ of $X^{(n)}$, there is a simplex of Y that contains both $(f \circ \zeta)(\sigma)$ and $g(\sigma)$. If $q > n$, a check of definitions shows that if f and g are contiguous, then so are f and $g \circ \zeta$. Similarly, if $q > 0$ and f and g are contiguous, then so are $f \circ \zeta$ and g , where now $\zeta: K^{(q)} \rightarrow K$. The relation of contiguity is reflexive and symmetric, but it is not transitive. We let \sim denote the equivalence relation generated by contiguity. Thus $f \sim g$ if there is a sequence of continuous maps $\{f = f_1, \dots, f_q = g\}$ such that f_i is contiguous to f_{i+1} for $i < q$.

PROPOSITION 8.5.2. *If $f: X \rightarrow Y$ and $g: X^{(n)} \rightarrow Y$ are contiguous maps between A-spaces, then $f \circ \zeta \simeq g: X^{(n)} \rightarrow Y$.*

The analogue for simplicial maps used the notion of simplicially close maps from an arbitrary space to a simplicial complex. We have an analogous notion for maps to A-spaces. The term “approximate map” is sometimes used for either of these notions.

DEFINITION 8.5.3. Let X be any space and let Y be an A-space. Two maps $f, g: X \rightarrow Y$ are *simplicially close* if for each $x \in X$ there is a simplex $\tau = \tau_x$ of Y such that $f(x)$ and $g(x)$ are both in τ .

Clearly contiguous maps between A-spaces are simplicially close in this sense. Therefore the following result implies [Proposition 8.5.2](#).

PROPOSITION 8.5.4. *At least if both X and Y are A-spaces, simplicially close maps $f, g: X \rightarrow Y$ are homotopic.*

PROOF. Define $h: X \times I \rightarrow Y$ by

$$\begin{aligned} h(x, t) &= f(x) \quad \text{if } 0 \leq t < 1/2 \\ h(x, 1/2) &= \begin{cases} g(x) & \text{if } f(x) \leq g(x) \\ f(x) & \text{if } g(x) \leq f(x). \end{cases} \\ h(x, t) &= g(x) \quad \text{if } 1/2 < t \leq 1 \end{aligned}$$

Since $f(x)$ and $g(x)$ are both in a simplex τ_x , either $f(x) \leq g(x)$ or $g(x) \leq f(x)$. Therefore h is well-defined, and it suffices to prove that h is continuous. One way to study the problem is to introduce the three point space $J = \{0, 1/2, 1\}$ whose proper open subsets are $\{0\}$, $\{1\}$, and their union $\{0, 1\}$. Define $\pi: I \rightarrow J$ by

$$\pi([0, 1/2)) = 0, \quad \pi(1/2) = 1/2, \quad \pi((1/2, 1]) = 1.$$

Certainly π is continuous, hence so is $\text{id} \times \pi: X \times I \rightarrow X \times J$. There is an obvious function $j: X \times J \rightarrow Y$ such that $h = j \circ (\text{id} \times \pi)$, namely

$$j(x, 0) = f(x), \quad j(x, 1/2) = h(x, 1/2), \quad j(x, 1) = g(x).$$

It suffices to prove that j is continuous. When X is an A -space, this can be done by giving $X \times J$ the product order, namely $(x, i) \leq (x', i')$ if and only if both $x \leq x'$ and $i \leq i'$, and checking that j is order-preserving since f and g are order preserving. Since both $0 < 1/2$ and $1 < 1/2$ and since $x \leq x'$ implies both $f(x) \leq f(x')$ and $g(x) \leq g(x')$, the check is easy and can be left to the reader. \square

Comparing our two definitions of simplicially close maps, for simplicial complexes and for Alexandroff spaces, we see the following properties of the constructions \mathcal{H} and \mathcal{X} .

PROPOSITION 8.5.5. *If $f: \mathcal{K}(X^{(m)}) \rightarrow \mathcal{K}(Y)$ and $g: \mathcal{K}(X^{(n)}) \rightarrow \mathcal{K}(Y)$ are contiguous maps of simplicial complexes, then $\zeta_Y \circ \mathcal{X}(f): X^{(m+1)} \rightarrow Y$ and $\zeta_Y \circ \mathcal{X}(g): X^{(n+1)} \rightarrow Y$ are contiguous maps of A -spaces. If $f: X^{(m)} \rightarrow Y$ and $g: X^{(n)} \rightarrow Y$ are contiguous maps of A -spaces, then $\mathcal{H}(f)$ and $\mathcal{H}(g)$ are contiguous maps of simplicial complexes.*

Now the simplicial results in ?? and ?? have the following immediate consequences.

PROPOSITION 8.5.6. *If $g: X^{(m)} \rightarrow Y$ and $g': X^{(n)} \rightarrow Y$ are finite approximations of the same map $f: |\mathcal{K}(X)| \rightarrow |\mathcal{K}(Y)|$, then g and g' are contiguous.*

THEOREM 8.5.7. *If f and f' are homotopic maps $|\mathcal{K}X| \rightarrow |\mathcal{K}Y|$ and g and g' are finite approximations to f and f' , then g is contiguous to g' . Therefore, for every pair of homotopic maps $f, f': |\mathcal{K}X| \rightarrow |\mathcal{K}Y|$, there is a sufficiently large n such that f and f' have contiguous finite approximations $X^{(n)} \rightarrow Y$.*

We have focused on understanding homotopy classes of maps between finite simplicial complexes in terms of contiguity classes of simplicial maps and contiguity classes of continuous maps between finite spaces, but one can also ask the relationship between homotopy classes and contiguity classes of maps between finite spaces. We have seen that contiguous maps are homotopic, but the converse is also true. To see that, we refine [Proposition 2.2.12](#), following [\[7, 2.1.1\]](#).

DEFINITION 8.5.8. Maps $f, g: X \rightarrow Y$ between Alexandroff spaces are very close if $f = g$ on all but one point $x \in X$, and either $f(x) < g(x)$ or $g(x) < f(x)$. The maps f, g are closely equivalent if there is a sequence of maps $\{f = f_1, f_2, \dots, f_q = g\}$ such that f_i is very close to f_{i+1} for $i < q$.

LEMMA 8.5.9. *If $f, g: X \rightarrow Y$ are very close, then they are contiguous.*

PROOF. Without loss of generality, we may assume that $f(x) < g(x)$ for the unique point x on which f and g differ. For a simplex σ of X that does not contain x , we have $f(\sigma) = g(\sigma)$, which is clearly contained in a simplex of Y . If x is in a

simplex $\sigma = \{x_0 < x_1 < \cdots < x_n\}$, then $x = x_i$ for some i and $f(\sigma) \cup g(\sigma)$ is the simplex obtained by deleting repetitions from the ordered set

$$\{f(x_0) \leq f(x_1) \leq \cdots \leq f(x_i) \leq g(x_i) \leq g(x_{i+1}) \leq \cdots \leq g(x_n)\} \quad \square$$

THEOREM 8.5.10. *If $f, g: X \rightarrow Y$ are homotopic maps between finite spaces, then f and g are very closely equivalent and are therefore contiguous.*

PROOF. By [Proposition 2.2.12](#), we may assume without loss of generality that $f \leq g$. Let $A \subset X$ be the set of points x such that $f(x) \neq g(x)$. Of course, we may assume that A is non-empty, and we let x be a maximal point in A , so that $x' > x$ implies $f(x') = g(x')$. Define f_2 by $f_2(x') = f(x')$ for $x' \neq x$ and $f_2(x) = g(x)$. Certainly f_2 is order-preserving and thus continuous. It differs from g at one less point than $f = f_1$ differs from g . Repeating the construction, we arrive at $f_q = g$ after finitely many steps since X and Y are finite. \square

Homotopy Theory for Subdivision and A -Space Models

The respective works of Clader and Thibault allow us to consider arbitrary topological spaces as represented by a system of subdivided models. Formally, our first result says that given a locally finite A -space X , its realization $|\mathcal{K}(X)|$ is homotopy equivalent to the inverse limit of the directed system:

$$\dots \xrightarrow{\text{inf}} (\text{Sd}^2 X)^{op} \xrightarrow{\text{inf}} (\text{Sd} X)^{op} \xrightarrow{\text{inf}} X^{op} .$$

After establishing this result, we turn our attention away from topological spaces and instead focus on modeling continuous functions. While we can associate to every CW complex a partially ordered set, the category of posets suffers from a “deficiency of morphisms” that limits our ability to model functions in the category of topological spaces. To remedy this issue, we follow the work of Hardie and Vermeulen as well as Thibault to formulate an elegant relationship between A -spaces and their geometric realizations using the subdivision functor.

The main result establishes that, given a finite A -space X and an arbitrary A -space Y , there is a natural bijection between the colimit of the system:

$$[X, Y] \xrightarrow{\text{sup}^*} [\text{Sd} X, Y] \xrightarrow{\text{sup}^*} [\text{Sd}^2 X, Y] \xrightarrow{\text{sup}^*} \dots$$

and $[|\mathcal{K}(X)|, |\mathcal{K}(Y)|]$. In plain terms, this result allows us to model continuous maps between topological spaces with order-preserving maps between posets, up to homotopy. These two results in conjunction effectively justify that the category of A -spaces is an appropriate place in which to study algebraic topology.

To add algebraic importance to these results, we develop the final piece of this result. In general, an isomorphism of sets is the strongest relationship we can hope for between this colimit and the homotopy classes of maps between the realizations of two A -spaces, since for arbitrary topological spaces X and Y , $[X, Y]$ need not be a group. However, if we restrict our attention to basepoint-preserving homotopy classes of the form $\langle \Sigma X, Y \rangle$, where ΣX is the reduced suspension of a space X , we have an important group structure. Here, we recover a *group isomorphism* between

$$\text{colim}_n \langle \text{Sd}^n(\mathbb{S}^{op} X), Y \rangle \text{ and } \langle \Sigma |\mathcal{K}(X)|, |\mathcal{K}(Y)| \rangle$$

by demonstrating an A -space model of the co-H-space structure on ΣX . We conclude by focusing on spheres as a particular class of suspensions that allow our system to model the group structure of homotopy groups, as well as return some results regarding the contractibility of particular spaces.

Recall [Definition 3.2.1](#). It is important to note that weak equivalence of topological spaces is *not* transitive, so we must refrain from a notion of weak equivalence

classes unless we undergo a process to formally invert these induced maps, which will not be necessary for our purposes.

DEFINITION 9.0.1. Given a topological space X , an A -space Y is said to be an A -space model of X if there exists a chain of weak equivalences from X to Y . That is, there is a finite collection consisting of both topological spaces and A -spaces X_i along with weak equivalences $f_i : X_i \rightarrow X_{i+1}$ that give the chain

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y$$

which in turn induces a bijection on all homotopy groups of X and Y .

PROPOSITION 9.0.2. *If Y is an A -space model of a CW-complex X , then $|\mathcal{K}(Y)| \simeq X$.*

This is a direct application of Whitehead's Theorem. In light of this result, we see that once we have found a model or, looking ahead, a collection of related models for a space, we may study them instead of our space directly, as they encode the same homotopy-invariant information of the original space after realization. Particularly, since every CW complex is homotopy equivalent to the geometric realization of some poset, these relationships are sufficient to study the homotopical properties of CW complexes. While many of the general theoretic results of this section exist independent of these facts, the results presented here would be of little use for studying recognizable topological spaces.

9.1. Modeling Topological Spaces

While the functors \mathcal{K} and \mathcal{S} give us a way to build subdivisions of posets and simplicial complexes, it will also be useful for us to define a way to go from the subdivision of an A -space back to the underlying space. With this in mind, we define the following map.

DEFINITION 9.1.1. Given an A -space X , let $\text{sup} : \text{Sd}X \rightarrow X$ be the map such that if $\sigma = \{x_0, x_1, \dots, x_k\}$ is an element of $\text{Sd}X$, then $\text{sup}(\sigma) = x_k$.

PROPOSITION 9.1.2. *The map sup is continuous.*

PROOF. Suppose $\sigma \leq \sigma'$. Since the ordering on $\text{Sd}X$ is induced by simplicial inclusion in $\mathcal{K}(X)$, we have $\sigma \subseteq \sigma'$ when regarded as simplices of $\mathcal{K}(X)$. Inclusion requires $x_i \in \sigma \implies x_i \in \sigma'$, so in particular $x_k \leq x'_{k'}$, and thus sup is continuous. \square

PROPOSITION 9.1.3. *The map sup is a weak homotopy equivalence.*

It is a straightforward check to see that $|\mathcal{K}(f)| : |\mathcal{K}(\text{Sd}X)| \rightarrow |\mathcal{K}(X)|$ is a homotopy equivalence. This implies f is a weak homotopy equivalence of A -spaces.

DEFINITION 9.1.4. Let X be an A -space. Define $\text{inf} : (\text{Sd}X)^{op} \rightarrow X^{op}$ to be the map that coincides pointwise with sup on $\text{Sd}X$ to X .

One should note that sup returns the largest element of a given chain, and inf , the smallest. We will use this descriptive language in proofs, as it illuminates the continuity of some maps and the commutativity of some diagrams that we will introduce in the coming pages.

To establish a concrete relationship between an A -space and its realization that will allow us to regard X as an A -space model of $|\mathcal{K}(X)|$, we present the following theorem.

DEFINITION 9.1.5. For an A -space X , let u be a point in $|\mathcal{K}(X)|$. Then u is in the interior of a unique simplex $\{x_0, x_1, \dots, x_k\}$ of $\mathcal{K}(X)$. Define the map $p : |\mathcal{K}(X)| \rightarrow X$ by $p(u) = x_0$.

THEOREM 9.1.6. [50] *For an A -space X , $p : |\mathcal{K}(X)| \rightarrow X$ is a natural weak homotopy equivalence.*

We can in fact extend this result to see that there is a weak homotopy equivalence $p_n : |\mathcal{K}(X)| \rightarrow (\text{Sd}^n X)^{op}$. While it may seem odd for us to pick the opposite topology, one should note that each p_n as defined above returns the maximal element of the unique simplex in $\text{Sd}^n X$ containing a point u . When composed with the inf map as above, we generate the following commutative diagram.¹

$$\begin{array}{ccccccc}
 & & |\mathcal{K}(X)| & & & & \\
 & \swarrow p_0 & \downarrow p_1 & \searrow p_2 & & & \\
 X^{op} & \xleftarrow{\text{inf}} & (\text{Sd}X)^{op} & \xleftarrow{\text{inf}} & (\text{Sd}^2 X)^{op} & \xleftarrow{\text{inf}} & \dots
 \end{array}$$

Given this diagram, we take the inverse limit of the system of the bottom row, given by

$$\tilde{X} = \prod_n (\text{Sd}^n X)^{op}/(\sim)$$

where equivalence is generated by the inf map. We would like to imagine that the increasingly fine subdivisions of a simplex “converge” to a point in $|\mathcal{K}(X)|$. To formalize this intuition, we offer the following setup:

We define the map $\tilde{p} : |\mathcal{K}(X)| \rightarrow \tilde{X}$ by $\tilde{p}(a) = (p_0(a), p_1(a), \dots)$, which associates to each point a in the realization of X the corresponding sequence of images of a under each p_n . One may note that this is a sequence of nested simplices all containing a . Since the maps p_i give a cone to our inverse system, so continuity is clear. In tandem, we offer the following map that will act as an inverse up to homotopy:

DEFINITION 9.1.7. Let $x = (x_0, x_1, \dots) \in \tilde{X}$, with each $x_i \in (\text{Sd}^i X)^{op}$. Pick some $a_i \in \tilde{p}_i^{-1}(x_i)$ for each $x_i \in x$. Then $\{a_n\}$ converges to a point $a \in |\mathcal{K}(X)|$. Let $G : \tilde{X} \rightarrow |\mathcal{K}(X)|$ denote this map.

The restriction in the following theorem that X be a *locally finite* A -space, i.e. an A -space where each element has finite closure and a finite neighborhood, is required to ensure G is continuous. These complete checks of continuity and well-definedness can be found in [67], which additionally correct a detail on the inherited topology of \tilde{X} in Clader’s original proof.

THEOREM 9.1.8. *Let X be a locally finite A -space. Then \tilde{X} and $|\mathcal{K}(X)|$ are homotopy equivalent and $|\mathcal{K}(X)|$ is a deformation retract of \tilde{X} .*

¹Note the importance of the opposite topology and the substitution of the inf map to make this a commutative diagram of continuous maps. For a more explicit construction of each p_n , which Thibault denotes \tilde{p}_n , his thesis gives full details that make the checks on this diagram and the inverse limit much more concrete. We elect to omit these here, and refer the skeptical and intrigued reader to [67].

PROOF. First, observe that $G \circ \tilde{p}$ is in fact the identity map on $|\mathcal{K}(X)|$, so we need only show $\tilde{p} \circ G \simeq id_{\tilde{X}}$. Suppose $x, x' \in \tilde{X}$ and say $x \sim x'$ if $G(x) = G(x')$. If E denotes a subset of \tilde{X} corresponding to one equivalence class partitioned by (\sim) , define a homotopy $h_E : E \times [0, 1] \rightarrow E$ by

$$h_E(x, t) = \begin{cases} x & t < 1 \\ \tilde{p}(G(x)) & t = 1. \end{cases}$$

Thibault proves that this homotopy is continuous in [67], so we have an explicit deformation retract of each equivalence class under (\sim) . These maps allow us to define globally $H : \tilde{X} \times [0, 1] \rightarrow \tilde{X}$.

To show H itself is continuous, consider $U \subset \tilde{X}$ open. Suppose $x \in E$, and let $y = G(x) \in |\mathcal{K}(X)|$. Because of our partition, y is the same for all $x \in E$. Observe that if $\tilde{p}(y) \notin (U \cap E)$, then $h_E^{-1}(U \cap E) = (U \cap E) \times [0, 1)$. Likewise, if $(U \cap E)$ does contain $\tilde{p}(y)$, then $h_E^{-1}(U \cap E) = (U \cap E) \times [0, 1]$.

Define $U' = \{x \in U \mid \tilde{p}(G(x)) \notin U\}$. Then $H^{-1}(U) = (U \times [0, 1]) \setminus (U' \times \{1\})$. Since \tilde{p} and G are continuous, U' is closed in U . So H is continuous on all of \tilde{X} , and gives a homotopy from the identity to $\tilde{p} \circ G(\tilde{X})$. Therefore, \tilde{X} deformation retracts onto $\tilde{p} \circ G(\tilde{X})$, and since G is surjective, $\tilde{p} \circ G(\tilde{X}) = \tilde{p}(|\mathcal{K}(X)|)$. \square

It should be surprising that the inverse limit and geometric realization act equivalently up to homotopy on a locally finite A -space X . This should not be a readily accepted result, as these functors typically do not behave nicely with one another. After all, the geometric realization functor is left adjoint, which in general does not preserve limits. While the intuition laid out in this proof makes the conclusion feel motivated, one should keep in mind the interesting mathematical relationships at play.

9.2. Motivation and Colimit Setup

Now, we turn our attention to another system that, while similar in style to the former, marks a departure from modeling topological spaces to instead modeling continuous functions. The benefit of a model built similarly to our inverse limit system is that with increasingly finer subdivisions of X , we are able to “fill in the gaps” between morphisms, so to speak.

Given the way we will construct our directed system for a finite A -space X and an arbitrary A -space Y , its colimit is given by

$$\coprod_n [\text{Sd}^n X, Y]/(\sim)$$

subject to the equivalence relation on homotopy classes generated by precomposition with the sup map. That is, elements, which for us are equivalence classes of maps, $[f] \in [\text{Sd}^i X, Y]$ and $[g] \in [\text{Sd}^j X, Y]$ are identified in the colimit if there exists an $N \geq 0$ such that $[f \circ \text{sup}^{N-i}] = [g \circ \text{sup}^{N-j}] \in [\text{Sd}^N X, Y]$. This relationship based on the sup map simply consolidates extraneous information; the crux of the result will come from the Simplicial Approximation Theorem.

To prove that this equivalence relation coincides precisely with those maps whose realizations are homotopic, we introduce a basic vocabulary to understand homotopy classes in the setting of A -spaces, and turn to the simplicial notion of

contiguity that provides us with precisely the tools necessary to prove that the above colimit is a useful model.

9.3. Revisiting contiguity

With the understanding gained of function spaces and homotopies (Section 2.2), we turn to a notion that is different from but related to homotopy that allows us to compare maps residing in different components of our colimit, as well as characterize the behavior of related maps under realization.

In order to create a formal relationship between maps with different subdivisions as their domains, we utilize the concept of contiguity of simplicial maps to generate a notion of equivalence of maps in distinct components in our colimit. We conclude many of our intermediate results in the setting of simplicial complexes, as the notion of contiguity does not extend perfectly to A -spaces. This will be sufficient for our purposes. We now present an alternate definition of contiguity using the supremum map.

DEFINITION 9.3.1. Let K and L be simplicial complexes. Two simplicial maps $u : K \rightarrow L$ and $v : \text{Sd}^m K \rightarrow L$ are *contiguous* if for each simplex $\sigma \in \text{Sd}^m K$, there is a simplex $\tau \in L$ such that $u(\text{sup}^{(m)}(\sigma)) \subset \tau$ and $v(\sigma) \subset \tau$.

If we have an A -space X such that $\mathcal{K}(X) = K$, we can somewhat extend this definition to maps between A -spaces. However, it should be clear by this definition that we are limited by the existence of simplices, which requires at least one subdivision of X to be sensible. Thus, we say that two maps f and g between A -spaces are contiguous if $\mathcal{K}(f)$ and $\mathcal{K}(g)$ are contiguous as simplicial maps.

Now that we have presented the notion contiguity as it pertains to simplicial and A -space maps, we offer the following relationships between contiguity and homotopy that link equivalence in the colimit to equivalence after realization.

PROPOSITION 9.3.2. *Suppose $\mathcal{K}(f) : \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ and $\mathcal{K}(g) : \mathcal{K}(\text{Sd}^n X) \rightarrow \mathcal{K}(Y)$ are contiguous. Then $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$.*

Since $\mathcal{K}(f)$ and $\mathcal{K}(g)$ are contiguous, they satisfy the definition of simplicial closeness given in [48]. Thus, there is a linear homotopy connecting their realizations.

THEOREM 9.3.3. [29] *For a finite A -space X and arbitrary A -space Y , if $f : |\mathcal{K}(X)| \rightarrow |\mathcal{K}(Y)|$ is continuous, then there exists an $n \geq 0$ and a map $g : \text{Sd}^n X \rightarrow Y$ with $|\mathcal{K}(g)| \sim f$.*

PROOF. By the classical Simplicial Approximation Theorem, there exists a simplicial map $u : \text{Sd}^{n-1} \mathcal{K}(X) \rightarrow \mathcal{K}(Y)$ such that $|u| \simeq f$. Noting $\text{Sd}^{n-1} \mathcal{K}(X) = \mathcal{K}(\text{Sd}^{n-1} X)$, define $g = \text{sup} \circ \mathcal{K}(u)$. Then g is a map from $\text{Sd}^n X$ to Y , and given $\sigma = \{S_1, S_2, \dots, S_k\}$, an arbitrary simplex of $\mathcal{K}(\text{Sd}^n(X))$, we have

$$(\mathcal{K}(\text{sup} \circ \mathcal{K}(u))(\sigma)) = \text{sup}\{u(S_1), u(S_2), \dots, u(S_k)\} = u(\text{sup}(\sigma)),$$

so by definition u and $\mathcal{K}(g)$ are contiguous. Following Proposition 9.3.2, this implies $|\mathcal{K}(g)| \simeq |u|$. Thus, $|\mathcal{K}(g)| \simeq f$. \square

We say any map g satisfying this condition is an *A -space approximation* of f .

PROPOSITION 9.3.4. [48] *Let X and Y be A -spaces. If $g : \text{Sd}^i X \rightarrow Y$ and $g' : \text{Sd}^j X \rightarrow Y$ are both simplicial approximations of $f : |\mathcal{K}(X)| \rightarrow |\mathcal{K}(Y)|$, then g and g' are contiguous.*

PROPOSITION 9.3.5. *Suppose f and g are maps from X to Y , where X is a finite A -space and Y is an arbitrary A -space. If $f \simeq g$, then $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$.*

PROOF. Since $f \simeq g$ implies that there is a sequence of comparable maps connecting the two, and since both $\text{Im}(f)$ and $\text{Im}(g)$ are finite, it suffices to show the desired result for f and g such that $f(x) = g(x)$ for all but one x' , where $f(x') \leq g(x')$. For a simplex σ of X that does not contain x' , we have $f(\sigma) = g(\sigma)$, which is clearly contained in a simplex of Y . If $x' \in \sigma$, then $x' = x_i$ for some i and both $f(\sigma)$ and $g(\sigma)$ are contained in the simplex given by

$$\{f(x_0), f(x_1), \dots, f(x'), g(x'), g(x_{i+1}), \dots, g(x_k)\}$$

after removing repetitions. By **Proposition 9.3.2**, $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$. \square

9.4. Modeling Homotopy Classes of Maps

We are ready to consider a method to model the homotopy classes of maps between topological spaces based on the framework we have laid.

THEOREM 9.4.1. *Let X be a finite A -space and let Y be an arbitrary A -space. Then there is a natural bijection between $[|\mathcal{K}(X)|, |\mathcal{K}(Y)|]$ and the colimit of the system*

$$[X, Y] \xrightarrow{\text{sup}^*} [\text{Sd}X, Y] \xrightarrow{\text{sup}^*} [\text{Sd}^2 X, Y] \xrightarrow{\text{sup}^*} \dots$$

This bijection, which we denote $K : \text{colim}_n [\text{Sd}^n X, Y] \rightarrow [|\mathcal{K}(X)|, |\mathcal{K}(Y)|]$, maps $[f] \in [\text{Sd}^i X, Y]$ to $[|\mathcal{K}(f)|] \in [|\mathcal{K}(X)|, |\mathcal{K}(Y)|]$.

PROOF. From **Proposition 9.3.5**, homotopies are preserved by each component map from $[\text{Sd}^n X, Y]$ to $[|\mathcal{K}(X)|, |\mathcal{K}(Y)|]$ so K is well-defined. Suppose $[f] \in [\text{Sd}^i X, Y]$ and $[g] \in [\text{Sd}^j X, Y]$. If $[f]$ and $[g]$ are identified in the colimit, then there exists an N such that $f \circ \text{sup}^{(N-i)} \simeq g \circ \text{sup}^{(N-j)}$. Again by **Proposition 9.3.5**, $|\mathcal{K}(f \circ \text{sup}^{(N-i)})| \simeq |\mathcal{K}(g \circ \text{sup}^{(N-j)})|$. Thus, we need only check that $|\mathcal{K}(f)| \simeq |\mathcal{K}(f \circ \text{sup}^{(N-i)})|$ to ensure $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$. Noting that f and $f \circ \text{sup}^{(N-i)}$ are tautologically contiguous, we have by **Proposition 9.3.2** that $|\mathcal{K}(f)| \simeq |\mathcal{K}(f \circ \text{sup}^{(N-i)})|$. Thus, $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$.

By **Theorem 9.3.3**, each homotopy class of maps has associated to it at least one A -space approximation, so K is surjective.

To prove injectivity, suppose $|\mathcal{K}(f)| \simeq |\mathcal{K}(g)|$. Then f and g are by definition A -space approximations for, say, $|\mathcal{K}(f)|$. Suppose $[f] \in [\text{Sd}^i X, Y]$ and $[g] \in [\text{Sd}^j X, Y]$. Then by **Proposition 9.3.4**, f and g are contiguous. This is sufficient to conclude $[f] = [g]$ in $\text{colim}_n [\text{Sd}^n X, Y]$. Thus, we have shown all necessary criteria for the existence of this canonical bijection. \square

While a bijection certainly creates a beautiful relationship here, the surjectivity of this map should be regarded as the most interesting component of this result; It means that given an arbitrary continuous map between $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$, only a finite number of subdivisions on the level of A -spaces are necessary to model this

map up to homotopy. The fact that we can transfer the study of continuous maps between topological spaces to that of order preserving maps between posets should be startling. The rest of the framework and proof simply collapses down maps redundant up to homotopy.

9.5. Suspensions and the Group Structure on $\langle X, Y \rangle$

In this section, we narrow our focus to a particular class of topological spaces that induce additional structure on homotopy classes of maps. For arbitrary topological spaces X and Y , $[X, Y]$ is in general simply a set, so a bijection is the strongest relationship we can hope for. However, the fact that under certain conditions based homotopy classes admit a group structure may raise the questions as to whether our bijection can also be extended to a group isomorphism in these cases. Keeping the motivating example of homotopy groups in mind, we widen our attention to suspensions, of which spheres are an example, to demonstrate a way to define a group structure on our colimit using subdivisions.

First, we show the bijection outlined in Section 7 can be modified to a correspondence, to borrow Hatcher's notation for pointed homotopy classes in [31], $\langle |\mathcal{K}(X)|, |\mathcal{K}(Y)| \rangle$ and the colimit of

$$\langle X, Y \rangle \xrightarrow{\text{sup}^*} \langle \text{Sd}X, Y \rangle \xrightarrow{\text{sup}^*} \langle \text{Sd}^2X, Y \rangle \xrightarrow{\text{sup}^*} \dots$$

by noticing that fixing basepoints $x_0 \in X$ and $y_0 \in Y$ in partially ordered set allows for a notion of based maps where we require $f(x_0) = y_0$. The corresponded totally ordered subsets $\{x_0\}$ and $\{y_0\}$ indeed realize to points in $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ respectively, since they denote 0-simplices and thus realize to 0-subcomplexes. Our choice of 0-cell for the basepoint is preserved as expected under subdivisions of X , since there is a canonical inclusion map $i : X \rightarrow \text{Sd}X$ that takes an element x to the corresponding single element subset of $\{x\}$ of X , which denotes an element of $\text{Sd}X$. Since under the sup map, $\{x_0\} \mapsto x_0$, taking pointed spaces is compatible with our colimit and our definition of contiguity, which importantly extends the Simplicial Approximation Theorem to this new result.

However, to make this variation play well with the sup map and contiguity, we must require that x_0 is a maximal point of X , which is no trouble since we have already assumed X is finite. This will agree with our methods in the following section.

Now that we have sufficiently restricted our homotopy class bijection, we may proceed with our group structure construction.

Recall that given a topological space X , we define the *suspension of X* , denoted SX , to be the quotient space $(X \times I)/(\sim)$, where $(x, t) \sim (y, s)$ if and only if $(x, t) = (y, s)$ or $s = t = 0$ or $s = t = 1$. We say the suspension is *reduced* if we additionally identify all points of the form (x_0, t) for $t \in [0, 1]$. For this, we write ΣX . Clearly, this collapsing map gives a homotopy equivalence from SX to ΣX .

DEFINITION 9.5.1. We define the finite analog of these constructions to be the *non-Hausdorff suspension* $\mathbb{S}X$, which is the resulting space after adding two points $+$ and $-$ to X such that the only open sets containing them are $\mathbb{S}X$ itself, $\{X \cup +\}$, and $\{X \cup -\}$.

In terms of posets, where we will set our following constructions, this amounts to adding two new maximal points to the poset of X .

THEOREM 9.5.2. [48] *For any space X , the map $\gamma : SX \rightarrow \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f : X \rightarrow Y$, the maps $Sf : SX \rightarrow SY$ and $\mathbb{S}f : \mathbb{S}X \rightarrow \mathbb{S}Y$ are weak homotopy equivalences. Therefore $\gamma^n : S^n X \rightarrow \mathbb{S}^n X$ is a weak homotopy equivalence for any space X .*

Further, this implies that there is a basepoint-preserving weak homotopy equivalence $\gamma^n : (\Sigma^n X, x_0) \rightarrow (\mathbb{S}^n X, x_0)$.

Unfortunately, the non-Hausdorff suspension of a space will not give quite the structure we need to make our desired map continuous. Thus, we define an alternative configuration:

DEFINITION 9.5.3. For a topological space X , the *non-Hausdorff opposite suspension* of X is given formally by $(\mathbb{S}(X^{op}))^{op}$, which we interpret as adding two new minimal points to X , and denote $\mathbb{S}^{op}X$.

From this will arise a nice space intimately related to the classical suspension that will allow us to model a defining property of suspensions of topological spaces.

There is a natural map from $\mathcal{K}(SX)$ to $\mathcal{K}(\mathbb{S}^{op}X)$ which, for example, takes a simplex $\sigma = \{x_0, x_1, \dots, x_k, +\}$ to the simplex $\{+, x_0, x_1, \dots, x_k\}$ in $\mathcal{K}(\mathbb{S}^{op}X)$ and preserves ordering. From this, it follows that $|\mathcal{K}(SX)| = |\mathcal{K}(\mathbb{S}^{op}X)|$, as the geometric realization functor acts independently of the order on the underlying poset.

The map γ in [Theorem 9.5.2](#) and the map p defined by McCord together generate a weak equivalence $\tilde{p} : \Sigma|\mathcal{K}(X)| \rightarrow SX$. This, coupled with the above equivalence, gives that $\Sigma|\mathcal{K}(X)| \simeq |\mathcal{K}(\mathbb{S}^{op}X)|$. Keeping this relationship in mind, we digress to lay some mathematical framework that will help us make use of this homotopy equivalence.

DEFINITION 9.5.4. [5] A pointed topological space (X, x_0) is a *co-H-space* if there exists a map $\psi : X \rightarrow X \vee X$ such that $p_1\psi \simeq id_X \text{ rel } x_0$ and $p_2\psi \simeq id_X \text{ rel } x_0$, where $p_1, p_2 : X \vee X \rightarrow X$ are the two projection maps. (X, x_0) is a *co-group* if it is co-associative and there is a co-inverse map $\xi : X \rightarrow X$ up to based homotopy such that the co-group axioms are satisfied.

Above, we require that $X \vee X$ be wedged at x_0 and call x_0 the identity of the co-H-space.

EXAMPLE 9.5.5. $(\Sigma X, x_0)$ is a co-group (and thus a co-H-space) for all pointed topological spaces (X, x_0) .

We are particularly interested in suspensions and their properties because of their presence in many important constructions in algebraic topology. For example, the n -sphere is homeomorphic to n iterative suspensions of the 0-sphere, generating a class of topological spaces central to algebraic topology. Additionally, the co-group structure of a suspension plays an important role in the Eckmann-Hilton duality, as understanding the loop spaces of certain topological spaces often gives a wealth of information.

Given that our motivation for looking at A -spaces is to encode information about topological spaces more generally, one may ask if we are able to model this co-H-space structure using finite A -spaces. Unfortunately, following a dual result for H-spaces from Stong, we have the following:

THEOREM 9.5.6. [33] *Suppose X is a finite connected space. Then X admits a co-H-space structure if and only if it is contractible.*

Thus, any finite model that is also a co-H-space is a model of a space with only trivial homotopy groups, which is both terribly narrow and uninteresting. This seems to imply that we have no way to model such a structure using finite spaces. For this reason, we turn to the colimit we have constructed for inspiration on how to model the co-H-space structure of a suspension that makes use of the subdivision functor in a way that strengthens the main result of Section 8.

THEOREM 9.5.7. *Suppose that X is a finite A -space. Then there exists an order-preserving map $\phi : \text{Sd}(\mathbb{S}^{op}X) \rightarrow \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$ such that $p_1|_{\mathcal{K}(\phi)} \simeq id_{\Sigma|\mathcal{K}(X)}$ and $p_2|_{\mathcal{K}(\phi)} \simeq id_{\Sigma|\mathcal{K}(X)}$, where p_1 and p_2 are the respective projection maps.*

CONSTRUCTION 9.5.8. We assume that the wedge of two copies of $\mathbb{S}X$ identifies different minimal points (i.e. $+ \sim -$) in each copy of X to make the description of our map cleaner, but the reader should check that a similar map exists regardless of choice of minimal basepoint on each copy.

Let $\sigma = \{x_0, x_1, \dots, x_n\}$ be an element of $\text{Sd}(\mathbb{S}^{op}X)$. Define an equivalence relation on $\text{Sd}(\mathbb{S}^{op}X)$ by the following two generating conditions:

- (i) Identify all such σ where x_0 was an original element of X (in other words, $\text{inf}(\sigma) \neq +$ or $-$).
- (ii) $\sigma \sim \sigma'$ if $\text{sup}(\sigma) = \text{sup}(\sigma')$ and $\text{inf}(\sigma) = \text{inf}(\sigma')$.

Call the identification space generated by this partition X' , equipped with the quotient topology (which corresponds to inheriting an order from $\mathbb{S}^{op}X$). Note that if X is a finite A -space, X' is a finite A -space. Let ρ denote the identification map from $\text{Sd}(\mathbb{S}^{op}X)$ to X' .

PROPOSITION 9.5.9. $X' \cong \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$.

PROOF. Define a map $f : X' \rightarrow \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$ as follows. Given an element σ of $\text{Sd}(\mathbb{S}^{op}X)$ such that $\text{inf}(\sigma)$ is not minimal in $\mathbb{S}^{op}X$, let $f(\sigma) = *$, the basepoint of the wedge. Otherwise, $\text{inf}(\sigma)$ is one of two minimal points of $\mathbb{S}^{op}X$. Let which minimal point σ contains be regarded as a coordinate to denote one of the copies of $\mathbb{S}^{op}X$. Then, restricted to that copy, let $f(\sigma) = \text{sup}(\sigma)$.

That this map is surjective follows from the fact that sup is surjective. To see that it is injective, consider σ and σ' such that $f(\sigma) = f(\sigma')$. Then either $f(\sigma) = f(\sigma') = *$, in which case neither contained a minimal element of $\mathbb{S}^{op}X$ so they were identified together in X' , or both the inf and sup map agree, which implies $\sigma \sim \sigma'$ in $\text{Sd } \mathbb{S}^n$, so $\sigma = \sigma'$ in X .

To prove f is continuous, suppose $\sigma \leq \sigma'$. Then $\rho(\sigma) \leq \rho(\sigma')$ by the quotient topology, and there are four cases to consider: first, that neither σ nor σ' have a minimal element. Then $f\rho(\sigma) = * = f\rho(\sigma')$. Suppose σ does not have a minimal element but σ' does. Then $f\rho(\sigma) = *$ which is tautologically less than or equal to $f\rho(\sigma')$. Then, suppose both contain a minimal element. If they do not agree on the minimal element, they were not comparable subsets to begin with. Then supposing they do, this means that ρ is determined by sup , which is continuous. So f preserves order and is therefore continuous.

Let $g : \mathbb{S}^{op} \vee \mathbb{S}^{op} \rightarrow X'$ be the map that sends the wedge point to the equivalence class of all chains not containing a minimal element. Suppose $x \in \mathbb{S}^{op} \vee \mathbb{S}^{op}$ is not the wedge point. Then define where x is sent by first noting which copy of \mathbb{S}^{op} x

belongs to (the copies correspond to the minimal points $+$ and $-$ respectively) and which point in \mathbb{S}^{op} maps to it under the inclusion $i : \mathbb{S}^{\text{op}} \rightarrow \mathbb{S}^{\text{op}} \vee \mathbb{S}^{\text{op}}$ into that copy. Let the resulting $+$ or $-$ association determine the minimal point of the chain and the preimage under inclusion the maximal point, completely determining an element of X' . This construction makes it a clear inverse, and checking continuity is simply another process of checking assignment cases while recognizing the ordering of X' is fully determined by that of $\text{Sd}(\mathbb{S}^{\text{op}}X)$. \square

We end by renaming the composition $f \circ \rho$ as ϕ for simplicity. For the next checks, we will use the construction above with instead the assumption that the points “ $+$ ” were identified for the basepoint of the wedge. Again, one should check that this choice is arbitrary and that the inconsistency in choice in this paper is instead to give the reader the simplest and most elegant constructions, as should be the point.

PROPOSITION 9.5.10. *Let*

$$\phi_* : (|\mathcal{K}(\text{Sd}(\mathbb{S}^{\text{op}}X))|, |\mathcal{K}(+)|) \rightarrow (|\mathcal{K}(\mathbb{S}^{\text{op}}X)| \vee |\mathcal{K}(\mathbb{S}^{\text{op}}X)|, |\mathcal{K}(*)|)$$

be the map induced by ϕ . Then $p_1\phi_* \simeq id_{\Sigma|\mathcal{K}(X)|} \text{ rel } |\mathcal{K}(+)|$ and $p_2\phi_* \simeq id_{\Sigma|\mathcal{K}(X)|} \text{ rel } |\mathcal{K}(+)|$, i.e. ϕ induces a co-H-space structure on $(\Sigma|\mathcal{K}(X)|, |\mathcal{K}(+)|)$.

PROOF. To show that ϕ induces a co-H-space structure on $\Sigma|\mathcal{K}(X)|$, we must show that the induced map $\phi_* = |\mathcal{K}(\phi)|$ satisfies the co-H-structure properties of ψ as above. Basepoint preservation comes from keeping track of its point-wise assignment, so we omit that tracking. Since our map is symmetric, we demonstrate the proof explicitly for only p_1 .

Since there is a canonical homeomorphism from $|\mathcal{K}(\text{Sd}(\mathbb{S}^{\text{op}}X))|$ to $|\mathcal{K}(\mathbb{S}^{\text{op}}X)|$ [26], the homotopy equivalence $|\mathcal{K}(\text{sup})| : |\mathcal{K}(\text{Sd}(\mathbb{S}^{\text{op}}X))| \rightarrow |\mathcal{K}(\mathbb{S}^{\text{op}}X)|$ is in fact a map homotopic to the identity. This simplifies our problem to showing that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} \text{Sd}^{k+1}(\mathbb{S}^{\text{op}}X) & \xrightarrow{\phi} & \text{Sd}^k(\mathbb{S}^{\text{op}}X) \vee \text{Sd}^k(\mathbb{S}^{\text{op}}X) & \xrightarrow{p_1} & \text{Sd}^k(\mathbb{S}^{\text{op}}X) \\ & & \text{sup} & & \end{array}$$

We begin by demonstrating that $p_1\phi \simeq \text{sup}$. Let $\sigma = \{x_0, x_2, \dots, x_k\}$ be an element of $\text{Sd}(\mathbb{S}^{\text{op}}X)$. We consider three cases. First, suppose x_0 is not a minimal element of $\mathbb{S}^{\text{op}}X$. Then $\phi(\sigma)$ is minimal in $\mathbb{S}^{\text{op}}X \vee \mathbb{S}^{\text{op}}X$, which is preserved under projection. If x_0 is a minimal element, but is the minimal element associated with the second copy of $\mathbb{S}^{\text{op}}X$, the projection map will ultimately crush σ to $*$, which is minimal. In both of these cases, it trivially follows that $p_1\phi(\sigma) \leq \text{sup}(\sigma)$. If σ contains the minimal point sent to “first” sphere, then $p_1\phi$ sends σ to $\text{sup}(\sigma)$, so we tautologically have that $p_1\phi(\sigma) \leq \text{sup}(\sigma)$.

It follows from ?? that if $p_1\phi(\sigma) \leq \text{sup}(\sigma)$ is true for all σ , then $p_1\phi \simeq \text{sup}$. By Proposition 9.3.5 we have that $p_1\phi \simeq \text{sup}$ implies $|\mathcal{K}(p_1\phi)| \simeq |\mathcal{K}(\text{sup})|$. Thus, $p_1\phi_* \simeq id \text{ rel } |\mathcal{K}(+)|$. \square

PROPOSITION 9.5.11. [5] *Let (X, x_0) and (Y, y_0) be arbitrary topological spaces. Then*

- (i) *There is a unital binary operation on $\langle X, Y \rangle$ if and only if X is a co-H-space.*
- (ii) *$\langle X, Y \rangle$ is a group if and only if X is a co-group.*

Given this result, we should expect $\langle |\mathcal{K}(\mathbb{S}^{op}X)|, |\mathcal{K}(Y)| \rangle$ to be a group since $|\mathcal{K}(\mathbb{S}^{op}X)|$ is homotopic to the reduced suspension of the geometric realization of X , a CW complex, and thus has a co-group structure. Below we will demonstrate how our finite model of the co-H-space structure of $\mathbb{S}^{op}X$ captures enough information to model the group structure of these homotopy classes of maps.

Suppose $[f] \in \langle \text{Sd}^k(\mathbb{S}^{op}X), Y \rangle$ and $[g] \in \langle \text{Sd}^m(\mathbb{S}^{op}X), Y \rangle$ with $k \leq m$. Then we may construct the following chain of maps:

$$\begin{array}{ccc} \text{Sd}^{m+1}(\mathbb{S}^{op}X) & \xrightarrow{\phi} & \text{Sd}^m(\mathbb{S}^{op}X) \vee \text{Sd}^m(\mathbb{S}^{op}X) \\ & & \downarrow \text{sup} \vee \text{id} \\ & & \text{Sd}^k(\mathbb{S}^{op}X) \vee \text{Sd}^m(\mathbb{S}^{op}X) \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y, \end{array}$$

where $\nabla : Y \vee Y \rightarrow Y$ denotes the codiagonal map. Define the homotopy class of the composition to be $[f] + [g]$. Thus, this co-H-space model structure allows us to define a unital binary operation in the colimit.

We would like our induced binary operation to be associative and have inverses, as to model the full group structure on these homotopy classes. For this, we use the construction of the bijection in the previous section to induce this additional structure from the modeled homotopy class of maps.

Given $[[f] + [g]] + [h]$ in the colimit, we have $[[f] + [g]] + [h] \mapsto K([[f] + [g]]) + K([h])$, which we know equals $K([f]) + K([[g] + [h]])$ by the associativity of the group structure on $[S^n, X]$. Then via the existence of K^{-1} , we get $[[f] + [g]] + [h] = [f] + [[g] + [h]]$, showing the operation is associative. Likewise, suppose we have $[f]$ in the colimit. Define $[f]^{-1}$ as the unique element $K^{-1}((K([f]))^{-1})$, whose existence and uniqueness is guaranteed by the existence and uniqueness of inverses in $\langle \Sigma|\mathcal{K}(X)|, |\mathcal{K}(Y)| \rangle$ and our one-to-one correspondence. Therefore we have constructed a model for the full group structure of $\langle \Sigma|\mathcal{K}(X)|, |\mathcal{K}(Y)| \rangle$ using the result of [Theorem 9.4.1](#).

9.6. Applications to Homotopy Groups

One very important place we can use the group structure induced from suspensions is in the homotopy groups of a topological space X . Homotopy groups are an essential element of algebraic topology, and the ability to model them using a system built from A -spaces should be incredibly motivating. Since for $n > 1$, the n -sphere is nothing more than the suspension of S^{n-1} , it should be unsurprising that we will conclude this paper by discussing applications to homotopy groups.

For this section, we will restrict our attention to connected (and hence path-connected) CW complexes X , and must pay some mind of based homotopy classes, since we have the relation $\pi_k(X) \cong \langle S^k, X \rangle$. Thus, applying the above sections to conclude homotopy-theoretic results is a natural extension.

Here, we omit the opposite suspension construction for the sphere, as $\mathbb{S}^{op}\mathbb{S}^n \cong \mathbb{S}\mathbb{S}^n$ because of the symmetric nature of the minimal finite model for S^n . Thus, we simplify our notation by dropping the opposites, and leave the reader to check that the same map ϕ suffices to give a map from $\text{Sd } \mathbb{S}^n$ to $\mathbb{S}^n \vee \mathbb{S}^n$ which induces a co-H-space structure on S^n . Given that we demonstrated both a bijection and group homomorphism in the previous sections, and under the assumption

that X is a connected A -space, we may conclude $\operatorname{colim}_n \langle (\operatorname{Sd}^k \mathbb{S}^n, *), (X, x_0) \rangle \cong \pi_n(|\mathcal{K}(X)|, |\mathcal{K}(x_0)|)$ as groups for $n \geq 2$.

We can say more about spheres, as we know that for $n \geq 2$, we have a cocommutative co-group that induces an abelian group structure on $\pi_n(X)$. We demonstrate this below:

PROPOSITION 9.6.1. [5] *Let (X, x_0) be a co- H -space. Then $\psi : X \rightarrow X \vee X$ is cocommutative if and only if $\langle X, Y \rangle$ is a cocommutative cogroup for all spaces Y .*

Given $[f] + [g]$ in the colimit, we have $[f] + [g] \mapsto K([f]) + K([g])$, which we know is equal to $K([g]) + K([f])$ in $\langle S^n, X \rangle$. Then $[f] + [g] = [g] + [f]$.

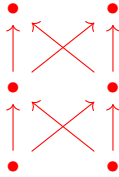
Applying the results of this paper to homotopy-theoretic results, we begin by showing that if a connected CW complex X admits an A -space model A with a sole maximal (or minimal) point, then X is contractible.

Let $*$ denote this maximal (or minimal) point. Then any based map $f : \operatorname{Sd}^k(\mathbb{S}^n) \rightarrow A$ for some values k and n , the image consists of finitely many points, and the constant map given by $g(x) \equiv *$ is always greater than or equal to f since $* \geq \operatorname{Im}(f)$ (or less than or equal to f since $* \leq \operatorname{Im}(f)$). Thus, $f \simeq g$ for all maps f based at $*$. This gives that $\langle S^n, X \rangle$ is trivial in every grading, giving trivial homotopy groups by inclusion of π_n . Since trivial homotopy groups imply contractibility for CW complexes by Whitehead's theorem, we conclude X is contractible.

Diverging to poset-theoretic results, this also implies that the geometric realization of a lattice is contractible. For a similar but perhaps more intriguing application, we set up the following definition:

DEFINITION 9.6.2. The *infinite sphere* S^∞ is defined to be the directed colimit of S^n , where the map from S^{n-1} to S^n is given by inclusion.

For our purposes, we would like to generalize the A -space model we have for S^n to an A -space model for S^∞ . As we understand \mathbb{S}^n as being n successive suspensions of S^0 , the same construction applies here. In particular, we typically represent the finite model of S^2 by the Hasse diagram below:



For S^∞ , we get much the same picture, except the tower stretches infinitely upwards. The A -space corresponding to this Hasse diagram we will denote \mathbb{S}^∞ .

PROPOSITION 9.6.3. S^∞ is contractible.

PROOF. Given our A -space model for S^∞ , we can consider $\operatorname{colim}_n \langle \operatorname{Sd}^n \mathbb{S}^k, \mathbb{S}^\infty \rangle$ to model the k th homotopy group of S^∞ . While working with a colimit does not seem necessarily advantageous compared to working with a topological space itself, it does allow us to conclude some important results here.

It is clear to see that for any n the image of a map $f : \operatorname{Sd}^n \mathbb{S}^k \rightarrow \mathbb{S}^\infty$ will be finite since $\operatorname{Sd}^n \mathbb{S}^k$ has only finitely many points. Since \mathbb{S}^∞ is infinite, and in particular never attains maximal points in its poset representation, there will always exist an

$x_K \in \mathbb{S}^\infty$ such that $f(x) < x_K$ for all $x \in \text{Sd}^n \mathbb{S}^k$. Then the constant map that sends every element of $\text{Sd}^n \mathbb{S}^k$ to x_K is homotopic to f by ???. Further, given maps f and g , we are ensured by the nonexistence of a maximal element that we can always find some x_K s.t. f and g are both dominated by the constant map to x_K . Thus, any two choices of f and g will be homotopic regardless of the number of subdivisions we take, and therefore by the same argument involving basepoints as above, $\pi_n(\mathbb{S}^\infty)$ is trivial for all n . \mathbb{S}^∞ is also a CW complex, so this is sufficient to conclude contractibility. \square

While the applications of this section involve little calculation, the potential applications of framing some homotopic information within the scope of combinatorial computation could be used more broadly. While the computation of these homotopy classes of maps between subdivided A -spaces remain unreasonably computable at present, there is no *theoretical* barrier to this method of finding the the homotopy classes of maps between spaces, particularly for finite homotopy groups. It remains to be seen whether an upper bound on subdivisions for the bijection to hold can be obtained in the case where $[X, Y]$ is a finite set, and additionally whether the Freudenthal Suspension Theorem can be combinatorially concluded.

Characterization of finite manifolds

We now describe a particularly well-behaved class of finite spaces and characterize them in dimension 2.

DEFINITION 10.0.1. A finite T_0 space X is a *finite n -manifold* if $|\mathcal{K}(X)|$ is a topological n -manifold.

The following definitions present useful towards this end. The familiar notions in the theory of simplicial complexes of links and pure complexes can be similarly connected to posets.

DEFINITION 10.0.2. The height of X is given by

$$ht(X) = \max_{C \in \mathcal{C}(X)} \{ht(C)\}.$$

Note that the height of a chain is equal to the dimension of its corresponding simplex. It is immediate from invariance of dimension that a finite n -manifold must be of height $n + 1$. This notion can be also extended to the realm of finite spaces.

DEFINITION 10.0.3. Let X be a finite T_0 space, and let $\mathcal{C}(X)$ be the set of non-empty chains of X . For $C \in \mathcal{C}(X)$, the *height* of C is given by

$$ht(C) = \#C - 1.$$

DEFINITION 10.0.4. Let X be a finite T_0 space, and let $x \in X$. The *level* of x in X is given by

$$\ell_X(x) = ht(\hat{U}_x^X) + 1.$$

Equivalently, the level of x is the maximum height of all chains in X with x as its greatest element.

DEFINITION 10.0.5. Let K be an abstract simplicial complex, and let σ be a face in K . Then the *link* of σ in K is given by

$$lk_K(\sigma) = \{x \in X \mid \tau \cup \sigma \in K, \tau \cap \sigma = \emptyset\}.$$

In other words, the link consists of all faces of K whose union with σ is a face of K , and whose intersection with σ is empty. Note that a link is always itself a simplicial complex.

We now define an analogous term for finite T_0 spaces.

DEFINITION 10.0.6. Let X be a finite T_0 space, and let C be a non-empty chain of X . Then the *link* of C in X is given by

$$lk_X(C) = \{x \in X \setminus C \mid C \cup \{x\} \text{ is a chain}\}.$$

We can easily see that these correspond in the expected manner.

PROPOSITION 10.0.7. *Let X be a finite T_0 space, and let C be a chain in X . Then $\mathcal{X}(lk_X(C)) = lk_{\mathcal{X}(X)}(C)$.*

PROOF. If D is a chain in $lk_X(C)$, then $D \cup C$ is a chain in X and $D \cap C = \emptyset$. Conversely, if v is a vertex in $lk_{\mathcal{X}(X)}(C)$, then $v \notin C$ and $C \cup \{v\}$ is a chain in X . \square

Finally, we introduce a related concept for individual vertices.

DEFINITION 10.0.8. Let X be a finite T_0 space, and let $x \in X$. The *lower link* of x in X is given by

$$\hat{U}_x^X = \{y \in X \mid y < x\}$$

The *upper link* of x in X is given by

$$\hat{F}_x^X = \{y \in X \mid y > x\}$$

When it is clear from context where the lower or upper link comes from, we write simply \hat{U}_x and \hat{F}_x .

We define lower and upper links for $\mathcal{X}(X)$ in the expected manner. Note that $\hat{U}_x \cup \hat{F}_x = lk_X(\{x\})$. Furthermore, we can extend x “upwards” into a chain C such that $\hat{U}_x = lk_X(C)$, and similarly “downwards” into a chain D such that $\hat{F}_x = lk_X(D)$. Finally, note that

$$\hat{U}_x^X = \hat{F}_x^{X^{OP}}$$

and similarly

$$\hat{F}_x^X = \hat{U}_x^{X^{OP}}$$

10.1. Pure complexes

The first of the following definitions is standard, while the second extends the idea of the first to finite T_0 spaces.

DEFINITION 10.1.1. An n -dimensional simplicial complex K is called *pure* if every simplex in K is contained in an n -simplex.

Equivalently, we require all maximal faces have the same dimension.

DEFINITION 10.1.2. A finite T_0 space X of height n is called *pure* if every maximal chain in X is of height n .

As suggested by the terminology, these notions are equivalent.

PROPOSITION 10.1.3. *A finite T_0 space X is pure if and only if $\mathcal{X}(X)$ is pure, and a finite simplicial complex K is pure if and only if $\mathcal{X}(K)$ is pure.*

PROOF. Suppose X is a finite T_0 space of height n , so $\mathcal{X}(X)$ is a simplicial complex of dimension $n - 1$. A k -simplex in $\mathcal{X}(X)$ is a chain of length $k + 1$, so every simplex in $\mathcal{X}(X)$ is contained in an $(n - 1)$ -simplex if and only if every chain in X is contained in a chain of length n .

Now suppose K is a finite simplicial complex of dimension n , so $\mathcal{X}(K)$ is a poset of height $n + 1$. The height of a maximal chain in $\mathcal{X}(K)$ is one greater than the dimension of the largest simplex it contains, so every maximal chain is of height $n + 1$ if and only if every simplex in K is contained in an n -simplex. \square

PROPOSITION 10.1.4. *If X is a pure finite T_0 space, then for all $x \in X$, the level $\ell_X(x)$ of x can be presented as follows:*

$$\ell_X(x) = ht(X) - \ell_{X \circ P}(x).$$

PROOF. Let $x \in X$ and let C be a maximal chain in X containing x . Since X is pure, $ht(C) = ht(X)$. Let $C_{\leq} = \{y \in C \mid y \leq x\}$, and $C_{\geq} = \{y \in C \mid y \geq x\}$. Then $ht(C) = ht(C_{\leq}) + ht(C_{\geq})$. It must be the case that $ht(C_{\leq}) = \ell_X(x)$ (otherwise, there would be some maximal chain longer than C), and using that $\hat{F}_x^X = \hat{U}_x^{X \circ P}$, we have that $ht(C_{\geq}) = \ell_{X \circ P}(x)$ for the same reasons. Our desired result immediately follows. \square

The reason for introducing pureness is that it plays an important role in the characterization of finite surfaces.

THEOREM 10.1.5. *A finite T_0 space X is a finite surface if and only if it satisfies the following conditions:*

- (1) X is pure of height 3;
- (2) For each height-2 point x , there are exactly two points greater than x and two points less than x ; and
- (3) For each maximal point x_m and each minimal point x_n , the set $(x_m, x_n) = \{x \in X \mid x_n < x < x_m\}$ contains either zero or two points.
- (4) For each extremal point x , the set of points other than x which are comparable to x is connected.

The bulk of the proof of this theorem is based on the corresponding result for simplicial complexes. Stating it requires the following standard definition.

DEFINITION 10.1.6. If v is a vertex in a simplicial complex K , the *link* of v , $Lk(v, K)$, is the undirected graph whose vertices are the 1-simplices of X with v as a face, and where there is an edge between two vertices if they are faces of a common 2-simplex.

LEMMA 10.1.7. *The geometric realization of a finite simplicial complex K is a surface if and only if K satisfies the following conditions:*

- (1) K is pure and 2-dimensional;
- (2) Each 1-simplex of K is a face of exactly two 2-simplices; and
- (3) For each vertex v of K , $|Lk(v, K)|$ is homeomorphic to S^1 .

PROOF. If (1) fails, $|K|$ is not a surface by invariance of dimension. If (2) fails, removing a line from any sufficiently small connected neighborhood of a point in the edge yields three components, so it is not locally Euclidean. If (3) fails, removing v from a sufficiently small connected neighborhood yields two components, so it is not locally Euclidean.

Suppose now that all three conditions hold. Then (1) guarantees that we only need to check the interior of 0-, 1-, and 2-simplices. The last is trivial. Since gluing together two polygons at an edge yields a Euclidean neighborhood for points on the edge, 1-simplices follow by (2). Finally, 0-simplices follow by (3), since it implies that at a 0-simplex v , the realization is locally homeomorphic to the disk obtained by gluing together triangles along their edges circularly. \square

Now we can prove the theorem.

PROOF. The first condition for the poset is equivalent to the first condition for the simplicial complex. Given pureness, the second and third poset conditions together are equivalent to the second simplicial complex condition, because (together with the pureness) they are equivalent to the statement that for any two comparable points p and q , there are exactly two ways of extending the 2-chain $\{p, q\}$ to a 3-chain. Finally, the second, third and fourth poset conditions together are equivalent to the third simplicial complex condition, since a graph is a circle if and only if it is connected and each vertex has degree 2. \square

There is an alternate characterization of finite surfaces which is also useful. While it is ultimately just a more compact rephrasing of Theorem 4.7, we will see that it is convenient for a number of purposes. The proof is given by point-counting together with the above criterion for a graph to be a circle, and comparing to the conditions of our original classification.

DEFINITION 10.1.8. Let X be a finite poset and $x \in X$. Then the *link* of x , $Lk(x)$, is the set of points other than x which are comparable to x .

COROLLARY 10.1.9. A finite T_0 space X is a finite surface if and only if for each $x \in X$, $|Lk(x)|$ is homeomorphic to S^1 .

One of the reasons this statement of the theorem is advantageous is that it can more easily describe the higher-dimensional version of the theorem. Although we have written it out specifically for finite surfaces, the proof of this theorem generalizes directly to higher dimensions¹, so we obtain the following.

COROLLARY 10.1.10. A finite T_0 space X is a finite n -manifold if and only if for each $x \in X$, $|\mathcal{K}(Lk(x))|$ is homeomorphic to S^{n-1} .

Another benefit of this form of the theorem is its relationship to the following result of A. Björner in [11].

THEOREM 10.1.11. Let P be a finite poset, and for each $x \in P$, denote the set of points less than x by \hat{U}_x . Then P is the face poset of a regular CW complex if and only if for each $x \in P$, $|\mathcal{K}(\hat{U}_x)|$ is homeomorphic to a sphere.²

This gives us a final characterization of finite surfaces which will be crucial in obtaining our bound in the next section.

THEOREM 10.1.12. A finite T_0 space X is a finite surface if and only if it is the face poset of a regular CW structure on some closed surface.

PROOF. Firstly, suppose $X = \mathcal{X}(Y)$, where Y is a regular CW structure on some closed surface. Then $|\mathcal{K}(X)|$ is nothing more than the cellular subdivision of Y , so the two are homeomorphic.

Suppose conversely that X is a finite surface, and let x be some point in X . If x is minimal, then by definition $|\mathcal{K}(\hat{U}_x)| \cong S^{-1}$. If x is on the second level, there are exactly two points below it by Theorem 4.7, so $|\mathcal{K}(\hat{U}_x)| \cong S^0$. Finally, if x is maximal, then $|\mathcal{K}(\hat{U}_x)| \cong S^1$ by Theorem 4.13. \square

¹In two dimensions, the conditions guarantee precisely that we have triangles glued in a circular fashion, which yields a Euclidean neighborhood of every point. The higher-dimensional equivalent is for n -simplices to be glued so as to form a ball in the neighborhood of a vertex, which is expressed via the condition that the indicated poset has order complex homeomorphic to S^{n-1} .

²We take the empty space to be the sphere of dimension -1 .

Before moving on, it is worth taking a moment to consider the common element between Corollary 4.11 (more generally Corollary 4.12) and Theorem 4.13 which allowed us to prove this relationship: subposets whose order complexes are simplicial spheres. It is generally a nontrivial problem to determine whether a finite poset has this property, although the case in dimension 1 is simple: a height-2 poset has geometric realization S^1 if and only if it is connected and every vertex has degree 2. Given this fact, the following theorem suggests the idea of taking an inductive approach to the problem.

THEOREM 10.1.13. *If X is a finite n -manifold, then $|\mathcal{K}(X)|$ is homeomorphic to S^n if and only if X is a finite model of S^n .*

PROOF. One direction is obvious: if $|\mathcal{K}(X)|$ is homeomorphic to S^n , then X is a finite model of S^n by Theorem 1.4.

To prove the other direction, suppose X is a finite n -manifold which is a finite model of S^n . Then $|\mathcal{K}(X)|$ is a CW space which is weak homotopy equivalent to S^n , and hence homotopy equivalent to it by the Whitehead theorem. Since $|\mathcal{K}(X)|$ is a closed n -manifold, the result follows by the Poincaré conjecture. \square

10.1.1. Bounds for finite surfaces. Throughout this section, we will denote the number of height 1, 2, and 3 points by ℓ , m , and n respectively.

The problem of finding absolutely minimal finite models amounts to minimizing the sum of the number of points at each level. As the following result shows, by restricting to finite surfaces, we need consider only one number rather than three or more.

PROPOSITION 10.1.14. *Let X be a finite surface which is a model of a closed surface S of genus g . If S is orientable, then $\#X = 2m + 2 - 2g$. If S is nonorientable, then $\#X = 2m + 2 - g$.*

PROOF. Because X is a finite surface, $\#X = \ell + m + n$. But we also know by Theorem 4.14 that X is the face poset of a regular CW complex structure on S , so $n - m + \ell = \chi(S)$. Thus, $\#X = \ell + m + n = 2m + \chi(S)$. The result follows by the standard formula for the Euler characteristic of a closed surface. \square

Using the fact that any finite model of a closed surface other than $\mathbb{R}P^2$ or S^2 must satisfy the (S2) splitting property and thus have at least three maximal and three minimal points, we can immediately derive from this the linear lower bounds $2g+10$ and $g+10$ for the size of finite surface models of orientable and nonorientable closed surfaces respectively. However, we can do slightly better than this.

THEOREM 10.1.15. *Let X be a finite surface modelling the closed surface S of genus g . If S is orientable, then $\#X \geq 2\lceil 4\sqrt{g} \rceil + 2g + 6$. If S is nonorientable, then $\#X \geq 2\lceil 2\sqrt{2g} \rceil + g + 6$.*

PROOF. Let c_i denote the degree of the i^{th} maximal point in the Hasse diagram of X . Then since each point in the middle level has up-degree 2, $\sum_i c_i = 2m$, and so for at least one i , $c_i \geq 2m/n$. Call this point x_i . Because \hat{U}_{x_i} is a finite model of S^1 , the number of minimal points less than x_i must be equal to the number of level 2 points less than x_i , which is just c_i . Thus, we get $c_i \leq \ell$, so $\lceil 2m/n \rceil \leq \ell$. The same argument for bottom points shows that $\lceil 2m/\ell \rceil \leq n$.

Adding these inequalities (and ignoring the ceilings), we get $2m(1/n + 1/\ell) \leq n + \ell = m + \chi(S)$, since $n - m + \ell = \chi(S)$. The smallest possible value of the

left side of the inequality is achieved when $n = \ell = (m + \chi(S))/2$, and we get $8m \leq (m + \chi(S))^2$. Solving, we get $m \geq 4\sqrt{g} + 2g + 2$ in the orientable case and $m \geq 2\sqrt{2g} + g + 2$ in the nonorientable case. The result follows from Proposition 5.1. \square

It is not clear that these inequalities are sharp, especially because we dropped the ceilings to derive them. However, there are some cases in which we can be certain they are achieved. To show this, we perform the following construction, illustrated in Figure 3.

PROPOSITION 10.1.16. *Let n and ℓ be positive even integers and set $2m = n\ell$. Then there is a finite orientable surface with n , m , and ℓ points in its third, second, and first levels respectively.*

PROOF. To construct this surface, take n ℓ -gons and identify them in the following way. Glue every other edge of the first ℓ -gon to every other edge of the second with coherent orientation, then glue the remaining edges of the second ℓ -gon to every other edge of the third (again with coherent orientation), and continue until the final ℓ -gon is glued back to the first. Because we have an even number of polygons, the final gluing will also have coherent orientation. Explicitly, we may embed the polygons in \mathbb{R}^3 centered at equal intervals along a circle and with parallel top edges, and glue them together via homotopies of \mathbb{R}^3 . Then each step of gluing switches the sides which are glued between containing and not containing the top edge, so having an even number of polygons guarantees that the first and last polygons will glue properly, so the space we have constructed admits an embedding in \mathbb{R}^3 . This construction also guarantees that the link of every vertex will be a circle (since it is connected and every vertex in the graph has degree two) and every edge will be adjacent to exactly two faces, so this will produce a closed orientable surface with a regular CW structure consisting of n faces, m edges, and ℓ vertices. We finish the construction by taking its face poset. \square

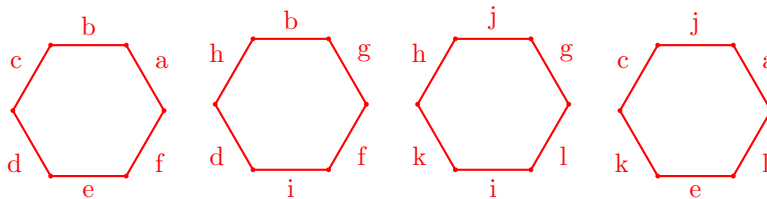


Figure 3: The polygons and identifications obtained by performing the construction with $\ell = 6$ and $n = 4$. All edges are oriented clockwise. Performing the gluing will yield the orientable surface of genus 2.

If we take $n = 4$, $\ell = 6$, this produces a model of the orientable surface of genus 2 with $n = 12$ (Figures 3,4). Geometrically, this is obtained by gluing together four hexagons in pairs to obtain two pairs of pants, then gluing together the pairs of pants to obtain the surface. By our bound above, this is minimal among finite orientable surfaces of genus 2.

It is unfeasible to explicitly construct every model individually to check if it achieves our bound. However, as the following theorem shows, for g with particularly nice number-theoretic properties, we don't need to.

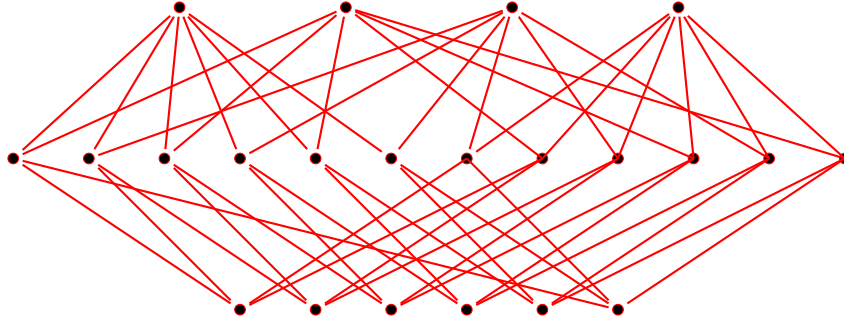


Figure 4: A minimal finite orientable surface of genus 2.

THEOREM 10.1.17. *If g is a perfect square, then performing this construction with $n = \ell = 2\sqrt{g} + 2$ yields a minimal finite orientable surface of genus g .*

PROOF. The resulting space has $m = 2g + 2 + 4\sqrt{g}$, so its Euler characteristic is $n - m + \ell = 2 - 2g$, which shows that it is indeed of genus g . Its cardinality is $\ell + m + n = 8\sqrt{g} + 2g + 6$, and since \sqrt{g} is an integer, this is precisely the lower bound derived above. \square

The simplest case is when g is a perfect square. However, the lower bound is more generally achieved by this construction when g is a product of two integers which are sufficiently close. For example, if g is of the form $(k-1)(k-2)$, then as long as k is at least 3, we get $4k - 7 < 4\sqrt{g} \leq 4k - 6$, so $\lceil 4\sqrt{g} \rceil = 4k - 6 = k - 1 + 2 + 4\sqrt{g}$, and setting $n = 2k$, $\ell = 2(k-1)$ yields a surface of the desired genus which achieves the bound. To further generalize this result is a problem of number theory.

Group actions and finite groups

We shall explain some of the results and questions in a beautiful 1978 paper [57] by Daniel Quillen. He relates properties of groups to homotopy properties of the simplicial complexes of certain posets constructed from the group. He does not explicitly think of these posets as finite topological spaces. He seems to have been unaware of the earlier papers of McCord [50] and Stong [65] that we have studied, and it is interesting to look at his work from their perspective. Stong himself first looked at Quillen's work this way [66], and we will include his results on the topic. We usually work with a finite group G , but the basic definitions apply more generally.

11.1. Equivariance and finite spaces

We begin with some general observations about equivariance and F -spaces, largely following Stong [66].

A topological group G is a group and a space whose product $G \times G \rightarrow G$ and inverse map $G \rightarrow G$ are continuous. An action of G on a topological space X is a continuous map $G \times X \rightarrow X$, written $(g, x) \mapsto gx$, such that $g(hx) = (gh)x$ and $ex = x$, where e is the identity element of G . A map $f: X \rightarrow Y$ of G -spaces is a continuous map f such that $f(gx) = gf(x)$ for $g \in G$ and $x \in X$.

For a space X , the automorphism group $\text{Aut } X$ is the topological group of homeomorphisms $X \rightarrow X$. The group operation is composition, and $\text{Aut } X$ is topologized as a subspace of the space of maps $X \rightarrow X$ with the compact open topology. Suppose a topological group G acts on X . Then the action of g on X gives a homeomorphism $g: X \rightarrow X$. This gives a group homomorphism $G \rightarrow \text{Aut } X$. At least if X is first countable, this map is also continuous. That is, it is a map of topological groups.

We say that G acts trivially on X if $gx = x$ for all g and x . We let G act diagonally on products $X \times Y$, $g(x, y) = (gx, gy)$. In particular, with G acting trivially on I , we have the notion of a G -homotopy, namely a G -map $h: X \times I \rightarrow Y$. There is a large subject of equivariant algebraic topology, in which one studies the algebraic invariants of G -spaces.

We begin with some basic ideas of equivalence in this context. We say that a G -map $f: X \rightarrow Y$ is a G -homotopy equivalence if there is a G -map $f': Y \rightarrow X$ and there are G -homotopies $f \circ f' \simeq \text{id}$ and $f' \circ f \simeq \text{id}$. For a subgroup H of G , define the H -fixed point space X^H of X to be $\{x \mid hx = x \text{ for } h \in H\}$. Say that a G -map f is an H -equivalence if $f^H: X^H \rightarrow Y^H$ is a nonequivariant homotopy equivalence. For nice G -spaces, the sort one usually encounters in classical algebraic topology, which are called G -CW complexes, a map f is a G -homotopy equivalence if and only if it is an H -equivalence for all subgroups H . Note that we have the

much weaker notion of an e -equivalence, namely a G -map which is a homotopy equivalence of underlying spaces, forgetting the action of G .

We also have weak notions. A G -map f is a weak G -homotopy equivalence if each $f^H: X^H \rightarrow Y^H$ is a weak homotopy equivalence in the nonequivariant sense. We also have the notion of a weak e -equivalence, meaning a G -map that is a weak homotopy equivalence of underlying spaces, forgetting the action of G .

In general, the notions of G -equivalence are very much stronger than the notions of e -equivalence. There are lots of G maps that are e -equivalences but are not G -equivalences. We show that cannot happen when G acts on a finite space. We start with some general observations.

LEMMA 11.1.1. *If an A -space G is a topological group, then it is discrete.*

PROOF. If $h \leq g$, then, by the continuity of the inverse map, $h^{-1} \leq g^{-1}$. By the continuity of left multiplication by h , $e \leq hg^{-1}$, and then, by the continuity of right multiplication by g , $g \leq h$. Since G is T_0 , $g = h$. Thus $U_g = \{g\}$ is open for all g and therefore every subset is open. \square

We have observed that if a topological group G acts on a space X , then we can view the action as given by a map of topological groups $G \rightarrow \text{Aut } X$. This homomorphism has a kernel K , and the action factors through the quotient group G/K , which is a topological group with the quotient topology. When X is an F -space, $\text{Aut } X$ is finite since there are only finitely many functions $X \rightarrow X$. But then G/K is finite and therefore discrete. Thus we lose no generality if we restrict our attention to finite discrete groups G acting on F -spaces. Therefore G will be finite from now on.

Recall the notion of upbeat and downbeat points in an F -space X . Note that if x is upbeat, so that there is a $y > x$ such that $z > x$ implies $z \geq y$, then y is uniquely determined by x .

THEOREM 11.1.2. *Let X be an F -space with an action by a group G . Then there is a core $C \subset X$ such that C is a sub G -space and equivariant deformation retract of X . We call C an equivariant core of X .*

PROOF. The orbit Gx of an element x is $\{gx | g \in G\}$. If x is upbeat, then gx is also upbeat, with gy playing the role of y . The inclusion $X - Gx \subset X$ is the inclusion of a sub G -space. Define $f: X \rightarrow X - Gx \subset X$ by $f(z) = z$ if $z \notin Gx$ and $f(gx) = gy$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \text{id}$ and thus $f \simeq \text{id}$. An explicit homotopy used to show this is given by $h(z, t) = z$ if $t < 1$ and $h(z, 1) = f(z)$, and this homotopy is a G -map. Removing upbeat and downbeat orbits successively until none are left, we reach an equivariant core. \square

COROLLARY 11.1.3. *If X is a contractible F -space with an action by a group G , then X is equivariantly contractible to a G -fixed point.*

PROOF. A core of X is a point, so an equivariant core must be a point with the trivial action by G . \square

COROLLARY 11.1.4. *If X is a contractible F -space, then X has a point that is fixed by every homeomorphism of X .*

PROOF. The finite group G of homeomorphisms of X acts on X , and an equivariant core is a fixed point. \square

THEOREM 11.1.5. *Let X and Y be F -spaces with actions by G and $f: X \rightarrow Y$ be a G -map. If f is an e -homotopy equivalence, then f is a G -homotopy equivalence.*

PROOF. Let C and D be equivariant cores of X and Y . Let $i_X: C \rightarrow X$ and $r_X: X \rightarrow C$ be the inclusion and retraction, and similarly for Y . Let p be the composite

$$C \xrightarrow{i_X} X \xrightarrow{f} Y \xrightarrow{r_Y} D, \quad p = r_Y \circ f \circ i_X.$$

Then p is a G -map and a homotopy equivalence between minimal finite spaces. The latter property implies that p is a homeomorphism, and p^{-1} is necessarily also a G -map. Define $g: Y \rightarrow X$ to be the composite

$$Y \xrightarrow{r_Y} D \xrightarrow{p^{-1}} C \xrightarrow{i_X} X, \quad g = i_X \circ p^{-1} \circ r_Y.$$

Then $g \circ f$ and $f \circ g$ are equivariantly homotopic to the respective identity maps. Indeed, we have the homotopies

$$gf = gf \operatorname{id}_X \simeq gf i_X r_X = i_X p^{-1} r_Y f i_X r_X = i_X p^{-1} p r_X = i_X r_X \simeq \operatorname{id}_X$$

and

$$fg = \operatorname{id}_Y fg \simeq i_Y r_Y fg = i_Y r_Y f i_X p^{-1} r_Y = i_Y p p^{-1} r_Y = i_Y r_Y \simeq \operatorname{id}_Y. \quad \square$$

11.2. The basic posets and Quillen's conjecture

Fix a finite group G and a prime p . We define two posets.

DEFINITION 11.2.1. Let $\mathcal{S}_p(G)$ be the poset of non-trivial p -subgroups of G , ordered by inclusion. An abelian p -group is *elementary abelian* if every element has order 1 or p . This means that it is a vector space over the field of p elements. Define $\mathcal{A}_p(G)$ to be the poset of non-trivial elementary abelian p -subgroups of G , ordered by inclusion and let $i: \mathcal{A}_p(G) \rightarrow \mathcal{S}_p(G)$ be the inclusion.

REMARK 11.2.2. Quillen calls a non-trivial elementary abelian p -group a *p -torus*, and he defines its rank to be its dimension as a vector space.

The reason these posets are interesting is that G acts on them in such a way that their topological properties relate nicely to algebraic properties of G . The action of G is by conjugation. If H is a subgroup of G and $g \in G$, write $H^g = gHg^{-1}$. The function f_g that sends P to P^g gives an automorphism of the posets $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$. Clearly $f_e = \operatorname{id}$, where e is the identity element of G , and $f_{g'g} = f_{g'} \circ f_g$. These automorphisms are what give these posets their interest: the poset together with its group action describe how the different p -subgroups are related under subconjugation in G .

In particular, a point P in $\mathcal{A}_p(G)$ is fixed under the action of G if and only if $P^g = P$ for all $g \in G$, and this means that P is a normal subgroup of G . Thus the poset $(\mathcal{A}_p(G))^G$ of fixed points is the poset of normal p -tori of G . We can therefore relate algebraic questions about the presence of normal subgroups to topological questions about the existence of fixed points. Of course, we may regard these posets as F -spaces with G actions, and the theory of the previous section applies.

REMARK 11.2.3. Some of Quillen's language for studying these posets is similar to the language we have been using, but it can be quite confusing. For example, he says that a subset S of a poset X is *closed* if $x \in S$ and $y \leq x$ implies $y \in S$. In our language, this means that $x \in S$ implies $U_x \subset S$, which says that S is *open*.

Compare with Bar-mak's book. Anything interesting further in there?

The posets $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ are both empty if p does not divide the order of G . At first sight, it might seem that $\mathcal{S}_p(G)$ is a lot more interesting and complicated than $\mathcal{A}_p(G)$, but that is not the case. To understand the discussion to follow, it is helpful to keep the following commutative diagram of spaces in mind, remembering that its vertical arrows are weak homotopy equivalences.

$$\begin{array}{ccc} |\mathcal{K} \mathcal{A}_p(G)| & \xrightarrow{|\mathcal{K}(i)|} & |\mathcal{K} \mathcal{S}_p(G)| \\ \psi \downarrow & & \downarrow \psi \\ \mathcal{A}_p(G) & \xrightarrow{i} & \mathcal{S}_p(G) \end{array}$$

We first consider p -groups.

PROPOSITION 11.2.4. *If P is a non-trivial p -group, then $\mathcal{A}_p(P)$ and $\mathcal{S}_p(P)$ are both contractible.*

PROOF. There is a central subgroup B of P of order p . We will be accepting as known some basic facts in the theory of finite groups, such as this one. But the proof is just an easy counting argument. We think of P as a P -set, with P acting on itself by conjugation. As is true for any finite P -set, P is isomorphic to a disjoint union of orbits, each isomorphic to some orbit P/Q . Unless the orbit consists of a single point, its number of elements is divisible by p , and the total number of elements is the order of P . Since the identity element is an orbit with a single point, there must be at least $p - 1$ other orbits with a single point, and such a point is a non-identity element in the center of P .

For any subgroup A of P , we have $A \subset AB \supset B$ where AB is the subgroup of P generated by A and B . If A is a p -torus, then so is AB since B is central. Define three maps $\mathcal{A}_p(P) \rightarrow \mathcal{A}_p(P)$: the identity map id , the map f that sends A to AB , and the constant map c_B that sends A to B . These are all continuous, and our inclusions say that $\text{id} \leq f \geq c_B$. This implies that $\text{id} \simeq f \simeq c_B$. Since the identity is homotopic to the constant map, $\mathcal{A}_p(G)$ is contractible. The proof for $\mathcal{S}_p(G)$ is the same. \square

Quillen calls a poset X *conically contractible* if there is an $x_0 \in X$ and a map of posets $f: X \rightarrow X$ such that $x \leq f(x) \geq x_0$ for all x . He was thinking in terms of associated simplicial complexes, but we are thinking in terms of F -spaces. The previous proof says that the F -spaces $\mathcal{A}_p(P)$ and $\mathcal{S}_p(P)$ are conically contractible. It is to be emphasized that conically contractible finite spaces are genuinely and not just weakly contractible. As we shall see, the difference is profound in the case at hand. In contrast with the previous result, we emphasize the word “weak” in the following result.

THEOREM 11.2.5. *The inclusion $i: \mathcal{A}_p(G) \rightarrow \mathcal{S}_p(G)$ is a weak homotopy equivalence. Therefore the induced map $|\mathcal{K}i|: |\mathcal{K} \mathcal{A}_p(G)| \rightarrow |\mathcal{K} \mathcal{S}_p(G)|$ is a weak homotopy equivalence and hence an actual homotopy equivalence.*

PROOF. We have the open cover of $\mathcal{S}_p(G)$ given by the U_P , where P is a non-trivial finite p -group. Clearly $i^{-1}U_P$ is the poset of p -tori of G that are contained in P , and this is the contractible space $\mathcal{A}_p(P)$. Our general theorem that weak homotopy equivalence is a local notion applies. \square

DEFINITION 11.2.6. Define the p -rank of G , denoted $r_p(G)$, to be the maximal rank of a p -torus in G . Observe that this is one greater than the dimension of the simplicial complex $\mathcal{K}\mathcal{A}_p(G)$. (We interpret the dimension of the empty complex to be -1).

EXAMPLE 11.2.7. If the p -Sylow subgroups of G are cyclic of order p and there are q of them, then $\mathcal{A}_p(G)$ is a discrete space with q points. For example, this holds for some q if G is the symmetric group on n letters, where p is a prime and $p \leq n < 2p$.

REMARK 11.2.8. Sylow's third theorem is relevant. The number of Sylow p -subgroups of G is congruent to 1 mod p and divides the order of G .

THEOREM 11.2.9. *The following statements are equivalent.*

- (i) G has a non-trivial normal p -subgroup.
- (ii) G has a non-trivial normal elementary abelian subgroup.
- (iii) $\mathcal{S}_p(G)$ is contractible.

Moreover, they are implied by the statement

- (iv) $\mathcal{A}_p(G)$ is contractible.

PROOF. Obviously (ii) implies (i). Conversely, as a matter of algebra, (i) implies (ii). To see that, let P be a non-trivial normal p -subgroup of G and let C be its center. For $g \in G$, $c \in C$, and $p \in P$,

$$g c g^{-1} p = g c g^{-1} p g g^{-1} = g g^{-1} p g c g^{-1} = p g c g^{-1}$$

since $g^{-1} p g$ is in P and therefore commutes with c . This shows that any conjugate of an element of C commutes with any element of P and is therefore in C , showing that C is normal in G . Now let B be the set of elements $b \in C$ such that $b^p = e$. Any conjugate of an element of B is in C and has p th power e , hence is in B . Therefore B is a non-trivial normal elementary abelian subgroup of G .

To see that (i) implies (iii), let P be a non-trivial normal p -subgroup of G . For any nontrivial p -subgroup Q of G , $Q \subset QP \supset P$, where QP denotes the subgroup of G generated by P and Q . Since P is normal in G , $QP = \{qp | q \in Q \text{ and } p \in P\}$. This implies that $\text{id} \leq f \geq c_P$, where $f(Q) = QP$ and $c_P(Q) = P$, hence $\mathcal{S}_p(G)$ is conically contractible, hence contractible. The same argument does not apply to show that (ii) implies (iv) since QP need not be abelian when Q and P are abelian.

Conversely, to see that (iii) implies (i) and (iv) implies (ii), we use [Corollary 11.1.3](#), which states that contractibility implies G -contractibility to a fixed point. A fixed point of $\mathcal{S}_p(G)$ is a normal p -subgroup and a fixed point of $\mathcal{A}_p(G)$ is a normal elementary abelian p -subgroup. \square

The inclusion $i: \mathcal{A}_p(G) \rightarrow \mathcal{S}_p(G)$ is not generally a homotopy equivalence. To see this, we use the following observation.

LEMMA 11.2.10. *Let $\mathcal{Q}_p(G) \subset \mathcal{S}_p(G)$ be the subset of nontrivial intersections of Sylow p -subgroups. Then $\mathcal{Q}_p(G)$ is a G -equivariant deformation retract of $\mathcal{S}_p(G)$.*

PROOF. For $P \in \mathcal{S}_p(G)$, let $f(P)$ be the intersection of the Sylow p -subgroups that contain P . Then $f: \mathcal{S}_p(G) \rightarrow \mathcal{Q}_p(G)$ is continuous and G -equivariant. Moreover, $f(P) = P$ if P is itself a p -Sylow subgroup. Let $j: \mathcal{Q}_p(G) \rightarrow \mathcal{S}_p(G)$ be the inclusion. Then $fj = \text{id}$. Since $P \leq f(P)$, $\text{id} \simeq fj$ via an equivariant homotopy. \square

EXAMPLE 11.2.11. Let $G = \Sigma_5$ be the symmetric group on five letters. Then $\mathcal{A}_2(G)$ and $\mathcal{S}_2(G)$ are not homotopy equivalent. There are 6 conjugacy classes of 2-subgroups of G , as follows.

- (i) Dihedral groups D_4 of order 8, the Sylow 2-subgroups.
- (ii) Cyclic groups C_4 of order 4.
- (iii) Elementary 2-groups $C_2 \times C_2$ generated by transpositions (ab) and (cd) .
- (iv) Elementary 2-groups $C_2 \times C_2$ generated by products of disjoint transpositions $(ab)(cd)$, $(ac)(bd)$, whose product in either order is $(ad)(bc)$.
- (v) Cyclic groups C_2 generated by a transposition.
- (vi) Cyclic groups C_2 generated by a product of two disjoint transpositions.

Of course, each $C_2 \times C_2$ contains three C_2 's. Each C_2 of type (v) is contained in three $C_2 \times C_2$'s of type (iii) and each C_2 of type (vi) is contained in one $C_2 \times C_2$ of type (iii) and one $C_2 \times C_2$ of type (iv). This information shows that $\mathcal{A}_2(G)$ is minimal, hence not homotopy equivalent to any space with fewer points. The intersections of Sylow 2-subgroups of G are the dihedral groups in (i), the groups $C_2 \times C_2$ of type (iv) and the subgroups C_2 of type (v). In fact, $\mathcal{Q}_2(G)$ is a core of $\mathcal{S}_2(G)$. Counting, one sees that there are fewer points in $\mathcal{Q}_2(G)$ than there are in the minimal F -space $\mathcal{A}_2(G)$, so these two F -spaces cannot be homotopy equivalent.

Quillen conjectured the following stronger version of the implication (iii) implies (i) of [Theorem 11.2.9](#), and he proved the conjecture for solvable groups.

CONJECTURE 11.2.12 (Quillen). *If $\mathcal{A}_p(G)$ or equivalently $\mathcal{S}_p(G)$ is weakly contractible, then G contains a non-trivial normal p -subgroup.*

The hypothesis holds if and only if $|\mathcal{H}\mathcal{A}_p(G)|$ or equivalently $|\mathcal{H}\mathcal{S}_p(G)|$ is weakly contractible and therefore contractible. We have seen that if G has a non-trivial normal p -subgroup, then $\mathcal{A}_p(G)$ is contractible and therefore weakly contractible. Quillen's conjecture is that, conversely, if $\mathcal{A}_p(G)$ is weakly contractible, then it is contractible and thus G has a non-trivial normal p -subgroup. In this form, we see that the conjecture can be thought of as a problem in the equivariant homotopy theory of F -spaces.

In particular, if G is simple and not isomorphic to C_p , then it has no non-trivial normal subgroups and the conjecture implies that $\mathcal{A}_p(G)$ cannot be weakly contractible. This consequence of the conjecture has been verified for many but not all finite simple groups, using the classification theorem and proving that the space $\mathcal{A}_p(G)$ has non-trivial homology. A conceptual proof would be a wonderful achievement!

11.3. Some exploration of the posets $\mathcal{A}_p(G)$

As an illustration of the translation of algebra to topology, we show how to compute $\mathcal{A}_p(G \times H)$ in terms of joins for finite groups G and H . We then see how the computation appears in Quillen's analysis of the poset $\mathcal{A}_p(\Sigma_{2p})$.

PROPOSITION 11.3.1. *The poset $\mathcal{A}_p(G \times H)$ is homotopy equivalent to the poset $C^- \mathcal{A}_p(G) \times C^- \mathcal{A}_p(H) - \{(c_G, c_H)\}$.*

PROOF. Let T be the subposet of $\mathcal{A}_p(G \times H)$ whose points are the p -tori in $G = G \times e$, the p -tori in $H = e \times H$, and the products $A \times B$ of p -tori A in G and B in H . (Remember that p -tori are non-trivial elementary abelian p -groups). Visibly, thinking of trivial groups as cone-points and therefore $<$ non-trivial subgroups, T

is isomorphic to $C^- \mathcal{A}_p(G) \times C^- \mathcal{A}_p(H) - \{(c_G, c_H)\}$. Let $i: T \rightarrow \mathcal{A}_p(G \times H)$ be the inclusion. The projections $\pi_1: G \times H \rightarrow G$ and $\pi_2: G \times H \rightarrow H$ induce a map $r: \mathcal{A}_p(G \times H) \rightarrow T$ such that $r \circ i = \text{id}$. Explicitly, for $C \in \mathcal{A}_p(G \times H)$, $r(C) = \pi_1(C) \times \pi_2(C)$. Then $i(r(C)) \supset C$, which means that $i \circ r \geq \text{id}$ and thus $i \circ r \simeq \text{id}$. \square

In view of ??, this has the following immediate consequence.

COROLLARY 11.3.2. *The space $|\mathcal{K}(\mathcal{A}_p(G \times H))|$ is homotopy equivalent to the space $|\mathcal{K}(\mathcal{A}_p(G))| * |\mathcal{K}(\mathcal{A}_p(H))|$.*

PROPOSITION 11.3.3. *Quillen's conjecture holds if $r_p(G) \leq 2$.*

PROOF. The hypothesis cannot hold if $r_p(G) = 0$, since $\mathcal{A}_p(G)$ is then empty and hence not weakly contractible. If $r_p(G) = 1$, then the space $\mathcal{A}_p(G)$ is discrete since there are no proper inclusions. It is weakly contractible if and only if it consists of a single point, and then its single point must be fixed by the action of G . This means that there is a unique p -torus in G , and it is a normal subgroup of order p . If $r_p(G) = 2$, then $|\mathcal{K}(\mathcal{A}_p(G))|$ is one dimensional and contractible, which means that it is a tree. According to Quillen, "one knows (Serre) that a finite group acting on a tree always has a fixed point". This means that G has a normal p -torus. The trees here are of a particularly elementary sort, but the conclusion is still not altogether obvious. The following problem gives a way of thinking about it. \square

PROBLEM 11.3.4. *Consider an F -space X such that $|\mathcal{K}(X)|$ is a tree (a contractible graph). Clearly X is weakly contractible. Prove that X is contractible. (Search for upbeat or downbeat points). It follows that if a finite group G acts on X , then X is G -contractible and therefore has a G -fixed point.*

Much of Quillen's paper is devoted to proving that the conjecture holds for solvable groups G . This means that there is a decreasing chain of subgroups of G , each normal in the next, such that the subquotients are cyclic of prime order. We shall not repeat the proof.

However, following Quillen, we shall work out the structure of $\mathcal{A}_p(G)$ when $G = \Sigma_{2p}$ is the symmetric group on $2p$ letters for an odd prime p . This is a first interesting case since $\mathcal{A}_p(\Sigma_n)$ is empty if $n < p$ and is a discrete space with one element for each cyclic subgroup of order p if $p \leq n < 2p$. (In fact, there are $n!/(n-p)!p(p-1)$ such subgroups.) The analysis shows just how non-trivial the posets $\mathcal{A}_p(G)$ are.

Let $g \in G = \Sigma_{2p}$ have order p . The group $\langle g \rangle$ it generates has order p , and its action on the set $S = \{1, \dots, 2p\}$ partitions S into two disjoint subsets, one given by the orbit generated by an element s such that $gs \neq s$ and the other given by its complement, on which $\langle g \rangle$ acts either freely or trivially. If $A \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is a maximal elementary abelian p -subgroup of G with generators g and g' , then since g and g' commute we can see that they give the same partition of S , so that each such A gives a unique partition of the set S into two A -invariant subsets, each with p elements. The set of such partitions of S into two subsets with p elements gives a corresponding decomposition of $\mathcal{A}_p(G)$ into disjoint subposets, each consisting of those A which partition S in the prescribed way.

Under the action of G , these partitions are permuted transitively, meaning that, given two partitions, there is an element of G that permutes one into the other. Consider for definiteness the partition into the first p and last p elements of S . Let

H be the subgroup of those elements of G that fix this partition. The corresponding subposet of $\mathcal{A}_p(G)$ is $\mathcal{A}_p(H)$. Here H is the *wreath product* $\Sigma_2 \wr \Sigma_p$, which is the semi-direct product of Σ_2 with $\Sigma_p \times \Sigma_p$ determined by the permutation action of Σ_2 on $\Sigma_p \times \Sigma_p$.

Since p is odd, $\mathcal{A}_p(H) = \mathcal{A}_p(\Sigma_p \times \Sigma_p)$, which, after passage to realizations of simplicial complexes, is the join $\mathcal{A}_p(\Sigma_p) * \mathcal{A}_p(\Sigma_p)$. Since Σ_p has $(p-2)!$ Sylow subgroups, each of order p , $\mathcal{A}_p(\Sigma_p)$ is the disjoint union of $(p-2)!$ points. After counting the number of partitions and inspecting the join of our two discrete spaces $\mathcal{A}_p(\Sigma_p)$, Quillen informs us, and we can work out for ourselves, that $|\mathcal{A}_p(\Sigma_{2p})|$ is a disconnected graph with $(2p)!/2(p!)^2$ components, each of which is homotopy equivalent to a one-point union of $((p-2)!-1)^2$ circles. For example, for $p=5$, there are 25 circles. The same analysis applies to the alternating groups A_n for $n \leq 2p$ since $\mathcal{A}_p(A_n) = \mathcal{A}_p(\Sigma_n)$. Of course, these $\mathcal{A}_p(G)$ are not weakly contractible.

11.4. The components of $\mathcal{S}_p(G)$

Let p be a prime which divides the order of G . We describe the set of components $\pi_0(\mathcal{S}_p(G))$, which of course is the same as $\pi_0(\mathcal{A}_p(G))$. Recall that two elements of a poset are in the same component if they can be connected by a chain of elements, each either \leq or \geq the next. In the poset $\pi_0(\mathcal{S}_p(G))$, each element is a p -group and is contained in a Sylow subgroup. Therefore there is at least one Sylow subgroup in each component. Since any one Sylow subgroup P generates all the others by conjugation by elements of G , G acts transitively on $\pi_0(\mathcal{S}_p(G))$, in the sense that there is a single orbit. If $N = N_P$ denotes the subgroup of G that fixes the component $[P]$ of P , then G/N is isomorphic to the G -set $\pi_0(\mathcal{S}_p(G))$ via $gN \mapsto [P^g]$. We want to determine the subgroup N . Let $\text{Syl}_p(G)$ denote the set of p -Sylow subgroups of G and let $N_G H$ denote the normalizer in G of a subgroup H . Recall that $H^g = gHg^{-1}$.

PROPOSITION 11.4.1. *The following conditions on a subgroup M of G are equivalent.*

- (i) *For some $P \in \text{Syl}_p(G)$, $M \supset N_P$.*
- (ii) *For some $P \in \text{Syl}_p(G)$, $M \supset N_G H$ for all $H \in \mathcal{S}_p(P)$.*
- (iii) *For some $P \in \text{Syl}_p(G)$, $M \supset N_G P$ and $K \subset M$ whenever K is a p -subgroup of G that intersects M non-trivially.*
- (iv) *p divides the order of M and $M \cap M^g$ is of order prime to p for all $g \notin M$.*

Moreover, $\mathcal{S}_p(G)$ is connected if and only if there is no proper subgroup M which satisfies these equivalent conditions.

PROOF. The last statement holds since G is connected if and only if $G = N_P$ for all $P \in \text{Syl}_p(G)$, in which case no proper subgroup can satisfy the stated conditions.

(i) \implies (ii): If $g \in N_G H$ with $H \subset P$, then $H^g = H$ is contained in both P and P^g , so that $[P] = [P^g] = g[P]$. This means that $g \in N_P \subset M$.

(ii) \implies (iii): Obviously $M \supset N_G P$. Since P is a p -Sylow subgroup of G , it is also a p -Sylow subgroup of M . Thus if H is a non-trivial p -subgroup of M , then H is conjugate in M to a subgroup, H^m say, of P . Since $M \supset N_G(H^m)$ and $(N_G H)^m = N_G(H^m)$, $M \supset N_G H$. Let K be a p -subgroup of G such that $K \cap M$ is non-trivial. We have

$$K \cap M \subset N_K(K \cap M) = K \cap N_G(K \cap M) \subset K \cap M.$$

Since K is a p -group, the first inclusion is proper if $K \cap M$ is a proper subgroup of K . Since this is a contradiction, we must have $K \cap M = K$ and $K \subset M$.

(iii) \implies (iv): Since $M \supset P$, p divides the order of M . Assume that p divides the order of $M \cap M^g$ for some $g \in G$. Then there is a non-trivial p -subgroup $H \subset M \cap M^g$. Let $H \subset Q$ for $Q \in \text{Syl}_p(G)$. Since $Q \cap M$ is non-trivial, we have $Q \subset M$. Since $H^{g^{-1}} \subset Q^{g^{-1}}$ and $H^{g^{-1}} \subset M$, we also have $Q^{g^{-1}} \subset M$. Since P, Q , and $Q^{g^{-1}}$ are p -Sylow subgroups of M , they are conjugate in M , say $Q^m = P$ and $Q^{g^{-1}} = P^n$ for $m, n \in M$. Then a quick check shows that $mgn \in N_G P \subset M$ and therefore $g \in M$, proving (iv).

(iv) \implies (i): Writing G as the disjoint union of double cosets MgM , one calculates that the index of M in G is the sum over double coset representatives g of the indices of $M \cap M^g$ in M . Since p divides the order of M and does not divide the order of $M \cap M^g$ if $g \notin M$, these indices are divisible by p except for the double coset represented by e . Thus the index of M in G is congruent to 1 mod p , hence M must contain some p -Sylow subgroup P . Let $N = N_P$. For $n \in N$, P and P^n are in the same component. Considering p -Sylow subgroups containing groups in a chain connecting them, we see that there is a sequence of p -Sylow subgroups $P = P_0, P_1, \dots, P_q = P^n$ such that $P_i \cap P_{i+1} \neq \{e\}$. There are elements g_i such that $P_{i-1}^{g_i} = P_i$, and we can choose g_q so that $g_q \cdots g_1 = n$. We have $P \subset M$, and we assume inductively that $P_{i-1} \subset M$. Then $P_{i-1} \cap P_i \subset M \cap M^{g_i}$, so this intersection contains a p -group and, by (iv), $g_i \in M$. This implies that $P_i \subset M$ and, inductively, we conclude that $n \in M$, so that $N \subset M$. \square

COROLLARY 11.4.2. N_P is generated by the groups $N_G H$ for $H \in \mathcal{S}_p(P)$.

PROOF. N_P contains all of these $N_G H$, so it contains the subgroup they generate, and it is the smallest such subgroup by the equivalence of (i) and (ii). \square

By the contrapositive, G is not connected if and only if there is a proper subgroup M of G that satisfies the equivalent properties of the proposition. For example, if $r_p(G) = 1$ and G has no non-trivial normal p -subgroup, then $\mathcal{A}_p(G)$ is discrete and not contractible, and is therefore not connected. Quillen gives a condition on G under which these are the only examples.

PROPOSITION 11.4.3. Let $H (= O_{p'}(G))$ be the largest normal subgroup of G of order prime to p and let $K (= O_{p',p}(G))$ be specified by requiring K/H to be the largest normal p -subgroup of the quotient group G/H . If K/H is non-trivial and $\mathcal{S}_p(G)$ is not connected, then $r_p(G) = 1$.

PROOF. If Q is a p -Sylow subgroup of K , then $K = QH$ since H is a p' -group and K/H is a p -group. This implies that H acts transitively on $\pi_0(\mathcal{S}_p(K))$ since it implies that any two p -Sylow subgroups are conjugate by the action of some $h \in H$. The intersection with K of a p -Sylow subgroup P of G is a p -Sylow subgroup of K . A p -subgroup of K is a p -subgroup of G , and the induced map $\pi_0(\mathcal{S}_p(K)) \rightarrow \pi_0(\mathcal{S}_p(G))$ is surjective since $P \cap K \subset P$ implies that $[P]$ is the image of $[P \cap K]$. Therefore H also acts transitively on $\pi_0(\mathcal{S}_p(G))$. Let A be a maximal p -torus of G . The map $\pi_0(\mathcal{S}_p(AH)) \rightarrow \pi_0(\mathcal{S}_p(G))$ is also surjective since H acts transitively on the target and the map is H -equivariant. Therefore $\mathcal{S}_p(AH)$ is not connected. The component $[A]$ is fixed by the centralizers $C_H(B)$ for all non-trivial subgroups B of A since $B^h = B \subset A$ for $h \in C_H(B)$. By [28,

6.2.4], if A is not cyclic (= rank one), then H is generated by these centralizers, which contradicts the fact that $\mathcal{S}_p(AH)$ is not connected. Therefore A is cyclic. \square

Really finite H -spaces

The circle is a *topological group*. If we regard it as the subspace of the complex plane consisting of points of norm one, then complex multiplication gives the product $S^1 \times S^1 \rightarrow S^1$. How can we model such a basic structure in terms of a map of finite spaces?

Stong proved a rather amazing *negative* result about this problem. We will not go into the combinatorial details of his proof, contenting ourselves with the statement.

12.0.1. Topological Groups. The interaction of group multiplication with a space's topology is captured in the following definition.

DEFINITION 12.0.1. A *topological group* is a group that is also a T_0 topological space in which the multiplication map given by $(x, y) \mapsto x \cdot y$ and the inverse map given by $x \mapsto x^{-1}$ are continuous.

PROPOSITION 12.0.2. Let H be a group that is also a T_0 topological space. Then H is a topological group if and only if the map $\rho : H \times H \rightarrow H$ given by $(x, y) \mapsto x \cdot y^{-1}$ is continuous.

PROOF. Suppose H is a topological group. The functions $f : H \times H \rightarrow H \times H$ where $(x, y) \mapsto (x, y^{-1})$ and $g : H \times H \rightarrow H$ sending $(a, b) \mapsto a \cdot b$ are then continuous, and so $\rho = g \circ f$ is as well.

Conversely, suppose ρ is continuous. First, the map v taking x to x^{-1} is equal to the composition of the continuous maps ρ and $h : H \rightarrow H \times H$ defined by $x \mapsto (e, x)$, and is therefore itself continuous. Second, the product map g is continuous because it equals the composition of the continuous functions ρ and f . \square

EXAMPLE 12.0.3. $(\mathbb{Z}, +)$

When equipped with the order topology, this is a T_1 space. Consider the open interval (a, b) , an arbitrary basis element for $(\mathbb{Z}, +)$. Define $\rho : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ by $(x, y) \mapsto x - y$. For the pre-image, we have $\rho^{-1}(a, b) = \{(x, y) \mid a < x - y < b\} = \{(x, y) \mid a + y < x < b + y\}$. This pre-image is the union over all y of the corresponding open sets $(a + y, b + y) \times (y - 1, y + 1)$, and is therefore open. Thus ρ is continuous.

EXAMPLE 12.0.4. $(\mathbb{R}, +)$

The continuity of $\rho : \mathbb{R} \rightarrow \mathbb{R}$ where $(x, y) \mapsto x - y$ in the usual topology is a standard fact of analysis.

EXAMPLE 12.0.5. (\mathbb{R}_+, \times)

The continuity of the quotient operation $q : \mathbb{R} \times (\mathbb{R} - \{0\}) \rightarrow \mathbb{R}$ is a standard fact of analysis. For \mathbb{R}_+ , construct the continuous ρ by restricting q 's domain to $\mathbb{R}_+ \times \mathbb{R}_+$ and its range to \mathbb{R}_+ .

Expository REU
paper? Research:
Alexandroff H-
spaces?

EXAMPLE 12.0.6. (S^1, \times)

The beauty of the algebra of these numbers is that their multiplication is the same as the addition of real numbers (complex numbers on the unit circle are written as exponentials, and their multiplication is given by the addition of the exponents). That (S^1, \times) is a topological group follows from the fact that $(\mathbb{R}, +)$ is.

We take S^1 as our main example. We are interested in finite models of S^1 that can be equipped with continuous multiplication.

12.0.2. Failure of the non-Hausdorff Suspension of S^1 . Our standard four-point model of S^1 , the non-Hausdorff suspension, is incompatible with continuous complex multiplication. The model in the complex numbers is pictured in the following diagram. An arrow pointing from one element to another says that the element being pointed to is greater than the other. The far-right and far-left points are identical.

$$i \longleftarrow -1 \longrightarrow -i \longleftarrow 1 \longrightarrow i$$

PROPOSITION 12.0.7. *In the complex numbers, the non-Hausdorff suspension S^1 of the zero-sphere gives discontinuous multiplication.*

PROOF. We have $(i, i) > (-1, i)$, but $i \cdot i = -1 < -1 \cdot i$. \square

12.0.3. Finite H -Spaces.

DEFINITION 12.0.8. Let (X, e) be a finite space with a basepoint e and let $\phi: X \times X \rightarrow X$ be a map. We say that X is an H -space of type I if multiplication by e on either the right or the left is homotopic to the identity. That is, the maps $x \rightarrow \phi(e, x)$ and $x \rightarrow \phi(x, e)$ are each homotopic to the identity. Say that X is an H -space of type II if the *shearing maps* $X \times X \rightarrow X \times X$ defined by sending (x, y) to either $(x, \phi(x, y))$ or $(y, \phi(x, y))$ are homotopy equivalences.

A topological group is an H -space of both types, but it is much less restrictive for a space to be an H -space than for a space to be a topological group. In particular, a topological group is an H -space in which multiplication by e is the identity map, so that e is an algebraic identity element. The definition of a type I H -space is often presented as the standard definition of an H -space. Henceforth, it will be the focus of the chapter.

By definition, the notion of H -space is homotopy invariant in the sense that if one defines an H -space structure on (X, e) to be a homotopy class of products ϕ , then one has the following result.

PROPOSITION 12.0.9. *If (X, e) and (Y, f) are homotopy equivalent, then H -space structures on (X, e) correspond bijectively to H -space structures on (Y, f) .*

This motivated Stong [65] to study H -space structures on minimal finite spaces covered in the following sections.

12.0.4. A combinatorial result. Before proving propositions about H -spaces, we modify the definitions of minimal finite spaces and cores to respect basepoints. Recall the given definition (Definition 2.4.2) of a base point of a finite space. We present the analogous notions for based spaces.

DEFINITION 12.0.10. A based finite space (X, x) is *minimal* if it satisfies the T_0 axiom and has no beat points except possibly x . A *core* of a finite space (X, x) is a subspace (Y, x) that is minimal and a deformation retract of X .

This modified definition ensures that when a based space is reduced to its core, the basepoint is not deleted.

The following fact will prove useful in the proof of following results.

PROPOSITION 12.0.11. *Let (X, e) be a minimal finite H -space. Then $\theta_1, \theta_2 : X \rightarrow X$ given by $\theta_1(x) = xe$ and $\theta_2(x) = ex$ are equal to the identity map.*

PROOF. Since (X, e) is a minimal finite space, any map from X to itself that is homotopic to the identity is the identity. \square

The following proposition provides the structure for the proof of Stong's major result, [Theorem 12.0.17](#). Recall that in a poset, an upbeat point x under y implies that y is the immediate successor of x . The opposite holds for downbeat points.

PROPOSITION 12.0.12. *Let (X, e) be a minimal finite space, $x \in X$. Then*

- (i) x is less than each of two distinct maximal points, or
- (ii) x is maximal, or
- (iii) x is upbeat under a maximal point (so $x = e$)

and

- (i') x is greater than each of two distinct maximal points, or
- (ii') x is minimal, or
- (iii') x is downbeat over a minimal point (so $x = e$)

PROOF. Suppose by way of contradiction that the set A of points that do not satisfy any of (i), (ii), (iii) is nonempty, and let a be a maximal element of A . Since a is not maximal in X , there exists $z \in X$ such that $z > a$. Let $B = \{x | x > a\} \subset X$. If B contains a point z' other than z , then a satisfies (i). If not, then a satisfies (iii). Either way, we have arrived at a contradiction with the fact that a is in A . We conclude that A must be empty. Similarly, every point must satisfy one of (i'), (ii'), (iii'). \square

We present the following combinatorial result on the relationship of general points of X to the point e , which leads Stong to his main result.

PROPOSITION 12.0.13. *Let (X, e) be a minimal finite H -space. Then e is both maximal and minimal under the associated order \leq .*

PROOF. Knowing that e must satisfy one of the conditions (i), (ii), (iii) and one of (i'), (ii'), (iii'), we proceed to eliminate from possibility all pairs of conditions except the pair consisting of (ii) and (ii').

REMARK 12.0.14. From the definition of the order on a finite T_0 space we can deduce the appropriate order on a product of two finite spaces. Let $(a, b), (c, d) \in X \times Y$ where X and Y are finite T_0 spaces. Then $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. If any of the two inequalities in the factor spaces is strict, then the inequality in the product space is strict as well.

The point e does not satisfy (i).

LEMMA 12.0.15. *Let m and m' be maximal points in X with $m, m' > e$. Then $m = m'$.*

PROOF. Since $m' > e$ and $m > e$, we have

$$(m, m') > (m, e) \text{ and } (m, m') > (e, m').$$

□

Applying the continuous (order-preserving) product ϕ to each, we obtain

$$mm' \geq me = m \text{ and } mm' \geq em' = m'$$

Because the right-hand sides of both inequalities above are maximal, we deduce that $m = mm' = m'$.

The point e does not satisfy (i').

This is true for perfectly symmetric reasons.

The point e does not satisfy both (ii') and (iii).

We show that if it did, then X would have infinitely many subsets, in contradiction to the fact that X is finite.

Suppose by way of contradiction that e satisfies (ii') and (iii), i.e., e is both minimal and upbeat under a maximal point.

Our claim is that for every integer $r \geq 0$, X contains a subset

$$(12.0.16) \quad D_r = \{e = u_0, u_1, \dots, u_r; m_0, \dots, m_{r-1}\}$$

with all u_i minimal in X and all m_i maximal in X , and such that all of the following conditions hold:

- (a) For i between 0 and $r - 1$ (inclusive), the only points in X less than m_i are u_i and u_{i+1} .
- (b) m_0 is the only point in X that is greater than u_0 .
- (c) For i between 1 and $r - 1$ (inclusive), the only points in X greater than u_i are m_{i-1} and m_i .
- (d) For i between 0 and $r - 1$ (inclusive), $xm_i = m_ix = m_i$ if x is m_k or u_k with $k \leq i$.
- (e) For i between 0 and r (inclusive), $xu_i = u_ix = u_i$ if $x = m_k$ with $k < i$ or $x = u_k$ with $k \leq i$.
- (f) For every $x \in X$ not in D_r , $xm_i = x = xu_i$ and $m_ix = x = u_ix$.

For $r = 0$, we have the set $D_0 = \{e = u_0\}$. It contains no m_i . Conditions (a) – (d) and (f) are vacuously satisfied. For condition (e), the first option (involving m_i) is vacuously satisfied, and the second demands only that we check $ee = e$. That equation is true in the minimal space (X, e) because multiplication by e is homotopic to the identity, but in fact it is true for any H -space. Multiplication by e is homotopic to the identity through maps from (X, e) to (X, e) . That is, the fact that e is the space's basepoint means the only allowed intermediate maps take e to itself.

Now assume X contains a set D_k of the form in (7.1.15). We show that there are an additional maximal point m_k and an additional minimal point u_{k+1} such that D_{k+1} (of the form (7.1.15)) satisfies (a) through (f).

First we show that it satisfies (a), (b), (c).

For $k = 0$, the assumption that e is upbeat under a maximal point gives a unique m_k , namely the point under which e is upbeat.

For $k > 0$, we know that $u_k \neq e$ and that u_k is not maximal (being less than m_{k-1}). So u_k is less than each of two distinct maximal points. By (a), only one of those is in D_k .

In order for D_{k+1} to satisfy (c), we cannot have a choice of multiple maximal points to call m_k . In the case $k = 0$, this needed uniqueness property has already been shown. In the case $k > 0$, in which we know that there exists a maximal point outside D_k , the uniqueness follows from the combination of (f) with the procedure of the previous subsection.

Existence and uniqueness of u_{k+1} now follow by the analogous argument, using $m_k \neq e$.

One can now see that D_{k+1} satisfies (a), (b), (c). We finally show that it satisfies (d), (e), (f).

To verify (d) for D_{k+1} , we substitute $x = m_k$ in the assumption (f) for D_k . Likewise, to verify (e), we substitute $x = u_{k+1}$.

Finally, let us verify (f) for D_{k+1} . We will show $xm_k = x = xu_k$. The derivation of the other identity uses the analogous argument.

Suppose x is not in D_{k+1} . We obtain immediately $xm_k \geq xu_k = x$. We now proceed by induction.

In the base case, where x is maximal, we find from the above that $xm_k = x$.

Now, for the inductive step, consider the point w , supposing that for every $y > w, ym_k = y$.

For any such y , by continuity of ϕ , we have $y = ym_k \geq wm_k$.

Thus, either w is upbeat or $wm_k = w$. The former is false because X is a minimal space (no point other than e can be upbeat) with $xe \neq e$ for $x \neq e$. So $wm_k = w = wu_k$. This completes the verification of (f).

We now see that if e satisfied (ii') and (iii), we would be able to construct infinitely many distinct subsets of X , contradicting the fact that X is finite.

The point e does not satisfy both (ii) and (iii').

This possibility is ruled out in the same way as the possibility (ii') and (iii).

The point e does not satisfy both (iii) and (iii').

This possibility is conceptually similar to the last, because it just replaces maximality by the situation of being upbeat under a maximal point. The technicalities of the demonstration are slightly different, but offer negligible additional insight.

The point e satisfies (ii) and (ii').

This is the only remaining possible pair of conditions. The proof of **Proposition 7.2** is now complete. \square

This means that e is a component of X . Stong shows that this implies the following conclusions for general finite H -spaces.

12.0.5. Inviability of finite H -space models of non-contractible connected spaces.

THEOREM 12.0.17. *Let X be a finite space and let $e \in X$. Then there is a product ϕ making (X, e) an H -space of type I if and only if e is a deformation*

retract of its component in X . Therefore X is an H -space for some basepoint e if and only if some component of X is contractible.

PROOF. Since (X, e) is homotopy equivalent to its core (Y, e) , **Proposition 4.3** says that there is an H -space structure on (X, e) only if there is one on (Y, e) .

Because (Y, e) is a minimal finite space, it is an H -space only if e is both maximal and minimal in Y under the associated order \leq , i.e., $\{e\}$ is a path component of Y .

In finite spaces, path components are the same as connected components. So, $\{e\}$ is a path component of Y only if it is a component of Y .

If $\{e\}$ is a component of Y (the core), then $\{e\}$ is the core of e 's component in X .

A core of a component is a deformation retract of the component. Thus the result is established. \square

THEOREM 12.0.18. *Let X be a finite space. Then there is a product ϕ making X an H -space of type II if and only if every component of X is contractible.*

COROLLARY 12.0.19. *A connected finite space X is an H -space of either type I if and only if X is contractible.*

So there is no way that we can model the product on S^1 by means of an H -space structure on some finite space X . Our standard model $\mathbb{T} = \mathbb{S}S^0$ of S^1 can be embedded in \mathbb{C} as the four point subgroup $\{\pm 1, \pm i\}$, but then the complex multiplication is not continuous. However, the multiplication can be realized as a map $(\mathbb{T} \times \mathbb{T})^{(n)} \rightarrow \mathbb{T}$ for some finite n , by the simplicial approximation theorem for finite spaces. Explicitly, it is implied that for an H -space X with product ϕ and finite model Y , there exist an integer n and a continuous map $\mu : (Y \times Y)^{(n)} \rightarrow Y$ such that $|\mathcal{K}(\mu)| \simeq \phi$. It is natural to expect that some small n works here.

The following result is proven in [30].

THEOREM 12.0.20. *Choosing minimal points e in \mathbb{T} and $f \in \mathbb{T}'$ as basepoints, there is a map*

$$\phi: \mathbb{T}' \times \mathbb{T}' \longrightarrow \mathbb{T}$$

such that $\phi(f, f) = e$ and the maps $x \rightarrow \phi(x, f)$ and $x \rightarrow \phi(f, x)$ from \mathbb{T}' to \mathbb{T} are weak homotopy equivalences.

That is, we can realize a kind of H -space structure after barycentric subdivision. The proof is horribly unilluminating. The space \mathbb{T}' has eight elements, the space \mathbb{T} has four elements. One writes down an 8×8 matrix with values in \mathbb{T} , choosing it most carefully so that when the 8 point and 4 point spaces are given the appropriate partial order, and the 64 point product space the product order, the function represented by the matrix is order preserving. Then one checks the row and column corresponding to multiplication by the basepoint.

Several other interesting spaces and maps are modelled similarly in the cited paper, for example $\mathbb{R}P^2$ and $\mathbb{C}P^2$.

Part 2

Topological spaces, simplicial sets, and categories

Simplicial sets

13.1. Motivation for the introduction of simplicial sets

Simplicial sets, and more generally simplicial objects in a given category, are central to modern mathematics. While I am not a mathematical historian, I thought I would describe in conceptual outline how naturally simplicial sets arise from the classical study of simplicial complexes. I suspect that something like this recapitulates the historical development.

We have described simplicial complexes in several different forms: abstract simplicial complexes, ordered simplicial complexes, geometric simplicial complexes, ordered geometric simplicial complexes and realizations of geometric simplicial complexes. It is possible to go directly from abstract simplicial complexes to realizations without passing through geometric simplicial complexes, but the construction is perhaps not as intuitive and will not be included.

An abstract simplicial complex is equivalent to a geometric simplicial complex, and neither of these notions involves anything about ordering the vertices. If one has a simplicial complex of either type, one can choose a partial ordering of the vertices that restricts to a linear ordering of the vertices of each simplex, and this gives the notion of an ordered simplicial complex. This can be done most simply, but not most generally, just by choosing a total ordering of the set of all vertices and restricting that ordering to simplices. However, there is no canonical choice.

We have seen in studying products of simplicial complexes that geometric realization behaves especially nicely only in the ordered setting. Both the category \mathcal{SC} of simplicial complexes and the category \mathcal{OSC} of ordered simplicial complexes have categorical products. Geometric realization preserves products when defined on \mathcal{OSC} , but it does not preserve products when defined on \mathcal{SC} . The functor \mathcal{K} is best viewed as a functor from the category \mathcal{P} of partially ordered sets to the category \mathcal{OSC} rather than just to the category \mathcal{SC} . Observe that there are generally many different ordered simplicial complexes with the same poset of vertices. The functor \mathcal{K} picks out the largest choice, the one in which every finite totally ordered subset of the set of vertices is a simplex.

The functor \mathcal{X} , on the other hand, starts in \mathcal{SC} and lands in \mathcal{P} , which can be identified with the category of A -spaces. The composite \mathcal{KX} is the barycentric subdivision functor $Sd: \mathcal{SC} \rightarrow \mathcal{OSC}$. It can be viewed as the construction of a canonical ordered simplicial complex SdK starting from a given unordered simplicial complex K , at the price of subdividing. Since the geometric realization functor gives a space $|SdK|$ that can be identified with $|K|$ there is no loss of topological generality working in \mathcal{OSC} instead of \mathcal{SC} .

The most important motivation for working with ordered rather than unordered simplicial complexes is that the ordering leads to the definition of an associated chain complex and thus to a quick definition of homology.

As noted earlier, a topological space X is called a polytope if it is homeomorphic to $|K|$ for a (given) simplicial complex K . Such a homeomorphism $|K| \rightarrow X$ is called a triangulation of X , and X is said to be triangulable if it admits a triangulation. Then we can define the homology of X to be the homology of K . This is a quick definition, and useful where it applies, but it raises many questions and is quite unsatisfactory conceptually. Not every space is triangulable, and triangulable spaces can admit many different triangulations. It is far from obvious that the homology is independent of the choice of triangulation.

Simplicial sets abstract the notion of ordered simplicial complexes, retaining enough of the combinatorial structure that homology can be defined with equal ease. The generalization allows myriads of examples that do not come from simplicial complexes. The original motivating example gives a functor from topological spaces to simplicial sets. Composing with the functor from simplicial sets to homology groups gives the quickest way of defining the homology groups of a space and leads to the proof that these groups depend only on the weak homotopy type of the space, not on any triangulation, and to the proofs that different triangulations, when they exist, give canonically isomorphic homology groups.

Perhaps the quickest and most intuitive way to motivate the definition of simplicial sets is to start from structure clearly visible in the case of ordered simplicial complexes. Let X denote the partially ordered set $V(K)$ of vertices of an ordered simplicial complex K . The reader might prefer to start with an ordered simplicial complex of the form $\mathcal{K}(X)$, where X is a poset. The reader may also want to insist that X is finite, but that is not necessary to the construction, and we later want to allow infinite sets.

Then an n -simplex σ of K is a totally ordered $n + 1$ -tuple of elements of X . Write such a tuple as (x_0, \dots, x_n) . When studying products, we saw that it can become essential to consider tuples (x_0, \dots, x_n) , where $x_0 \leq x_1 \leq \dots \leq x_n$. Of course, (x_0, \dots, x_n) is no longer a simplex, but one can obtain a simplex from it by deleting repeated entries. When there are repeated entries, we think of (x_0, \dots, x_n) as a “degenerate” n -simplex. Let K_n denote the set of such generalized n -simplices, degenerate or not. For $0 \leq i \leq n$, define functions

$$d_i: K_n \rightarrow K_{n-1} \quad \text{and} \quad s_i: K_n \rightarrow K_{n+1},$$

called face and degeneracy operators, by

$$d_i(x_0, \dots, x_n) = (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

and

$$s_i(x_0, \dots, x_n) = (x_0, \dots, x_i, x_i, \dots, x_n).$$

Of course, the d_i and s_i just defined also depend on n , but it is standard not to indicate that in the notation. In words, d_i deletes the i^{th} entry and s_i repeats the i^{th} entry. If $i < j$ and we first delete the j^{th} entry and then the i^{th} entry, we get the same thing as if we first delete the i^{th} entry and then delete the (new) $(j - 1)^{\text{st}}$ entry. Similarly, elementary inspections give commutation relations between the d_i and s_j and between the s_i . Here is a list of all such relations:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i = j + 1 \\ s_j \circ d_{i-1} & \text{if } i > j + 1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j$$

The reader can easily check that these identities really do follow immediately from the definition of the K_n , d_i , and s_i above.

The K_n are defined in terms of the partially ordered vertex set $V(K)$ of K , but there are many examples of precisely similar structure that arise differently.

13.2. The definition of simplicial sets

We obtain our first definition of simplicial sets by formalizing structure that, as we have just seen, is implicit in the definition of an ordered simplicial complex.

DEFINITION 13.2.1. A simplicial set K is a sequence of sets K_n , $n \geq 0$, and functions $d_i: K_n \rightarrow K_{n-1}$ and $s_i: K_n \rightarrow K_{n+1}$ for $0 \leq i \leq n$ that satisfy the identities just displayed. The elements of the set K_n are called n -simplices, following the historic precedent of simplicial complexes. Just as if K were a simplicial complex, a map $f: K \rightarrow L$ of simplicial sets is a sequence of functions $f_n: K_n \rightarrow L_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$. With these objects and morphisms, we have the category $s\mathcal{S}et$ of simplicial sets.

Now our motivating example can be recapitulated in the following statement.

PROPOSITION 13.2.2. *There is a canonical functor $i: \mathcal{O}\mathcal{S}\mathcal{C} \rightarrow s\mathcal{S}et$ from the category of ordered simplicial complexes to the category of simplicial sets. It assigns to an ordered simplicial complex K the simplicial set K^s given by the sequence of sets K_n^s and the functions d_i and s_i defined above. It assigns to a map $f: K \rightarrow L$ of ordered simplicial complexes the map $f^s: K^s \rightarrow L^s$ induced by its map of vertex sets:*

$$f_n^s(x_0, \dots, x_n) = (f(x_0), \dots, f(x_n)).$$

It is a full embedding, meaning that the maps $K \rightarrow L$ of ordered simplicial complexes map bijectively to the maps $K^s \rightarrow L^s$ of simplicial sets.

The identities listed above are hard to remember and do not appear to be very conceptual. The definition admits a conceptual reformulation that may or may not make things clearer, depending on personal taste, but definitely allows many arguments and constructions to be described more clearly and conceptually than would be possible without it. We define the category Δ of finite ordered sets.

DEFINITION 13.2.3. The objects of Δ are the finite ordered sets $[n]$ with $n + 1$ elements $0 < 1 < \dots < n$. Its morphisms are the monotonic functions $\mu: [m] \rightarrow [n]$. This means that $i < j$ implies $\mu(i) \leq \mu(j)$. Define particular monotonic functions

$$\delta_i: [n-1] \rightarrow [n] \quad \text{and} \quad \sigma_i: [n+1] \rightarrow [n]$$

for $0 \leq i \leq n$ by

$$\delta_i(j) = j \text{ if } j < i \quad \text{and} \quad \delta_i(j) = j + 1 \text{ if } j \geq i$$

and

$$\sigma_i(j) = j \text{ if } j \leq i \quad \text{and} \quad \sigma_i(j) = j - 1 \text{ if } j > i.$$

In words, δ_i skips i and σ_i repeats i .

There are identities for composing the δ_i and σ_i that are “dual” to those for composing the d_i and s_i that appear in the definition of a simplicial set. Precisely, the duality amounts to reversing the direction of arrows. The following pair of commutative diagrams should make clear how to interpret this, where $i < j$.

$$\begin{array}{ccc}
 K_n & \xrightarrow{d_j} & K_{n-1} \\
 d_i \downarrow & & \downarrow d_i \\
 K_{n-1} & \xrightarrow{d_{j-1}} & K_{n-2}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 [n] & \xleftarrow{\delta_j} & [n-1] \\
 \delta_i \uparrow & & \uparrow \delta_i \\
 [n-1] & \xleftarrow{\delta_{j-1}} & [n-2]
 \end{array}$$

A moment’s reflection should convince the reader that every monotonic function $\mu: [m] \rightarrow [n]$ can be written as a composite of monotonic functions δ_i and σ_j for varying i and j . That is, μ can be obtained by omitting some of the i ’s and repeating some of the j ’s. Just as a group can be defined by specifying a set of generators and relations, so a category can often be specified by a set of generating morphisms and relations between their composites. The category Δ is generated by the δ_i and σ_i subject to our “dual” relations. This leads to the proof of the following reformulation of the notion of a simplicial set. Recall that a contravariant functor F assigns a morphism $FY \rightarrow FX$ of the target category to each morphism $X \rightarrow Y$ of the source category.

PROPOSITION 13.2.4. *The category of simplicial sets can be identified with the category of contravariant functors $K: \Delta \rightarrow \mathcal{S}et$ and natural transformations between them.*

PROOF. The correspondence is given by viewing the functions d_i and s_i that define a simplicial set as the morphisms of sets induced by the morphisms δ_i and σ_i of the corresponding functor $\Delta \rightarrow \mathcal{S}et$. It is convenient to write $\mu^*: K_n \rightarrow K_m$ for the function induced by contravariance from a morphism $\mu: [m] \rightarrow [n]$, and then $d_i = \delta_i^*$ and $s_i = \sigma_i^*$. For a map f , the corresponding natural transformation is given on the object $[n]$ by the function f_n . \square

While we do not want to emphasize abstraction in the first instance, we nevertheless cannot resist the temptation to generalize the definition of simplicial sets to simplicial objects in a perfectly arbitrary category. The generalization has a huge number of applications throughout mathematics, and we shall use it when defining homology.

DEFINITION 13.2.5. A simplicial object in a category \mathcal{C} is a contravariant functor $K: \Delta \rightarrow \mathcal{C}$. A map $f: K \rightarrow L$ of simplicial objects in \mathcal{C} is a natural transformation $K \rightarrow L$; it is given by morphisms $f_n: K_n \rightarrow L_n$ in \mathcal{C} . We have the category $s\mathcal{C}$ of simplicial objects in \mathcal{C} . By composition of functors and natural transformations, any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a functor $sF: s\mathcal{C} \rightarrow s\mathcal{D}$. By duality, a *covariant* functor $\Delta \rightarrow \mathcal{C}$ is called a cosimplicial object in \mathcal{C} .

13.3. Standard simplices and their role

We explain a general conceptual way to relate simplicial sets to “standard simplices”. Standard simplices exist in many categories. We have standard simplices in topological spaces, simplicial sets, and even posets and categories. In general, fixing a category \mathcal{V} , we often have a standard cosimplicial object in \mathcal{V} , that is a

certain covariant functor $\Delta[\bullet]^v: \Delta \rightarrow \mathcal{V}$. The superscript v is meant as a reminder that the functor is assigning objects in \mathcal{V} to objects in Δ ; it should also help to distinguish the functor $\Delta[\bullet]^v$ from the category Δ . On objects, we write the functor $\Delta[\bullet]^v$ as $[n] \mapsto \Delta[n]^v$, but we agree to write μ_* rather than $\Delta[\mu]^v$ for the map $\Delta[m]^v \rightarrow \Delta[n]^v$ in \mathcal{V} obtained by applying our functor to a morphism μ in Δ . For each object V of \mathcal{V} we obtain a *contravariant* functor, denoted $SV: \Delta \rightarrow \mathcal{S}et$, by letting the set $S_n V$ of n -simplices be the set $\mathcal{V}(\Delta[n]^v, V)$ of morphisms $\Delta[n]^v \rightarrow V$ in the category \mathcal{V} . The faces and degeneracies are induced by precomposition with the maps

$$\delta_i: \Delta[n-1]^v \rightarrow \Delta[n]^v \quad \text{and} \quad \sigma_i: \Delta[n+1]^v \rightarrow \Delta[n]^v$$

obtained by applying the functor $\Delta[\bullet]^v$ to the generating morphisms δ_i and σ_i of Δ . That is, for a morphism $\nu: \Delta[n]^v \rightarrow V$ in \mathcal{V} ,

$$d_i(\nu) = \nu \circ \delta_i \quad \text{and} \quad s_i(\nu) = \nu \circ \sigma_i.$$

Before turning to the motivating examples, in which \mathcal{V} is the category \mathcal{U} of topological spaces or the category $\mathcal{C}at$ of small categories, we apply this construction to the case $\mathcal{V} = s\mathcal{S}et$.

DEFINITION 13.3.1. Define the standard simplicial n -simplex $\Delta[n]^s$ to be the contravariant functor $\Delta \rightarrow s\mathcal{S}et$ represented by $[n]$. This means that the set $\Delta[n]_q^s$ of q -simplices is the set of all morphisms $\phi: [q] \rightarrow [n]$ in Δ . For a morphism $\nu: [p] \rightarrow [q]$ in Δ , the function $\nu^*: \Delta[n]_q^s \rightarrow \Delta[n]_p^s$ is given by composition, $\nu^*(\phi) = \phi \circ \nu: [p] \rightarrow [q]$.

DEFINITION 13.3.2. We define a covariant functor $\Delta[\bullet]^s$ from Δ to the category $s\mathcal{S}et$ of simplicial sets. On objects, the functor sends $[n]$ to the standard simplicial n -simplex $\Delta[n]^s$. On morphisms $\mu: [m] \rightarrow [n]$ in Δ , define $\mu_*: \Delta[m]_q^s \rightarrow \Delta[n]_q^s$ by $\mu_*(\psi) = \mu \circ \psi: [q] \rightarrow [m] \rightarrow [n]$. Thus the simplicial set $\Delta[n]^s$ is defined using pre-composition with morphisms of Δ , and then the covariant functoriality of $\Delta[\bullet]^s$ is defined using post-composition with morphisms of Δ . The object $\Delta[\bullet]^v$ is a cosimplicial simplicial set, that is, a cosimplicial object in the category of simplicial sets.

We may identify the set of all non-degenerate simplices of $\Delta[n]^s$ with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n]^s = (\mathcal{K}([n]))^s$ is the ordered simplicial set determined by the simplicial complex $\mathcal{K}([n])$.

Although we shall give a direct proof, the following result is an application of the Yoneda lemma. Let $\iota_n \in \Delta[n]_n^s$ be the identity map $\text{id}: [n] \rightarrow [n]$.

PROPOSITION 13.3.3. *Let K be a simplicial set. For $x \in K_n$, there is a unique map of simplicial sets $Y(x): \Delta[n]^s \rightarrow K$ such that $Y(x)(\iota_n) = x$. Therefore K is naturally isomorphic to the simplicial set whose n -simplices are the maps of simplicial sets $\Delta[n]^s \rightarrow K$.*

PROOF. The map $Y(x)$ is a natural transformation from the contravariant functor $\Delta[n]^s$ to the contravariant functor K from Δ to $\mathcal{S}et$. Since a q -simplex $\phi: [q] \rightarrow [n]$ is $\phi^*(\iota_n)$, we can and must specify $Y(x)$ at the object $[q] \in \Delta$ by the function $\Delta[n]_q^s \rightarrow K_q$ that sends ϕ to the q -simplex $\phi^*(x)$. \square

We can vary the construction in a way that may look unnatural but that will lend itself to generalization to other examples. We show how to reconstruct K directly from the $\Delta[n]^s$.

CONSTRUCTION 13.3.4. For a set J and a simplicial set L , one can form a new simplicial set $J \times L$ by setting $(J \times L)_q = J \times L_q$ and letting the faces and degeneracies be induced by those of L . Said another way, we think of J as a “discrete” simplicial set with each $J_q = J$ and all faces and degeneracies the identity map of J , and we then take the product $J \times L$ of simplicial sets. We apply this with $J = K_n$ and $L = \Delta[n]^s$ as n varies to obtain a simplicial set

$$\overline{K} = \coprod_{n \geq 0} K_n \times \Delta[n]^s.$$

We define an equivalence relation \simeq on \overline{K} by requiring

$$(13.3.5) \quad (\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma))$$

for $k \in K_n$, $\sigma \in \Delta[m]_q^s$, and $\alpha: [m] \rightarrow [n]$ in Δ . Here $\alpha^*(k) \in K_m$ is given by the fact that K is a contravariant functor from Δ to sets and $\alpha_*(\sigma) \in \Delta[n]_q^s$ is given by the fact that $\Delta[-]^s$ is a covariant functor from Δ to simplicial sets. With the simplicial structure induced from the simplicial structure on the $\Delta[n]^s$, passage to equivalence classes gives us a new simplicial set that we shall denote by $T^s K$ for the moment. Then T^s is a functor from simplicial sets to simplicial sets.

PROPOSITION 13.3.6. *The simplicial set $T^s K$ is naturally isomorphic to K .*

PROOF. We claim that an arbitrary pair (k, τ) in $K_n \times \Delta[n]_q^s$ is equivalent to the pair $(\tau(k), \iota_q)$ in $K_q \times \Delta[q]_q^s$ where, as above, $\iota_q: [q] \rightarrow [q]$ is the identity map viewed as a canonical q -simplex in $\Delta[q]^s$. Viewing $\tau: [q] \rightarrow [n]$ as a morphism of Δ , we have $\tau = \tau_*(\iota_q)$, and the claim follows. Identifying equivalence classes of q -simplices with elements of K_q in this fashion, we find that the faces and degeneracies agree. Indeed, for $\xi: [p] \rightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p). \quad \square$$

13.4. The total singular complex SX and the nerve $N\mathcal{C}$

We turn to the historical motivating example $\mathcal{V} = \mathcal{U}$ by constructing the total singular complex SX of a topological space X . We need a covariant functor $\Delta[\bullet]^t: \Delta \rightarrow \mathcal{U}$, and that is given by the standard topological simplices $\Delta[n]^t$.

DEFINITION 13.4.1. Recall that the standard topological n -simplex $\Delta[n]^t$ is the subspace

$$\{(t_0, \dots, t_n) \mid 0 \leq t_i \leq 1 \text{ and } \sum_i t_i = 1\}$$

of \mathbb{R}^{n+1} . Define

$$\delta_i: \Delta[n-1]^t \rightarrow \Delta[n]^t \quad \text{and} \quad \sigma_i: \Delta[n+1]^t \rightarrow \Delta[n]^t$$

by

$$\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n)$$

and

$$\sigma_i(t_0, \dots, t_{n+1}) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}).$$

Then the δ_i and σ_i satisfy the commutation relations required to specify a covariant functor $\Delta[\bullet]^t$ from Δ to the category \mathcal{U} of topological spaces, that is, a cosimplicial object in the category of topological spaces.

DEFINITION 13.4.2. The *total singular complex* SX of a space X is the simplicial set whose set $S_n X$ of n -simplices is the set of continuous maps $\Delta[n]^t \rightarrow X$ and whose faces d_i and degeneracies s_i induced by precomposition with δ_i and σ_i . By composition of continuous maps, a map $f: X \rightarrow Y$ induces the map $f_* = Sf: SX \rightarrow SY$ of simplicial sets that sends an n -simplex $s: \Delta[n]^t \rightarrow X$ to the n -simplex $f \circ s$. This defines the total singular complex functor S from topological spaces to simplicial sets.

We shall return to this example after giving an analogue that may seem astonishing at first sight. Although it has become a standard and commonplace construction, its importance and utility were only gradually recognized. Recall that a poset can be viewed as a category with at most one arrow between any pair of objects: either $x \leq y$, and then there is a unique arrow $x \rightarrow y$, or $x \not\leq y$, and then there is no arrow $x \rightarrow y$. Composition is defined in the only possible way. By definition $[n]$ is a totally ordered set, hence of course it is a partially ordered set. We can view it as a category and then the monotonic functions $\mu: [m] \rightarrow [n]$ are precisely the functors $[m] \rightarrow [n]$: monotonicity says that if there is an arrow $i \rightarrow j$, then there is an arrow $i \leq j$, which must be the value of the functor μ on that arrow.

DEFINITION 13.4.3. Let $\mathcal{C}at$ denote the category whose objects are small categories and whose morphisms are the functors between them. Define a covariant functor $\Delta[\bullet]^c: \Delta \rightarrow \mathcal{C}at$ by sending the ordered set $[n]$ to the corresponding category $[n]$ and sending a morphism $\mu: [m] \rightarrow [n]$ to the corresponding functor $\mu_*: [m] \rightarrow [n]$. Thus $\Delta[\bullet]^c$ is a cosimplicial category. When necessary for clarity, we write $[n]^c$ for the ordered set $[n]$ regarded as a category.

It is consistent with our previous notations to write $\Delta[n]^c$ for the poset $[n]$ regarded as a category. With that notation, the analogy with the definition of the total singular complex becomes especially obvious.

DEFINITION 13.4.4. Let \mathcal{C} be a small category. We define a simplicial set $N\mathcal{C}$, called the nerve of \mathcal{C} . Its set $N_n \mathcal{C}$ of n -simplices is the set of covariant functors $\phi: [n]^c \rightarrow \mathcal{C}$. The function $\mu^*: N_n \mathcal{C} \rightarrow N_m \mathcal{C}$ induced by $\mu: [m] \rightarrow [n]$ is given by $\mu^*(\phi) = \phi \circ \mu$, where μ is viewed as a functor $[m]^c \rightarrow [n]^c$. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a function $F_n = N_n F: N_n \mathcal{C} \rightarrow N_n \mathcal{D}$ by composition of functors, $F_n(\phi) = F \circ \phi$. These functions specify a map $F_* = NF: N\mathcal{C} \rightarrow N\mathcal{D}$ of simplicial sets. Thus we have the nerve functor N from $\mathcal{C}at$ to the category $\mathcal{S}et$ of simplicial sets.

The definition can easily be unravelled. The category $[0]^c$ has one object and its identity morphism, hence a functor $\phi: [0]^c \rightarrow \mathcal{C}$ is just a choice of an object of \mathcal{C} . That is, if we write $\mathcal{O}\mathcal{C}$ for the set of objects of \mathcal{C} , then $N_0 \mathcal{C} = \mathcal{O}\mathcal{C}$. For $n \geq 1$, a functor $\phi: [n]^c \rightarrow \mathcal{C}$ is a choice of n composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n.$$

Denoting such a string by (f_1, \dots, f_n) , the faces and degeneracies are given by

$$(13.4.5) \quad d_i(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) & \text{if } i = 0 \\ (f_1, \dots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \dots, f_n) & \text{if } 0 < i < n \\ (f_1, \dots, f_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(f_1, \dots, f_n) = (f_1, \dots, f_{i-1}, \text{id}, f_i, \dots, f_n)$$

In words, the 0^{th} and n^{th} faces send (f_1, \dots, f_n) to the $(n-1)$ -simplex obtained by deleting f_1 or f_n ; when $n = 1$ this is to be interpreted as giving the object c_1 or c_0 . For $0 < i < n$, the i^{th} face composes f_{i+1} with f_i . The i^{th} degeneracy operation inserts the identity morphism of c_i . The ordering may look unnatural, since $f_{i+1} \circ f_i$ means first f_i and then f_{i+1} , and many authors prefer to reverse the ordering in a composable sequence so that for $n \geq 1$, a functor $\phi: [n]^c \rightarrow \mathcal{C}$ is a choice of n composable morphisms

$$c_0 \xleftarrow{f_1} c_1 \longleftarrow \quad \cdots \longleftarrow c_{n-1} \xleftarrow{f_n} c_n.$$

This amounts to replacing the categories $\Delta[n]^c$ by their opposite categories. It is the choice taken in the following hugely important example.

EXAMPLE 13.4.6. Let G be a group regarded as a category with a single object $*$; the elements of the group are the morphisms $* \rightarrow *$, and every pair of morphisms is composable. The nerve NG is often written B_*G and called the bar construction. It is the simplicial set with $B_nG = G^n$, with n -tuples of elements written $[g_1 | \cdots | g_n]$ (hence the name “bar”) and with faces and degeneracies specified for $0 \leq i \leq n$ by

$$d_i[g_1 | \cdots | g_n] = \begin{cases} [g_2 | \cdots | g_n] & \text{if } i = 0 \\ [g_1 | \cdots | g_{i-1} | g_i g_{i+1} | g_{i+2} | \cdots | g_n] & \text{if } 0 < i < n \\ [g_1 | \cdots | g_{n-1}] & \text{if } i = n. \end{cases}$$

$$s_i[g_1 | \cdots | g_n] = [g_1 | \cdots | g_{i-1} | e | g_i | \cdots | g_n]$$

However $N\mathcal{A}$ is written, in general it looks nothing like our original example of the simplicial set associated to an ordered simplicial complex! In one important case, which we will find is far more common than one might reasonably expect, it does look like that.

EXAMPLE 13.4.7. Let X be a poset. We can obtain a simplicial set by regarding X as a category and taking its nerve. Alternatively, we can take the ordered simplicial complex $\mathcal{K}X$ and then take the simplicial set associated to that. It is an instructive exercise to check that we get the same simplicial set via either route. That is, NX is naturally isomorphic to $(\mathcal{K}X)^s$.

13.5. The geometric realization of simplicial sets

We have observed that the category Δ is generated by the injections δ_i and surjections σ_i . Decomposing a morphism $\mu: [m] \rightarrow [n]$ as a composite of δ_i 's and σ_j 's records which elements of the target $[n]$ are not in the image of μ and which elements of the source $[m]$ have the same image under μ . It is helpful to be more precise about this. Let i_1, \dots, i_q in reverse order $0 \leq i_q < \cdots < i_1 \leq n$ be the elements of $[n]$ that are not in the image $\mu([m])$. Let j_1, \dots, j_p in order $0 \leq j_1 < \cdots < j_p < m$ be the elements $j \in [m]$ such that $\mu(j) = \mu(j+1)$. With these notations, $m - p + q = n$ and

$$(13.5.1) \quad \mu = \delta_{i_1} \cdots \delta_{i_q} \sigma_{j_1} \cdots \sigma_{j_p}.$$

That is, we record duplications in such a manner that the indices record the repeated and skipped elements in a sensible canonical order. The sequences of i 's and j 's in this description of μ are uniquely determined.

Using this canonical decomposition implicitly, we can be precise about the definition and description of the geometric realization of a simplicial set K . The construction is precisely analogous to [Construction 13.3.4](#) and might well be denoted by $T^t K$.

CONSTRUCTION 13.5.2. For a set J and a space L , we regard J as a discrete topological space and obtain the space $J \times L$. Applying this with $J = K_n$ and $L = \Delta[n]^t$ for $n \geq 0$, we obtain the space

$$\bar{K} = \coprod_{n \geq 0} K_n \times \Delta[n]^t$$

with the topology of the union. That is, we take the union of one topological simplex for each n -simplex $k \in K_n$. Say that an n -simplex k is degenerate if $k = s_i \ell$ for some $(n-1)$ -simplex ℓ and some i and nondegenerate otherwise. We shall glue the simplices together in such a way that we obtain a space with one “ n -cell” for each nondegenerate n -simplex of K . That means in particular that in the resulting space every point will be the interior point of the image of exactly one simplex $\{k\} \times \Delta[n]^t$, where k is nondegenerate. Note that the unique point of $\Delta[0]^t$ is an interior point. We say that a point (k, u) of \bar{K} is nondegenerate if k is nondegenerate and u is interior.

Define an equivalence relation \approx on \bar{K} by letting

$$(\mu^* k, u) \approx (k, \mu_* u)$$

for each $k \in K_n$, $u \in \Delta[m]$, and $\mu: [m] \rightarrow [n]$. This equivalence relation is generated by the relations obtained by specializing to $\mu = \delta_i$ or $\mu = \sigma_i$. These can be rewritten as

$$(d_i k, u) \approx (k, \delta_i u) \quad \text{and} \quad (s_i k, u) \approx (k, \sigma_i u).$$

Each n -simplex k_n can be written uniquely in the form $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$, where k_{n-p} is nondegenerate and $0 \leq j_1 < \cdots < j_p < n$. Define a function $\lambda: \bar{K} \rightarrow \bar{K}$ by

$$\lambda(k_n, u_n) = (k_{n-p}, \sigma_{j_1} \cdots \sigma_{j_p} u_n)$$

where $u_n \in \Delta[n]^t$. Similarly, every $u_n \in \Delta[n]^t$ can be written uniquely in the form $u_n = \delta_{i_q} \cdots \delta_{i_1} u_{n-q}$, where u_{n-q} is interior and $0 \leq i_q < \cdots < i_1 \leq n$. Define a function $\rho: \bar{K} \rightarrow \bar{K}$ by

$$\rho(k_n, u_n) = (d_{i_q} \cdots d_{i_1} k_n, u_{n-q}).$$

LEMMA 13.5.3. *The composite $\lambda \circ \rho$ carries each point of \bar{K} into the unique nondegenerate point that is equivalent to it.*

Define the geometric realization of K , which is usually denoted $|K|$ but which we shall usually denote by TK , to be the set of equivalence classes $\bar{K}/(\approx)$. Define $F_p TK$ to be the image of $\coprod_{0 \leq n \leq p} K_n \times \Delta[n]^t$ in TK and give it the quotient space topology. Then topologize TK by giving it the topology of the union of the $F_p TK$. This means that a subset C is closed if and only if it intersects each $F_p TK$ in a closed subset. We shall shortly give an equivalent description of this topology.

13.6. The adjoint relationship between S and T

It has long been known that we can use simplicial sets pretty much interchangeably with topological spaces when studying homotopy theory. We sketch how this is seen through the categorical eyes of an adjunction. For a simplicial set K , we have defined a space $|K| = TK$, called the geometric realization of K . We write $|k, u|$ for the image of (k, u) in TK , where $k \in K_n$ and $u \in \Delta[n]$. For a space X , we have defined a simplicial set SX , called the total singular complex of X , whose n -simplices are the continuous maps $f: \Delta[n]^t \rightarrow X$. The homotopical behavior is studied through an adjunction: T and S are left and right adjoint functors in the sense that we have just defined. That is, there is a bijection, natural in both variables, between morphism sets

$$\mathcal{U}(TK, X) \cong s\mathcal{S}et(K, SX).$$

It is specified by letting $f: TK \rightarrow X$ correspond to $g: K \rightarrow SX$ if

$$f(|k, u|) = g(k)(u).$$

There is an equivalent way of saying this. Define $\gamma: TSX \rightarrow X$ by

$$\gamma|f, u| = f(u) \quad \text{for } f: \Delta_n \rightarrow X \quad \text{and } u \in \Delta_n.$$

It is a fact that γ is a weak homotopy equivalence for every space X , although we shall not prove that here. There is also a map $\iota: K \rightarrow STK$ of simplicial sets specified by $\iota(k)(u) = |k, u|$ for $k \in K_n$ and $u \in \Delta_n$. Again, as we also shall not prove, $|\iota|: |K| \rightarrow |STK|$ is a homotopy equivalence. These facts are proven, for example, in [45]. The natural composite

$$SX \xrightarrow{\iota S} STSX \xrightarrow{S\gamma} SX$$

is the identity map of SX . The natural composite

$$TK \xrightarrow{T\iota} TSTK \xrightarrow{\gamma T} TK$$

is the identity map of TK . Here ιS means first apply the functor S and then the natural map γ , and similarly for γT . The natural maps ι and γ are the unit and the counit of the adjunction. This means that, in the correspondence above, $f = \gamma \circ Tg$ and $g = Sf \circ \iota$.

13.7. CW complexes

We explain the nature of the space TK by introducing two equivalent definitions of a CW complex. We start with the original 1949 definition of J.H.C. Whitehead [68], which explains the name. We then observe that TK satisfies the specifications of that definition. Finally, we give the more modern and now standard definition of a CW complex. Let D^n be the disc $\{x \mid |x| \leq 1\} \subset \mathbb{R}^n$.

The capitalized C and W are the source of the name ‘‘CW complex’’, but this form of the definition is so rarely used nowadays that younger experts often have no idea where the name came from. However, it is convenient for describing TK .

THEOREM 13.7.1. *The space TK is a CW complex with one n -cell for each nondegenerate n -simplex $k_n \in K_n$.*

PROOF. The n -cells e^n of TK are the images of the subspaces $\{k_n\} \times \Delta[n]$, and the map $j: \Delta[n] \rightarrow \bar{e}^n$ is the restriction of the map $\bar{K} \rightarrow TK$ to $\{k_n\} \times \Delta[n]$. The topology of the union we prescribed before is in fact the “weak topology”. It is “weak” in the sense that in general it has more open sets than the quotient space topology, but the novice may not want to worry about the verification, preferring to simply accept that our original definition of the topology gives what once upon a time was called the weak topology. \square

Here is the modern redefinition of a CW complex.

DEFINITION 13.7.2. A CW complex is a space X that is the union of an expanding sequence of subspaces X^n , where X^n is called the n -skeleton of X . It is required inductively that

- (1) X^0 is a set with the discrete topology.
- (2) X^{n+1} is constructed from X^n as a “pushout”

$$\begin{array}{ccc} \coprod S^n & \xrightarrow{j} & X^n \\ \cap \downarrow & & \downarrow \\ \coprod D^{n+1} & \xrightarrow{\bar{j}} & X^{n+1}. \end{array}$$

This means that X^{n+1} is the quotient space

$$X^n \cup_{\coprod S^n} (\coprod D^{n+1}) \equiv X^n \amalg (\coprod D^{n+1}) / (\approx)$$

specified by the equivalence relation $s \approx j(x)$ for $s \in S^n \subset D^{n+1}$.

The space X is given the topology of the union; equivalently, a subset is closed if its intersection with each closed cell $\bar{j}(D^n)$ is closed.

We leave it as an exercise for the reader to see that the two definitions of a CW complex give exactly the same spaces. The compactness of the spheres that are the domains of attaching maps ensures that a CW complex with the second definition is closure finite, as required in the first definition.

The intuition is that we glue discs D^{n+1} to X^n as dictated by attaching maps defined on their boundaries S^n . The attaching maps can be quite badly behaved. For an ordered simplicial complex K , the classical geometric realization $|K|$ is homeomorphic to the geometric realization $T(K^s)$ of its associated simplicial set K^s . This is visually apparent since each has an n -cell for each n -simplex of K . Remember that the n -simplices of K itself are of the form $\{x_0 < \cdots < x_n\}$ whereas the elements of K_n are of the form $\{x_0 \leq \cdots \leq x_n\}$. The degeneracy identifications in the construction of TK^s serve to eliminate the degenerate elements in which some of the vertices are repeated.

In $T(K^s)$ the closed cells are homeomorphic to $\Delta[n]$ and the attaching maps are homeomorphisms on boundaries. Spaces can be “triangulated” as CW complexes using many fewer cells than are required for polyhedral triangulations. For example, we can triangulate the n -sphere S^n as a CW complex with just two cells. Clearly S^0 is a CW complex with two 0-cells, or vertices. For $n > 0$, we start with a single 0-cell $*$, take $(S^n)^{n-1} = *$ and attach a single n -cell with attaching map the trivial map $S^{n-1} \rightarrow *$. Then the n -skeleton is $* \cup_{S^{n-1}} D^n = D^n / S^{n-1}$, which is already homeomorphic to S^n .

REMARK 13.7.3. Earlier we neglected to give a precise definition of $|K|$ for a geometric simplicial complex with a possibly infinite number of vertices and thus with possibly infinite dimension: while every simplex has a finite dimension, simplices of all finite dimensions can occur. When K is ordered, we now have such a definition. We just take the geometric realization of the associated simplicial set; the result is a functor from the category of ordered simplicial sets to the category of spaces. When K is finite, TK^s is homeomorphic to $|K|$ as we defined it originally. We can also start with A -spaces, alias posets X . Then $T\mathcal{K}(X)^s$ gives a composite functor from the category of posets to the category of spaces.

Remember that the product $K \times L$ of ordered simplicial complexes K and L has simplices all subsets of products $\sigma \times \tau$ of simplices, where the ordering on vertices is given by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

DEFINITION 13.7.4. Define the product $K \times L$ of simplicial sets K and L by letting $(K \times L)_n = K_n \times L_n$, with $d_i = (d_i, d_i)$ and $s_i = (s_i, s_i)$, which implies that $\mu^* = (\mu^*, \mu^*)$ for all morphisms μ in Δ .

This definition is forced by two considerations. First, it ensures the consistency statement $(K \times L)^s \cong K^s \times L^s$. That is, if we start with ordered simplicial complexes K and L , then the simplicial set $(K \times L)^s$ is naturally isomorphic to the product simplicial set $K^s \times L^s$. Second, the definition is dictated by the universal property that we require of products in any category. Recall that the n -simplices of $K \times L$ involve repeated vertices of K and L . These correspond to the use of degeneracy operators in the factors K^s and L^s of the associated simplicial set. It clarifies matters to be precise about this. We state the following lemma for general simplicial sets K and L , but the reader should think about what it is saying when we apply it to K^s and L^s for ordered simplicial complexes K and L .

LEMMA 13.7.5. *Let K and L be simplicial sets. The nondegenerate n -simplices of $K \times L$ can be written uniquely in the form*

$$(s_{i_p} \cdots s_{i_1} k, s_{j_q} \cdots s_{j_1} \ell),$$

where k is a nondegenerate $(n-p)$ -simplex of K , ℓ is a nondegenerate $(n-q)$ -simplex of L , $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, and the sets $\{i_a\}$ and $\{j_b\}$ are disjoint.

The set $\{i_a\} \cup \{j_b\}$ has $p+q$ elements and corresponds to a (p, q) shuffle permutation of a set with $p+q$ elements. The term “shuffle” comes from thinking of a permutation of a deck of $p+q$ cards that starts with a cut into p cards and q cards, which are kept in order by the permutation. The reader will easily see that when we started with posets X and Y and showed that $\mathcal{K}(X \times Y)$ is a subdivision of $\mathcal{K}(X) \times \mathcal{K}(Y)$, we were actually verifying an instance of essentially this lemma. From here, the reader will have no trouble believing the following result, the proof of which amounts to appropriately subdividing topological simplices $\Delta[n]^t$.

THEOREM 13.7.6. *For simplicial sets K and L , the map*

$$T(K \times L) \longrightarrow TK \times TL$$

whose coordinates are the maps $T\pi_1$ and $T\pi_2$ induced by the projections of $K \times L$ on K and L is a homeomorphism.

We shall not repeat the proof, which adds precision and decreases intuition, referring the reader, for example, to [45, 14.3] or [26, 4.3.15] for details. The latter

book is especially recommended as a very good and relatively recent treatment of CW complexes, simplicial complexes, and simplicial sets.

13.8. The fundamental category functor Π

It is also known, although this is more recent, that we can use categories pretty much interchangeably with topological spaces when studying homotopy theory. We are going to say quite a lot about this later. This comparison again starts with an adjunction. We have constructed a simplicial set $N\mathcal{C}$ called the nerve of \mathcal{C} . We define $B\mathcal{C} = TN\mathcal{C}$. This is called the *classifying space* of the category \mathcal{C} . When G is a group regarded as a category with a single object, BG is called the classifying space of the group G . The space BG is often written as $K(G, 1)$. It is called an Eilenberg-Mac Lane space. It is characterized (up to homotopy type) as a connected space with $\pi_1(K(G, 1)) = G$ and with all higher homotopy groups $\pi_q(K(G, 1)) = 0$. A concise summary of how that works is in [48, §16.5]. More generally, a detailed study of the classifying spaces of topological groups and what they classify is in [46]. These are fundamentally important constructions in topology and its applications.

The nerve functor N is accompanied by a functor $\Pi: s\mathcal{S}et \rightarrow \mathcal{C}at$, called the “fundamental category” functor.¹ It is left adjoint to N , meaning that

$$\mathcal{C}at(\Pi K, \mathcal{C}) \cong s\mathcal{S}et(K, N\mathcal{C}).$$

This means that it is conceptually sensible, but, in contrast to such functors as S and T , it does not have good homotopical properties, as we shall see.

For a simplicial set K , the objects of the category ΠK are the vertices (that is, the 0-simplices) of K . To construct the morphisms, one starts by thinking of the 1-simplices y as maps $d_1y \rightarrow d_0y$. One forms all words (formal composites) that make sense, that is, whose targets and sources match up. One then imposes the relations on morphisms determined by

$$s_0x = \text{id}_x \text{ for } x \in K_0 \text{ and } d_1z = d_0z \circ d_2z \text{ for } z \in K_2.$$

We use the relations $d_id_j = d_{j-1}d_i$ for $i < j$ when (i, j) is $(0, 1)$, $(1, 2)$, and $(0, 2)$ to see that sources and targets match up. This makes good sense since if $K = N\mathcal{C}$, then a 0-simplex is an object x of \mathcal{C} , a 1-simplex y is a map $d_1y \rightarrow d_0y$, the 1-simplex s_0x is id_x , and a 2-simplex z is given by a pair of composable morphisms d_2z and d_0z together with their composite d_1z .

Therefore there is a natural map $\varepsilon: \Pi N\mathcal{C} \rightarrow \mathcal{C}$ that is the identity on objects (the zero simplices of $N\mathcal{C}$) and is induced by the identity map from the generating morphisms of $\Pi N\mathcal{C}$ (the 1-simplices on $N\mathcal{C}$) to the morphisms of \mathcal{C} . In fact, ε is an isomorphism of categories: it is the identity on objects, and it presents the category in terms of generators given by the morphism sets modulo relations determined by the category axioms.

For the adjunction, a functor $F: \Pi K \rightarrow \mathcal{C}$ is constructed from a map of simplicial sets $g: K \rightarrow N\mathcal{C}$ by letting F be the unique functor that agrees with g on objects (= 0-simplices) and equivalence classes of morphisms (= 1-simplices). Applying the adjunction to the identity map of ΠK , we obtain a natural map $\eta: K \rightarrow \Pi K$, which is the unit of the adjunction, and the counit is the isomorphism ε .

¹There is no fully standard notation for this category. I’ve seen it denoted τ_1 , π_1 , π , and C .

13.9. Tensor products of functors?

Add in: see §9 examples

Give the idea, relate to geometric realization of simplicial spaces and $K \cong K \otimes_{\Delta} \Delta^s$. Motivate by coming analogy with subdivision.

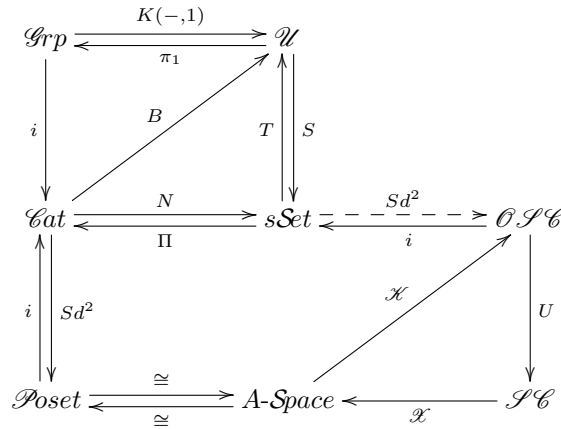
The big picture: a schematic diagram and the role of subdivision

The n -skeleton K^n of a simplicial set K is the subsimplicial set generated by the q -simplices for all $q \leq n$. Visibly, ΠK depends only on the 2-skeleton K^2 . Therefore the inclusion $K^2 \rightarrow K$ of simplicial sets induces an isomorphism of categories $\Pi K^2 \rightarrow \Pi K$ for any K . In particular, Π takes the inclusion $\iota: \partial\Delta[n]^s \rightarrow \Delta[n]^s$ of the boundary of the n -simplex to the identity functor when $n > 2$. Thus Π loses homotopical information: upon realization, $|\iota|$ is equivalent to the inclusion $S^{n-1} \rightarrow D^n$. What is amazing is that this extreme loss of information disappears after subdividing twice. This is something I have been trying to better understand for quite some time.

The reader will find it easy to believe that there is a subdivision functor on simplicial sets that generalizes the subdivision functor Sd on simplicial complexes in the sense that $(SdK)^s \cong Sd(K^s)$ for a simplicial complex K . This allows one to define a subdivision functor on categories by setting $Sd\mathcal{C} = \Pi SdN\mathcal{C}$. One can iterate subdivision, forming functors Sd^2 on both simplicial sets and categories. What is mind blowing at first is that the iterated subdivision $Sd^2\mathcal{C}$ is actually a poset whose classifying space $BSd^2\mathcal{C}$ is homotopy equivalent to $B\mathcal{C}$. I will start from a more combinatorial definition of $Sd\mathcal{C}$, and I will use it to give what I hope the reader will find an easy combinatorial proof that $Sd^2\mathcal{C}$ is indeed a poset.

However, before heading for that, let us summarize a schematic and technically oversimplified global picture of all of the big categories that we are constructing and comparing by functors. This is the same diagram as in the introduction, and it gives an interesting picture of lots of kinds of mathematics that come together with a focus on simplicial sets.

Add left adjoint to i ,
from Cat to Poset ?



Our earlier work focused on finite spaces, but the basic theory generalizes with the finiteness removed, provided we understand simplicial complexes to mean abstract simplicial complexes. As noted above, we didn't define geometric realization in general earlier, but we have done so now. The equivalence of posets with A -spaces and the constructions \mathcal{K} and \mathcal{X} that we worked out in detail for finite spaces work in exactly the same way when we no longer restrict ourselves to the finite case. The functors i in the diagram are thought of as inclusions of categories. Remember that we write $i(K) = K^s$ for the simplicial set associated to an ordered simplicial complex. We have defined all of the categories and functors exhibited in the diagram except for Sd^2 , which is second subdivision.

Describe features of the diagram: posets vs ordered simplicial complexes (latter: some but not all totally ordered subsets of the poset of vertices. [Said earlier]) Remember no canonical ordering, u cannot be a right adjoint, etc.

Subdivision and Properties A, B and C in the Category of Sets

We shall define three properties of a simplicial set, called Properties *A*, *B*, and *C*. We say that a category satisfies property *A*, *B*, or *C* if its nerve satisfies that property. Remember that the nerve functor N is a right adjoint whose left adjoint is the fundamental category functor Π . We shall define the subdivision of a simplicial set in such a way as to generalize the subdivision of simplicial complexes that plays such a fundamental role in our study of finite spaces. We shall define the companion notion of the subdivision of a category in the next chapter. We write Sd^s for the subdivision functor on simplicial sets and Sd^c for the subdivision functor on categories when necessary for clarity. These are the main characters in our story. We want to understand the relationships between these functors and the rest of the categories and functors in our big picture. There are a number of surprising and interesting implications.

15.1. Properties A, B and C of simplicial sets

DEFINITION 15.1.1. We define and name three properties that a simplicial set might have.

- (A) Property *A*, the *nondegenerate simplex property*: K has property *A* if every face of a nondegenerate simplex x of K is nondegenerate.
- (B) Property *B*, the *distinct vertex property*: K has property *B* if the $n+1$ vertices of any nondegenerate n -simplex x of K are distinct.
- (C) Property *C*, the *unique simplex property*: K has property *C* if for any set of $n+1$ distinct vertices of K , there is at most one nondegenerate n -simplex of K whose vertices are the elements of that set.

REMARK 15.1.2. In Property *A*, we mean that all faces $d_i x$ are nondegenerate. But then all faces of all $d_i x$ are also nondegenerate. Iterating, all of the face q -simplices of x for $q < n$ are nondegenerate.

In line with this remark, there is a less succinct but useful characterization of Property *B*. We express it with a notation that we shall use frequently later.

NOTATION 15.1.3. For a simplex $x \in K_n$ and a (nonempty) subset S of the set $[n] = \{0, 1, \dots, n\}$, let S^*x denote the simplex $\mu^*x \in K_m$, where $\mu: [m] \rightarrow [n]$ is the unique injection in Δ with image S . Then the cardinality of S , which we write as $|S|$, is $m+1$.

PROPOSITION 15.1.4. *A simplicial set K has Property B if and only if for every n and every nondegenerate simplex $x \in K_n$, μ^*x and ν^*x are distinct simplices of K for every pair μ and ν of distinct injections with target $[n]$ in Δ ; equivalently, $S^*x \neq T^*x$ for every pair of distinct subsets S and T of $[n]$.*

PROOF. Property B is the case when μ and ν have source $[0]$, so it is clear that the new property implies Property B . For the converse, suppose that K satisfies Property B and that $S^*x = T^*x$ for a nondegenerate simplex $x \in K_n$ and nonempty subsets S and T of $[n]$. This clearly implies that $|S| = |T| = m + 1$, say, where $0 \leq m \leq n$. Write $S = \{s_0, \dots, s_m\}$ and $T = \{t_0, \dots, t_m\}$, each in strictly increasing order. Consider the singleton subsets $\{i\} \subset [m]$, $\{s_i\} \subset [n]$, and $\{t_i\} \subset [n]$, where $0 \leq i \leq m$. Using the language of [Notation 15.1.3](#), we have

$$\{s_i\}^*x = \{i\}^*S^*x = \{i\}^*T^*x = \{t_i\}^*x.$$

Since these are vertices of x , they are equal by Property B . This implies that $s_i = t_i$ and thus $S = T$. \square

It is natural to ask if there are implications among Properties A , B , and C .

THEOREM 15.1.5. *Property B implies Property A , but there are no other implications between these properties.*

PROOF. Suppose that K does not have Property A . There is an $n \geq 1$ and a nondegenerate n -simplex with a degenerate face. Using the commutation relations between faces and degeneracies, we see that any degenerate simplex has a degenerate 1-simplex as one of its 1-faces. Since both vertices of a degenerate 1-simplex s_0x are x , our original nondegenerate n -simplex cannot have distinct vertices. The non-implications are proven by exhibiting counterexamples. We choose nerves of categories, so that these non-implications will also be clear for categories. \square

EXAMPLE 15.1.6. Here are some examples which exhibit various non-implications.

- (i) Let $K = N\mathcal{C}$ where \mathcal{C} is the category with one object x and one non-identity morphism p , with $p \circ p = p$. Then K satisfies Property A but not Property B .
- (ii) Let $K = N\mathcal{C}$, where \mathcal{C} is the category with two vertices x and y , two non-identity morphisms $x \rightarrow y$, and no morphisms $y \rightarrow x$. Then K satisfies Properties A and B but not Property C .
- (iii) Let $K = NC_2$, where C_2 is the cyclic group of order 2 regarded as a category with one object. Then K satisfies Property C but not Properties A or B . For each q , K has a unique nondegenerate q -simplex (g, \dots, g) , where g is the generator of C_2 . Since $g^2 = e$, that simplex has a degenerate face when $q \geq 2$.
- (iv) More generally, if $K = NC_n$, where C_n is the cyclic group of order $n > 2$ with generator g , the simplices $x = (g, \dots, g) \in K_q$ have all faces $d_i x$ nondegenerate, but iterated face operations reach degenerate simplices when $q \geq n$.

Suspect

Here is a thought exercise. Consider the simplicial set K^s associated to an ordered simplicial complex K . Clearly it has all three properties. What about a converse? Recall that there is a natural order on the set of vertices of the standard n -simplex $\Delta[n]^s$. After all, they are the i with $0 \leq i \leq n$. Since the set K_n can be identified with the set of simplicial maps $\Delta[n]^s \rightarrow K_n$, each simplex has an induced ordering of its vertices. It need not be consistent as the simplices vary. We can try to give the set of vertices a partial order that restricts to a total order on each simplex by setting $v \leq w$ if [and only if] v and w are vertices of some simplex x in some K_n and $v \leq w$ in the ordering of the vertices of that simplex [and taking the partial order generated by this relation [to get transitivity]?]

EXERCISE 15.1.7. Suppose that a simplicial set K satisfies Properties B and C . Then \leq is a well-defined partial order on the set $V = K_0$ that restricts to a

This is not so easy! Transitivity? FALSE!

total order on the vertices of each non-degenerate simplex of K . With simplices those finite sets of vertices that are the vertices of some nondegenerate $x \in K_n$, we obtain a simplicial complex L , and K is isomorphic to L^s . Conversely, if K does not satisfy either Property B or Property C , then it cannot be isomorphic to L^s for any simplicial complex L .

By abuse of language, we say that a simplicial set is a simplicial complex if it is isomorphic to L^s for some ordered simplicial complex L . In fact, L is canonically determined by K in the manner that we have described. The exercise proves the following result.

THEOREM 15.1.8. *A simplicial set is a simplicial complex if and only if it satisfies Properties B and C .*

15.2. The definition of the subdivision of a simplicial set

For both simplicial sets and categories, there is both a conceptual definition and an equivalent combinatorial definition. For simplicial sets, we begin with the perhaps ugly looking and hard to grasp combinatorial definition and then show that it is equivalent to a conceptual definition that is closely analogous to the definition of geometric realization.

DEFINITION 15.2.1. We define the subdivision $\text{Sd}K = \text{Sd}^s K$ of a simplicial set K . The q -simplices of $\text{Sd}K_q$ are the equivalence classes of tuples

$$(x; S_0, \dots, S_q),$$

where, for some $n \geq 0$, $x \in K_n$, each S_i is a subset of $[n]$, and $S_i \subset S_{i+1}$ for $0 \leq i < q$. The equivalence relation is specified by

$$(\mu^*x; S_0, \dots, S_q) \sim (x; \mu_*(S_0, \dots, S_q))$$

for a morphism $\mu: [m] \rightarrow [n]$ in Δ , where $x \in K_n$, hence $\mu^*x \in K_m$; here $\{S_i\}$ is an increasing sequence of subsets of $[m]$ and

$$\mu_*(S_0, \dots, S_q) = (\mu(S_0), \dots, \mu(S_q)).$$

The simplicial operations are induced by

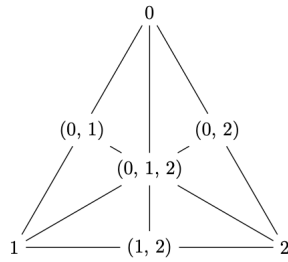
$$\nu^*(x; S_0, \dots, S_q) = (x; S_{\nu(0)}, \dots, S_{\nu(p)})$$

for a map $\nu: [p] \rightarrow [q]$ in Δ , where $x \in K_n$ and $\{S_i\}$ is an increasing sequence of subsets of $[p]$ for some n . Subdivision is functorial. For a map $f: K \rightarrow L$ of simplicial sets, $f_* = \text{Sd}f: \text{Sd}K \rightarrow \text{Sd}L$ is induced by

$$f_*(x; S_0, \dots, S_q) = (f(x); S_0, \dots, S_q).$$

This definition is convenient for doing combinatorics and is directly motivated by the following comparison, which we will prove in §15.3.

EXAMPLE 15.2.2. We can denote the vertices of a geometric n -simplex using elements in $[n]$ and denote the barycenter of each q -simplex with vertices $v_{i_0} < \dots < v_{i_q}$ using the q -tuple $(v_{i_0}, \dots, v_{i_q})$. The following illustrates this in the case of a 2-simplex.



REMARK 15.2.3. Notice that barycenters of higher dimensional simplices have higher orders. In the example above for instance, we have the following order on the vertices:

$$0 \leq (0, 1) \leq (0, 1, 2)$$

THEOREM 15.2.4. *If K is an ordered simplicial complex, then the simplicial sets $\text{Sd}(K^s)$ and $(\text{Sd}K)^s$ are naturally isomorphic.*

However, it obscures the idea behind the definition, which we now elucidate. The conceptual definition parallels Constructions Construction 13.3.4 and Construction 13.5.2. The parallel with the geometric realization functor is particularly useful, but the parallel with the reconstruction functor $T^s K$ is especially illuminating.

Recall that $\Delta[n]^s$ is the represented simplicial set with q -simplices the maps $\alpha: [q] \rightarrow [n]$ in Δ . Its nondegenerate simplices are the injections. It is a simplicial complex. That is, it can be viewed as $(\mathcal{K}[n])^s$. As a simplicial complex it has the subdivision studied earlier, which we now regard as a simplicial set and denote by $\text{Sd}\Delta[n]^s$. Then the nondegenerate q -simplices of $\text{Sd}\Delta[n]^s$ are the ordered q -tuples $\underline{\alpha} = \{\alpha_0, \dots, \alpha_q\}$ of $\Delta[n]^s$, where α_i is a face of α_{i+1} , so that α_i is obtained from α_{i+1} by precomposition with an injection in Δ . For a map $\nu: [p] \rightarrow [q]$ in Δ , the simplicial operation ν^* on $\text{Sd}\Delta[n]$ is given by

$$\nu^*(\underline{\alpha}) = (\alpha_{\nu(0)}, \dots, \alpha_{\nu(p)}).$$

As n varies, the subdivisions $\text{Sd}\Delta[n]$ define a covariant functor

$$\text{Sd}\Delta[\bullet]^s: \Delta \rightarrow s\mathcal{S}et,$$

that is, a cosimplicial simplicial set. For $\mu: [m] \rightarrow [n]$, $\mu_*: \text{Sd}\Delta[m]^s \rightarrow \text{Sd}\Delta[n]^s$ is given by

$$\mu_*\underline{\alpha} = (\mu \circ \alpha_0, \dots, \mu \circ \alpha_q).$$

Strictly speaking, to write simplices in terms of injections only, we must interpret $\mu \circ \alpha_i$ as the injective part δ of the canonical decomposition of $\mu \circ \alpha_i$ as the composite $\delta\sigma$ of a surjection σ and an injection δ . Here is the conceptual definition of $\text{Sd}K$.

CONSTRUCTION 15.2.5. As in the construction of $T^s K$ given in Construction 13.3.4, regard each set K_n as just a set, or as a discrete simplicial set with each $(K_n)_q = K$ and all faces and degeneracies the identity map. Then form the product simplicial sets $K_n \times \text{Sd}\Delta[n]^s$ and take their disjoint union to obtain the simplicial set

$$\overline{\text{Sd}K} = \coprod_{n \geq 0} K_n \times \text{Sd}\Delta[n].$$

Again as in the construction of $T^s K$, define an equivalence relation on \overline{SdK} . For $\mu: [m] \rightarrow [n]$ in Δ , we let

$$(\mu^* x, \underline{\alpha}) \sim (x, \mu_* \underline{\alpha}).$$

where $x \in K_n$ and $\underline{\alpha} \in Sd\Delta[m]^s$. We suppress from the notation that this defines an equivalence relation on q -simplices for each q . Now $(SdK)_q$ is the set of equivalence classes of q -simplices. The simplicial operations on the simplicial sets $K_n \times Sd\Delta[n]^s$ are of the form $\text{id} \times \nu^*$. They induce the simplicial operations on SdK .

REMARK 15.2.6 (Categorical remark). The definitions of $T^s K$, SdK and TK are all examples of “tensor products of functors”, often written $K \otimes_{\Delta} L$ for a contravariant functor K and a covariant functor L defined on Δ (which could be replaced by any other small category) but we shall not go into the general categorical framework. However, as a specialization of a general categorical result about such categorical tensor products, there is an associativity isomorphism of simplicial sets

$$(K \otimes_{\Delta} L) \otimes_{\Delta} M \cong K \otimes_{\Delta} (L \otimes_{\Delta} M)$$

where K is a simplicial set and L and M are cosimplicial simplicial sets. Inductively, this implies that

$$Sd^n K \cong K \otimes_{\Delta} Sd^n \Delta[-] = \coprod_n K_n \times Sd^n \Delta[n]/(\sim),$$

where the equivalence relation is defined exactly as in [Construction 15.2.5](#). This gives a good hold on these functors, since $Sd^n \Delta[-] = (\mathcal{X}^{(n)} \Delta[-])^s$ is just the classical iterated barycentric subdivision, regarded as a simplicial set.

To reconcile the combinatorial and conceptual definitions of SdK , observe that injective maps α in Δ are uniquely determined by their images. The q -tuples $(\alpha_0, \dots, \alpha_q)$ of injections above can just as well be viewed as the q -tuples (S_0, \dots, S_q) of the images of the α_i , which are increasing sequences of subsets of $[n]$ for some n . After this replacement, the two definitions coincide. Observe that the degenerate simplices of $Sd\Delta[n]^s$ are those for which $S_i = S_{i+1}$ for some i .

The conceptual definition is the one best suited for the proof of the following basic result.

THEOREM 15.2.7. *The geometric realization of a simplicial set K is homeomorphic to the geometric realization of SdK , but there is no natural simplicial map between the two that realizes the homeomorphism. There is a natural map of simplicial sets $SdK \rightarrow K$ that induces a homotopy equivalence $TSdK \rightarrow TK$.*

PROOF. We compare SdK with the simplicial set isomorphic to K given by [Proposition 13.3.6](#). That simplicial set is constructed from K and the $\Delta[n]$ rather than from K and the $Sd\Delta[n]$. The standard homeomorphisms between the $|\Delta[n]|$ and the $|Sd\Delta[n]|$ induce the claimed homeomorphism between $|K|$ and $|SdK|$.

The standard maps of simplicial sets $\xi: Sd\Delta[n]^s \rightarrow \Delta[n]^s$ given by [Definition 6.4.13](#) together specify a map $\xi: Sd\Delta[\bullet]^s \rightarrow \Delta[\bullet]^s$ of cosimplicial simplicial sets since they are natural, as observed in [Remark 6.4.18](#). Using the conceptual definition of SdK and the description of K as $T^s K$ in [Proposition 13.3.6](#), we see that ξ induces a natural map of simplicial sets $\xi: SdK \rightarrow K$. The geometric realization of the maps $\xi: Sd\Delta[n]^s \rightarrow \Delta[n]^s$ are homotopy equivalences by [Proposition 6.4.12](#). It follows that the induced map $T\xi: TSdK \rightarrow TK$ is a homotopy equivalence. The proof of the implication is just a bit beyond the scope of this

Will we?

book; an old reference is [48, A.4(ii)]. The idea is that application of the maps ξ gives a map that by inspection of the filtrations of $TSdK$ and TK can be proven to be a local weak homotopy equivalence, so that [Theorem 3.3.1](#) gives that $T\xi$ is a weak homotopy equivalence. Since it is a map between CW complexes, it is a homotopy equivalence. \square

15.3. Combinatorial properties of subdivision

We use the combinatorial definition to derive some basic combinatorial properties of subdivision.

DEFINITION 15.3.1. A q -simplex $(x; S_0, \dots, S_q)$ of SdK is in *minimal form* if $x \in K_n$ is nondegenerate and $S_q = [n]$.

PROPOSITION 15.3.2. *Every simplex of SdK is equivalent to a unique simplex in minimal form. When so written, a simplex is degenerate if and only if $S_i = S_{i+1}$ for some i .*

PROOF. Conceptually, this is analogous to the description of the points of the geometric realization TK in nondegenerate form. We think of q -simplices of $Sd\Delta[n]^s$ as “interior” if $S_q = [n]$, and we then use the same canonical form for morphisms of Δ as composites of σ ’s and δ ’s that we used to prove the analogue for realization. If we start with an element $(y; T_1, \dots, T_q)$ with $y \in K_p$, $T_i \subset [p]$ and $|T_q| = m+1$, we have a unique injection $\delta: [m] \rightarrow [p]$ such that $\delta([m]) = T_q$. There are unique subsets R_i of $[m]$ such that $\delta(R_i) = T_i$, and $(y; T_1, \dots, T_q)$ is equivalent to $(\delta^*y; R_1, \dots, R_q)$, where $R_q = [m]$. Now there is a surjection $\sigma: [m] \rightarrow [n]$ and a nondegenerate simplex x of K_n such that $\sigma^*x = \delta^*y$. Then $(\delta^*y; R_1, \dots, R_q)$ is equivalent to $(x; S_1, \dots, S_q)$ where $S_i = \sigma^*(R_i)$. By the surjectivity of σ , $S_q = [n]$. It is left as an exercise to check that this process reaches the unique minimal element equivalent to the element we started with.

Now suppose that $z = (x; S_1, \dots, S_q)$ is in minimal form. If $S_i = S_{i+1}$, then z is certainly degenerate. We must show that if z is degenerate, then some $S_i = S_{i+1}$. The assumption means that z is equivalent to $z' = (y; T_0, \dots, T_q)$, where $T_j = T_{j+1}$ for some j . However, unlike z , z' might not be in minimal form. Just as above, let $y \in K_p$, so that the T_i are contained in $[p]$. Let $|T_q| = m+1$ and choose an injection $\delta: [m] \rightarrow [p]$ such that $\delta([m]) = T_q$. Define $R_i = \delta^{-1}(T_i)$ for all i and note that $R_q = [m]$. Then z' is equivalent to $z'' = (\delta^*y; R_0, \dots, R_q)$. Now let $\delta^*y = \sigma^*w$ where σ is a surjection and $w \in K_n$ is nondegenerate. Then z'' is equivalent to $(w; \sigma(R_0), \dots, \sigma(R_q))$. This simplex is in minimal form since $\sigma([m]) = [n]$, so it must be z . Thus $x = w$ and $S_i = \sigma(R_i) = \sigma_i \delta^{-1}(T_i)$. Since $T_j = T_{j+1}$, $S_j = S_{j+1}$. This proves the result. \square

COROLLARY 15.3.3. *Let $x \in K_n$ be nondegenerate. Then there is a nondegenerate q -simplex y_q in SdK with q th vertex $(x; [n])$ if and only if $q \leq n$.*

PROOF. If $q \leq n$, set $y_q = (x; [n-q], [n-q+1], \dots, [n])$. Then y_q is in minimal form and nondegenerate, and its q th vertex is $(x; [n])$. Conversely, if we have a nondegenerate y_q with q th vertex $(x; [n])$, then, in minimal form, we must have $y_q = (x; S_0, \dots, S_{q-1}, S_q)$ with S_i strictly contained in S_{i+1} for $0 \leq i < n$ and $S_q = [n]$. Clearly that implies $q \leq n$. \square

PROOF OF [THEOREM 15.2.4](#). The nondegenerate q simplices of the barycentric subdivision SdK are the strictly increasing chains $\sigma_0 \subset \dots \subset \sigma_q$ of faces of a

simplex. If σ_q has cardinality $n + 1$, its elements specify a nondegenerate n -simplex x of K^s . Viewing x as a map $\Delta[n] \rightarrow K^s$ via [Proposition 13.3.3](#), the inverse images of the σ_i specify an increasing sequence of subsets S_i of $[n]$ with $S_q = [n]$. The rest is an exercise about the description of elements of $Sd^s(K^s)$ in minimal form. \square

15.4. Subdivision and Properties A, B, and C of simplicial sets

Here is how subdivision relates to Properties A, B, and C.

THEOREM 15.4.1. *Subdivision of simplicial sets has the following properties.*

- (i) *K has Property A if and only if SdK has Property A.*
- (ii) *K has Property A if and only if SdK has Property B.*
- (iii) *K has Property B if and only if SdK has Property C.*

The following two corollaries are immediate.

COROLLARY 15.4.2. *If K does not have Property A, then $Sd^n K$ does not have any of the three properties for any $n \geq 1$. If K does have property A, then $Sd^n K$ has all three properties for all $n \geq 2$.*

COROLLARY 15.4.3. *K has Property A if and only if $Sd^2 K$ has Property C, and then $Sd^2 K$ also has Property B.*

Now the following very satisfactory theorem follows directly from [Theorem 15.1.8](#).

THEOREM 15.4.4. *A simplicial set K satisfies Property A if and only if $Sd^2 K$ is a simplicial complex.*

We might also ask whether our properties shed light on the question of whether or not a simplicial complex is the nerve of a category. We have the following complement to the previous result. It is an analogue of the fact that the subdivision of a simplicial complex is a poset. We will prove it later, in [§16.6](#).

THEOREM 15.4.5. *A simplicial set satisfies Property A if and only if SdK is the nerve of a category, namely the category ΠSdK .*

The last clause is a consequence of the following general observation.

PROPOSITION 15.4.6. *If a simplicial set K is isomorphic to $N\mathcal{C}$ for some category \mathcal{C} , then the category \mathcal{C} is isomorphic to ΠK .*

PROOF. If $K \cong N\mathcal{C}$, then $\Pi K \cong \Pi N\mathcal{C} \cong \mathcal{C}$. \square

Since ordered simplicial complexes satisfy Property A when regarded as simplicial sets, [Theorem 15.4.5](#) has the following result as a special case. It says that the subdivision of a simplicial complex is the nerve of a category. Remarkably, this appears to be a new result.

THEOREM 15.4.7. *If K is an ordered simplicial complex, then $Sd(K^s)$ is isomorphic to $\Pi Sd(K^s)$.*

15.5. The proof of [Theorem 15.4.1](#)

Since Property *B* implies Property *A*, by [Theorem 15.1.5](#), the following two implications prove both (i) and (ii) of [Theorem 15.4.1](#).

PROOF THAT IF $\text{Sd}K$ HAS PROPERTY *A*, THEN SO DOES K . Suppose for a contradiction that we have a nondegenerate $x \in K_n$ with a degenerate face $d_i x = s_j z$, where $z \in K_{n-2}$. Recall that $d_j s_j = \text{id}$. In $\text{Sd}K$, we have the 2-simplex¹

$$(x; \delta_i \delta_j [n-2], \delta_i [n-1], [n]).$$

It is written in minimal form and is nondegenerate. Its last face is the 1-simplex

$$(x; \delta_i \delta_j [n-2], \delta_i [n-1]) \sim (d_i x; \delta_j [n-2], [n-1]) = (s_j z; \delta_j [n-2], [n-1]) \sim (z; [n-2], [n-2])$$

since $\sigma_j \delta_j = \text{id}$ and $\sigma_j: [n-2] \rightarrow [n-2]$ is a surjection. This simplex is in minimal form and degenerate, which contradicts the assumption that $\text{Sd}K$ has Property *A*. \square

PROOF THAT IF K HAS PROPERTY *A*, THEN $\text{Sd}K$ HAS PROPERTY *B*. Consider a nondegenerate q -simplex $y = (x; S_0, \dots, S_q)$ written in minimal form. For some n , $x \in K_n$ is nondegenerate and the S_i give a strictly increasing sequence of subsets of $[n]$, with $S_q = [n]$. The vertices of y are the $(x; S_i)$. Suppose that $(x; S_i) \sim (x; S_j)$ where $0 \leq i < j \leq q$. Let $\mu: [m_i] \rightarrow [n]$ and $\nu: [m_j] \rightarrow [n]$ be injective maps in Δ with images S_i and S_j , respectively. Then

$$(\mu^* x; [m_i]) \sim (x; S_i) \sim (x; S_j) \sim (\nu^* x; [m_j]).$$

Since K has Property *A*, the faces $\mu^* x$ and $\nu^* x$ are nondegenerate. Therefore, by the uniqueness of the minimal form, we must have $m_i = m_j$. Since $S_i \subset S_j$, this implies that $S_i = S_j$. The contradiction proves that $\text{Sd}K$ has Property *B*. \square

Finally, the following two implications prove (iii) of [Theorem 15.4.1](#).

PROOF THAT IF K HAS PROPERTY *B*, THEN $\text{Sd}K$ HAS PROPERTY *C*. Let

$$z_1 = (x; S_0, \dots, S_q) \quad \text{and} \quad z_2 = (y; T_0, \dots, T_q)$$

be nondegenerate q -simplices of $\text{Sd}K$ that have the same set of $q+1$ distinct vertices. We must show that $z_1 = z_2$. We may assume without loss of generality that z_1 and z_2 are in minimal form, with $x \in K_m$, $S_q = [m]$, $y \in K_n$, and $T_q = [n]$ for some m and n . Let $m_i + 1 = |S_i|$ and $n_i + 1 = |T_i|$ and note that $m_0 < \dots < m_q = m$ and $n_0 < \dots < n_q = n$. Using [Proposition 15.1.4](#), we see that the vertices of z_1 and z_2 , in minimal form, are the $(S_i^* x; [m_i])$ and the $(T_i^* y; [n_i])$, respectively.

We are assuming that these two sets of vertices are the same. We claim that they are the same as *ordered* sets. That is, $(S_i^* x; [m_i]) = (T_i^* y; [n_i])$ for $0 \leq i \leq q$. Suppose not. Then $(S_i^* x; [m_i]) = (T_j^* y; [n_j])$ for some $i \neq j$, and we may assume $i < j$. Since these are both in minimal form, $m_i = n_j$. By the pigeonhole principle, we must have some $j' < j$ and $i' > i$ such that $m_{i'} = n_{j'}$. But then we have $m_i < m_{i'} = n_{j'} < n_j = m_i$, which is a contradiction.

Thus $m_i = n_i$ and $S_i^* x = T_i^* y$ for all i . Since $S_q = [m] = [n] = T_q$, we have $x = S_q^* x = T_q^* y = y$. Then, by [Proposition 15.1.4](#) again, S_i and T_i must be defined by the same injection and so must be equal. Therefore $z_1 = z_2$ and $\text{Sd}K$ has Property *C*. \square

¹Here and below, we write $\alpha[n]$ to denote the set $\alpha([n])$.

PROOF THAT IF $\text{Sd}K$ HAS PROPERTY C , THEN K HAS PROPERTY B . Suppose that K does not have Property B . Let $x \in K_n$, $n > 0$, be nondegenerate with repeated vertices α^*x and β^*x for injections $\alpha, \beta: [0] \rightarrow [n]$. By the uniqueness of the minimal form, $(x; \alpha[0], [n])$ and $(x; \beta[0], [n])$ are distinct 1-simplices of $\text{Sd}K$. However, these 1-simplices have the same vertex sets since one of the vertices of each is $(x; [n])$ and the other is

$$(x; \alpha[0]) \sim (\alpha^*x; [0]) = (\beta^*x; [0]) \sim (x; \beta[0]).$$

Thus $\text{Sd}K$ does not have Property C . □

15.6. Isomorphisms of subdivisions

We saw that if X and Y are posets, then the subdivisions of $X * Y$ and $(X * Y)^-$ are isomorphic, hence so are their associated simplicial sets. However, the posets $X * Y$ and $(X * Y)^-$ are not isomorphic, and neither are their associated simplicial sets. We round out the picture with the following rather strange looking result, which puts this example in a more general context.

PROPOSITION 15.6.1. *If K and L are simplicial sets such that $\text{Sd}K$ and $\text{Sd}L$ are isomorphic, then although K and L need not be isomorphic, for each n there is a bijection of sets $f_n: K_n \cong L_n$ such that the faces of a simplex $x \in K_n$ correspond bijectively under f_{n-1} to the faces of $f(x)$.*

PROOF. Let $g: \text{Sd}K \rightarrow \text{Sd}L$ be an isomorphism of simplicial sets. For a nondegenerate n -simplex $x \in K_n$, we have the vertex $(x; [n])$ in $\text{Sd}K$. Write $g(x; [n]) = (y; [m])$ in minimal form. Using Corollary 15.3.3, we see that $m = n$, and we define $f_n(x) = y$. If $x \in K_n$ is degenerate, there is a unique surjection σ and nondegenerate simplex z such that $x = \sigma^*z$. Define $f_n(x) = \sigma^*f(z)$. If we apply the same construction starting from $g^{-1}: \text{Sd}L \rightarrow \text{Sd}K$, we obtain an inverse function f_n^{-1} to f_n . The $(n+1)$ faces $d_i x$ of a nondegenerate $x \in K_n$ correspond to the $(n+1)$ 1-simplices $y_i = (x; \delta_i[n-1], [n])$ of $\text{Sd}K$, counted with multiplicities in case of repetitions. The vertices of y_i are $d_0 y_i = (x; \delta_i[n-1]) \sim (d_i x; [n-1])$ and $d_1 y_i = (x; [n])$ in minimal form. Since the nondegenerate faces of L admit a similar description, we see that these faces correspond under f_{n-1} to the faces of $f_n(x)$. The following example shows that K and L need not be isomorphic. □

Not worth a section?

Missing!

15.7. Regularity and non-singularity of simplicial sets and CW complexes

Property A of a simplicial set is an analogue of the classical notion of regularity for a CW complex X . The results of this section are peripheral to our main interests here, but they help contrast simplicial sets with CW complexes.

DEFINITION 15.7.1. A CW complex X is *regular* if its closed cells are homeomorphisms onto their images so that each cell map $(D^n, S^{n-1}) \rightarrow (e^n, \partial e^n)$ is a homeomorphism.

Incomplete section, see Piccinini?
Or expository REU paper project

DEFINITION 15.7.2. A nondegenerate simplex $x \in K_n$ is *regular* if the following diagram is a pushout, where $[x]$ denotes the subsimplicial set generated by x .

$$\begin{array}{ccc} \Delta[n-1] & \xrightarrow{d_n x} & [d_n x] \\ \delta^n \downarrow & & \downarrow \\ \Delta[n] & \xrightarrow{x} & [x]; \end{array}$$

K is *regular* if all of its nondegenerate simplices are regular.

THEOREM 15.7.3. *For any K , SdK is regular.*

THEOREM 15.7.4. *If K is a regular simplicial set, then $|K|$ is a regular CW complex.*

THEOREM 15.7.5. *If X is a regular CW complex, then X is triangulable; that is X is homeomorphic to $|K^s|$ for some simplicial complex K .*

PROOF. This can be proven using induction. For a regular simplicial complex X , let the triangulation on the vertices X_0 be the vertices themselves. Supposing that the triangulation up to the skeleton X_{n-1} has been defined, we construct X_n in the following way. The boundary of the closure of each n -cell \bar{e}^n lies in X_{n-1} , and thus we turn our attention to the triangulation of \bar{e}^n . Let b denote the barycenter of \bar{e}^n and notice that since X is regular, b does not lie in X_{n-1} . Let the cone space $|C(\partial\bar{e}^n, b)|$ be the triangulation of \bar{e}^n with $|C(\partial\bar{e}^n, b)| \cong \bar{e}^n$. Inducting over all n -cells in X_n gives the desired triangulation. \square

COROLLARY 15.7.6. *If K is a regular simplicial set, then $|K|$ is triangulable.*

PROOF. This follows directly from Theorem 15.7.4 and Theorem 15.7.5 above. \square

Now we turn to a special case of regular simplicial sets: nonsingular simplicial sets.

DEFINITION 15.7.7. A simplicial set is nonsingular if for each nondegenerate simplex x , the map \bar{x} is degreewise injective, i.e. the induced map on the geometric realization $|\Delta[n]| \rightarrow |[x]|$ is an embedding including the boundary.

PROPOSITION 15.7.8. *If a simplicial set is nonsingular, then it is regular.*

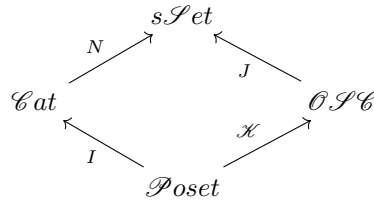
PROOF. Suppose that K is a nonsingular simplicial set and $x \in K_n$ is a simplex. Each face is then a nondegenerate $(n-1)$ -simplex and so can be identified as $\Delta[n]$ attaching along the n th face, giving regularity as desired. \square

REMARK 15.7.9. If K is nonsingular, then the induced map on the geometric realization $|\Delta[n]|$ together with its boundary is an embedding to $|[x]|$. Nonsingular simplicial set is a structure in between ordered simplicial complex and simplicial set. The main difference between nonsingular simplicial sets and those induced by ordered simplicial complexes is that nondegenerate simplicies in nonsingular simplicial sets are not uniquely determined by the vertices. Hence we have the following inclusion relations between the three types of simplicial sets:

$$\mathcal{OSCS} \subset \text{nonsingular } sSet \subset sSet$$

15.8. Characterizing nerves of posets

It is natural then to ask which simplicial sets are the nerves of posets, and equivalent condition to which are in the image of $N \circ I$. As we will find, the intersection of the images of functors N and J is exactly the subcategory of the nerves of posets.



To be the nerve of a poset, the left side of the commutative diagram above shows that the following three conditions are necessary:

- (1) its vertices must have a partial ordering
- (2) it must be the nerve of a category
- (3) there is at most one nondegenerate n -simplex for every set of distinct $(n + 1)$ vertices

The right side of the commutative diagram indicates the following necessary conditions:

- (1) its vertices must have a partial ordering and
- (2) it must be the maximal $\mathcal{O}\mathcal{S}\mathcal{C}$ realization based on the ordering of the vertices.

These observations give us some intuition about what properties the nerves of a poset need to satisfy. First of all we characterize a property that gives partial ordering on the vertices.

DEFINITION 15.8.1. Let X be a simplicial set, then

- (1) Define $v < w$ for vertices v and w in X if there is a *nondegenerate* n -simplex $x \in X$ such that the i^{th} vertex is v and the j^{th} vertex is w , with $i < j$.
- (2) Define $v \leq w$ for vertices v and w in X if there is a 1-simplex x such that $d_0x = w$ and $d_1x = v$.

X satisfies *Property D* if the ordering based on (1) is a strict partial ordering and that of (2) is a partial ordering.

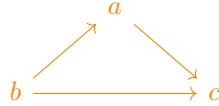
REMARK 15.8.2. By definition the vertices of a simplicial set X that satisfies *Property D* form a poset.

LEMMA 15.8.3. *Property D implies Property B.*

PROOF. Given a simplicial set X , if there exists a nondegenerate n -simplex that has a repeated vertex v , then by *Property D* we have $v < v$, which is a contradiction to strict partial ordering. \square

One may ask how near *Property D* is to being a characterization of the nerves of posets and whether there are simplicial sets satisfying *Property D* that are not nerves of posets. The following example shows that there exists simplicial sets K that satisfy *Property D* but are not the nerve of a poset.

EXAMPLE 15.8.4. Consider the following simplicial set with nondegenerate 0-simplices and 1-simplices as shown below, and with no nondegenerate 2-simplices.



This satisfies Property D but is not the nerve of a category.

Now we consider the following theorem which characterizes those simplicial sets which are the nerves of categories.

THEOREM 15.8.5. *Let K be a simplicial set. Then the following conditions are equivalent:*

- K is isomorphic to the nerve of a category
- Every inner horn of K has a unique filler

REMARK 15.8.6. The second condition in Theorem 15.8.5 can be explicitly characterized as the following: for any $n \geq 2$ and any n -tuple of simplices $\{x_i \in K_1 \mid 1 \leq i \leq n\}$, such that $d_0 x_{i-1} = d_1 x_i$ for $2 \leq i \leq n$, there is a unique $y \in K_n$ such that $v_i^* y = x_i$, where $v_i : [1] \rightarrow [n]$ with the image as $\{i-1, i\}$.

In the following we denote the condition in Theorem 15.8.5 as the *unique inner horn filling condition*. One might ask whether Property D and the unique inner horn filling condition imply Property C , another condition necessary to be a nerve of a poset. The following counterexample proves this to be false.

EXAMPLE 15.8.7. Consider the simplicial set K with $K_0 = \{0, 1\}$ and two nondegenerate 1-simplices with 0, 1 as vertices and the ordering as $0 < 1$. It has the geometric realization in the form:



K satisfies Property D and the unique inner horn filling condition, but it does not satisfy Property C because there are two 1-simplices with the same vertices. The key to this contradiction lies in the fact that the unique inner horn filling is defined only for simplices with dimension greater than or equal to 2.

THEOREM 15.8.8. *K is the nerve of a poset if and only if it satisfies the following conditions:*

- (1) Property C
- (2) Property D
- (3) Unique inner horn filling condition

PROOF. \Rightarrow If K is the nerve of a poset P , then the partial ordering on the vertices implies Property D , and the uniqueness of morphisms between objects implies Property C . Being the nerve of a category implies the unique inner horn filling condition.

\Leftarrow For a given simplicial set K which satisfies Property C , D and the unique inner horn filling condition, we construct a poset P using K_0 along with their partial orderings in K . By Property C and the unique inner horn filling condition, there exists a bijection between

$$\{\text{nondegenerate } n\text{-simplex in } K\} \leftrightarrow \{\text{chains of length } n \text{ in } P\}$$

By the construction of nerves, there's also a bijection between

$$\{\text{chains of length } n \text{ in } P\} \leftrightarrow \{\text{non-degenerate } n\text{-simplices in } N(P)\}$$

Hence there is a bijective simplicial map from K to $N(P)$ by mapping the corresponding vertices, which shows K is the nerve of a poset. \square

It sometimes proves quite difficult to check whether a given simplicial complex satisfies property D . Thus, we develop another equivalent, easier-to-check condition that characterizes nerves of posets.

LEMMA 15.8.9. *Property B, Property C and the unique inner horn filling condition imply Property D.*

PROOF. Suppose X is a simplicial set satisfying the three conditions, we first check the partial ordering property. Given any vertex v , there always exists the degenerate 1-simplex σ_0^*v such that it has both of its vertices as v , hence reflexivity is satisfied;

Suppose by way of contradiction that there exist two 1-simplices with v, w as its vertices with $v \leq w$ and $w \leq v$. Then there exist two nondegenerate simplices with the same vertex set, contrary to Property C . This then gives antisymmetry.

Transitivity is given by the unique inner horn filling condition.

We claim vertices are also strictly partial ordered. Property B ensures that anti-reflexivity (since any nondegenerate 1-simplex must have distinct vertices). Property C guarantees antisymmetry, and transitivity is given by the unique inner horn filling property. \square

REMARK 15.8.10. Picking any two of the previous three conditions cannot give Property D . This shows that Property D is indeed a very delicate condition.

COROLLARY 15.8.11. *K is the nerve of a poset if and only if it satisfies the following conditions:*

- (1) Property B
- (2) Property C
- (3) Unique inner horn filling condition

PROOF. \Rightarrow This is given by [Theorem 15.8.5](#) and [Lemma 15.8.3](#).

\Leftarrow This is given by [Theorem 15.8.5](#) and [Lemma 15.8.9](#). \square

REMARK 15.8.12. These three conditions are easier to check than the one using Property D . Also, careful readers should also realized that we can characterize the nerves of posets as the simplicial sets coming from some maximal realization of posets based on the right side of the commutative diagram at the beginning of the section. However, this characterization provides little insight on the nerves of poset and is comparatively difficult to check.

With the characterization in [Corollary 15.8.11](#), we can derive the following interesting result.

COROLLARY 15.8.13. *The intersection of the image of the nerve functor N and the inclusion functor J is exactly the nerves of posets.*

$$\begin{array}{ccc} & s\mathcal{S}et & \\ N \nearrow & & \nwarrow J \\ \mathcal{C}at & & \mathcal{O}S\mathcal{C} \end{array}$$

PROOF. By [Theorem 15.8.5](#), being the nerve of a category is equivalent to the unique inner horn filling condition. Since $(\mathcal{O}\mathcal{S}\mathcal{C})^s$ satisfies Property B and C, [Corollary 15.8.11](#) gives the desired result. \square

PROPOSITION 15.8.14. *Any simplicial set in $J(\mathcal{O}\mathcal{S}\mathcal{C}) = \mathcal{O}\mathcal{S}\mathcal{C}^s$ satisfies Properties A, B, C and D.*

PROOF. Recall that [Theorem 15.1.5](#) shows that Property B implies Property A. The result then simply follows since $J(\mathcal{O}\mathcal{S}\mathcal{C})$ satisfies Property B and C and any element in $(\mathcal{O}\mathcal{S}\mathcal{C})^s$ satisfies Property D given that the vertices of ordered simplicial complexes always satisfy the two partial ordering conditions. \square

PROPOSITION 15.8.15. *The nerves of posets satisfy Property A, B, C, and D.*

PROOF. This follows from [Proposition 15.8.14](#) and the fact that $N \circ I = J \circ \mathcal{H}$. \square

REMARK 15.8.16. Note that the main difference between the nerves of posets and $\mathcal{O}\mathcal{S}\mathcal{C}^s$ is that a nerve of a poset must satisfy the unique inner horn filling condition. This will be shown to be the only difference.

PROPOSITION 15.8.17. *A simplicial set X belongs to $\mathcal{O}\mathcal{S}\mathcal{C}^s$ if and only if X satisfies Property C and D.*

Before proving this proposition, we introduce first a short lemma.

LEMMA 15.8.18. *Let K be a simplicial set and x be an arbitrary n -simplex with vertex set V . For any subset $S \subset V$ such that $|S| = m+1$, there exists an m -simplex with S as the set of vertices.*

PROOF. Regard the simplicial set K as a contravariant functor $K : \Delta \rightarrow \mathcal{S}et$. Define $g : [m] \rightarrow [n]$ which maps each element i to the position of the i^{th} element of S in V . Consider the map $f_i : [0] \rightarrow [n]$ defined by $0 \mapsto g(i)$. Denote the $g(i)^{th}$ vertex of x as v_i , then this means:

$$K(f_i)(x) = v_i$$

Define morphism $\delta_i : [0] \rightarrow [m]$ by $0 \mapsto i$, then $f_i = g \circ \delta_i$. Therefore any function in the collection $\{f_i\}_{i \in [m]}$ factors through g , as shown in the following diagram for f_i :

$$\begin{array}{ccc} [0] & \xrightarrow{f_i} & [n] \\ & \searrow \delta_i & \nearrow g \\ & & [m] \end{array}$$

This corresponds to the following commutative diagram in the category of simplicial sets:

$$\begin{array}{ccc} K_0 & \xleftarrow{X(f_i)} & K_n \\ & \swarrow X(\delta_i) & \nwarrow X(g) \\ & & K_m \end{array}$$

Therefore for given $x \in K_n$, $X(g)(x)$ gives a m -simplex with vertex set S . \square

Equipped with the previous lemma, we return ready to prove [Proposition 15.8.17](#).

PROOF. \Rightarrow follows from [Proposition 15.8.14](#). \Leftarrow By [Lemma 15.8.3](#), we know X also satisfies Property B . For any simplicial set X that satisfies Property B and C , all of its nondegenerate simplices are uniquely determined by its vertices. For any nondegenerate n -simplex x with vertex set V , for any subset $S \subset V$, there exists a simplex with vertex set S based on [Lemma 15.8.18](#). Hence the collection of nondegenerate simplices in X form an abstract simplicial complex. Since Property D ensures a partial ordering on the vertices, we know the collection of nondegenerate simplices in X form an \mathcal{OSC} , which means X is in $(\mathcal{OSC})^s$. \square

After identifying when a simplicial set is the nerve of a poset, we venture to explore when the barycentric subdivision of a simplicial set is the nerve of a poset. To this end, the following theorem proves as a classification.

THEOREM 15.8.19. *A simplicial set X has its subdivision $Sd(X)$ as the nerve of a poset if and only if X satisfies Property B .*

PROOF. Recall the following facts:

- (1) $Sd(X)$ satisfies Property C if and only if X satisfies Property B .
- (2) $Sd(X)$ satisfies Property B if and only if X satisfies Property A .
- (3) $Sd(X)$ is the nerve of a category if and only if X satisfies Property A .

Since Property B implies Property A , we can deduce that X satisfies Property B if and only if $Sd(X)$ satisfies Property B , C , and the unique inner horn filling condition. \square

COROLLARY 15.8.20. *A simplicial set X has its double subdivision $Sd^2(X)$ as the nerve of a poset if and only if X satisfies Property A .*

PROOF. A simplicial set X satisfies Property A if and only if $Sd(X)$ satisfies Property B . Applying [Theorem 15.8.19](#), we know $Sd(X)$ satisfies Property B if and only if $Sd^2(X)$ is the nerve of a poset. \square

Subdivision and Properties A , B , and C in \mathcal{Cat}

16.1. Properties A , B , and C of categories

Categories are implicitly small unless they are obviously large, like the categories of spaces, simplicial sets, or (small) categories. We may interpret properties A , B , and C of the simplicial set $N\mathcal{C}$ as properties of a category \mathcal{C} .

DEFINITION 16.1.1. A (small) category \mathcal{C} has Property A , B , or C if the simplicial set $N\mathcal{C}$ has Property A , B , or C .

THEOREM 16.1.2. *Let \mathcal{C} be a category. The following statements hold.*

- (i) $N\mathcal{C}$ has property A if and only if \mathcal{C} has the no retracts property, meaning that retractions are identity maps: if we have morphisms $i: a \rightarrow b$ and $r: b \rightarrow a$ in \mathcal{C} such that $r \circ i = \text{id}_a$, then $a = b$ and $i = r = \text{id}$.
- (ii) $N\mathcal{C}$ has property B if and only if \mathcal{C} has the no loops property, meaning that loops are identity maps: if we have morphisms $f: a \rightarrow b$ and $g: b \rightarrow a$ in \mathcal{C} , then $a = b$ and $f = g = \text{id}$.
- (iii) $N\mathcal{C}$ has property C if and only if \mathcal{C} has the one way property: there is at most one sequence of nonidentity morphisms $f_i: C_i \rightarrow C_{i+1}$ connecting any finite ordered set of objects $\{C_i\}$.
- (iv) \mathcal{C} is a poset if and only if $N\mathcal{C}$ has properties B and C .

PROOF. A nondegenerate n -simplex of $N\mathcal{C}$ is a composable sequence

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n$$

of nonidentity morphisms. It has a degenerate face if and only if one of the composites $f_{i+1} \circ f_i$ is an identity map. This proves (i).

For (ii), Property B says that the objects c_i of a nondegenerate n -simplex are distinct, which clearly implies the no loops property. Conversely, if $c_i = c_j$ for some $i < j$, the composite of f 's from c_i to c_j is a loop $c_i \rightarrow c_i$. We can write the composite as $g \circ f_i$. The no loops property implies that f_i and g are identity maps, so that our simplex is degenerate. This proves (ii)

Statement (iii) is immediate from the definition of Property C .

For (iv), it is immediate from (ii) and (iii) that \mathcal{C} satisfies Properties A and B if and only if there is at most one morphism between any pair of objects of \mathcal{C} . That is precisely the characterization of posets regarded as categories. \square

16.2. The definition of the subdivision of a category

Let \mathcal{C} be a category. We start with a combinatorial definition of $\text{Sd}\mathcal{C} = \text{Sd}^c\mathcal{C}$. It may be hard to assimilate, but it is the right definition to start with. We will eventually see that Sd is actually nothing but the composite functor $\text{IISd}^s N$, but that will require a fair amount of proof.

The intuition is that $\text{Sd}\mathcal{C}$ has objects all chains of non-identity maps, and the set of morphisms from (f_i, n) to (g_i, m) is the set of all ways that (f_i, n) can be mapped injectively to a subchain of (g_i, m) . These ways are to be distinct after accounting for degeneracies, which motivates the definition of the equivalence relation in the following definition.

To define $\text{Sd}\mathcal{C}$ rigorously, we first define a category $\mathcal{D}\mathcal{C}$. The objects of $\mathcal{D}\mathcal{C}$ are the chains of composable arrows in \mathcal{C} . To abbreviate notation, we sometimes write $A = (f_i, m)$ as shorthand for a chain

$$a_0 \xrightarrow{f_1} a_1 \longrightarrow \cdots \longrightarrow a_{m-1} \xrightarrow{f_m} a_m.$$

We may think of such a chain as an m -simplex of $N\mathcal{C}$.

The morphisms from (f_i, m) to (g_i, n) are the equivalence classes of maps $\mu: [m] \rightarrow [n]$ in Δ such that $\mu^*(g_i, n) = (f_i, m)$ in $N\mathcal{C}$. The equivalence relation is generated under composition by the following basic equivalences. For a surjective map $\sigma: [q] \rightarrow [p]$ in Δ and for right inverses $\alpha, \beta: [p] \rightarrow [q]$ to σ , so that $\sigma\alpha$ and $\sigma\beta$ are both the identity morphism of $[p]$, set $\alpha \sim \beta: (h_i, p) \rightarrow \sigma^*(h_i, p)$ for any object (h_i, p) . This makes sense since $\alpha^*\sigma^* = \text{id} = \beta^*\sigma^*$. Composition in $\mathcal{D}\mathcal{C}$ is induced by composition in Δ . Then define $\text{Sd}\mathcal{C}$ to be the full subcategory of $\mathcal{D}\mathcal{C}$ whose objects are the non-degenerate chains. A functor $F: \mathcal{C} \rightarrow \mathcal{C}'$ induces a functor $NF: N\mathcal{C} \rightarrow N\mathcal{C}'$, which in turn induces a functor $\text{Sd}F: \text{Sd}\mathcal{C} \rightarrow \text{Sd}\mathcal{C}'$. With these definitions, Sd is a functor $\mathcal{CAT} \rightarrow \mathcal{CAT}$.

There is another way to view the definition, which may be easier to grasp. The letter \mathcal{D} above is meant to indicate that we allow degenerate chains as objects of the category $\mathcal{D}\mathcal{C}$. We can instead start with the smaller category $\mathcal{C}\mathcal{C}$ whose objects (f_i, m) are the nondegenerate chains, so that no f_i is an identity map. The maps from (f_i, m) to (g_i, n) in $\mathcal{C}\mathcal{C}$ are the maps $\nu: [m] \rightarrow [n]$ in Δ such that $\nu^*(g_i, n) = (f_i, m)$. Notice that such a map ν must be an injection since (f_i, m) is nondegenerate. Now define $\text{Sd}\mathcal{C}$ to be the quotient category of $\mathcal{C}\mathcal{C}$ with the same objects but with equivalence classes of morphisms under the equivalence relation generated by setting $\nu\alpha \sim \nu\beta$ when

$$\nu^*(g_i, n) = (f_i, m) = \sigma^*(h_i, q)$$

for some surjection $\sigma: [m] \rightarrow [q]$ with right inverses $\alpha, \beta: [q] \rightarrow [m]$.

The difference is whether we choose to first restrict to nondegenerate simplices and then impose an equivalence relation or to first impose an equivalence relation and then restrict to nondegenerate simplices. We get the same category either way.

REMARK 16.2.1. It is useful to observe that if \mathcal{C} has Property A, then no $\nu^*(g_i, n)$ can be degenerate and therefore $\mathcal{C}\mathcal{C} = \text{Sd}\mathcal{C}$.

16.3. Subdivision and Properties A, B, and C of categories

Despite the analogy with simplicial sets, the conclusions here read rather differently.

THEOREM 16.3.1. *Subdivision of categories has the following properties.*

- (i) For any category \mathcal{C} , $\text{Sd}\mathcal{C}$ has Property B.
- (ii) A category \mathcal{C} has Property B if and only if $\text{Sd}\mathcal{C}$ is a poset.

Again, the following remarkable theorem follows directly. Since this result applies to any category \mathcal{C} , it does not make sense to ask for a converse.

THEOREM 16.3.2. *For any category \mathcal{C} , $\text{Sd}^2\mathcal{C}$ is a poset.*

EXAMPLE 16.3.3. The nerve of a poset need not be the subdivision of a simplicial set. The poset \mathbb{Z} of integers with its usual ordering provides a counterexample. If $N\mathbb{Z} \cong \text{Sd}K$ and 0 corresponds to $(x; [n])$ in minimal form, then for any nondegenerate q -simplex $(y; S_0, \dots, S_q)$ in minimal form that has q th vertex $(x; [n])$, we have $S_q = [n]$ and thus $q \leq n$. However, in $N\mathcal{C}$ there are nondegenerate simplices $(-r, -r+1, \dots, 0)$ for arbitrarily large r .

Since we have subdivision functors on both categories and simplicial sets, it is natural to ask how these functors relate to the adjoint pair (Π, N) . The following result is either a theorem or a definition, depending on whether one chooses to start with the combinatorial or the conceptual definition of the subdivision of a category. We shall take it as a theorem and prove it in §16.5.

THEOREM 16.3.4. *For any category \mathcal{C} , $\text{Sd}^c\mathcal{C}$ is isomorphic to $\Pi\text{Sd}^sN\mathcal{C}$.*

This implies another characterization of categories having Property A.

COROLLARY 16.3.5. *A category \mathcal{C} has Property A if and only if $\text{Sd}^sN\mathcal{C}$ is isomorphic to $N\text{Sd}^c\mathcal{C}$.*

PROOF. If \mathcal{C} has Property A, then [Theorem 15.4.5](#) implies that $\text{Sd}^sN\mathcal{C}$ is isomorphic to $\Pi\text{Sd}^sN\mathcal{C}$. By [Theorem 16.3.4](#), the latter is isomorphic to $N\text{Sd}^c\mathcal{C}$. For the converse, $N\text{Sd}^c\mathcal{C}$ has Property B and therefore Property A by [Theorems 16.3.1\(i\)](#) and [??\(ii\)](#). If $\text{Sd}^sN\mathcal{C} \cong N\text{Sd}^c\mathcal{C}$, then \mathcal{C} has Property A by [??\(i\)](#). \square

REMARK 16.3.6. For posets X , we obtain naturally isomorphic simplicial sets if we regard X as a category and take its nerve or if we regard X as the simplicial complex $\mathcal{K}X$ and take the associated simplicial set $(\mathcal{K}X)^s$. It is natural to ask whether $N\text{Sd}^cX$ is isomorphic to $\text{Sd}^s(\mathcal{K}X)^s$. Since X satisfies Property A (and B and C), the previous result gives that

$$N\text{Sd}^cX \cong \text{Sd}^sNX \cong \text{Sd}^s(\mathcal{K}X)^s.$$

Remarkably, [Theorem 16.3.4](#) also implies that the categorical analogue of [Theorem 15.2.7](#) is a direct implication of that result.

THEOREM 16.3.7. *There is a njk on passage to classifying spaces.*

PROOF. We apply the natural map of simplicial sets of [Theorem 15.2.7](#) and the fact that the composite ΠN is isomorphic to the identity functor to obtain the required map as the composite

$$\text{Sd}^c\mathcal{C} \cong \Pi\text{Sd}^sN\mathcal{C} \longrightarrow \Pi N\mathcal{C} \cong \mathcal{C}.$$

\square

gap? Does Π preserve the property in question?

16.4. The proof of [Theorem 16.3.1](#)

We have three implications to prove.

PROOF THAT $\text{Sd}\mathcal{C}$ HAS PROPERTY B. We first prove that $\mathcal{C}\mathcal{C}$ has Property B. Let $A = (f_i, m)$ and $B = (g_i, n)$ be objects of $\mathcal{C}\mathcal{C}$ and suppose that we have morphisms $\mu: A \rightarrow B$ and $\nu: B \rightarrow A$. Since these morphisms are given by injections in Δ , $m = n$. Since the only injection $[n] \rightarrow [n]$ is the identity map, we have $A = B$ and $\mu = \text{id} = \nu$. Thus $\mathcal{C}\mathcal{C}$ has the no loops property, which is

equivalent to Property B. This property is inherited by the quotient category $\text{Sd}\mathcal{C}$. If we have maps $\bar{\mu}: A \rightarrow B$ and $\bar{\nu}: B \rightarrow A$ in $\text{Sd}\mathcal{C}$, they must be represented by maps μ and ν in $\mathcal{C}\mathcal{C}$, but these maps are identity maps by what we have just shown, hence $\bar{\mu}$ and $\bar{\nu}$ are identity maps. \square

PROOF THAT IF \mathcal{C} HAS PROPERTY B, THEN $\text{Sd}\mathcal{C}$ IS A POSET. Since Property B implies Property A, $\mathcal{C}\mathcal{C} = \text{Sd}\mathcal{C}$ by Remark 16.2.1. We must show that $\mathcal{C}\mathcal{C}$ is a poset. Let A and B be objects of $\mathcal{C}\mathcal{C}$. We must show that there is at most one morphism between A and B . Suppose there is a morphism $\mu: A \rightarrow B$. Since we have just shown that $\mathcal{C}\mathcal{C}$ has the no loops property, there is no morphism $B \rightarrow A$ unless $A = B$ and $\mu = \text{id}$. Suppose there is another morphism $\nu: A \rightarrow B$. We must show that $\mu = \nu$. Since $A = \mu^*B = \nu^*B$, we have $a_i = b_{\mu(i)} = b_{\nu(i)}$ for all i , where the a_i and b_j are the objects appearing in the chains A and B . Since B must be nondegenerate when thought of as an element of $N\mathcal{C}$ and \mathcal{C} has the no loops property, we have $b_i \neq b_j$ for $i \neq j$. Therefore $\mu(i) = \nu(i)$ for all i and $\mu = \nu$. \square

PROOF THAT IF $\text{Sd}\mathcal{C}$ IS A POSET, THEN \mathcal{C} HAS PROPERTY B. Suppose that \mathcal{C} does not have Property B. Then there are objects A and B (possibly the same) and non-identity maps $f: A \rightarrow B$ and $g: B \rightarrow A$. Consider the objects $A \xrightarrow{f} B \xrightarrow{g} A$ and A in $\text{Sd}\mathcal{C}$. Let $\alpha, \gamma: [0] \rightarrow [2]$ be the maps with images $\{0\}$ and $\{2\}$, respectively. Then

$$\alpha^*(A \xrightarrow{f} B \xrightarrow{g} A) = A = \gamma^*(A \xrightarrow{f} B \xrightarrow{g} A).$$

Since no degeneracy operator on A is a face of $A \xrightarrow{f} B \xrightarrow{g} A$, we cannot have $\alpha \sim \gamma$; that is, they represent distinct morphisms of $\text{Sd}\mathcal{C}$. But that contradicts the assumption that $\text{Sd}\mathcal{C}$ is a poset. \square

16.5. Relations among Sd^s , Sd^c , N , and Π

We are heading towards the proof of Theorem 16.3.4. We recall that ΠK has objects the vertices $x \in K$, morphisms generated by the 1-simplices $y \in K$, and relations dictated by the 2-simplices z . For a vertex x , s_0x is the identity map of x . For a 1-simplex y , d_1y is the source of y and d_0y is the target of y . For a 2-simplex z , $d_1z = d_0z \circ d_2z$. The functor Π is left adjoint to N , and the counit of the adjunction is a natural isomorphism $\Pi N\mathcal{C} \cong \mathcal{C}$. We start work with the following understanding of the category $\Pi\text{Sd}^s K$ for simplicial sets K .

PROPOSITION 16.5.1. *Every morphism of the category $\Pi\text{Sd}^s K$ can be represented by a 1-simplex in $\text{Sd}^s K$, and the category $\Pi\text{Sd}^s K$ has Property B.*

PROOF. By definition, every morphism is a formal composite of 1-simplices, say $y_q \circ \cdots \circ y_1$. Since $y_{i+1} \circ y_i$ is defined, the target d_0y_i is equal to the source d_1y_{i+1} . We will show that such a formal composite of length q is equivalent to a formal composite of length $q - 1$. By induction, it must be equivalent to a formal composite of length 1, which is just a 1-simplex.

Write y_i in minimal form $(x_i; S_i, [n_i])$, where $x_i \in K_{n_i}$ is nondegenerate. Let $|S_i| = m_i \leq n_i$ and let $\alpha_i: [m_i] \rightarrow [n_i]$ be the injection with image S_i . Since

$$(x_q; S_q) = d_1(x_q; S_q, [n_q]) = d_0(x_{q-1}; S_{q-1}, [n_{q-1}]) = (x_{q-1}; [n_{q-1}]),$$

there must be some surjection $\sigma: [m_q] \rightarrow [n_{q-1}]$ in Δ such that $\alpha_q^* x_q = \sigma^* [x_{q-1}]$. Let $\beta: [n_{q-1}] \rightarrow [m_q]$ be a right inverse to σ . Then

$$(x_q; \alpha_q \beta [n_{q-1}], S_q) \sim (\sigma^* x_{q-1}; \beta [n_{q-1}], [m_q]) \sim (x_{q-1}; [n_{q-1}], [n_{q-1}]),$$

which is degenerate and thus an identity morphism in $\Pi\text{Sd}K$. Consider the 2-simplex $z = (x_q; \alpha_q \beta [n_{q-1}], S_q, [n_q])$. The relation $d_1 z = d_0 z \circ d_2 z$ gives that

$$(x_q; \alpha_q \beta [n_{q-1}], [n_q]) = (x_q; S_q, [n_q]) = y_q$$

as morphisms in $\Pi\text{Sd}K$. Now use that $\beta^* \sigma^* = \text{id}$ on $[n_{q-1}]$ to see that

$$y_{q-1} = (x_{q-1}; S_{q-1}, [n_{q-1}]) \sim (x_q; \alpha_q \beta S_{q-1}, \alpha_q \beta [n_{q-1}]).$$

Finally, consider the 2-simplex $w = (x_q; \alpha_q \beta S_{q-1}, \alpha_q \beta [n_{q-1}], [n_q])$. The relation $d_1 w = d_0 w \circ d_2 w$ gives that $(x_q; \alpha_q \beta S_{q-1}, [n_q]) = y_q \circ y_{q-1}$ in $\Pi\text{Sd}K$. This gives the claimed reduction from word length q to word length $q - 1$.

To prove that $\Pi\text{Sd}^s K$ has Property B , we must verify the no loop condition. Thus suppose that $f: (x; [m]) \rightarrow (y; [n])$ and $g: (y; [n]) \rightarrow (x; [m])$ are morphisms in $\Pi\text{Sd}^s K$, where $x \in K_m$ and $y \in K_n$ are nondegenerate simplexes. We have just shown that f and g can be represented by 1-simplices. It suffices to show that both are degenerate, so that they are identity morphisms in $\Pi\text{Sd}^s K$. We have

$$d_0 f = d_1 g = (y; [n]) \quad \text{and} \quad d_0 g = d_1 f = (x; [m]).$$

By the conditions on d_0 , we can write $f = (y; T, [n])$ and $g = (x; S, [m])$ in minimal form. By the conditions on d_1 , we then have $(y; T) \sim (x; [m])$ and $(x; S) \sim (y; [n])$. Choose injections $\alpha: [p] \rightarrow [m]$ and $\beta: [q] \rightarrow [n]$ with images S and T . We then have

$$(x; [m]) \sim (y; T) \sim (\beta^* y; [p]) \quad \text{and} \quad (y; [n]) \sim (x; S) \sim (\alpha^* x; [q]).$$

Write $\alpha^* x = \sigma^* u$ where $u \in K_j$ is nondegenerate and $\sigma: [q] \rightarrow [j]$ is a surjection. Then

$$(y; [n]) \sim (\alpha^* x; [q]) = (\sigma^* u; [q]) \sim (u; [j]).$$

Since these are both in minimal form, $n = j \leq q$. Similarly $m \leq p$. Since α and β are injections, $n = q$, $m = p$, and α and β are identity maps. Thus $S = [m]$ and $T = [n]$, showing that f and g are degenerate. \square

PROOF OF THEOREM 16.3.4. We shall prove that the categories $\text{Sd}^c \mathcal{C}$ and $\Pi\text{Sd}^s N \mathcal{C}$ are isomorphic by exhibiting inverse functors between these categories. Moreover, these inverse isomorphisms of categories will be natural in \mathcal{C} .

We first define $F: \text{Sd} \mathcal{C} \rightarrow \Pi\text{Sd}^s N \mathcal{C}$ and its inverse G on objects. The objects $A = (f_i, m)$ of $\text{Sd} \mathcal{C}$ are the nondegenerate simplices of $N \mathcal{C}$. The objects of $\Pi\text{Sd}^s N \mathcal{C}$ are the vertices of $\text{Sd}^s N \mathcal{C}$. We may write these in minimal form as $(A; [m])$, where A is an object of $\text{Sd}^c \mathcal{C}$. We define F and G on objects by

$$F(A) = (A; [m]) \quad \text{and} \quad G(A; [m]) = A.$$

Visibly, $FG = \text{Id}$ and $GF = \text{Id}$ on objects.

We next define F on morphisms and we first define it on the morphisms of \mathcal{C} , which has the same objects as $\text{Sd} \mathcal{C}$. For objects $A = (f_i, m)$ and $B = (g_i, n)$, a morphism $\nu: A \rightarrow B$ is an injection $\nu: [m] \rightarrow [n]$ such that $\nu^* B = A$. We let $F(\nu)$ be the morphism of $\Pi\text{Sd}^s N \mathcal{C}$ represented by the 1-simplex $\bar{\nu} = (B; \nu[m], [n])$ of $\text{Sd}^s N \mathcal{C}$. It is straightforward and left to the reader to check that F is indeed a functor, respecting composition and identities.

To see that F induces a functor $Sd^c\mathcal{C} \rightarrow \Pi Sd^s N\mathcal{C}$, we must show that F respects the equivalence relation used to define morphisms in $Sd^c\mathcal{C}$ from morphisms in \mathcal{C} . Thus suppose that we have an injection $\nu: [m] \rightarrow [n]$ and a surjection $\sigma: [m] \rightarrow [q]$ such that $\nu^*B = A = \sigma^*C$ for some object C . Let $\alpha, \beta: [q] \rightarrow [m]$ be right inverses to σ . Then $\nu\alpha \sim \nu\beta$ and we must show that $\overline{\nu\alpha} = \overline{\nu\beta}$ in $\Pi Sd^s N\mathcal{C}$. Observe first that

$(B; \nu\alpha[q], \nu[q]) \sim (\sigma^*A; \alpha[q], [m]) \sim (A; [q], [q]) \sim (\sigma^*A; \beta[q], [m]) \sim (B; \nu\beta[q], \nu[q])$ are degenerate 1-simplices of $SdN\mathcal{C}$. Therefore they are identity morphisms of $\Pi SdN\mathcal{C}$. We now use the definition of Π to see that

$$\overline{\nu\alpha} = (B; \nu\alpha[q], [n]) = (B; \nu\beta[q], [n]) = \overline{\nu\beta}$$

$\Pi Sd^s N\mathcal{C}$. In fact, both are equivalent to $(B; \nu[m], [n])$, as we see by considering the relations of the form $d_1z = d_0zd_2z$ induced by the 2-simplices

$$(B; \nu\alpha[q], \nu[m], [n]) \quad \text{and} \quad (B; \nu\beta[q], \nu[m], [n])$$

of $N Sd^s\mathcal{C}$. Therefore F induces a well-defined functor $Sd^c\mathcal{C} \rightarrow \Pi Sd^s N\mathcal{C}$.

We next define $G: \Pi Sd^s N\mathcal{C} \rightarrow Sd^c\mathcal{C}$ on morphisms. We claim that every morphism $(A; [m]) \rightarrow (B; [n])$ in $\Pi Sd^s N\mathcal{C}$ is of the form $\bar{\nu}$, and we define $G(\bar{\nu}) = \nu$. Visibly this will ensure that $FG = \text{Id}$ and $GF = \text{Id}$ on morphisms. By [Proposition 16.5.1](#), a morphism $(A; [m]) \rightarrow (B; [n])$ in $\Pi Sd^s N\mathcal{C}$ can be represented by some 1-simplex $(D; S, [r])$ in $Sd^s N\mathcal{C}$. Inspection of source and target shows that we must have

$$d_1(D; S, [r]) = (D; S) \sim (A; [m]) \quad \text{and} \quad d_0(D; S, [r]) = (D; [r]) \sim (B; [n]).$$

By the uniqueness in minimal form $r = n$ and $D = B$. Then $(B; S) \sim (A; [m])$. Let S be the image of an injection $\nu: [p] \rightarrow [n]$, and note that ν is uniquely determined by S . Then $(B; S) \sim (\nu^*B; [p])$. By the uniqueness in minimal form, $[p] = [m]$ and $\nu^*B = A$. Thus our morphism is given in minimal form by the 1-simplex $\bar{\nu} = (B; \nu[m], [n])$, where $\nu^*B = A$. We have effectively used the defining relations for $\Pi Sd^s N\mathcal{C}$ in the reduction to 1-simplices of [Proposition 16.5.1](#), and G is well-defined.

We have not checked that G is actually a functor, but fortunately we don't have to. It is a familiar observation that a homomorphism of groups that is a bijection of sets is an isomorphism of groups. In our situation, the same argument works to prove that G preserves identity morphisms and respects composition. Indeed

$$G(\text{id}_{(A; [m])}) = GF(\text{id}_A) = \text{id}_A$$

and, for composable morphisms $\bar{\mu}$ and $\bar{\nu}$ of $\Pi Sd^s N\mathcal{C}$,

$$G(\bar{\nu} \circ \bar{\mu}) = G(F(\nu) \circ F(\mu)) = GF(\nu \circ \mu) = \nu \circ \mu$$

and

$$G(\bar{\nu}) \circ G(\bar{\mu}) = GF(\nu) \circ GF(\mu) = \nu \circ \mu. \quad \square$$

16.6. Horn-filling conditions and nerves of categories

There are special kinds of simplicial sets that appear ubiquitously and are central to the applications of simplicial sets to other areas of mathematics. They are closely related to our focus on the relationship between simplicial sets and categories, and understanding them leads to several equivalent characterizations of those simplicial sets which are the nerves of categories.

Define Λ_n^k to be the subsimplicial set of $\Delta[n]^s$ generated by the faces $d_i \iota_n$ for all $i \neq k$. The name horn comes from the picture that one sees after passage to geometric realization. The realization of $\Delta[n]^s$ is $\Delta[n]^t$, and the realization of Λ_n^k is the “horn” that one sees after removing one of the faces of the boundary $\partial\Delta[n]^t$. If one has a map f from the realization $T\Lambda_n^k$ to a space X , then one can extend the map to $T\Delta[n]^s = \Delta[n]^t$. In fact, the topological n -simplex retracts onto any of its horns, as one sees by pushing in along the missing face. Composing f with such a retraction extends f over the simplex. This leads to the following definition and example.¹

DEFINITION 16.6.1. A simplicial set K is a *Kan complex* if every map of simplicial sets $\Lambda_n^k \rightarrow K$ extends to a map $\Delta[n]^s \rightarrow K$. There is a concrete combinatorial way to rephrase the condition. For every set of n -simplices $x_i \in K_{n-1}$, $0 \leq i \leq n$ and $i \neq k$ that satisfy the necessary compatibility condition $d_i x_j = d_{j-1} x_i$ for $i < j$ with neither $i = k$ nor $j = k$, there must exist an n -simplex $x \in K_n$ such that $d_i x = x_i$ for $i \neq k$.

The equivalence of the two formulations is immediate from [Proposition 13.3.3](#).

PROPOSITION 16.6.2. *For every space X , the simplicial set SX is a Kan complex.*

One might ask whether the extensions in [Definition 16.6.1](#) are unique. If they are, we say that K has the *unique horn filling property*. Looking at the definition of the faces of the nerve of a category, ([\(13.4.5\)](#)), we see that not all horns are created equal. We say that Λ_n^k is an *inner horn* if $0 < k < n$; the outer horns are those with $k = 0$ or $k = n$.

Looking at $N\mathcal{C}$ or at ΠK , one sees that the inner horns play a special role. If we have faces $d_0 z$ and $d_2 z$, their composite is $d_1 z$. In a category, if we are given morphisms f_0 and f_2 such that the source of f_2 is the target of f_0 , they define a map $\Lambda_2^1 \rightarrow N\mathcal{C}$, and the composable pair (f_0, f_2) gives a 2-simplex that extends the horn. This doesn’t work if we are given f_0 and f_1 or f_1 and f_2 , since we cannot compose those. We can use inverses to fill these outer horns when \mathcal{C} is a groupoid. This leads to the following result whose meaning should I hope be clear. We leave some details of proof to the reader. For $1 \leq i \leq n$, let $\nu_i: [1] \rightarrow [n]$ denote the injection with image $\{i-1, i\}$.

THEOREM 16.6.3. *Let K be a simplicial set. The following conditions are equivalent.*

- (i) K is isomorphic to the nerve of a category.
- (ii) Every inner horn of K has a unique filler.
- (iii) For any $n \geq 2$ and any n -tuple of simplices $x_i \in K_1$, $1 \leq i \leq n$, such that $d_0 x_{i-1} = d_1 x_i$ for $2 \leq i \leq n$, there is a unique $y \in K_n$ such that $\nu_i^* y = x_i$.

K is isomorphic to the nerve of a groupoid if and only if every horn of K , inner or outer, has a unique filler.

SKETCH PROOF. First suppose that $K \cong N\mathcal{C}$. We deduce (ii) and (iii). It helps to recall the formulas for the faces and degeneracies of $N\mathcal{C}$ as given in ([\(13.4.5\)](#)).

¹These are so basic that they appear on pages 2 and 3 of my book [\[45\]](#).

If we have an inner horn $\Lambda_n^k \rightarrow K$ given by compatible $(n-1)$ -simplices x_i for $i \neq k$, then we can reconstruct from these simplices a unique string (f_1, \dots, f_n) of composable arrows, and they give a filler for the given inner horn. One way of seeing this is to look at the ordered string of $n-1$ 1-simplices obtained from x_0 and x_n by applying all iterated face operations. Applied to x_0 , we obtain 1-simplices in order that we denote by f_i , $2 \leq i \leq n$. Applied to x_n , we obtain 1-simplices that we also denote by f_i , but now for $1 \leq i \leq n-1$. The duplicate f_i for $2 \leq i \leq n-1$ are equal by the assumed compatibility condition, and the required y is the n -simplex (f_1, \dots, f_n) . If we have simplices $x_i \in K_1$ as in (iii), they are a string of composable morphisms (f_1, \dots, f_n) , and that string is the required simplex y .

If \mathcal{C} is a groupoid, we can use inverses to modify the proof of (ii) so that it applies to outer as well as inner horns.

Conversely, assume (ii) or (iii). We claim that either suffices to prove that the unit $\eta: K \rightarrow N\Pi K$ of the (N, Π) -adjunction is an isomorphism. The meaning is that the formal words of length n in the 1-simplices that appear in the definition of ΠK are all realized uniquely by simplices in K_n . We show that η is an isomorphism on n -simplices for all n by induction on n . The induction starts with $n=0$ and $n=1$, where there is nothing to prove. Assume that η is an isomorphism on $(n-1)$ -simplices. Let y be an n -simplex of $N\Pi K$. Its faces give inner horns Λ_n^k in K , and they also give the data of (iii). With either hypothesis, a filler gives an n -simplex x of K such that y and $\eta(x)$ have the same faces. This means $\eta(x)$ is the same composite of 1-simplices as y , so that $\eta(x) = y$. If also $\eta(x') = y$, then x and x' have the same faces and so are equal by the uniqueness assumed in (ii) or (iii).

If we have fillers for all horns, then $K \cong N\Pi K$ and the fillers for the outer horns defined on Λ_2^0 and Λ_2^2 give left and right inverses for all morphisms. Just as for groups, the left and right inverses must be equal, and $N\Pi K$ must be a groupoid. \square

We use this characterization to prove [Theorem 15.4.5](#).

PROOF OF THEOREM 15.4.5. Suppose that K has Property A. We show that SdK satisfies condition (iii) of [Theorem 16.6.3](#). Thus let $(x_i; S_i, [q_i])$, $1 \leq i \leq n$, be 1-simplices of SdK in minimal form such that

$$d_0(x_{i-1}; S_{i-1}, [q_{i-1}]) = d_1(x_i; S_i, [q_i])$$

for $2 \leq i \leq n$. Choose an injection $\alpha_i: [p_i] \rightarrow [q_i]$ with image S_i for $0 \leq i \leq n$. Note that $p_1 = q_0$, where $q_0 = |S_0|$. The compatibility condition is equivalent to

$$(x_{i-1}, [q_{i-1}]) \sim (x_i; S_i) \sim (\alpha_i^* x_i; [p_i])$$

for $2 \leq i \leq n$. Since K has Property A, the faces $\alpha_i^* x_i$ are nondegenerate. By the uniqueness in minimal form, $q_{i-1} = p_i$ and $x_{i-1} = \alpha_i^* x_i$ for $2 \leq i \leq n$. Letting $x_0 = \alpha_1^* x_1$, this still holds for $i=1$. The composite $\alpha_n \cdots \alpha_1: [p_1] \rightarrow [q_n]$ is defined. Let

$$y = (x_n; \alpha_n \cdots \alpha_1 [p_1], \alpha_n \cdots \alpha_2 [p_2], \dots, \alpha_n [p_n], [q_n]).$$

Then $\nu_n y = (x_n; S_n, [q_n])$ and, for $1 \leq i < n$,

$$\nu_i^* y = (x_n; \alpha_n \cdots \alpha_i [p_i], \alpha_n \cdots \alpha_i [p_{i+1}]) \sim (x_i; S_i, [q_i])$$

For the uniqueness, suppose that we have another extension $z = (w; T_0, \dots, T_n)$ in minimal form such that $\nu_i z = (x_i; S_i, [q_i])$ for $1 \leq i \leq n$. The n th vertex $(w; T_n)$ of z must be $(x_n; [q_n])$, so that $(w; T_n) \sim (x_n; [q_n])$. Since K satisfies Property

A and w is nondegenerate, it follows from the uniqueness in minimal form that $w = x_n$ and $T_n = [q_n]$. Similarly, for $0 \leq i < n$, the i th vertex of z must be the i th vertex of y , hence

$$(x_n; T_i) \sim (x_n; \alpha_n \cdots \alpha_{i+1}[p_{i+1}]).$$

Therefore T_i must be $\alpha_n \cdots \alpha_{i+1}[p_{i+1}]$ and $z = y$.

We shall prove a strengthened form of the converse statement in [Proposition 16.7.3](#) below. \square

REMARK 16.6.4 (Categorical remark). The functor Sd is a left adjoint. Its right adjoint is denoted Ex . Iterating it leads to an endofunctor Ex^∞ on $s\mathcal{S}et$ that assigns a Kan complex $Ex^\infty K$ to a simplicial set K . The composite ST is another such functor. They fit into a more sophisticated context of Quillen model category theory. One recent reference is [\[49, 17.5\]](#).

16.7. Quasicategories, subdivision, and posets

Looking at the definition of Kan complexes and the characterization of nerves of categories, one sees that they have a natural common generalization.

DEFINITION 16.7.1. A simplicial set is a *quasicategory* if and only if every inner horn has a filler, not necessarily unique.

The idea is that compositions are defined, but they need not be unique. This is a very fashionable notion, and in much current literature the rather grandiose terms “ ∞ -category” or “ $(\infty, 1)$ -category” are used for quasicategories. To go with this, the term “ ∞ -groupoid” is then often used for Kan complexes. There is even some motivation for the terminology. In view of their importance, it seems reasonable to ask how these concepts behave with respect to subdivision and our Properties A , B , and C .

PROPOSITION 16.7.2. *If SdK is a Kan complex, then K is discrete, meaning that it has no nondegenerate simplices other than vertices.*

PROOF. Suppose that K has a nondegenerate n -simplex, where $n > 0$. Let v be a vertex of x and let $\alpha: [0] \rightarrow [n]$ be an injection such that $\alpha^*x = v$. Define an outer horn $\Lambda_2^2 \rightarrow SdK$ by sending the vertices $0, 1, 2$ to the vertices $(x; [n])$, $(v; [0])$, $(x; [n])$ of SdK and sending the 1-simplices $(1, 2)$ and $(0, 2)$ to $(x; \alpha[0], [n])$ and $(x; [n], [n])$. Since $v \in K_0$, there is clearly no 1-simplex $(y; S, [m])$ with vertices $(x; [n])$ and $(v; [0])$, so SdK cannot be a Kan complex. \square

PROPOSITION 16.7.3. *If SdK is a quasicategory, then K satisfies Property A .*

PROOF. Assume that K does not satisfy Property A . We construct an inner horn $f: \Lambda_3^2 \rightarrow SdK$ that cannot be extended to a map $\Delta[3] \rightarrow K$, thus showing that SdK cannot be a quasicategory. Since Property A fails for K , we can choose a nondegenerate simplex $x \in K_n$, an injection $\alpha: [m] \rightarrow [n]$, and a surjection $\sigma: [m] \rightarrow [p]$, $m > p$, such that $\alpha^*x = \sigma^*y$ in K_m for some nondegenerate simplex $y \in K_p$. Choose a right inverse $\beta: [p] \rightarrow [m]$ to σ . The three 2-faces of $\Lambda_3^2 \subset \Delta[3]$ are $d_0\iota_3$, $d_1\iota_3$, $d_3\iota_3$, where ι_3 is the identity simplex that generates $\Delta[3]$. We specify f on these three 2-simplices by sending them to

$$(x; \alpha\beta[k], \alpha[m], [n]), \quad (x; \alpha[m], \alpha[m], [n]), \quad (y; [p], [p], [p])$$

respectively. It is straightforward to check that they satisfy the required consistency on 1-faces of the horn. However, f cannot be extended to the last 2-face $d_2\iota_3$. Any possible image would have a minimal form $(x; S, T, [n])$. For consistency with the prescribed faces, we would have

$$(x; S, [n]) \sim (x; \alpha[m], [n]) \quad \text{and} \quad (x; T, [n]) \sim (x; \alpha\beta[p], [n]).$$

By the uniqueness of the minimal form, $S = \alpha[m]$ and $T = \alpha\beta[p]$. Thus, since $p < m$, T is a proper subset of S . Since $S \subset T$ by definition, $S = T$. This contradicts the choice of β as a non-identity injection. \square

REMARK 16.7.4. There is a curious analogue for quasicategories of the result that a simplicial set is a simplicial complex if and only if it satisfies Properties B and C . If K is the nerve of a poset, then it satisfies Properties B and C by Theorem 16.3.1, and of course it is a category and thus a quasicategory. It is reasonable to ask whether a quasicategory K that satisfies Properties B and C is a poset. By Theorem 15.1.8, K is the simplicial set associated to a simplicial complex, and we now write K for the latter. The set of vertices of K is a poset, and its order restricts to a total order on each simplex, so that we can write simplices in the form $\{x_0 < \cdots < x_n\}$ for vertices x_i . Then K is isomorphic to the nerve of the poset K_0 if and only if every finite totally ordered set $\{x_0 < \cdots < x_n\}$ is a simplex.

The example of $\partial\Delta[1]^s$ shows that for two vertices $x_0 < x_1$, $\{x_0 < x_1\}$ need not be a simplex of K . However, suppose that all such sets $\{x_0 < x_1\}$ are 1-simplices. Then K is a poset. To see this assume by induction that all totally ordered subsets of K_0 with at most n elements are simplices. Suppose for a contradiction that $\{x_0 < \cdots < x_n\}$ is totally ordered but not a simplex. Since all faces of this missing simplex are simplices, it is easy to construct an inner horn $f: \Lambda_n^k \rightarrow K$, in fact one for each $0 < k < n$, from all but one of the faces. A filler is an n -simplex of K , hence a totally ordered set $\{y_0, \dots, y_n\}$; it must be totally ordered since otherwise it would have degenerate faces, which it clearly does not have; that its vertices must be the x_i follows from the fact that the map $\Delta[n] \rightarrow K$ determined by $\{y_0, \dots, y_n\}$ extends f , and f maps onto the vertices.

We also remark that Properties B and C clearly fail to imply that K is a quasicategory. The inner horn Λ_2^1 is a simplicial complex, and its identity map does not extend to a simplex $\Delta[2] \rightarrow \Lambda_2^1$.

Part 3

Appendices

Cores of Alexandroff spaces

This appendix is taken from an REU paper written by Xi (Cathy) Chen in 2015. Her paper is based on work of Kukiela[42]. We have made only relatively minor editorial changes. All spaces are A -spaces throughout.

We first introduce some classes of A -spaces, including finite-chains spaces, locally finite spaces, finite-paths spaces, and bounded-paths spaces. Next, we present Kukiela's generalizations. If an infinite A -space is sufficiently well-behaved, then we get a core by recursively removing sets of beat points until no more beat points are left, just as for finite spaces. We have the following results. Every bounded-paths space or countable finite-paths space has a core, and if X is a minimal finite-paths space, then the connected component of $\text{id}(X)$ in the space $C(X, X)$ of self maps of X is a singleton. Moreover, if X and Y are fp -spaces that both have cores, then X is homotopy equivalent to Y if and only if their respective cores are homeomorphic.

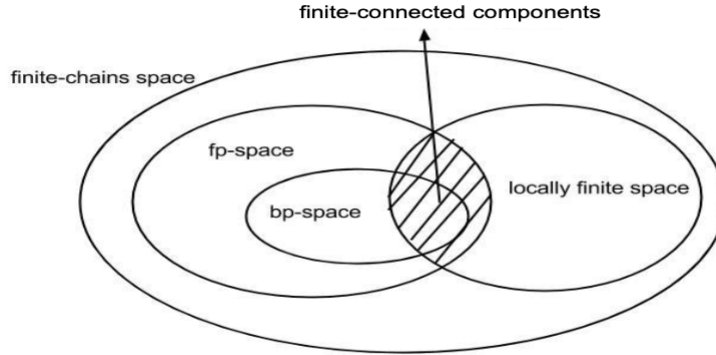
DEFINITION 17.0.1. Given a poset X , we define a *chain* of X to be a sequence $\{x_n\}$ of points of X such that $x_i < x_{i+1}$ for all i .

DEFINITION 17.0.2. Let X be an A -space. A (finite or infinite) sequence (x_n) of elements of X is an *s-path* if $x_i \neq x_j$ for $i \neq j$ and $x_{i-1} \sim x_i$ for all $i > 0$. Given a finite s-path $k = (x_0, \dots, x_m)$, we say m is the *length* of k and call k an *s-path from x_0 to x_m* .

DEFINITION 17.0.3. An A -space X is:

1. a *finite-chains space* if every chain in X is finite,
2. a *locally finite space* if for every $x \in X$, the set $\{y \in X \mid y \sim x\}$ is finite,
3. a *finite-paths space (fp-space)* if every s-path of elements of X is finite,
4. a *bounded-paths space(bp-space)* if there exists an $n \in \mathbb{N}$ such that every s-path of elements in X has less than n elements.

REMARK 17.0.4. Bp-spaces form a strict subclass of fp-spaces and both fp-spaces and locally finite spaces are strict subclasses of finite-chains spaces. Moreover, the connected components of the spaces, which are both fp-spaces and locally finite, are finite. Finite connected components can be visualized as the intersection of the following Venn diagram.



We saw that an F -space can be reduced to its core through the removal of beat points. We shall see a similar notion, which the following reduction techniques help to define. For an upbeat point x , we write u_x for the minimal point above x . For a downbeat point x , we write d_x for the maximal point below x .

DEFINITION 17.0.5. Let X be an A -space. A retraction $r : X \rightarrow r(X)$ is called:

1. a *comparative retraction* if $r(x) \sim x$ for every $x \in X$.
2. an *up-retraction* if $r(x) \geq x$ for every $x \in X$.
3. a *down-retraction* if $r(x) \leq x$ for every $x \in X$.
4. a *retraction removing a beat point* if there exists an $x \in X$ that is an upbeat point under some $u_x \in X$ or a downbeat point over some $d_x \in X$ such that $r(x) = u_x$ or $r(x) = d_x$, and $r(y) = y$ for all $y \neq x$.

REMARK 17.0.6. Every comparative retraction can be written as a composition of an up-retraction and a down-retraction. If $r : X \rightarrow A$ is a comparative retraction, then $r = r_d \circ r_u$, where

$$r_u(x) = \begin{cases} r(x) & \text{if } r(x) \geq x \\ x & \text{if } r(x) \leq x \end{cases}$$

and

$$r_d(x) = \begin{cases} r(x) & \text{if } r(x) \leq x \\ x & \text{if } r(x) \geq x \end{cases}$$

DEFINITION 17.0.7. Let X be an A -space. Let \mathcal{C} be the class of all comparative retractions and \mathcal{I} be the class of {retractions removing a beat point} \cup {identity maps}. The space X is called a \mathcal{C} -minimal space (or an \mathcal{I} -minimal space) if there is no retraction $r : X \rightarrow r(X)$ in \mathcal{C} (or \mathcal{I}) other than id_X . The space X is called a \mathcal{C} -core (or an \mathcal{I} -core) if X is a \mathcal{C} -minimal subspace (or an \mathcal{I} -minimal subspace) that is a strong deformation retract of X .

PROPOSITION 17.0.8. A space X is \mathcal{I} -minimal if and only if it has no beat points.

PROOF. (\Rightarrow) This direction follows from the definition above. Since in the class of \mathcal{I} , there is no retraction removing a beat point other than id_X , it follows that there are no beat points in X .

(\Leftarrow) If X has no beat points, then the retractions of removing a beat point are the same as the identity maps. This means id_X is the only retraction in \mathcal{I} , which implies X is \mathcal{I} -minimal. \square

COROLLARY 17.0.9. *Suppose X is a finite-chains space. Then X is \mathcal{C} -minimal if and only if X is \mathcal{I} -minimal.*

PROOF. (\Leftarrow): Suppose X is \mathcal{I} -minimal and that $r : X \rightarrow r(X)$ is a \mathcal{C} -retraction. Factor r as $r_d \circ r_u$, which gives that $r_d \leq id_X$ and $r_u \geq id_X$. Since X is a finite-chains space, X contains no strictly decreasing infinite sequence and we can therefore use induction. Take $y \in X$ and suppose $r_d(x) = x$ for all $x < y$. We will show that if $r_d(y) < y$, then y is a downbeat point over $r_d(y)$, contradicting the \mathcal{I} -minimality of X . Hence, we must have $r_d(y) = y$ and by the induction argument, $r_d = id_X$. So, suppose $r_d(y) < y$. For any $x < y$, $x = r_d(x) \leq r_d(y) < y$ by induction and monotonicity. This means y is a downbeat point over $r_d(y)$, a contradiction. By previous remarks, it follows $r_d = id_X$. A similar argument shows that if $r_d \geq id_X$, then $r_d = id_X$. Using the same arguments for r_u gives that $r_u = id_X$. Therefore X is \mathcal{C} -minimal.

(\Rightarrow): A retraction removing a beat point is also a comparative retraction. So if X is \mathcal{C} -minimal, then there is no comparative retraction, and hence no \mathcal{I} -retraction, other than id_X . Therefore X is \mathcal{I} -minimal. \square

DEFINITION 17.0.10. [42, Defn. 5.9] Let γ be an ordinal and X be an A -space. Let $\{r_\alpha | X_\alpha \rightarrow X_{\alpha+1}\}_{\alpha < \gamma}$ be a family of retractions from \mathcal{C} (or \mathcal{I}) such that $X_0 = X$, $X_{\alpha+1} = r_\alpha(X_\alpha)$ for all $\alpha < \gamma$ and $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$ for limit ordinals $\alpha < \gamma$. By transfinite recursion, we define a family of retractions $\{R_\alpha | X \rightarrow X_\alpha\}_{\alpha \leq \gamma}$ such that:

1. $R_0 = id_X$,
2. $R_{\alpha+1} = r_\alpha \circ R_\alpha$,
3. for a limit ordinal α and an $x \in X$, if there exists $\beta_0 < \alpha$ such that $R_\beta(x) = R_{\beta_0}(x)$ for all $\beta_0 \leq \beta < \alpha$, then $R_\alpha(x) = R_{\beta_0}(x)$, and if not, we leave $R_\alpha(x)$ undefined.

The recursion ends when R_γ is defined or when R_α cannot be totally defined for some limit ordinal α . In the first case we say the family $\{r_\alpha\}_{\alpha < \gamma}$ is *infinitely composable* and X is \mathcal{C} -*dismantlable* (or \mathcal{I} -*dismantlable*) to X_γ (in γ steps). In the second case we say the family $\{r_\alpha\}_{\alpha < \gamma}$ is *not infinitely composable*.

DEFINITION 17.0.11. Let X be a finite-chains space. Let $u_X : X \rightarrow X$ be given by:

$$u_X(x) = \begin{cases} u_x & \text{if } x \text{ is upbeat under } u_x \\ x & \text{otherwise} \end{cases}$$

Since $u_X(x) \geq x$ for every $x \in X$ and X is a finite-chains space, it follows that for every $x \in X$ there exists an $N_x \in \mathbb{N}$ such that $(u_X)^n(x) = (u_X)^{N_x}(x)$ for every $n \geq N_x$. Let $U_X : X \rightarrow U_X(X)$ be an up-retraction given by $U_X(x) = (u_X)^{N_x}(x)$. Similarly we define the down-retraction $D_X : X \rightarrow D_X(X)$.

REMARK 17.0.12. We check that u_x and U_X are order-preserving, as well as d_X and D_X . Given $x, y \in X$ such that $x < y$, we will show $u_X(x) \leq u_X(y)$. Note that we can assume $x < y$ here because if $x = y$, then $u_X(x) = u_X(y)$.

- If neither x nor y is an upbeat point, then $u_X(x) = x < y = u_X(y)$.
- If x is an upbeat point under u_x and y is not an upbeat point, then $u_X(x) = u_x \leq y = u_X(y)$.
- If y is an upbeat point under u_y and x is not an upbeat point, then $u_X(x) = x < y < u_y = u_X(y)$.
- If both x and y are upbeat points, then $u_X(x) = u_x \leq y < u_y = u_X(y)$.

Now we check U_X is order-preserving. Note that for any pair $x \leq y$, there is some $N \gg 0$ such that $U_X(x) = u_X^N(x)$ and $U_X(y) = u_X^N(y)$. Since u_X is monotone, $U_X(x) = U_X(y)$ by induction.

Similarly, we can check d_X and D_X are order-preserving as well.

DEFINITION 17.0.13. Given an ordinal γ and a finite-chains space X , we define a sequence of retractions $\{r_\alpha|X_\alpha \rightarrow X_{\alpha+1}\}$ by transfinite recursion. Let $X_0 = X$, $X_{\alpha+1} = r_\alpha(X)$ and $X_\alpha = \bigcap_{\beta < \alpha} X_\beta$ if α is a limit ordinal. For $\alpha = 0$ or α a limit ordinal and n a finite ordinal, let

$$r_{\alpha+n} = \begin{cases} D_{X_{\alpha+n}} & \text{if } n \text{ is even} \\ U_{X_{\alpha+n}} & \text{if } n \text{ is odd} \end{cases}$$

We call this sequence of retractions $\{r_\alpha|X_\alpha \rightarrow X_{\alpha+1}\}_{\alpha < \gamma}$ *the standard sequence of X (of length γ)*.

THEOREM 17.0.14. [42, Thm. 4.18] *Let X, Y be A -spaces and $\{f_\alpha|X \rightarrow Y\}_{\alpha \leq \gamma}$, where γ is a countable ordinal, be a family of continuous maps such that:*

- (1) *if $\alpha = \beta + 1$, then $f_\alpha \sim f_\beta$,*
- (2) *if α is a limit ordinal, then for every $x \in X$, there exists $\beta_x^\alpha < \alpha$ such that $f_{\beta_x^\alpha}(x) \leq f_\alpha(x)$ for all $\beta_x^\alpha \leq \beta \leq \alpha$.*

The f_0 is homotopic to f_γ .

DEFINITION 17.0.15. An A -space X is *countably \mathcal{C} -dismantlable* (or *countably \mathcal{I} -dismantlable*) to $X' \subseteq X$ if it is \mathcal{C} -dismantlable (or \mathcal{I} -dismantlable) to X' in γ steps, where γ is a countable ordinal.

The above theorem and definition imply that when an A -space X is countably \mathcal{C} -dismantlable (or \mathcal{I} -dismantlable) to a \mathcal{C} -minimal subspace (or an \mathcal{I} -minimal subspace), we can build a strong deformation retraction from X . By Corollary 17.0.9, these two notions of minimality coincide. We call such a minimal subspace of X a *core of X* .

We now present the main theorems on cores from Kukiela's paper[42].

THEOREM 17.0.16. *Every bp -space or countable fp -space X has a core. Moreover, if X is a bp -space with path length bounded by some $n \in \mathbb{N}$, then X can be \mathcal{C} -dismantled to a core in fewer than $2n + 2$ steps.*

Recall that in the finite case, we can construct a core by removing beat points one by one until we obtain a minimal space. Since removing a beat point is a strong deformation retract, this produces a core. However, in the infinite case, we

use the standard sequence to remove many beat points at a time, and repeat. After countably many steps, X is \mathcal{C} -dismantled to a core. The following is the sketch of the proof, and details can be found in [42, Thm. 5.14].

PROOF. (Sketch) Assume X is an infinite A -space. Let Ω be the first ordinal of cardinality greater than X . Let $\{r_\alpha | X_\alpha \rightarrow X_{\alpha+1}\}_{\alpha < \Omega}$ be the standard sequence of X of length Ω .

First, we claim that if X is an fp-space, then the standard sequence is infinitely composable. If not, then for some limit ordinal α , r_α could not be totally defined and we could construct an infinite s-path in X , using a point that moves infinitely often. This would contradict that X is an fp-space. Since the standard sequence of X is infinitely composable, it will be constant beginning with some $\alpha_0 < \Omega$. If not, then X would have cardinality at least Ω , which is a contradiction. Thus we obtain an \mathcal{I} -minimal space at α_0 . If X is countable, then $\Omega = \omega_1$, the first uncountable ordinal. Therefore $\alpha_0 < \omega_1$ is countable, and we can construct a strong deformation retract to X_{α_0} by Theorem 17.0.14. Thus X_{α_0} is a core of X .

If X is a bp-space with path length bounded by some $n \in \mathbb{N}$, one can show that the standard sequence is constant after $2n + 2$ steps. For if not, then X would contain an s-path of length greater than n , which is a contradiction. \square

Recall that $C(X, X)$ denotes the space of all continuous maps $X \rightarrow X$ in the compact open topology, and that $W(C, U) = \{f | f(C) \subset U\}$ are the canonical subbasis elements of $C(X, Y)$. We have the following theorem [42, Thm. 5.16].

THEOREM 17.0.17. *If X is a \mathcal{I} -minimal fp-space, then the connected component of id_X in $C(X, X)$ is a singleton.*

PROOF. (Sketch) One first shows that for every $x \in X$, there exists a subspace $A_x \subseteq X$ such that:

- (1) A_x is finite,
- (2) if $y \in A_x$ is not maximal in X , then $|A_x \cap \max\{z \in X | z < y\}| \geq 2$,
- (3) if $y \in A_x$ is not minimal in X , then $|A_x \cap \min\{z \in X | z > y\}| \geq 2$.

A_x can be thought of as the image of a tree (but the order on the tree is not the same as the order on X). If A_x is not finite, we could construct a tree A_x , where at each node, there are at most 4 new branches. König's Lemma¹ would imply that if A_x is infinite, then X has an infinite s-path, which contradicts that X is an fp-space.

Since for all $y \in A_x \subseteq X$, $id_X(y) = y \leq y$ it follows that $id_X \in \bigcap_{y \in A_x} W(\{y\}, U_y)$, which is an open neighborhood of id_X . We can show that this $\bigcap_{y \in A_x} W(\{y\}, U_y)$ is also closed. Thus $\bigcap_{y \in A_x} W(\{y\}, U_y)$ is a clopen set containing id_X . From point set topology, the connected component of id_X is a subset of the intersection of all clopen sets $\bigcap_{y \in A_x} W(\{y\}, U_y)$ containing id_X , therefore the component of id_X is contained in $\bigcap_{x \in X} \bigcap_{y \in A_x} W(\{y\}, U_y)$.

Next, one can show that for every $x \in X$, if $f \in \bigcap_{y \in A_x} W(\{y\}, U_y)$, then $f|_{A_x} = id_{A_x}$. If not, then one may inductively construct an infinite, strictly decreasing sequence in A_x , which is a contradiction as well. Thus the connected component

¹König's Lemma: Let P be a well-founded poset, and $S(x) = \min\{y \in P | y > x\}$ be the set of immediate successors of x . If for all $x \in P$, $S(x)$ is finite, and there exists an $x \in P$ such that the set $\{y | y \geq x\}$ is infinite, then there exists an infinite ascending chain in P .

of id_X is contained in $\bigcap_{x \in X} \bigcap_{y \in A_x} W(\{y\}, U_y) = \{id_X\}$, and hence the connected component of id_X is exactly $\{id_X\}$. \square

COROLLARY 17.0.18. *Suppose X and Y are fp-spaces, and suppose that they both have cores X^C and Y^C . Then X is homotopy equivalent to Y if and only if X^C is homeomorphic to Y^C .*

Lastly, we introduce the concept of chain-complete posets. Although they do not belong to one of those classes of infinite A -spaces considered in [Definition 17.0.5](#), we still have similar results.

DEFINITION 17.0.19. A poset P is called *chain-complete* if every chain in P has both a supremum and an infimum in P .

DEFINITION 17.0.20. An *antichain* in a poset P is a subset $A \subseteq P$ such that no two elements in A are comparable.

THEOREM 17.0.21. [[42](#), Thm. 5.8] *Every chain-complete poset X with no infinite antichains has a finite core.*

REMARK 17.0.22. In [Corollary 17.0.18](#), instead of requiring X and Y to be fp-spaces, we only need X^C and Y^C to be fp-spaces. Also note that if X^C is a finite core, then it is an \mathcal{I} -minimal fp-space, so we can use [Theorem 17.0.17](#) above. In this case, it is straightforward to prove that if any two chain-complete posets X , Y without infinite antichains have finite cores X^C and Y^C respectively, then X is homotopy equivalent to Y if and only if X^C is homeomorphic to Y^C .

The enumeration of homotopy classes of F -spaces

As promised in [Section 2.5](#), we here give the results on the enumeration of homotopy types of F -spaces that appeared in the 2008 REU paper of Alex Fix and Stephen Patrias. We follow their exposition with minor edits.

18.0.1. Constructing Posets. Intuitively, we expect that as the number of points in a poset grows large, the number of neighbors of each point in the graph should grow large as well, and that cases where a point has exactly one neighbor should be very rare. We will examine this probabilistic reasoning rigorously in the final section, but for now, it seems a good heuristic that the large majority of graphs will be minimal once n grows large enough, and that non-minimal graphs will be the exception. Thus, it makes sense to try to count the number of minimal graphs by first enumerating all posets of a given size, and then checking to see whether each such generated graph is minimal.

As a reminder, by [Corollary 2.5.7](#) we are interested in enumerating the minimal spaces up to homeomorphism, and by [Corollary 2.5.4](#), homeomorphism of spaces is equivalent to graph isomorphism of the constructed Hasse diagrams.

DEFINITION 18.0.1. Since an isomorphism between graphs is equivalent to relabeling the vertices in a consistent fashion, an equivalence class of graphs under graph isomorphism is called an *unlabeled graph*.

Since any relabeling of a minimal graph produces another minimal graph (as it does not change the in or out degree of any of the vertices), we can treat an unlabeled minimal graph as the equivalence class of a minimal graph under graph isomorphism. This represents the same object as the equivalence class of a minimal space under homeomorphism, so our task is to produce exactly one representative for each unlabeled minimal graph.

Fortunately, a fast algorithm for producing exactly one representative of each unlabeled Hasse diagram has already been proposed by Brinkmann and McKay[[13](#)], and has been used to enumerate all unlabeled posets on up to 16 points. The remainder of this section will be a summary of these results.

The algorithm works by a method called the canonical construction path which, for every unlabeled poset P on n points, gives a canonical unlabeled poset Q on $n - 1$ points such that Q can be obtained from P by deleting a point from the top level. This essentially turns the set of all unlabeled posets into a tree, whereby each poset on n points has a unique parent with $n - 1$ points, turning the task of enumeration into a search on this tree.

In order for this construction to work, it is necessary to be able to reconstruct all children of a given poset, and to only construct exactly one example of each child graph (so that we do not produce two different labelings of the same graph, and consider them as different children). It is relatively straightforward to construct the

set of all possible children for a graph. However, to reject possible isomorphisms between these candidates we require a device called a canonical choice function.

DEFINITION 18.0.2. Let C be a set of candidates, each of which is a poset on n points, with vertex set $[n] = \{1, 2, \dots, n\}$. Then a function $f : C \rightarrow 2^{[n]}$ (from candidates to subsets of $[n]$) is a *canonical choice function* if

- (1) For each candidate G , the set $f(G)$ is an orbit under the automorphisms of G consisting of vertices on the highest level of G .
- (2) For any two candidates G, G' , if $\sigma : G \rightarrow G'$ is an isomorphism of graphs, then σ maps $f(G)$ onto $f(G')$.

DEFINITION 18.0.3. The *parent* of a graph G is the unlabeled graph formed by removing a point v in $f(G)$ from the graph.

DEFINITION 18.0.4. Conversely, a graph G' is a *candidate child* of a graph G if we can add a point v to G to obtain G' , and so that v is on the highest level of G' .

Since the point removed will be on the highest level, we will remove only downwards pointing edges from the graph, so we cannot create any shortcuts or cycles. Thus the parent of a Hasse diagram is again a Hasse diagram.

Also, the parent of a graph is uniquely defined, regardless of which point we remove from $f(G)$ to obtain it. Since $f(G)$ is an orbit of G , if v, w are both in $f(G)$ then there is an automorphism σ such that $\sigma(v) = w$. But then, the two parents, $G \setminus \{v\}$ and $G \setminus \{w\}$ are isomorphic by σ , so they are actually the same unlabeled graph.

DEFINITION 18.0.5. If G' is a candidate child of G , formed by adding a point v , we say that f *accepts* G' if and only if v is in $f(G')$, where f is a canonical choice function. If we have fixed some f beforehand, we say that G' is an (actual) *child* of G if f accepts G' .

This definition allows us to use the canonical choice function to distinguish between the children of a graph so as to accept only one representative from the unlabeled children of a graph.

LEMMA 18.0.6. *If H and H' are distinct children of a graph G , i.e., both are accepted by some canonical choice function f , then H and H' are not isomorphic.*

The only remaining task is to ensure that we actually construct all possible candidate children of a graph, and accept at least one from each isomorphism class. To do this, we must consider all ways in which we can add a point to G such that the new point is now on the highest level.

First, note that if G has ℓ levels, then the new point must have an edge to some point on level $\ell - 1$ or level ℓ , or else the new point would not be on the highest level of G' .

Second, the new edges we add between our new point and its neighbors cannot create any shortcuts, since G' must be a Hasse diagram. So, if x and y are both neighbors of our new point, we cannot have $x > y$ or $y > x$. Thus, the neighbors of our new point must be pairwise incomparable. In graph theory, we call such a set an antichain. Each antichain with a point on the highest or next-highest level gives a valid set of neighbors for a new point on the top level, so these antichains describe all ways of connecting a new point to a graph to get a point at the highest level.

Finally, if we pick two antichains A and A' such that there is a graph automorphism σ that sends A to A' , then the resulting graphs formed by connecting a new point to each of A and A' will be isomorphic by the same permutation σ (extended to send the new vertex to itself). Thus, it suffices to consider only one representative from each orbit of the antichains under group automorphism.

From the above considerations, we have the following algorithm:

THEOREM 18.0.7. *To construct all children of an unlabeled poset P with ℓ levels:*

- (1) *Find a representative from each orbit of antichains that contains a point on level ℓ or $\ell - 1$.*
- (2) *Connect a new point v to each antichain computed in step (1) in turn.*
- (3) *Compute the canonical choice function for each candidate constructed in step (2). A candidate is a child of P if and only if the new point v is in $f(P)$.*

To actually enumerate all unlabeled posets with at most n points, begin with the graphs consisting of no more than n points all on the first row, and then perform a depth-first search on the children of each graph that we find.

The proof of the correctness of this algorithm is due to Brinkmann and McKay [13], but for now, the assertion that it does generate exactly one example of each unlabeled poset should suffice to justify our modifications to count minimal graphs.

18.0.2. Constructing Minimal Graphs. Since we are not in fact trying to count all posets, but only a subset of them, we really only need to generate graphs which are minimal, or some of whose children will eventually be minimal. If we can determine that a given graph will never have minimal descendants, then we can prune that node from our search, and not have to waste computation on branches which will never bear fruit. We can do this most easily by considering a slightly larger collection than the set of all minimal graphs.

DEFINITION 18.0.8. We say that a graph is *non-downbeat* if there are no points with out-degree equal to 1. This is equivalent to the statement that the underlying topology has no downbeat points.

All minimal graphs are of course non-downbeat, so if we can construct all non-downbeat graphs and then check whether each one is non-upbeat as well, we will have accomplished our task of counting all minimal graphs.

The categorization of graphs as non-downbeat is useful primarily because it is a hereditary property:

LEMMA 18.0.9. *If a graph G' is non-downbeat, then its parent G is non-downbeat as well.*

PROOF. Let v be the vertex that we remove from G' to obtain G . Remember that v is on the top level, so there cannot be any edges $w \rightarrow v$, or else w would be on a higher level; thus in removing v from G' , we do not change the out-degree of any point $w \neq v$. Thus since no points in $G' \setminus \{v\}$ have out-degree equal to 1, no points in G have out-degree 1 either. Thus G is non-downbeat. \square

We can also categorize which children of a non-downbeat graph will also be non-downbeat (allowing us to not construct the other children in the first place).

LEMMA 18.0.10. *If G is non-downbeat, and G' is obtained from G by adding a point v on the highest level, then G is non-downbeat if and only if v has two or more neighbors.*

PROOF. Again, by adding a point at the top level, we do not change the out-degree of any of the points in G , so G' is non-downbeat if and only if v is not a downbeat point. Then, it is clear that v will not be a downbeat point if and only if it has two or more neighbors. \square

Finally, we can identify a special case of child which will never produce any minimal descendants, even though the child itself is non-downbeat.

LEMMA 18.0.11. *If G has exactly one point on the top level ℓ , and G' is obtained from G by adding a point to a new level $\ell + 1$, then no descendant of G' will ever be minimal.*

PROOF. We claim that all descendants of G' will have exactly one point on level ℓ , but have a highest level $\ell' > \ell$. By Proposition 2.5.11, such graphs cannot be minimal.

We proceed by structural induction on the tree of descendants of G' . As a base case, this is trivially true of G' . Now, let H be a descendant of G' with exactly one point on level ℓ and with highest level $\ell' > \ell$. Then all children of H are formed by adding a point on level ℓ' or $\ell' + 1$, so all children of H still have exactly one point on level ℓ . \square

These three Lemmas allow us to make the following changes to the above algorithm which will prune dead-ends. We call all children which are not known to be dead-ends by the above lemmas *useful children*.

THEOREM 18.0.12. *To construct all useful children of a graph G with highest level ℓ :*

- (1) *Find a representative from each orbit of antichains that contains a point on level $\ell - 1$. If G has more than one point on level ℓ , also find representatives from each orbit of antichains with a point on level ℓ .*
- (2) *Connect a new point v to each antichain computed in step 1 whenever the antichain contains at least two vertices.*
- (3) *Compute the canonical choice function for each candidate constructed in step 2. A candidate is a child of P if and only if the new point v is in $f(P)$.*
- (4) *If the canonical choice function accepts, then verify that the graph is non-upbeat as well by checking that no point has in-degree 1. If the graph is non-upbeat, then increment our count of minimal graphs encountered. Even if the graph contains upbeat points, it is still a useful child of G and could have minimal descendants, so we must recursively find its children as well.*

By the above Lemmas, the children which we ignore are all such that they are not minimal, and will never have minimal descendants, so we can ignore those branches and still find representatives of all minimal graphs.

18.0.3. Computational Results. The above algorithm was actually implemented and run to obtain the exact counts of unlabeled minimal graphs with small

numbers of points. Various optimizations described in [13] were implemented to expedite the computation of the canonical choice function, and in the construction of antichains. Canonical labeling of graphs (needed for the canonical choice function) was achieved by using the graph isomorphism library `nauty` [51]. This is the same library used by Brinkmann and McKay in their original library [13].

Points	Minimal graphs	Homotopy classes	Unlabeled posets
1	1	1	1
2	1	2	2
3	1	3	5
4	2	5	16
5	4	9	63
6	11	20	318
7	36	56	2045
8	160	216	16999
9	954	1170	183231
10	7929	9099	2567284
11	92092	101191	46749427
12	1493102	1594293	1104891746

TABLE 1. Counts of minimal graphs and homotopy classes

To ensure the correctness of these results, we used the C preprocessor to compile two different versions of the algorithm, one with our changes as described above, and one functionally identical to the original algorithm for enumerating all unlabeled posets. The unmodified algorithm successfully reproduced the counts for all unlabeled posets up to 11 points, but could not be run on higher inputs since it takes far longer to run than the modified version (This was the purpose of pruning branches in the first place). Since the code for the two versions is 99% identical, it is much more feasible for a human to check that the changes we implemented actually produce the desired result. Furthermore, at the beginning of researching this topic, one of the authors enumerated all minimal graphs up to 8 points by hand, and these counts were verified by the algorithm.

Table 18.0.3 gives the counts for the number of unlabeled minimal graphs with up to 12 points. Since the number of homotopy classes with n points is the number of unlabeled graphs with at most n points, their number is simply the sum of the counts of minimal graphs with at most n points. We also provide the number of unlabeled graphs (equal to the number of F -spaces up to homeomorphism) from [13] for reference.

18.0.4. Asymptotic Enumeration. Kleitman and Rothschild's paper [38] has been used to describe the asymptotic behavior of posets as consisting of graphs with exactly three levels with 'roughly' $n/4, n/2$ and $n/4$ points on each of the three levels. However, the exact statement of the result will prove much more useful in describing the asymptotic behavior of minimal graphs.

Their paper describes a set of posets on a vertex set V of n points which formalizes this notion of three-leveled posets. The collection, $Q(V)$ consists of the posets P such that

- (1) The vertices of P are the disjoint union, $S_1 \amalg S_2 \amalg S_3$ where points in S_i only have edges going to points in S_{i-1} or S_{i-2}
- (2) The size of the partition is such that
 - (a) $||S_i| - n/4| < (n-1)^{\frac{1}{2}} \log(n-1)$
 - (b) $||S_2| - n/2| < \log(n-1)$
- (3) For every $u \in S_1 \cup S_3$, $||N(u) \cap S_2| - n/4| < (n-1)^{7/8}$, where $N(u)$ is the set of neighbors of u .
- (4) For every $u \in S_2$, $||N(u) \cap S_i| - n/8| < (n-1)^{7/8}$ for $i = 1$ or $i = 3$

By a collection of logarithmic bounds given by their lemma, they find that the number of posets on n points, P_n , is asymptotically equivalent to the number of posets in $X(V)$, and that this is asymptotically equivalent to the number of posets in $Q(V)$. Specifically, if Q_n counts the number of posets in $Q(V)$ with n points, then $P_n = (1 + O(1/n))Q_n$.

In our enumeration we have been concerned with non-isomorphic, minimal, leveled digraphs (equivalently unlabeled, minimal Hasse diagrams) as these define the homotopy classes of F -spaces, yet Kleitman and Rothschild's result is using *labeled* Hasse Diagrams, which gives the number of all F -spaces. To make use of their result, we need to know the relation between the number of unlabeled graphs and labeled graphs. For this we make use of an exceedingly general result from Prömel [56], which states that in any large enough collection of labeled objects, the fraction of objects with non-trivial automorphism group goes to 0, and thus asymptotically, the ratio of labeled objects to unlabeled objects approaches $\frac{1}{n!}$.

LEMMA 18.0.13. *Let \mathcal{C} be a class of finite labeled structures (i.e., a finite labeled set with a single binary relation) which is closed under substructures and isomorphisms. Let $C(n)$ count the number of such structures on sets with n points, and let $C^u(n)$ count the number of unlabeled structures on n points. If (C) satisfies the growth condition*

$$C(n) = cn^2 + dn + o(n)$$

where $c > 0$ and d is arbitrary, then

$$C^u(n) \sim \frac{C(n)}{n!}$$

Applied to the case of classes of posets, this lemma states that as long as our collection of labeled posets is large enough, we can directly derive asymptotic bounds on the growth of the collection of unlabeled posets. Since this condition is satisfied both by the set of all posets and by the set of posets in $Q(V)$ we have the immediate corollary:

COROLLARY 18.0.14. *The number of unlabeled posets in $Q(V)$, Q_n^u , is asymptotically equal to the number of unlabeled posets, P_n^u .*

PROOF. We know, by Kleitman and Rothschild's result [38], that the number, P_n , of all labeled posets, is such that $\log(P_n) = \frac{n^2}{4} + \frac{3n}{2} + O(\log(n))$. So by the above lemma, $P_n^u \sim \frac{1}{n!}P_n$. Similarly, since $P_n \sim Q_n$, we have that $Q(V)$ satisfies the growth condition as well, so $Q_n^u \sim \frac{1}{n!}Q_n$. Also, $P_n \sim Q_n$ implies that $\frac{P_n}{n!} \sim \frac{Q_n}{n!}$ so

$$Q_n^u \sim \frac{Q_n}{n!} \sim \frac{P_n}{n!} \sim P_n^u$$

□

An asymptotic enumeration of the homotopy classes of finite F -spaces follows directly from this.

COROLLARY 18.0.15. *The number of homotopy classes of finite T_0 topological spaces is asymptotically equivalent to the number of all T_0 spaces up to homeomorphism.*

PROOF. By definition, graphs in $Q(V)$ have the property that

- (1) For every $u \in S_1 \cup S_3$, the number of neighbors of u in S_2 is greater than $n/4 - (n - 1)^{7/8}$
- (2) For every $u \in S_2$, the numbers of neighbors of u in S_1 and S_3 are each greater than $n/8 - (n - 1)^{7/8}$

Thus, for n large enough, every point in the top row has out-degree at least 2, every point in the middle row has out-degree and in-degree at least 2, and every point in the bottom row has in-degree at least 2. Thus, every graph in $Q(V)$ with enough points is a minimal graph.

But then, every unlabeled graph in $Q(V)$ is an unlabeled minimal graph, so if we let M_n^u be the number of unlabeled minimal graphs with n points, then we have that $Q_n^u \leq M_n^u \leq P_n^u$. Since $Q_n^u \sim P_n^u$, by the squeeze theorem we have $M_n^u \sim P_n^u$.

But remembering that M_n^u also counts the number of homotopy classes of finite spaces up to homotopy, and P_n^u counts the number of finite spaces up to homeomorphism, we have that almost every unlabeled graph on n vertices is minimal and therefore the number of homotopy classes of F -spaces is asymptotically equal to the number of all F -spaces. \square

Before considering the implications of this, it is worth noting that the above method is not the only way to prove this result; instead, one only needs that almost every poset has three levels and that these levels monotonically increase in size as the poset grows.

LEMMA 18.0.16. *Almost all graphs with 3 levels are minimal.*

PROOF. Let $P = L_1 \amalg L_2 \amalg L_3$ be an unlabeled digraph with three levels, and let $|L_3| = j$, $|L_2| = k$, and $|L_1| = l$.

To determine the probability of this graph being minimal, consider that P is formed by taking the complete tri-partite graph on its levels, randomly deleting some number of edges, and possibly adding edges from L_3 to L_1 .

So $x \in L_3$ has between 1 and k edges leading to L_2 , by definition of the levels of a graph; for $y \in L_2$ y has between 0 and j edges to it from L_3 . A point in L_3 might have edges going to L_1 in addition to its edges going to L_2 , so for any $x \in L_3$ $\text{prob}(\text{outdegree}(x) > 2) \geq 1 - \frac{1}{k}$. This bound is from the fact that there are k ways for x to have one edge, but also k ways for it to have any degree up to $k - 1$ and so we get a very conservative bound by considering only one possibility for each possible degree that x may have.

Each event (placing edges from a point in L_3 to points in L_2) is independent from the others, so

$$\text{prob}(\forall x \in L_3, \text{outdegree}(x) \geq 2) \geq \left(1 - \frac{1}{k}\right)^j = \left(\frac{k-1}{k}\right)^j = \frac{k^j - jk^{j-1} + \dots - (-1)^j k + (-1)^j}{k^j}$$

Therefore, for a given j ,

$$\lim_{k \rightarrow \infty} (\text{prob}(\forall x \in L_3, \text{outdegree}(x) \geq 2)) = 1.$$

Then, we have that for any $x \in L_3$, $\text{prob}(\text{indegree}(x) \geq 2) > (1 - \frac{1}{j})^2$ and

$$\text{prob}(\forall y \in L_2, \text{outdegree}(y) \geq 2) > \left(1 - \frac{1}{j}\right)^{2k} = \left(\frac{j-1}{j}\right)^{2k} = \frac{j^{2k} - 2kj^{2k-1} + \dots - k + 1}{k^j}$$

Therefore, for a given k ,

$$\lim_{j \rightarrow \infty} (\text{prob}(\forall y \in L_2, \text{outdegree}(y) \geq 2)) = 1.$$

Similarly

$$\lim_{l \rightarrow \infty} (\text{prob}(\forall y \in L_2, \text{outdegree}(y) \geq 2)) = 1$$

and

$$\lim_{k \rightarrow \infty} (\text{prob}(\forall z \in L_1, \text{outdegree}(z) \geq 2)) = 1.$$

These events are not probabilistically independent, so we cannot just multiply the individual probabilities to obtain the probability of all 4 events happening simultaneously. However, we can take the union bound on the complement of these events, giving $\text{prob}(P \text{ is not minimal}) \leq \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$ where

$$\epsilon_1 = \text{prob}(\exists x \in L_3, \text{outdegree}(x) < 2),$$

$$\epsilon_2 = \text{prob}(\exists y \in L_2, \text{indegree}(x) < 2),$$

$$\epsilon_3 = \text{prob}(\exists y \in L_2, \text{outdegree}(x) < 2),$$

and

$$\epsilon_4 = \text{prob}(\exists z \in L_1, \text{outdegree}(z) < 2).$$

Then almost all such graphs P are minimal, provided that the size of each level increases as the graph itself grows, meaning graphs on n vertices $P = L_1 \amalg L_2 \amalg L_3$ with $|L_3| = an$ $|L_2| = bn$ $|L_1| = cn$ such that $a + b + c = 1$. \square

REMARK 18.0.17. The graphs in $Q(V)$ are of this form, but this proof is perhaps more intuitive.

Let us go back and consider this result. In some ways it is unsurprising to find this behavior; given a large space, the digraph representing it is large and thus has many more possible edges between vertices. In this way it makes sense that with enough edges on the graph, there is a good probability that every vertex has in-degree and out-degree at least 2. However, with respect to the topology, this result is startling; homotopy equivalence does not narrow down the classification of F -spaces any more than homeomorphism for large F -spaces. Nevertheless, when we look at the actual, numerical counts for number of spaces up to homotopy and homeomorphism, we see a large gap between the relative growth rates. For example, for spaces with 12 points, there are 1,104,891,746 spaces up to homeomorphism, with only 1,594,293 distinct spaces up to homotopy equivalence (a factor of 70 difference). Thus, even though the asymptotic behavior of these two numbers is the same, the convergence for small values is very slow.

An outline summary of point set topology

We have implicitly given a quick outline of a bare bones introduction to point set topology in Chapter 1. The focus was on basic concepts and definitions rather than on the usual examples that give substance to the subject. We thought the reader might like to see a brief summary of some of the most basic parts of point-set topology that were not discussed in Chapter 1, including but not limited to those results we that we have used in our exposition.

19.1. Metric spaces

The intuition for and the most important examples in point-set topology come from metric spaces, where the topology is defined in terms of a distance function.

DEFINITION 19.1.1. A metric d on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ such that

- (i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- (ii) $d(x, y) = d(y, x)$.
- (iii) $d(x, y) + d(y, z) \geq d(x, z)$.

The basis \mathcal{B} determined by a metric d consists of the sets $B(x, r) = \{y \mid d(x, y) < r\}$. The topology generated by \mathcal{B} is called the metric topology on X determined by d . A topological space X is *metrizable* if its topology is determined by a metric.

A subset A of a metric space X has an induced metric, and the metric and subspace topologies coincide. Any metric space is Hausdorff.

Of course, \mathbb{R}^n has the standard metric

$$d(x, y) = \left(\sum (y_i - x_i)^2 \right)^{1/2}.$$

The metric topology that it determines coincides with the product topology. The product of countably many copies of \mathbb{R} is metrizable, but the product of uncountably many copies of \mathbb{R} is not. There is a metric topology on any product of copies of \mathbb{R} , called the uniform topology, but it is finer than the product topology when the product is infinite.

For metric spaces, Lemma 1.5.8 leads to the familiar ε, δ formulation of continuity.

LEMMA 19.1.2. A function $f : X \rightarrow Y$ between metric spaces is continuous if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon);$$

that is, if the distance from x to y is less than δ , then the distance from $f(x)$ to $f(y)$ is less than ε .

Moreover, we can characterize continuity in terms of convergent sequences.

DEFINITION 19.1.3. A sequence $\{x_n\}$ of points in a space X *converges* to a point x if every neighborhood of x contains all but finitely many of the x_n . We then write $\{x_n\} \rightarrow x$. If X is Hausdorff, then the limit of $\{x_n\}$ is unique if it exists.

Observe that if $\{x_n\} \subset A$ and $\{x_n\} \rightarrow x$, then $x \in \bar{A}$. The converse does not hold for general topological spaces, but it does hold for metric spaces. Actually, what is relevant is not the metric but something it implies.

defined and used earlier

DEFINITION 19.1.4. A space X is *first countable* if for each $x \in X$, there is a countable set of neighborhoods U_n of x such that any neighborhood of x contains at least one of the U_n ; X is *second countable* if its topology has a countable basis.

Using the neighborhoods $B(x, 1/n)$, we see that a metric space is first countable.

LEMMA 19.1.5. *Let X be first countable. Then $x \in \bar{A}$ if and only if there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \rightarrow x$.*

Using Lemma 1.5.2 this leads to the promised characterization of continuity.

PROPOSITION 19.1.6. *Let $f : X \rightarrow Y$ be a function, where X is first countable and Y is any space. Then f is continuous if and only for every convergent sequence $\{x_n\} \rightarrow x$ in X , $\{f(x_n)\} \rightarrow f(x)$ in Y .*

19.2. Compact and locally compact spaces

DEFINITION 19.2.1. A space X is *compact* if every open cover contains a finite subcover. That is, if X is the union of open sets U_i , then there are finitely many indices i_j , such that X is the union of the U_{i_j} .

Using standard facts about complements, one can reformulate the notion of compactness as follows. Say that a set of subsets of X has the finite intersection property if any finite subset has nonempty intersection.

defined and used earlier

PROPOSITION 19.2.2. *A space X is compact if and only if any set of closed subsets of X with the finite intersection property has nonempty intersection. In particular, if X is compact and if $\{C_n\}$ is a nested sequence of closed subsets of X , $C_n \supset C_{n+1}$, then $\cap C_n$ is nonempty.*

A metric space X is *bounded* if $d(x, y) \leq D$ for some fixed D and all $x, y \in X$; the least such D is called the *diameter* of X . Boundedness is not a “topological” property, since it depends on the choice of metric: different metrics can define the same topology but have very different bounded subsets. With the standard Euclidean metric, we have the following result.

THEOREM 19.2.3 (Heine-Borel). *A subspace of \mathbb{R}^n is compact if and only if it is closed and bounded.*

In general, we have the following observations about the compactness of subspaces. For a subset A of a space X , a cover of A in X is a set of subsets of X whose union contains A .

PROPOSITION 19.2.4. *Let A be a subspace of a space X . Then A is compact if and only if every cover of A in X has a finite subcover. If X is compact, then every closed subspace of X is compact.*

For compact Hausdorff spaces, the second statement has a converse.

PROPOSITION 19.2.5. *Every compact subspace of a compact Hausdorff space is closed.*

PROPOSITION 19.2.6. *If $f : X \rightarrow Y$ is a continuous function and X is compact, then the image of f is a compact subspace of Y . In particular, any quotient space of a compact space is compact.*

THEOREM 19.2.7. *Let X be compact and Y be Hausdorff. Then a continuous bijection $f : X \rightarrow Y$ is a homeomorphism (hence X is Hausdorff and Y is compact).*

PROOF. If C is closed in X , then C is compact, hence $f(C)$ is compact, hence $f(C)$ is closed in Y . This proves that f^{-1} is continuous. \square

The results above give the behavior of compactness with respect to subspaces and quotient spaces. The behavior with respect to products is deeper than anything that we have stated so far.

THEOREM 19.2.8 (Tychonoff). *Any product of compact spaces is compact.*

The case of finite products is not difficult, but the general case is.

For metric spaces, compactness can be characterized in terms of limit points and convergent sequences.

THEOREM 19.2.9. *Consider the following conditions on a space X .*

- (i) X is compact.
- (ii) Every infinite subset of X has a limit point.
- (iii) Every sequence in X has a convergent subsequence.

In general, (i) \Rightarrow (ii) \Rightarrow (iii). If X is a metric space, the three conditions are equivalent.

We say that X is *sequentially compact* if it satisfies (iii). The following important fact is used in proving that (iii) \Rightarrow (i) when X is a metric space.

LEMMA 19.2.10 (Lebesgue Lemma). *Let \mathcal{O} be an open cover of a sequentially compact metric space X . Then there is a $\delta > 0$ such that if $A \subset X$ is bounded with diameter less than δ , then A is contained in some $U \in \mathcal{O}$.*

PROOF. If not, then for each n we can choose a subset A_n of diameter less than $1/n$ which is not contained in any $U \in \mathcal{O}$. Choose a point $x_n \in A_n$ for each n . Suppose that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ that converges to some x . Certainly $x \in O$ for some $U \in \mathcal{O}$. For small enough ε and large enough n_i , $B(x, 2\varepsilon) \subset U$, $d(x, x_{n_i}) < \varepsilon$ and $1/n_i < \varepsilon$. It follows easily that $A_{n_i} \subset U$, which is a contradiction. \square

DEFINITION 19.2.11. A space X is *locally compact* if each point of X has a neighborhood that is contained in a compact subspace of X .

Clearly \mathbb{R}^n is locally compact but not compact.

LEMMA 19.2.12. *Let X be a Hausdorff space. Then X is locally compact if and only if for any point x and any neighborhood U of x , there is a smaller neighborhood V of x such that \bar{V} is compact and $\bar{V} \subset U$.*

This criterion is needed to prove the second part of the following result.

LEMMA 19.2.13. *Let A be a subspace of a locally compact subspace X . If A is closed or if A is open and X is Hausdorff, then A is locally compact.*

Locally compact Hausdorff spaces admit a canonical compactification, as we now make precise.

DEFINITION 19.2.14. A *compactification* of a space X is an inclusion of X as a dense subspace in a compact Hausdorff space Y . Observe that a compactification of a compact Hausdorff space must be a homeomorphism. Two compactifications Y and Y' are equivalent if there is a homeomorphism $Y \rightarrow Y'$ which restricts to the identity map on X .

Compactifications are of fundamental importance in topology and algebraic geometry. The most naive example is the one-point compactification. The construction applies to any Hausdorff space, but it only gives a Hausdorff space when X is locally compact.

CONSTRUCTION 19.2.15. Let X be a Hausdorff space and let Y be the union of X and a disjoint point denoted ∞ . Then Y is a topological space whose open sets are the open sets in X together with the complements of the compact sets in X . The space Y is called the *one point compactification of X* .

If X is itself compact, then $\{\infty\}$ is open and closed in Y and Y is the union of its components X and $\{\infty\}$.

PROPOSITION 19.2.16. *If X is a locally compact Hausdorff space that is not compact, then the one point compactification Y of X is in fact a compactification: Y is compact Hausdorff and X is a dense subspace.*

Since X is itself one of the open sets in Y , Lemma 19.2.13 gives the following implication.

COROLLARY 19.2.17. *A space X is locally compact and Hausdorff if and only if it is homeomorphic to an open subset of a compact Hausdorff space.*

19.3. Further separation properties

We have defined T_0 , T_1 spaces and T_2 , or Hausdorff spaces. We give three analogous definitions, and we describe various implications relating these separation properties to each other and to local compactness.

DEFINITION 19.3.1. Let X be a T_1 space (points are closed), let $x \in X$, and let A and B be closed subsets of X .

- (i) X is *regular* if whenever $x \notin A$, there are open subsets U and V such that $x \in U$ and $A \subset V$.
- (ii) X is *completely regular* if whenever $x \notin A$, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(a) = 1$ for $a \in A$.
- (iii) X is *normal* if whenever $A \cap B = \emptyset$, there are open subsets U and V such that $A \subset U$ and $B \subset V$.

Together with Lemma 19.2.12, the following result makes clear that these separation properties are closely related to local compactness.

LEMMA 19.3.2. *Let X be a T_1 space.*

- (i) X is regular if and only if for any point x and any neighborhood U of x , there is a smaller neighborhood V of x such that $\bar{V} \subset U$.
- (ii) X is normal if and only if for any closed set A contained in an open set U , there is an open set V such that $A \subset V$ and $\bar{V} \subset U$.

Language varies. The terms regular, completely regular, and normal are often defined without assuming that X is T_1 . Then what we call regular and normal spaces are called T_3 and T_4 spaces and what we call completely regular spaces are called Tychonoff spaces. (As already noted, the T_i notation goes back to a 1935 paper of Alexandroff and Hopf [3], but some later references confuse things further by forgetting history and using T_i differently).

LEMMA 19.3.3. *The following implications hold: A normal space is completely regular. A completely regular space is regular. A regular space is Hausdorff.*

$$\text{normal} \Rightarrow \text{completely regular} \Rightarrow \text{regular} \Rightarrow \text{Hausdorff}$$

The implications $\text{normal} \Rightarrow \text{regular} \Rightarrow \text{Hausdorff}$ are obvious. The implication $\text{normal} \Rightarrow \text{completely regular}$ is a consequence of the following important result.

THEOREM 19.3.4 (Urysohn's lemma). *If X is normal and A and B are disjoint closed subsets of X , then there is a continuous function $f : X \rightarrow I$ such that $f(a) = 0$ if $a \in A$ and $f(b) = 1$ if $b \in B$.*

The proof is non-trivial, and the closely analogous assertion that regular implies completely regular is false. Urysohn's lemma can be used to prove the following equally important result.

THEOREM 19.3.5 (Tietze extension theorem). *If A is a closed subspace of a normal space X and $f : A \rightarrow I$ is a continuous function, then f can be extended to a continuous function $X \rightarrow I$.*

Normality is the most desirable separation property, but it is much less nicely behaved than our other separation properties.

PROPOSITION 19.3.6. *A subspace of a Hausdorff, regular, or completely regular space is again Hausdorff, regular, or completely regular. A product of Hausdorff, regular, or completely regular spaces is again Hausdorff, regular, or completely regular. Neither of these assertions is true in general for normal spaces.*

For example, the product of uncountably many copies of \mathbb{R} is not normal. Since \mathbb{R} is homeomorphic to the open interval $(0, 1)$ and Tychonoff's theorem implies that the product of uncountably many copies of I is compact Hausdorff, this example also shows that a subspace of a normal space need not be normal. Nevertheless, most spaces of interest are normal.

THEOREM 19.3.7. *If X is metrizable or compact Hausdorff, then X is normal.*

Some indication of the importance of complete regularity is given by the following sequence of results, the second of which should be compared with [Corollary 19.2.17](#).

THEOREM 19.3.8. *If X is completely regular, then it can be embedded as a subspace of a product of copies of the unit interval.*

COROLLARY 19.3.9. *The following conditions on a space X are equivalent.*

- (i) X is completely regular.
- (ii) X is homeomorphic to a subspace of a compact Hausdorff space.
- (iii) X is homeomorphic to a subspace of a normal space.

COROLLARY 19.3.10. *A space X admits a compactification if and only if it is completely regular.*

PROOF. If Y is a compactification of X , then X is a subspace of the compact Hausdorff space Y and is thus completely regular. Conversely, if X is completely regular and thus homeomorphic to a subspace of some compact Hausdorff space Z , then the closure of the image of X in Z is a compactification of X , called the compactification induced by the inclusion of X in Z . \square

The very definition of complete regularity leads to a canonical compactification.

CONSTRUCTION 19.3.11. Let X be completely regular. Let $F = F(X)$ be the set of all continuous functions $f : X \rightarrow I$, let $Z = Z(X)$ be the product of copies of I indexed on the set F , and let $i : X \rightarrow Z$ be the map whose f th coordinate is the map f . Then i is an inclusion. The induced compactification is denoted βX and called the *Stone-Čech compactification* of X .

The Stone-Čech compactification is characterized as the unique compactification (up to equivalence) that satisfies the following “universal property”.

PROPOSITION 19.3.12. *Let X be a completely regular space. A map $f : X \rightarrow Y$, where Y is a compact Hausdorff space, extends uniquely to a map $\tilde{f} : \beta X \rightarrow Y$.*

PROOF. Uniqueness holds by [Lemma 1.5.3](#). When $Y = I$, the existence is immediate from the construction: f is one of the coordinate maps, and the projection from $Z(X)$ to this coordinate restricts to $\tilde{f} : \beta X \rightarrow I$. In general, Y is homeomorphic to $\beta Y \subset Z(Y)$. The map $f_g : X \xrightarrow{f} Y \cong \beta Y \subset Z(Y) \xrightarrow{\pi_g} I$ obtained from the g th coordinate projection $\pi_g, g \in Z(Y)$, extends to a map $\tilde{f}_g : \beta X \rightarrow I$, and \tilde{f}_g is the g th coordinate of a map $\beta X \rightarrow Z(Y)$. This map sends X into the closed set βY , hence it sends the closure βX into $\beta Y \cong Y$, giving \tilde{f} . \square

19.4. Metrization theorems and paracompactness

Since we are much more comfortable with metric spaces than with general spaces, it is important to be able to recognize when the topology on a given space is that induced by some metric. The simplest criterion is the following. Metrization theorems are proven by embedding a given space as a subspace of a space that is known to be metrizable. Let I^ω denote the product of countably many copies of I . It is a metric space, which would be false for an uncountable product.

THEOREM 19.4.1 (Urysohn metrization theorem). *The following conditions on a T_1 space X are equivalent.*

- (1) X is regular and second countable.
- (2) X is homeomorphic to a subspace of I^ω .
- (3) X is metrizable and has a countable dense subset.

Remember that second countable means that there is a countable basis for the topology. This ensures the following analogue of compactness.

LEMMA 19.4.2. *If X is second countable, then any open cover of X has a countable subcover and X has a countable dense subset.*

Second countability is a strong condition, and a weaker countability condition, plus regularity, is necessary and sufficient for metrizable.

DEFINITION 19.4.3. A set \mathcal{V} of subsets of X is *locally finite* if each $x \in X$ has a neighborhood that intersects at most finitely many subsets of \mathcal{V} . A cover \mathcal{O} of X is *σ -locally finite* if it is the union of countably many locally finite subsets.

THEOREM 19.4.4 (Nagata-Smirnov metrization theorem). *A space is metrizable if and only if it is regular and has a σ -locally finite basis.*

The “ σ ” here is essential: if a Hausdorff space has a locally finite cover, then it is discrete.

There is another characterization of metrizable that is perhaps more intuitive.

DEFINITION 19.4.5. A space X is *locally metrizable* if every point $x \in X$ has a neighborhood U such that U (with its subspace topology) is metrizable.

Clearly any metric space is locally metrizable. There is a property, called paracompactness, that is very often used to patch local conditions to obtain a global condition, and Stone proved that any metric space is paracompact.

THEOREM 19.4.6 (Smirnov metrization theorem). *A space is metrizable if and only if it is paracompact and locally metrizable.*

We explain paracompactness. A *refinement* of a cover \mathcal{O} of X is a collection of subspaces each of which is contained in at least one of the spaces in \mathcal{O} .

DEFINITION 19.4.7. A space X is *paracompact* if every open cover of X has a locally finite refinement that is again an open cover of X .

Clearly a compact Hausdorff space is paracompact. The following sharpening of part of [Theorem 19.3.7](#) holds.

THEOREM 19.4.8. *A paracompact space X is normal.*

Like normality, paracompactness is not preserved by standard constructions. For this reason, Stone’s theorem that metrizable \Rightarrow paracompact seems more useful than the converse implication of Smirnov’s metrization theorem.

PROPOSITION 19.4.9. *A closed subspace of a paracompact space is paracompact. In general, subspaces of paracompact spaces and products of paracompact spaces need not be paracompact.*

The point of paracompactness is that it ensures the existence of particularly convenient open covers. This is very important in the theory of fiber bundles in algebraic topology.

DEFINITION 19.4.10. An open cover \mathcal{O} of X is *numerable* if it is locally finite and for each $U \in \mathcal{O}$ there is a continuous function $\phi_U : X \rightarrow I$ such that $\phi_U(x) > 0$ only if $x \in U$. A numerable cover \mathcal{U} is a *partition of unity* if $\sum_U \phi_U(x) = 1$ for each $x \in X$.

Given a numerable cover \mathcal{O} , we can define $\phi(x) = \sum_U \phi_U(x)$ and $\psi_U(x) = \phi_U(x)/\phi(x)$, thereby obtaining a partition of unity.

PROPOSITION 19.4.11. *If X is paracompact, then any open cover of X has a numerable refinement.*

DEFINITION 19.4.12. An n -manifold M is a second countable Hausdorff space each point of which has a neighborhood homeomorphic to \mathbb{R}^n .

By the Urysohn metrization theorem, an n -manifold is metrizable. By Stone's theorem, it is therefore paracompact. The following theorem can be proven by use of a numerable cover of M .

THEOREM 19.4.13. *Any n -manifold M can be embedded as a subspace of \mathbb{R}^N for a sufficiently large N .*

19.5. Finite metric spaces and their embedding into Lebesgue spaces

A finite metric space is a finite collection of points with a real distance defined between each pair. From the perspective of algebraic topology, they have no interest as discrete spaces. Although relaxing metrics to pseudometrics appears to provide finite metric spaces with more interest, pseudometric spaces are homotopically equivalent to the discrete space formed when they are passed through the Kolmogorov quotient. Despite their uninteresting topological structure, finite metric spaces have applications to computer science. Many physical systems can be modeled with finite points and distances between them, so computer scientists are motivated to embed finite metric spaces into host spaces like \mathbb{R}^N where detailed analysis can be done. Perfect embeddings cannot always be achieved, so the study of the distortion needed for embeddings and when isometric embeddings exist is a rich area.

19.6. Finite Metric Spaces

Finite spaces have different metrization and pseudometrization conditions and their metrics can be represented in convenient ways.

19.6.1. Pseudometrizing Metrics on Finite Spaces.

DEFINITION 19.6.1. A *pseudometric* is a function $d : X \times X \rightarrow \mathbb{R}$ which satisfies the following properties:

- (1) $d(x, x) = 0$ for all $x \in X$
- (2) $d(x, y) \geq 0$
- (3) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (4) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

This definition is a weakening of the standard metric. Two distinct points may have a distance of zero. Pseudometrics are sometimes referred to as *semimetrics*.

DEFINITION 19.6.2. A space X is *pseudometrizable* if there is a pseudometric d on X that induces the topology of X .

DEFINITION 19.6.3. A space is R_0 if each pair of topologically distinct points (i.e. points which do not have the same set of neighborhoods) have some neighborhood not containing the other point.

THEOREM 19.6.4. *A finite topological space is pseudometrizable iff it is R_0 .*

PROOF. Given a topological space X and points x and y in X , define $x \equiv y$ to mean that x and y are topologically indistinguishable.

Define the *standard discrete pseudometric* to be:

$$d(x, y) = \begin{cases} 0 & \text{if } x \equiv y \\ 1 & \text{if } x \not\equiv y \end{cases}$$

Given $x \not\equiv y$, take neighborhoods $B(x, (\frac{1}{2}))$ and $B(y, (\frac{1}{2}))$ of x and y so that

$$B\left(x, \left(\frac{1}{2}\right)\right) \cap B\left(y, \left(\frac{1}{2}\right)\right) = \emptyset$$

This metric induces a topology on X where every topologically distinguishable pair is separated.

If a finite space is R_0 with its given topology, then it can be given this topology which separates topologically distinguishable points, satisfying the R_0 condition as well as inducing a topology which puts families of points equivalent to the given topology into the same neighborhoods.

Take a space X to be pseudometrizable. Then its metric topology forms open balls around topologically distinguishable points which can be separated.

If no points in the space have distinct neighborhoods (i.e. the pseudometric outputs 0 given any two points), then there are no topologically distinguishable points, so the space is vacuously R_0 . \square

19.6.2. Representing Metrics on Finite Spaces. A metric on a finite space can be explicitly defined by $\binom{n}{2}$ non-negative numbers, where each number corresponds to a distance between two points. This property of finite metric spaces allows them to be represented in convenient ways, most importantly with matrices and graphs.

19.6.2.1. *Matrix Representation.* Take a finite metric space (X, d) with points (x_0, x_1, \dots, x_n) . Construct an $n \times n$ matrix with entries $(a_{i,j})$ giving the distance between point i and point j in the space. Then the following characteristics can be observed.

- (1) $d(x_i, x_j) \geq 0$ for all $0 \leq i, j \leq n$ so the matrix is comprised of nonnegative real numbers.
- (2) $d(x_i, x_i) = 0$ for all $0 \leq i \leq n$ so the diagonal of the matrix is 0.
- (3) $d(x_i, x_j) = d(x_j, x_i)$ for all $0 \leq i, j \leq n$ so the matrix equals its transpose.

Thus any finite metric space has a real, positive, symmetric matrix containing all the information of its metric.

19.6.2.2. *Graph Representation.* The matrix defined by the finite metric space can be translated to an undirected, no loop, weighted, finite graph. Given a finite metric space (X, d) with points (x_0, x_1, \dots, x_n) , a graph G with n vertices and $\binom{n}{2}$ weighted edges giving the distance between vertices can be constructed to represent it.

The distance function defines a distance between any two points of the space, so each vertex of the graph connects to every other vertex, forming a complete graph. Metrics satisfy the triangle inequality, so all edges may not be necessary if the shortest path metric is used on the graph.

DEFINITION 19.6.5. Given a weighted graph G , the *shortest path metric* is a metric which defines the distance between two vertices to be the length of the

shortest path between them. If the two vertices are not connected, the distance is said to be infinite.

THEOREM 19.6.6. *A graph G with n vertices and the shortest path metric represents an n point finite metric space (X, d) iff it is undirected, no loop, weighted and connected.*

PROOF. Set each vertex in G to represent a distinct point in the underlying set X . The properties of a metric give rise to the conditions necessary for the graph.

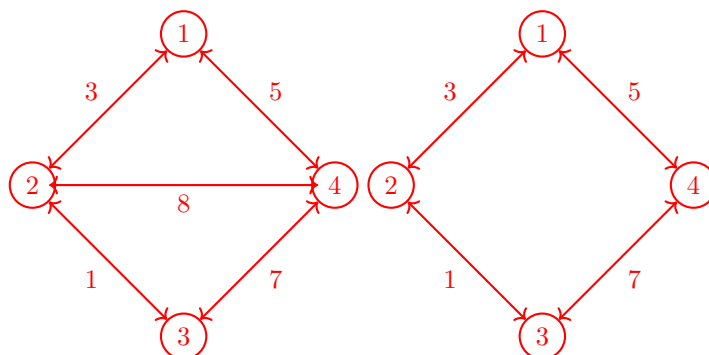
- (1) $d(x_i, x_j) = d(x_j, x_i)$ for all $0 \leq i, j \leq n$ (G must be undirected).
- (2) $d(x_i, x_i) = 0$ for all $0 \leq i \leq n$ (G must have no loops).
- (3) $d(x_i, x_j) \geq 0$ for all $0 \leq i \leq n$ (G must be weighted with nonnegative real values).
- (4) $d(x_i, x_j) < \infty$ for all $0 \leq i, j \leq n$ (G must be connected).

The triangle inequality means that the shortest path metric must be used.

Conversely, a graph fulfilling the above properties can be made into a finite metric space if the vertices are made into the underlying set and the shortest path metric is made into the metric on that set. \square

DEFINITION 19.6.7. It may be possible to obtain a graph with fewer than $\binom{n}{2}$ (i.e. not a complete graph) to represent the finite metric space. When all edges which do not alter the output of the shortest path metric are dropped, the *critical graph* is obtained.

EXAMPLE 19.6.8. Where the triangle inequality is satisfied by an equality an edge can be removed. In this case a critical graph is obtained.



19.7. The Problem with Finite Metric Spaces

Finite metric spaces are of no interest to algebraic topologists as they induce the discrete topology on the space. This section illustrates why this is the case and how an indiscrete pseudometric space can be made into a discrete space when it is made T_0 through the Kolmogorov Quotient.

19.7.1. The Discrete Topology. Recall that the discrete topology is the finest topology possible on a set. Every subset is an open set, and therefore every subset is also a closed set. The fact that finite metric spaces have the discrete topology can be proved directly, or illustrated through Lipschitz equivalence of metrics.

THEOREM 19.7.1. *Any metric on a finite space induces the discrete topology.*

PROOF. Take a finite metric space (X, d) . If every point in the space is open, then all of their possible unions are open, giving the discrete topology.

For any $x \in X$, find $r = \min_{y \in X} (d(x, y))$. This r exists and is nonzero as X is finite and $d(x, y) > 0$ for $x \neq y$. Then the open ball of radius r about x contains only x . Thus, the set $\{x\}$ is open. \square

THEOREM 19.7.2. *A finite space is metrizable iff it is discrete.*

PROOF. Given a finite space with the discrete topology, the discrete metric ensures that every point is in a singleton open set (any open ball of radius less than 1) and so the finite space can be metrized.

Conversely, any finite space can be metrized in order to give the discrete topology. In fact, as proved above, the discrete topology is the only possible metric topology given to a finite space. \square

19.7.2. The Kolmogorov Quotient. Finite pseudometric spaces allow distinct points to have the same open neighborhoods in the induced topology. This seems to give them greater topological interest as they are not necessarily discrete. The Kolmogorov quotient $K(X)$ of a space X identifies points with the same open neighborhoods, and allows for a way to form a T_0 space. In this case, the T_0 space would be a metric space. This process of converting a pseudometric space into a metric space through a Kolmogorov quotient is called *metric identification*.

19.7.2.1. Metric Identification. Suppose (X, d) is a pseudometric space with $x, y \in X$, and let $x \sim y$ if $d(x, y) = 0$. Define $X^* = X / \sim$. If we construct a metric d^* on X^* by setting $d^*([x], [y]) = d(x, y)$, then (X^*, d^*) is a metric space.

PROPOSITION 19.7.3. *Metric $d^*([x], [y]) = d(x, y)$ is well-defined.*

PROOF. It is clear that d^* is a metric as it inherits properties from metric d . We show that for $x_1, x_2 \in [x]$ and $y \in [y]$, $d^*(x_1, y) = d^*(x_2, y) = d(x, y)$. Take $d^*(x_1, y) = d(x, y)$. By the triangle inequality on d^* , $d^*(x_1, x_2) + d^*(x_2, y) \geq d^*(x_1, y)$. Because $x_1 \sim x_2$, $d^*(x_1, x_2) = 0$, so $d^*(x_2, y) = d^*(x_1, y)$. Thus d^* is independent of choice of representative from the equivalence class, and hence is well-defined. \square

THEOREM 19.7.4. *Metric identification preserves the metric induced topology.*

PROOF. We show the set $A \subset X$ is open iff set $[A]$ (the set of all $[x]$ where x is in A) is open in (X^*, d^*) .

Take $A \subset (X, d)$, open. Then for all $x \in A$, there is an open ball around x which is contained in A . Identify all x, y such that $d(x, y) = 0$. These equivalence classes are made of points distance zero from each other, so the set of open balls $[B(x, \epsilon)]$ for a given $[x]$, all overlap. \square

19.7.2.2. Kolmogorov Quotient of Pseudometric Spaces.

THEOREM 19.7.5. *The topology induced by metric identification forms a quotient space that is the Kolmogorov quotient.*

PROOF. Take (X, d) a pseudometric space with metric identified as above. It must be shown that the relation \sim is an equivalence relation and that topology induced by d^* on X / \sim forms $K(X)$.

- (1) The relation \sim is an equivalence relation
- (a) Reflexivity: $d(x, x) = 0$ for all $x \in X$, so $x \sim x$.
 - (b) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$, so if $d(x, y) = 0$, then $d(y, x) = 0$. Thus, if $x \sim y$, then $y \sim x$.
 - (c) By the triangle inequality, $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. If $x \sim y$ and $y \sim z$, then $d(x, y) + d(y, z) = 0$, $d(x, z) \geq 0$, and so $d(x, z) = 0$.
- (2) For the topology induced by d^* on X/\sim to be $K(X)$, the equivalence classes must be comprised of topologically indistinguishable points. Take $x, y \in X$, with x and y topologically distinguishable. Then there is an open subset U of X where $x \in U$ but $y \notin U$. This means that there is an open ball of some radius about x that does not contain y , so $d(x, y) > 0$, so $x \not\sim y$.

Conversely, if x and y are topologically indistinguishable, then there is no open ball containing only one of the points. Then each $B(x, \frac{1}{n})$ must contain both x and y , so $d(x, y)$ must be zero. This means that the topology induced by d^* on X/\sim is putting only topologically indistinguishable points into equivalence classes. This, taken with [Theorem 19.7.5](#) above, shows that this quotient forms $K(X)$.

□

19.7.2.3. *Homotopy Equivalence of the Kolmogorov Quotient.* Finite pseudometric spaces (in fact all finite spaces) are homotopy equivalent to their Kolmogorov Quotient $K(X)$.

THEOREM 19.7.6. *Every finite space is homotopically equivalent to a T_0 space, $K(X)$.*

COROLLARY 19.7.7. *Any finite pseudometric space X is homotopically equivalent to its Kolmogorov Quotient, $K(X)$, with $K(X)$ being a finite metric space.*

19.8. Embedding Finite Metric Spaces

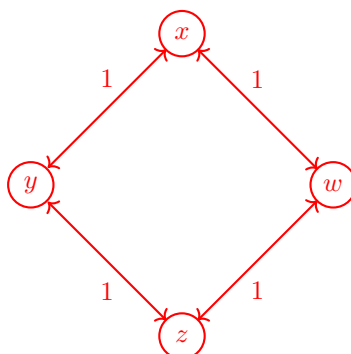
Despite the properties explored above, finite metric spaces are of interest to fields other than algebraic topology. In fields like microbiology, large tables of numbers are generated and need to be analyzed. It can be difficult to work with large tables, meaning that a representation in Euclidean space is desirable. An embedding would offer a way to see the distribution and behavior of the points of the metric space. In addition, a metric space with n points could be described in $2n$ numbers instead of $\binom{n}{2}$ numbers.

The interest in representing combinatorial objects like finite metric spaces in this way comes from a wider interest in the *geometrization of combinatorial objects*, which is a method used to transform large amounts of information into a usable form. Considering the equivalence between linear graphs and finite metric spaces given above, it would seem that all finite metric spaces could be represented in \mathbb{R}^N for some finite N . This is not the case.

The distance metric on the weighted graph representing the finite metric space is the shortest path metric. In \mathbb{R}^N , the shortest path between two points is a straight line, so if equality holds in the triangle equality, those three points lie on the same line in \mathbb{R}^N . This fact will mean that not all finite metric spaces can be

embedded without distorting the distances between points. This is illustrated in the following example.

EXAMPLE 19.8.1. Take finite metric space (X, d) with 4 points represented by the weighted graph below with distance given by the shortest path metric.



This is a simple 4 cycle with edges of uniform length. Note that

$$d(x, z) = d(x, y) + d(y, z) = 2 \text{ and } d(x, z) = d(x, w) + d(w, z) = 2$$

This fact will give a contradiction when an embedding is done. Embed this metric space in \mathbb{R}^N . There are then two minimal paths between x and z and both obtain equality with the triangle inequality. As explained above, the fact that

$$d(x, z) = d(x, y) + d(y, z) \text{ and } d(x, z) = d(x, w) + d(w, z)$$

implies that points x, y, z are collinear, as are x, w, z . Line segments xyz and xwz are the same as they have the same endpoints. Because y and w are both distance 1 away from x on the same line, they are distance zero from each other. This implies that $y = w$, contradicting the fact that X has 4 points.

The graph must be *distorted* to be represented in \mathbb{R}^N .

DEFINITION 19.8.2. Take metric spaces (X, d_X) and (Y, d_Y) and a function $f : X \rightarrow Y$. Then the *distortion* of f can be realized by its Lipschitz constants. The *expansion* of f is defined as

$$\|f\|_{Lip} = \sup_{x, y \in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)}$$

The *contraction* of f is given by

$$\|f\|_{Lip}^{-1} = \sup_{x, y \in X} \frac{d_X(x, y)}{d_Y(f(x), f(y))}$$

The *distortion* of f is given by

$$\text{distortion}(f) = \text{contraction}(f) * \text{expansion}(f) = \|f\|_{Lip}^{-1} * \|f\|_{Lip}$$

This is equivalent to finding the closest $a, b \in \mathbb{R}$ such that

$$a \geq \frac{d_Y(f(x), f(y))}{d_X(x, y)} \geq b$$

and defining $\text{distortion}(f) := \frac{a}{b}$.

REMARK 19.8.3. A mapping $f : X \rightarrow Y$ is an *isometry* if $\frac{a}{b} = 1$. That is, all distances are preserved up to scaling.

DEFINITION 19.8.4. Take metric spaces (X, d) and (Y, d') . Then (X, d) is *isometrically embeddable* into (Y, d') if there is a map $f : X \rightarrow Y$ such that $d(x, y) = d'(f(x), f(y))$ for all x and y in X .

As Example 4.1 illustrates, distortion is often necessary for embedding to occur. In that particular case, the distances can be distorted by a factor of $\sqrt{2}$ in order to form the square cycle.

Embedding a metric space in \mathbb{R}^N is a useful case of embedding, but embedding can be described in general settings.

DEFINITION 19.8.5. For $0 < p < \infty$, ℓ_p *space* is the set of all real sequences $\{x_n\}$ such that $\sum_n |x_n|^p < \infty$.

The norm of this space is given by

$$\|x\|_p = \left(\sum_n |x_n|^p \right)^{\frac{1}{p}}$$

Note that when $p = 2$ this is the Euclidean norm.

DEFINITION 19.8.6. A metric space (X, d) is ℓ_p *embeddable* if (X, d) is isometrically embeddable into ℓ_p^n for some natural number n . This number n is the ℓ_p dimension of (X, d) .

19.8.1. Embedding in ℓ_2 . Embedding in ℓ_2 attracts special attention. To those looking to analyze large amounts of data, translating data points into a finite metric space and then into a representation can be useful. In ℓ_2 there are extremely well developed tools in analysis and geometry to aid in the analysis of the data, so obtaining a good representation is important.

For its usefulness, ℓ_2 is very strict in its behavior, making embeddings difficult. The general theory of Banach spaces gives additional insight into why this is the case and additional motivation to consider ℓ_2 embeddings.

DEFINITION 19.8.7. The *Banach-Mazur distance* is a measure of distance on the set of n -dimensional normed spaces. Take two normed spaces X and Y of dimension n and $GL_{X,Y}$, the set of linear isomorphisms from X to Y .

The *Banach-Mazur distance* between X and Y is defined to be

$$\delta(X, Y) = \log \left(\inf_{T \in GL_{X,Y}} \text{distortion}(T) \right)$$

This is a metric on the space of n -dimensional normed spaces. For many purposes (including ours) the *multiplicative Banach-Mazur distance*

$$d(X, Y) = e^{\delta(X, Y)} = \inf_{T \in GL_n} \text{distortion}(T)$$

will be used. Because $\delta(X, Y)$ is a metric, the multiplicative Banach-Mazur distance obeys the multiplicative triangle inequality, $d(X, Z) \leq d(X, Y)d(Y, Z)$. For convenience, this will be referred to as the *Banach-Mazur distance*.

The Banach-Mazur distance gives a sense of how close two normed spaces are to one another. If the distance is small, then the space needs little distortion for there to be a linear isomorphism between them. The following theorem, Dvoretzky's

theorem, is a classical theorem which gives a quantitative sense of how close ℓ_2 space is to arbitrary normed spaces.

THEOREM 19.8.8. (*Dvoretzky's Theorem* [43]) *For every $n \in \mathbb{N}$ and $\epsilon > 0$, every n -dimensional normed space contains a subspace X of dimension $m = \Omega(\epsilon^2 \log(n))$ such that $d(X, \ell_2) \leq 1 + \epsilon$. Ω denotes that m is bounded asymptotically by $\epsilon^2 \log(n)$ as $n \rightarrow \infty$.*

19.8.1.1. *Bourgain's Theorem.* [12]. Motivated by this property of ℓ_2 , in 1986, Jean Bourgain developed an algorithm which describes embedding in ℓ_2 .

THEOREM 19.8.9. *Any metric space (X, d) with n points can be embedded in ℓ_2 with distortion $\leq O(\log n)$.*

PROOF. Bourgain's proof gives an efficient randomized algorithm for the embedding in ℓ_2 with distortion $\leq O(\log n)$. Take a metric space (X, d) with n points.

- (1) Take m and q to be integers $m = \lfloor \log 2 \rfloor$ and $q = \lfloor C \log(n) \rfloor$ where C is a constant.
- (2) Construct an embedding into ℓ_2^{mq} with coordinates $i = 1, \dots, m$ and $j = 1, \dots, q$.
- (3) Construct subsets of X , A_{ij} by putting each $x \in X$ into A_{ij} with probability 2^{-j} .
- (4) Now embed with function $f(x)_{ij} = d(x, A_{ij})$.

This is an embedding in $\ell_2^{O(\log)^2 n}$. It has distortion $O(\log n)$. □

19.8.1.2. *Tightness of Bound.* The construction of this algorithm raises the question whether a better embedding can be achieved. A paper by Nathan Linial (2002) shows that this bound is tight. He considers a specific type of graph that has a shortest path metric which is as far from the ℓ_2 metric as possible in order to guarantee a large distortion, giving a lower bound on distortion of graphs. To state his theorem, some definitions from graph theory are needed.

DEFINITION 19.8.10. The *girth* of a graph is the shortest cycle contained in the graph. The girth of an acyclic graph is defined to be infinite.

DEFINITION 19.8.11. An *expander graph* is a connected graph in which every "small" subset of vertices has a "large" boundary. That is, the graph cannot be disconnected without removing many edges.

This quality can be quantified in the notion of an ϵ *edge expander*. A graph with n vertices is an ϵ edge expander if every set of K vertices with $0 \leq K \leq \frac{n}{2}$ has $\epsilon|K|$ edges connected to K^c (the set of vertices not in K).

DEFINITION 19.8.12. A *k-regular graph* is a graph where each vertex is of degree k .

THEOREM 19.8.13. *Linial's Lower Bound* [43] *Take G , a k -regular graph, with $k \geq 3$, and girth g . Then every embedding $f : G \rightarrow \ell_2$ has distortion $\Omega(\sqrt{g})$.*

PROOF. Sketch. This proof uses a random walk on the graph. Knowing the girth of the graph and that all vertices are connected to k other vertices, it can be proven that the walk moves away from where it started at constant speed at a

time bounded asymptotically by g . The geometry of Euclidean space means that this class of random walks is at time T expected to be $O(\sqrt{T})$ from its origin. This difference must be accounted for by a distortion in the metric if it is to be embedded in ℓ_2 . Comparing the two walks on the graph at time $O(g)$ gives a distortion of $\Omega(\sqrt{g})$. \square

The triangle inequality is satisfied by equality many times, necessitating significant distortion.

19.8.1.3. *Isometric Embedding in ℓ_2* . I. J. Schoenberg's 1937 paper [61] outlines the necessary and sufficient conditions for an isometric embedding in ℓ_2 . In particular, he addresses separable pseudometric spaces and characterizes embeddable metrics in terms of positive definite functions.

DEFINITION 19.8.14. A real function $f = f(x_1, x_2, \dots, x_n)$ is a *positive definite function* if it is defined for all real values, and if for any real numbers x_1, x_2, \dots, x_n the $n \times n$ matrix A where $A = (a_{i,j})$ and $a_{i,j} = f(x_i - x_j)$ is a positive, semi-definite matrix (that is, $x^t A x \geq 0$ for all real numbers x).

A similar notion of positive definite functions can be defined for real-valued functions which take as input distances on a pseudometric space (X, d) .

A real function $g(t)$ is positive definite if g is continuous, even, defined on the range of distances in the pseudometric space, and satisfies the inequality

$$\sum_{i,j=1}^n g(d(x_i, x_j)) \geq 0$$

Examples of positive definite functions in ℓ_2 are $f(t) = e^{-t^2}$, and more generally, $f(t) = e^{-\lambda t^2}$ for all $\lambda \in \mathbb{R}$.

THEOREM 19.8.15. *Schoenberg's Embedding*

A separable pseudometric space (X, d) is isometrically embeddable in ℓ_2 if and only if the functions $e^{-\lambda t^2}$ are positive definite in (X, d) .

PROOF. Sketch. The idea of this proof is to note that $e^{-\lambda t^2}$ for $(\lambda \in \mathbb{R})$ is a family of positive definite functions in ℓ_2 . It is only necessary to consider $\lambda > 0$ as $\lambda = 0$ is an accumulation point of this family and the cases where $\lambda < 0$ follow by symmetry. The proof uses ideas from analysis about positive definite functions to show that if the given characteristics of positive definite functions are preserved on embedding into ℓ_2 , then all distances must have been preserved and if the given family of functions are positive definite in the metric space, then the metric of the space will allow isometric embedding into ℓ_2 . \square

19.8.2. Embedding in ℓ_1 . Following the formula given for ℓ_p space ℓ_1 is the set of all real sequences $\{x_n\}$ such that $\sum_n |x_n| < \infty$. The distance metric on ℓ_1 is defined to be $d_{\ell_1}(x, y) := \sum_n |x_n - y_n| < \infty$.

To consider isometric embedding in ℓ_1 , the *cut semimetric* will be used.

DEFINITION 19.8.16. The *cut semimetric* is a pseudometric d on a set X . Given partitions A and B of X , define $d(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B \\ 1 & \text{otherwise} \end{cases}$.

Every cut semimetric is clearly isometrically embeddable in ℓ_1 .

The set of all linear combinations of semimetrics on a set forms a special class of metrics on that set. These are exactly the ℓ_1 metrics on the set (that is, the metrics which can be isometrically embedded in ℓ_1) [21].

19.8.3. Embedding in ℓ_∞ .

DEFINITION 19.8.17. ℓ_∞ space is defined to be the set of all real bounded sequences. It takes on the norm $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|$.

THEOREM 19.8.18. [27, Ch 8.1.3] *Every finite metric space (X, d) with n points can be embedded in ℓ_∞^n .*

PROOF. Take a finite metric space (X, d) with $X = \{x_1, x_2, \dots, x_n\}$ and define an embedding function $f : X \rightarrow \ell_\infty^n$ by $f(x_i)_j = d(x_i, x_j)$ for all $1 \leq i$ and $j \leq n$. \square

Embeddings into lower dimensional ℓ_∞^k spaces exist.

DEFINITION 19.8.19. Take a metric space (X, d) and every subset $S \subset X$. Then define a mapping $f_S : X \rightarrow \mathbb{R}$ for each S by

$$f_S(x) = d(x, S) = \min_{s \in S} (d(x, s))$$

A *Frechet Embedding* is a map $f : X \rightarrow \mathbb{R}^k$ where each coordinate in \mathbb{R}^k is a scaled f_S mapping. Then f is a Frechet Embedding if, for some $\beta_S \in \mathbb{R}$,

PROPOSITION 19.8.20. [58] *When $\beta_S = 1$ for all $S \subset V$, $\|f(x) - f(y)\|_\infty \leq d(x, y)$. That is, Frechet embeddings are contraction mappings in the ℓ_∞ metric.*

PROOF. Let S_x denote the point in $S \subset X$ closest to some point $x \in X$. Then both

$$d(x, S) - d(y, S) \leq d(x, S_x) - d(y, S_x) \leq d(x, y), \text{ and}$$

$$d(y, S) - d(x, S) \leq d(y, S_x) - d(x, S_x) \leq d(x, y)$$

This implies that $\|f(x) - f(y)\|_\infty = |d(x, S) - d(y, S)| \leq d(x, y)$. \square

A 1996 paper by Jiri Matousek uses these mappings to do distorted mappings into lower dimension ℓ_∞^k space.

THEOREM 19.8.21. [44] *Take an n -point metric space (X, d) and integer D . Then (X, d) can be embedded into $\ell_\infty^{O(Dn^{2/D} \log(n))}$.*

PROOF. The idea of this proof is to divide X into $O(Dn^{2/D} \log(n))$ subsets, each of which will correspond to a dimension in the range ℓ_∞ space.

Construct the embedding function $\psi : (X, d) \rightarrow \ell_\infty^{O(Dn^{2/D} \log(n))}$ to be a Frechet embedding with j th coordinate of $\psi(x)$ to be $d(x, S)$. Noting the proposition above, function ψ must be a contraction mapping. The rest of the proof uses an algorithm and probability to show that its contraction is limited. \square

19.8.4. Embedding in \mathbb{R}^N [53]. A paper by C.L. Morgan published in 1974 proved necessary and sufficient conditions for embedding a metric space in \mathbb{R}^N . His theorem applies to arbitrary metric spaces, not only finite ones. It holds special interest for embedding finite metric spaces. His theorem makes the computation necessary to determine whether embeddability is feasible. His proof also shows that for any metric space, embedding into \mathbb{R}^N is a very strong condition, but it is one that is determined by a finite number of points in the metric space.

In order to state and prove the embedding theorem, some special definitions will be needed, as well as some general results about inner products, metrics, and linear algebra.

DEFINITION 19.8.22. An *inner product* on a vector space V over a field \mathbb{F} with characteristic 0 is a bilinear map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$. This function satisfies conjugate symmetry and positive definiteness.

For a vector space V with element $x \in V$, define a norm $\|x\| = \sqrt{\langle x, x \rangle}$.

THEOREM 19.8.23. For a vector space V over characteristic 0 field F with inner product $\langle \cdot, \cdot \rangle$, and norm $\|x\| = \sqrt{\langle x, x \rangle}$, a metric $d(x, y) = \|x - y\|$ is induced by the norm.

DEFINITION 19.8.24. Let (X, d) be a metric space and for points $x, y, z \in X$ define a function from $X \times X \times X \rightarrow \mathbb{R}$ by:

$$\langle x, y, z \rangle = \frac{1}{2} (d(x, z)^2 + d(y, z)^2 - d(x, y)^2)$$

If we define X to be a subset of some vector space V such that metric d is induced by an inner product on V , then $\langle x, y, z \rangle$ is the inner product of $x - z$ and $y - z$.

DEFINITION 19.8.25. Take metric space (X, d) . Then define Y to be a *metric subspace* of X if $Y \subset X$ and Y has the distance function $d|_{Y \times Y}$.

Finite metric subspaces of X are *n-simplices* in X . In particular, a metric subspace of $n + 1$ elements is an *n-simplex* in X .

If (X, d) is a subspace of Euclidean space, then these simplices have a clear notion of volume. The following function will begin to generalize this idea to arbitrary metric spaces.

DEFINITION 19.8.26. Define a function $D : X^{n+1} \rightarrow \mathbb{R}$ as follows: Construct an $n \times n$ matrix A from (x_0, x_1, \dots, x_n) with real entries $(a_{i,j}) = \langle x_i, x_j, x_0 \rangle$ and let $D(x_0, x_1, \dots, x_n) = \det(A)$. This function D is a real valued function on the n -simplices of X .

PROPOSITION 19.8.27. The function D is symmetric.

PROOF. In Euclidean space, the entry $(a_{i,j})$ in the above matrix is

$$\begin{aligned} \langle x_i, x_j, x_0 \rangle &= \frac{1}{2} \left(\left(\sqrt{(x_i - x_0)^2} \right)^2 + \left(\sqrt{(x_j - x_0)^2} \right)^2 - \left(\sqrt{(x_i - x_j)^2} \right)^2 \right) \\ &= \frac{1}{2} \left((x_i - x_0)^2 + (x_j - x_0)^2 - (x_i - x_j)^2 \right) \\ &= \frac{1}{2} \left(-2x_jx_0 - (-2x_0x_i) + 2x_ix_j + 2x_0^2 \right) \\ &= -x_jx_0 - x_0x_i + x_jx_i + x_0^2 \\ &= (x_i - x_0) * (x_j - x_0) \end{aligned}$$

The determinant of a matrix with these entries is the square of the volume of a parallelepiped spanned by the set of n vectors (x_1, \dots, x_n) based at x_0 . \square

With this machinery, it is possible to find the volume of the simplex (x_1, \dots, x_n) .

PROPOSITION 19.8.28. *The volume of the n -simplex $Y = (x_1, \dots, x_n)$ in Euclidean space is*

$$\text{Vol}_n(Y) = \frac{1}{n!} \sqrt{D(x_0, x_1, \dots, x_n)}$$

Having computed this volume in Euclidean space, define the volume of an n -simplex Y in any metric space to be the formula given by $\text{Vol}_n(Y)$. We can now provide two definitions which will describe which metric spaces can be embedded in \mathbb{R}^N .

DEFINITION 19.8.29. A metric space (X, d) is *flat* if for each n -simplex Y in X , $\text{Vol}_n(Y)$ is real.

DEFINITION 19.8.30. If (X, d) is a flat metric space, the dimension of (X, d) is the largest $n \in \mathbb{N}$ where there exists an n -simplex of X with positive volume.

These characteristic of metric spaces will determine which can be isometrically embedded in \mathbb{R}^N . To prove Morgan's main theorem, some results from linear algebra are quickly cited.

LEMMA 19.8.31. *Any real n -dimensional inner product space is linearly isometric to Euclidean n -space.*

LEMMA 19.8.32. *Let M be an $m \times m$ real symmetric matrix with all non-negative eigenvalues. If $D[i, j]$ is the determinant of the $(m-1) \times (m-1)$ minor of M obtained by deleting its i th row and j th column, then $D[i, j]^2 \leq D[i, i]D[j, j]$.*

THEOREM 19.8.33. *Morgan's Embedding in \mathbb{R}^N . A metric space can be isometrically embedded in Euclidean n -space iff the metric space is flat and has dimension less than or equal to n .*

PROOF. Take a metric space (X, d) which can be isometrically embedded in Euclidean n -space. Isometries preserve volume, so the simplices must have real volume in (X, d) (as they have real volume in \mathbb{R}^N), so (X, d) is flat. Because volume is preserved, the simplices of positive volume in (X, d) have positive volume in \mathbb{R}^N . Since there cannot be any simplices of positive volume in \mathbb{R}^N with greater than $n+1$ points, (X, d) must have dimension less than or equal to n .

Suppose (X, d) is flat and of dimension n with n -simplex $Y = (x_0, x_1, \dots, x_n)$ such that Y has positive volume.

If a map $f : X \rightarrow \mathbb{R}^N$ can be constructed such that f embeds X isometrically in \mathbb{R}^N with some inner product, then (X, d) can be embedded in Euclidean n -space because any real n -dimensional inner product space is linearly isometric to Euclidean n -space.

Let $f : X \rightarrow \mathbb{R}^N$ be the map defined by $f(x) := (\langle x, x_1, x_0 \rangle, \dots, \langle x, x_n, x_0 \rangle)$, and construct bilinear form on \mathbb{R}^N as follows: Let L be an $n \times n$ matrix with entries $(a_{i,j}) = \langle x_i, x_j, x_0 \rangle$, and let

$$\langle u, v \rangle = u^t L^{-1} v \text{ for all } u, v \in \mathbb{R}^N$$

If the eigenvalues of matrix L are positive, this bilinear form is an inner product on \mathbb{R}^N and f embeds (X, d) isometrically into this inner-product space.

The roots of the polynomial $\det(xI+L)$ are the negatives of the eigenvalues of L . Thus, we can look at the coefficient of the term of degree $n - k$ in this polynomial, which is the sum of the $k * n$ minors that lie along the main diagonal. These minors are all non-negative because they are volumes of k -simplicial complexes (these volumes are all real, nonnegative as (X, d) is flat and dimension n). These make the polynomial positive, so it must have no positive roots, so there cannot be negative eigenvalues of L . L being symmetric and non-singular (as (X, d) has non-zero dimension) ensures that its eigenvalues are positive.

We show the inner product given on \mathbb{R}^N preserves the structure of all of the n -simplexes of (X, d) , and that therefore f is an isometry, by showing that

$$\langle f(x), f(y) \rangle = \langle x, y, x_0 \rangle \text{ for all } x, y \in X$$

Construct a $(n + 2) \times (n + 2)$ matrix M with entries $\langle x_j, x_i, x_0 \rangle$. By the same reasoning used on the similarly constructed matrix L , M has all non-negative eigenvalues.

Set $D[i, j]$ to be the determinant of the $(n + 1) \times (n + 1)$ of the matrix obtained by deleting the i th row and j th column of M .

Recall the lemma stating that $D[i, j]^2 \leq D[i, i]D[j, j]$. $D[i, i]$ is the determinant corresponding to the volume of a $(n + 1)$ -simplex squared and scaled by a factor of $(n + 1)!$ and since (X, d) is n -dimensional, the volume of any $(n + 1)$ -simplex must be zero, and therefore $D[i, i] = 0$. By the lemma, this also means that $D[i, j] = 0$.

Setting $i = n$ and $j = n + 1$ shows that, in particular, $D[n, n + 1] = 0$. Consider the minor of M with the n th row and $(n + 1)$ st columns deleted.

$$\begin{pmatrix} \langle x_1, x_1, x_0 \rangle & \dots & \dots & \langle x_n, x_1, x_0 \rangle & \langle x_{n+2}, x_1, x_0 \rangle \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \langle x_1, x_{n-1}, x_0 \rangle & \dots & \dots & \langle x_n, x_{n-1}, x_0 \rangle & \langle x_{n+2}, x_{n-1}, x_0 \rangle \\ \langle x_1, x_{n+1}, x_0 \rangle & \dots & \dots & \langle x_n, x_{n+1}, x_0 \rangle & \langle x_{n+2}, x_{n+1}, x_0 \rangle \\ \langle x_1, x_{n+2}, x_0 \rangle & \dots & \dots & \langle x_n, x_{n+2}, x_0 \rangle & \langle x_{n+2}, x_{n+2}, x_0 \rangle \end{pmatrix}$$

Since in $\langle f(x), f(y) \rangle = f(x)^t L^{-1} f(y)$ in general, the condition for isometry is

$$\langle f(x), f(y) \rangle = \langle x, y, x_0 \rangle$$

Set $x := x_{n+1}$ and $y := x_{n+2}$ so that

$$f(x) = (\langle x_{n+1}, x_1, x_0 \rangle, \dots, \langle x_{n+1}, x_n, x_0 \rangle), f(y) = (\langle x_{n+2}, x_1, x_0 \rangle, \dots, \langle x_{n+2}, x_n, x_0 \rangle)$$

Note that by deleting one row and one column from the matrix above, and dividing by the determinant of L , the matrix becomes the L^{-1} (when assigning the correct cofactor signs).

Expand the above matrix by the last row to calculate the determinant, using the minors

$$\begin{pmatrix} \langle x_1, x_1, x_0 \rangle & \dots & \dots & \langle x_n, x_1, x_0 \rangle \\ \vdots & \ddots & \ddots & \dots \\ \langle x_1, x_{n-1}, x_0 \rangle & \dots & \dots & \langle x_n, x_{n-1}, x_0 \rangle \\ \langle x_1, x_{n+1}, x_0 \rangle & \dots & \dots & \langle x_n, x_{n+1}, x_0 \rangle \end{pmatrix} = \begin{pmatrix} \langle x_2, x_1, x_0 \rangle & \dots & \dots & \langle x_{n+2}, x_1, x_0 \rangle \\ \vdots & \ddots & \ddots & \dots \\ \langle x_2, x_{n-1}, x_0 \rangle & \dots & \dots & \langle x_{n+2}, x_{n-1}, x_0 \rangle \\ \langle x_2, x_{n+1}, x_0 \rangle & \dots & \dots & \langle x_{n+2}, x_{n+1}, x_0 \rangle \end{pmatrix}$$

Taking the appropriate sign changes and summing their determinants gives zero (as $D[n, n + 1] = 0$). So dividing by $\det(L)$ still yields zero.

Continue the calculation to get that

$$\langle x_{n+1}, x_{n+2}, x_0 \rangle = f(x_{n+1})^t L^{-1} f(x_{n+2})$$

This means that $\langle f(x), f(y) \rangle = \langle x, y, x_0 \rangle$ for all $x, y \in X$ and thus, f is an isometry. \square

These characterizations of metric spaces provides a useful way to analyze examples of metric spaces.

THEOREM 19.8.34. [53] *For $n \geq 2$, \mathbb{R}^N with the ℓ^p metric is flat iff $p = 2$.*

PROOF. Morgan gives the two examples used below for his proof of this theorem without additional argument. However, working through the process to show why these examples work illustrates why the case when $p = 2$ is special.

Given \mathbb{R}^N with the ℓ^2 metric, the previous theorem proves that it is flat (i.e. (\mathbb{R}^N, ℓ^2) can embed in itself). The example given in [Example 19.8.1](#) of a non-embeddable metric space suggests how to construct simplices of imaginary volume in (\mathbb{R}^N, ℓ^p) when $p \neq 2$. It is only necessary to find examples in \mathbb{R}^2 as $\mathbb{R}^2 \subset \mathbb{R}^N$ for $n \geq 2$.

Consider (\mathbb{R}^N, ℓ^p) for $p < 2$.

If $1 \leq p$, the ℓ^p is induced by the norm

$$\|x\|_p = \left(\sum_n |x_n|^p \right)^{\frac{1}{p}}$$

Take the example of the 3-simplex Y in (\mathbb{R}^N, ℓ_2) with $Y = \{(0, 0), (1, 0), (1, 1), (0, 1)\}$. Observe that for any value of $p \geq 1$, the horizontal and vertical distances on this simplex are the same.

If $p \geq 1$,

$$d((a, b), (a, c)) = \|(a, b) - (a, c)\|_p = (|(a - a)|^p + |(b - c)|^p)^{\frac{1}{p}} = |b - c|$$

The same argument applies, by symmetry, when the second coordinates are equal. This means that distortion would occur in the distance between two non-adjacent points in this simplex. By the triangle inequality, for any $p \geq 1$,

$$d((0, 0), (1, 1)) \leq d((0, 0), (0, 1)) + d((0, 1), (1, 1)) = 1 + 1 = 2$$

$$d((0, 0), (1, 1)) \leq d((0, 0), (1, 0)) + d((1, 0), (1, 1)) = 1 + 1 = 2$$

$$d((0, 0), (1, 1)) = \|(0, 0) - (1, 1)\|_p = (|(0 - 1)|^p + |(0 - 1)|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

As $p \rightarrow \infty$, the quantity $d((0, 0), (1, 1)) \rightarrow 1$, so this square in (\mathbb{R}^N, ℓ_2) collapses to a line as p increases.

Now consider the matrix constructed to compute function $D(Y)$:

$$A = \begin{pmatrix} \langle (0, 0), (1, 0), (1, 0) \rangle & \langle (0, 0), (1, 0), (1, 1) \rangle & \langle (0, 0), (1, 0), (0, 1) \rangle \\ \langle (0, 0), (1, 1), (1, 0) \rangle & \langle (0, 0), (1, 1), (1, 1) \rangle & \langle (0, 0), (1, 1), (0, 1) \rangle \\ \langle (0, 0), (0, 1), (1, 0) \rangle & \langle (0, 0), (0, 1), (1, 1) \rangle & \langle (0, 0), (0, 1), (0, 1) \rangle \end{pmatrix}$$

Notice an entry on the diagonal takes the form

$$\langle x, y, y \rangle = \frac{1}{2} (d(x, y)^2 + d(y, y)^2 - d(x, y)^2) = 0,$$

and therefore A has a zero diagonal. Then since $d((0, 0), (0, 1)) = d((0, 0), (1, 0)) = d((1, 0), (1, 1)) = d((0, 1), (1, 1)) = 1$ for any p , matrix A can be simplified to

$$A = \begin{pmatrix} 0 & \frac{1}{2}[d((0, 0), (1, 1))^2] & \frac{1}{2}[d((1, 0), (0, 1))^2] \\ 1 - \frac{1}{2}[d((0, 0), (1, 1))^2] & 0 & 1 - \frac{1}{2}[d((0, 0), (1, 1))^2] \\ \frac{1}{2}[d((0, 1), (1, 0))^2] & \frac{1}{2}[d((0, 0), (1, 1))^2] & 0 \end{pmatrix}$$

We find $D(Y) := D((0, 0), (1, 0), (1, 1), (0, 1))$ can be calculated:

$$\begin{aligned} D(Y) &= \left[\frac{1}{2}d((0, 0), (1, 1))^2 \right] \left[\frac{1}{2}d((0, 1), (1, 0))^2 \right] \left[1 - \frac{1}{2}d((0, 0), (1, 1))^2 \right] \\ &\quad + \left(\left[\frac{1}{2}d((1, 0), (0, 1))^2 \right] \left[1 - \frac{1}{2}d((0, 0), (1, 1))^2 \right] \left[\frac{1}{2}d((0, 0), (1, 1))^2 \right] \right) \\ &= d((1, 0), (0, 1))^2 d((0, 0), (1, 1))^2 \left(\frac{1}{2} - \frac{1}{4}d((0, 0), (1, 1))^2 \right) \end{aligned}$$

The term $d((1, 0), (0, 1))^2 d((0, 0), (1, 1))^2$ is always positive. Then this value of $D(Y)$ is negative (and so the volume of Y imaginary) only when

$$\frac{1}{2} < \frac{1}{4}d((0, 0), (1, 1))^2$$

Solving this inequality gives that the volume is imaginary when $\sqrt{2} < d((0, 0), (1, 1))$. If $0 < p < 1$ then ℓ^p has the metric $d_p(x, y) = \sum_{i=1}^n |x_i - y_i|^p$ so

$$d((0, 0), (1, 1)) = \sum_{i=1}^2 |0 - 1|^p = 1^p + 1^p = 2$$

Then since $D(Y)$ is negative for $0 < p < 1$, $Vol(Y)$ is imaginary, and therefore (\mathbb{R}^N, ℓ^p) is not flat for $0 < p < 1$. If $1 \leq p < 2$, then this distance takes the form

$$d((0, 0), (1, 1)) = \|(0, 0) - (1, 1)\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

If $p < 2$, then the inequality is satisfied, meaning that (\mathbb{R}^N, ℓ^p) is not flat for $1 \leq p < 2$.

Consider (\mathbb{R}^N, ℓ^p) for $p > 2$. Take example of the 3-simplex Y in (\mathbb{R}^N, ℓ^p) with $Y = \{(0, 1), (1, 0), (-1, 0), (0, -1)\}$. This simplex has vertical and horizontal distances of 2 which are preserved in all (\mathbb{R}^N, ℓ^p) for all p . It is the distances which are not preserved which will cause this simplex to have imaginary volume for $p > 2$. This example's invariant distances are larger than the changing distances, so by repeating the same computation as above, the inequality is reversed, giving that the volume of Y is imaginary when

$$\sqrt{2} > d((-1, 0), (0, 1))$$

This is an equality when $p = 2$. By the same analysis as above, as p becomes greater than 2, this inequality is satisfied, showing that Y has an imaginary volume when $p > 2$, and therefore (\mathbb{R}^N, ℓ^p) is not flat for $p > 2$. \square

19.8.5. Embeddings of the ℓ_2 Metric. In [Section 19.8.2](#) it was shown that ℓ_2 is close to other normed spaces. That is, there is a linear isomorphism between them which requires little distortion of the spaces. It is then natural to ask when there is an isometric embedding from ℓ_2 to other spaces.

19.8.5.1. *Dimension reduction in ℓ_2 .* Given a metric space (X, ℓ_2) in \mathbb{R}^N , it is useful to ask whether the dimension of the host space, ℓ_2 , can be reduced in exchange for distortion. A paper by William Johnson and Joram Lindenstrauss quantified the possible dimension reduction.

THEOREM 19.8.35. (*Johnson and Lindenstrauss Dimension Reduction [36]*) *Given any n -point metric space $(X, \ell_2) \subset \mathbb{R}^N$ and $\epsilon > 0$, there is an embedding of*

distortion of at most $1 + \epsilon$ such that

$$(X, \ell_2) \longrightarrow \ell_2^{O\left(\frac{\log n}{\epsilon^2}\right)}$$

The proof of this dimension reduction theorem and other proofs of isometric embedding from ℓ_2 to ℓ_p uses a technique in theoretical computer science, *random projection*.

DEFINITION 19.8.36. Take vectors $r_1, \dots, r_k \subset \mathbb{R}^N$ which have been obtained by some random process. Then define map $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ as follows:

$$\psi : v \longrightarrow (\langle v, r_1 \rangle, \dots, \langle v, r_k \rangle)$$

The map ψ is a *random projection* from $\mathbb{R}^N \longrightarrow \mathbb{R}^k$.

Random projection ψ can be conveniently expressed as a $k \times n$ matrix A whose rows are r_1, \dots, r_k so that $\psi(v) = Av$. This means that random projections are linear.

There are three notable examples of random process used to generate the r_1, \dots, r_k . All three have been used to prove the Johnson-Lindenstrauss Theorem.

- EXAMPLE 19.8.37. (1) Set each $r_i = (r_i^1, \dots, r_i^n)$ and obtain values for each r_i^j from a normal probability distribution centered at 0 with variance 1. This is labeled ψ_N and was used to prove Johnson-Lindenstrauss [35].
- (2) Set each $r_i = (r_i^1, \dots, r_i^n)$ and obtain values for each r_i^j by choosing either +1 or -1, each with probability $\frac{1}{2}$. This method is called binary coins and is labeled ψ_B . This is the simplest method used to prove Johnson-Lindenstrauss [1].
- (3) Take r_1, \dots, r_k to be a set of k orthogonal vectors from S^{n-1} . This is labeled ψ_S and was originally used by Johnson and Lindenstrauss [36].

19.8.5.2. *Isometric Embedding from ℓ_2 to ℓ_1* . Two interesting cases of ℓ_p spaces are ℓ_2 and ℓ_1 , so the existence of an isometric embedding of a n -point metric space in ℓ_2^n to some finite dimensional ℓ_1^k is an important one. In order to prove that there does exist such an embedding, the space $\ell_1^{S^{n-1}}$ will be explored. The definition of this space and the proof of an embedding theorem is given in lecture 12 of the series on finite metric spaces given at TTIC [58].

DEFINITION 19.8.38. Space $\ell_1^{S^{n-1}}$ is a ℓ_1 metric space with a coordinate for each vector in S^{n-1} . Each point in $\ell_1^{S^{n-1}}$ is given by a function $f : S^{n-1} \longrightarrow \mathbb{R}$. The ℓ_1 norm is given by

$$\|f\|_1 = \int_{r \in S^{n-1}} |f(r)| dr$$

LEMMA 19.8.39. *There exists an isometric embedding of every n -point metric space in ℓ_2^n to $\ell_1^{S^{n-1}}$.*

With this embedding lemma, it only need be shown that there is an isometric embedding from to isometric embeddings from $\ell_1^{S^{n-1}}$ into a finite dimensional ℓ_1 . This result can also be generalized to isometric embeddings from $\ell_p^{S^{n-1}}$ to finite dimensional ℓ_p .

THEOREM 19.8.40. [58] *Every n -point metric space in ℓ_2^n can be isometrically embedded in ℓ_1^n .*

PROOF. Sketch. Isometrically embed space metric space $X = \{x_1, \dots, x_n\}$ in ℓ_2^n by the above lemma. S^{n-1} is partitioned into $n!$ regions and each region is assigned an x_i and x_j . Each region is defined in such a way that the sign of $\langle x_i, r \rangle - \langle x_j, r \rangle$ is constant within it. It can then be shown that this produces an isometric embedding from ℓ_2^n to $\ell_1^{S^{n-1}}$ and into $\ell_1^{n!}$. \square

Reduction methods of finite spaces

The manipulation of a finite space through removal of points presents a space weakly homotopy equivalent to the original. The exposition presented in the next section follows the work of Sharon Zhou, in her 2020 REU paper on homotopy types of finite spaces and simplicial complexes.

As observed, if two finite spaces X and Y are homotopy equivalent, then so are their corresponding order complexes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$. In fact, T. Osaki [55] showed that $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are actually simple homotopy equivalent, which is a more refined notion of homotopy equivalence in the world of simplicial complexes.

In order to understand this result, the following definition of simple homotopy is presented:

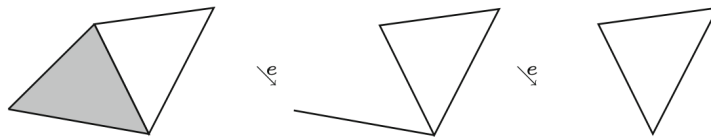
DEFINITION 20.0.1. Let K be a finite simplicial complex and $L \subset K$ be a subcomplex. We say that K collapses to L via an *elementary simplicial collapse* and write $K \searrow^e L$ if there exists a simplex $S \in K$ and a vertex $a \in K$ that is not contained in S such that

$$K = L \cup aS \text{ and } L \cap aS = a\partial S.$$

In other words, K collapses to L via an elementary simplicial collapse if there are only two simplices $S, S' \in K$ disjoint from L such that S is a free face of S' , i.e., S' is the only simplex disjoint from L that contains S as a face.

DEFINITION 20.0.2. We say that K (*simplicially*) *collapses* to L or L (*simplicially*) *expands* to K if L can be obtained from K via a sequence of elementary collapses. We denote this by $K \searrow L$ or $L \nearrow K$. Two complexes K and L have the same *simple homotopy type* if there exists a sequence of simplicial complexes $K = K_1, K_2, \dots, K_n = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for all $1 \leq i \leq n$.

For a concrete example, consider the sequence of elementary collapses below, which can be found in [7].



We say that a simplicial complex K is *collapsible* if it collapses to one of its vertices. For example, any simplicial cone aK is collapsible. The key observation here is that simple homotopy equivalence is a special case of homotopy equivalence, as we show below.

PROPOSITION 20.0.3. *If two simplicial complexes are simple homotopy equivalent, then they are homotopy equivalent.*

PROOF. Let K, L be two simplicial complexes. Without loss of generality, let $L \subset K$ be a subcomplex and suppose that K collapses to L via an elementary simplicial collapse. Then there exists some simplex $S \in K$ and a vertex $a \in K, a \notin S$ such that $K = L \cup aS$ and $L \cap aS = a\partial S$. Note that the inclusion $i : L \cap aS \hookrightarrow aS$ is a homotopy equivalence. Applying the gluing theorem (Theorem A.2.5 in [7]) to the diagram below,

$$\begin{array}{ccc} L \cap aS & \xrightarrow{i} & aS \\ \cap \downarrow & & \downarrow \\ L & \xrightarrow{I} & K \end{array}$$

we see that the inclusion $I : L \hookrightarrow K$ is also a homotopy equivalence. \square

This yields the following proper containment of types of homotopies between simplicial complexes, where \mathcal{S} denotes the set of simple homotopy equivalences. A theorem by Whitehead shows that a homotopy equivalence between simplicial complexes is a simple homotopy equivalence precisely when the Whitehead torsion τ vanishes (see [52] for details).

$$\mathcal{S} \subset \{\text{Homotopy equivalence}\} = \{\text{Weak Homotopy Equivalence}\}$$

In finite spaces, as we will soon show, a different relation holds:

$$\{\text{Homotopy equivalence}\} \subset \mathcal{S} \subset \{\text{Weak equivalence}\}$$

In both cases, the containment is proper. A natural question to ask is whether there exists some kind of homotopy equivalence between simple homotopy and homotopy equivalence of CW complexes. In other words, one might hope to define a new class of homotopy equivalences that will “fill in” the first chain of set containment. The close correspondence between finite spaces and simplicial complexes suggests that we may find an answer by examining the hierarchy of homotopy equivalences of finite spaces.

The following question may then be presented: *Can we further refine the notion of homotopy equivalence in the world of simplicial complexes to obtain some formal class of homotopy equivalence between simple homotopy equivalence and general homotopy equivalence by using the homotopy theory of finite spaces?*

Although this remains an open problem, several methods of examination are presented in Section 4 of the source paper.

What follows presents the effect of one-point reductions on the order complex of the space. In particular, results will show that removing beat points from a finite space X does not affect the homotopy type of either X or $\mathcal{K}(X)$.

20.0.1. One-point reduction of finite spaces. In this section, we study three types of one-point reductions of finite spaces, namely the removal of beat points, weak points, and γ -points, and consider what kind of homotopy equivalences they induce on the corresponding order complexes.

20.0.1.1. *Beat points.* Recall that two finite spaces are homotopy equivalent if and only if one can be obtained from another by successively removing or adding beat points. We have the following useful corollary, directly implied from

[Theorem 2.4.4.](#)

COROLLARY 20.0.4. *A finite space X is contractible if and only if one can remove beat points from X one at a time to obtain a space consisting of only one point.*

We now consider what kind of homotopy equivalence a beat point removal will induce on the order complex associated to the original finite space. As one would reasonably expect, removing a beat point from a finite space X does not change the simple homotopy type of $\mathcal{K}(X)$. This result was first proved by Osaki [55].

THEOREM 20.0.5 (Osaki). *If x is a beat point, then $\mathcal{K}(X)$ collapses to $\mathcal{K}(X \setminus \{x\})$.*

Since two finite spaces are homotopy equivalent if and only if one can be obtained from another by successively removing and adding beat points, this theorem generalizes readily to the following corollary.

COROLLARY 20.0.6. *If X and Y are homotopy equivalent, then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type.*

There is one thing unsatisfactory, however, about this corollary: its converse is false. To see that, consider the following example given by Barmak and Minian [7].

EXAMPLE 20.0.7. The finite space W (inspired by its resemblance to a wallet), which we draw below, has no beat points and is therefore non-contractible. Nevertheless, if one follows the definition of an order complex and draws out $\mathcal{K}(W)$, one sees that $\mathcal{K}(W)$ is contractible. In fact, it will soon be shown that $\mathcal{K}(W)$ is simple homotopy equivalent to a point.

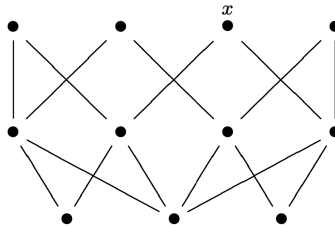


FIGURE 1. W

This example suggests that homotopy equivalence of finite spaces is a “stronger” relation than simple homotopy equivalence of simplicial complexes. To put it more precisely, the set of homotopy equivalences in simplicial complexes that are induced by removal of beat points from finite spaces, which we sometimes call *strong homotopy equivalence*, is a proper subset of the set of simple homotopy equivalences of simplicial complexes. Accordingly, the removal of a beat point from a finite space is a “stronger” move than an elementary collapse in simplicial complexes.

This observation naturally gives rise to the following question: does there exist an “elementary move” in finite spaces that would precisely correspond to an elementary collapse in simplicial complexes? It is for precisely this reason that Barmak and Minian [7] introduced the notion of a weak point. In particular, we will show that the point $x \in W$ in the above example is a weak point, and that $\mathcal{K}(W)$ is homotopically trivial.

20.0.1.2. *Weak points.*

DEFINITION 20.0.8. Let X be an F -space. We say that $x \in X$ is an *up weak point* if \hat{F}_x is contractible and a *down weak point* or \hat{U}_x is contractible. A point is a *weak point* if it is either an up weak point or a down weak point.

Note that a beat point is necessarily a weak point, since for any beat point x , either \hat{U}_x has a maximum or \hat{F}_x has a minimum, which makes at least one of these two sets contractible. To lighten the notation, we make the following definitions.

DEFINITION 20.0.9. Given an F -space X , the *link* of $x \in X$ is defined as $lk(x) = \hat{C}_x = \hat{U}_x * \hat{F}_x$.

The following lemma gives us an alternative way to characterize weak points.

LEMMA 20.0.10. *Let X, Y be F -spaces. Then the join $X * Y$ is contractible if and only if either X or Y is contractible.*

PROOF. Without loss of generality, suppose that X is contractible with point $\{+\}$. By Corollary 20.0.4, we can find a decreasing sequence of spaces

$$X = X_n \supset X_{n-1} \supset \dots X_1 = \{+\},$$

where we remove beat points from X one by one such that each X_i contains i points and $x_i \in X_i$ is a beat point. Note that x_i is also a beat point of $X_i * Y$, so $X * Y$ inductively deformation retracts to $\{+\} * Y$, which has a minimum and is therefore contractible. The argument where Y is contractible is exactly analogous if one replaces minimum by maximum at the end.

Conversely, suppose that $X * Y$ is contractible. Again by Corollary 20.0.4, there exists a decreasing sequence of spaces

$$X * Y = (X * Y)_n \supset (X * Y)_{n-1} \supset \dots (X * Y)_1 = \{+\},$$

where $(X * Y)_i = \{z_1, z_2, \dots, z_i\}$ such that z_i is a beat point of $(X * Y)_i$.

Fix some $2 \leq i \leq n$, and suppose that $z_i \in X_i$. Then z_i is a beat point of X_i unless it is a maximal point of X_i , Y_i has a minimum, and $X_i \setminus \{z_i\}$ has no maximum. Similarly, if $z_i \in Y_i$, then either z_i is a beat point of Y_i or X_i has a maximum and $Y_i \setminus \{z_i\}$ has no minimum. Thus for every i , at least one of the following statements is true: (1) either $X_{i-1} \hookrightarrow X_i$ or $Y_{i-1} \hookrightarrow Y_i$ is a deformation retract, and (2) one of X_i and Y_i is contractible. Hence X or Y is contractible, as desired. \square

PROPOSITION 20.0.11. *Let X be an F -space. Then $x \in X$ is a weak point if and only if $lk(x) = \hat{C}_x$ is contractible.*

As shown, if x is a beat point of X , then $X \setminus \{x\}$ is homotopy equivalent to X . This is no longer true if we replace beat points with weak points. Nevertheless, a weaker version of this result holds.

PROPOSITION 20.0.12. *Let X be an F -space, and let $x \in X$ be a weak point. Then the inclusion $i : X \setminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

The proof of this proposition makes use of Theorem 3.3.1. Note that for any F -space X , the minimal basis $\{U_x\}_{x \in X}$ is a basis like open cover.

PROOF. Without loss of generality, suppose that x is a up weak point. Then \hat{F}_x is contractible. Let $y \in X$. Then the set $i^{-1}(F_y) = F_y \setminus \{x\}$ has a minimum if $y \neq x$, and is contractible if $y = x$. Hence the restricted map

$$i|_{i^{-1}(F_y)} = i^{-1}(F_y) \rightarrow F_y,$$

is a weak homotopy equivalence, since the map $\pi_n(i^{-1}(F_y), y) \rightarrow \pi_n(F_y, y)$ is an isomorphism for all n . As remarked above, the minimal basis of X is a basis like open cover of X . Now applying Theorem 3.15 to the minimal basis of X shows that the restricted inclusion is a weak homotopy equivalence.

The case where x is a down weak point follows immediately by applying the above argument to X^{op} , noting that $\mathcal{K}(X^{op}) = \mathcal{K}(X)$. \square

To illustrate this proposition, let us return to Example 20.0.7, as promised. To see that the point x is a weak point, we draw out the subspace \hat{U}_x as follows.

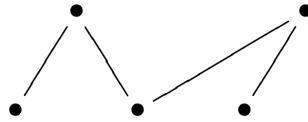


FIGURE 2. \hat{U}_x

Clearly, \hat{U}_x is contractible, so x is a weak point. Hence Proposition 20.0.12 tells us that W is weak homotopy equivalent to $W \setminus \{x\}$, whose Hasse diagram looks like the following:

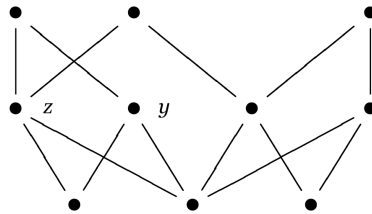


FIGURE 3. $W \setminus \{x\}$

$W \setminus \{x\}$ is contractible because we can remove beat points one by one (starting with the point y as labeled in the diagram, then proceed to z , and so on), eventually obtaining a space consisting of a single point. This motivates the following definition.

DEFINITION 20.0.13. Let X be an F -space and $Y \subset X$ a subspace. We say that X collapses to Y by an *elementary collapse* (or that Y expands to X by an *elementary expansion*) if Y is obtained from X by removing a weak point. In this case, we denote $X \searrow_e Y$ or $Y \nearrow_e X$.

In general, given two F -spaces X and Y , we say that X collapses to Y (or Y expands to X) if there is a sequence of F -spaces $X = X_1, X_2, \dots, X_n = Y$ such that for each $1 \leq i < n$, $X_i \searrow^e X_{i+1}$. In this case, we write $X \searrow Y$ or $Y \nearrow X$. Two F -spaces X and Y are *simply equivalent* if one can be obtained from another via a sequence of elementary collapses and expansions.

Before stating the following corollary from [Proposition 20.0.12](#), we make a quick note on convention: adopting the terminology of Barmak and Minian, we will say that two F -spaces are simply equivalent and two simplicial complexes are simple homotopy equivalent (or have the same simple homotopy type). The crux of [Theorem 20.0.15](#) is that these two definitions are really describing the same relation for two kinds of objects.

COROLLARY 20.0.14. *Let X, Y be two simply equivalent F -spaces. Then they are weakly equivalent.*

The next theorem, which was proved by Barmak and Minian [\[7\]](#) as the main result of simple homotopy theory of finite spaces and simplicial complexes, essentially says that weak points do exactly what we want them to do. That is, removal of weak points is the F -space counterpart to an elementary simplicial collapse in simplicial complexes.

THEOREM 20.0.15 (Barmak and Minian).

- (1) *Let X and Y be F -spaces. Then X and Y are simply equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. In particular, if $X \searrow Y$, then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*
- (2) *Let K and L be finite simplicial complexes. Then K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simply equivalent. In particular, if $K \searrow L$, then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.*

We say that an F -space is *collapsible* if it collapses to a point. Similarly, a simplicial complex is said to be *collapsible* if it simplicially collapses to a single point. Since every weak point is a weak point, the set of contractible F -spaces is a proper subset of collapsible spaces. For example, the wallet W as constructed above is a collapsible space that is not contractible.

20.0.1.3. γ -points. Recall that our goal is to define a formal class of homotopy equivalences of simplicial complexes that are not simple homotopy equivalences. Having seen that removing weak points induces simple homotopy equivalences in simplicial complexes, we want to relax the condition even further.

This motivates the definition of a γ -point.

DEFINITION 20.0.16. Let X be an F -space. Then $x \in X$ is a γ -point if \hat{C}_x is homotopically trivial. That is, $\pi_n(\hat{C}_x) = 0$ for all $n \geq 0$.

This definition gives us a new method of reduction of finite spaces.

DEFINITION 20.0.17. We say that X γ -collapses to $X \setminus \{x\}$ by an *elementary γ -collapse* if $x \in X$ is a γ -point. More generally, an F -space X γ -collapses to a subspace $Y \subset X$ if there is a sequence of spaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_k = Y \quad (n > k)$$

such that X_i γ -collapses to X_{i-1} via an elementary γ -collapse for all $k \leq i \leq n$. In this case, we write $X \nearrow^\gamma Y$. If X γ -collapses to a point, we say that X is *γ -collapsible*.

Note that every weak point is a γ -point, since a contractible space necessarily has all trivial homotopy groups. To see what kind of homotopy equivalence a γ -point reduction will induce on simplicial complexes, we first consider the relationship between $X \setminus \{x\}$ and X where $x \in X$ is a γ -point.

PROPOSITION 20.0.18. *If $x \in X$ is a γ -point, then the inclusion $i : X \setminus \{x\} \rightarrow X$ is a weak homotopy equivalence.*

The proof for [Proposition 20.0.12](#) does not apply directly because neither \hat{F}_x nor \hat{U}_x is necessarily contractible. Nevertheless, the following pushout diagram still holds:

$$\begin{array}{ccc} |\mathcal{K}(\hat{C}_x)| & \xrightarrow{\varphi} & |\mathcal{K}(C_x)| \\ \downarrow \psi & & \downarrow \\ |\mathcal{K}(X \setminus \{x\})| & \longrightarrow & |\mathcal{K}(X)| \end{array}$$

Note that $\varphi : |\mathcal{K}(\hat{C}_x)| \rightarrow |\mathcal{K}(C_x)|$ is a homotopy equivalence, and that $\psi : |\mathcal{K}(\hat{C}_x)| \rightarrow |\mathcal{K}(X \setminus \{x\})|$ satisfies the homotopy extension property. Hence the map $|\mathcal{K}(X \setminus \{x\})| \rightarrow |\mathcal{K}(X)|$ is a homotopy equivalence. This implies that $i : X \setminus \{x\} \rightarrow X$ is a weak homotopy equivalence. The converse to this proposition, however, is true only when x is neither maximal nor minimal ([Theorem 3.13](#) in [\[7\]](#)).

If $x \in X$ is a γ -point, one can show that the map $\mathcal{K}(X \setminus \{x\}) \rightarrow \mathcal{K}(X)$ is a simple homotopy equivalence (the proof uses the relativity principle of simple homotopy theory; see [\[19\]](#)). In fact, Barmak and Minian [\[7\]](#) proved the following more general result, which says that this is the case whenever we have a weak homotopy equivalence between finite spaces.

THEOREM 20.0.19. *Let X be an F -space, and let $x \in X$. Suppose that the inclusion $i : X \setminus \{x\} \rightarrow X$ is a weak homotopy equivalence. Then the induced simplicial map $\mathcal{K}(X \setminus \{x\}) \rightarrow \mathcal{K}(X)$ is a simple homotopy equivalence.*

This theorem essentially shows that one-point reductions do not generate all weak homotopy types of finite spaces. We might then look beyond one-point reductions, a discussion for the following section. Before proceeding, we briefly discuss how some of the previous results can be generalized to a broader class of topological spaces.

While we cannot directly take these results for granted in general CW complexes, we can consider them on subsets called *regular* and *h -regular* CW complexes.

DEFINITION 20.0.20. Let K be a CW complex. We say that K is *regular* if, for each open cell e^n , the characteristic map $D^n \rightarrow e^n$ is a homeomorphism. Equivalently, the attaching map $S^{n-1} \rightarrow K$ is a homeomorphism onto its image ∂e^n .

For a regular CW complex K , the closure $\overline{e^n}$ of each cell is a subcomplex of K . There is also a more general notion of *h -regular* CW complex, where one only requires the attaching map of each cell to be a homotopy equivalence with its image and that the closed cells e^n are subcomplexes of K .

[Theorem 20.0.15](#) fails even when we consider only regular CW complexes (see page 60 of [\[7\]](#) for a counterexample). Nevertheless, a weaker version of the second part

of Theorem 20.0.15, as proved in the same book, shows that simplicial collapses of h -regular CW complexes do induce γ -collapses in the corresponding finite spaces.

CHAPTER 21

Epilogue

We conclude the book with a few remarks on an old list of problems.

Revisit

We give a few problems that spring immediately to mind. To the best of my knowledge, these have not been studied, at least not thoroughly. The original 2003 list was considerably longer, but a number of people around the world have since solved many of its problems. Some of their solutions are sprinkled through the book.

PROBLEM 21.0.1. *For small n , determine all homotopy types and weak homotopy types of spaces with at most n elements.*

ADDENDUM 21.0.1. We have given the answer or left it as an exercise when $n \leq 6$. Most finite spaces with so few points are disjoint unions of (weakly) contractible spaces, but we have seen several more interesting examples. I'd like to see the answer for larger n .

PROBLEM 21.0.2. *Is there an effective algorithm for computing the homotopy groups of X in low degrees in terms of the increasing chains in X ? An REU paper of Weng described in §Section 4.3 elaborated on the computation of the fundamental group by Barmak [7].*

Not written yet

REMARK 21.0.3. The dimension of the simplicial complex $\mathcal{K}(X)$ is the maximal length of a sequence $x_0 < \cdots < x_n$ in X . A map $g: K \rightarrow L$ of simplicial complexes of dimension less than n is a homotopy equivalence if and only if it induces an isomorphism of homotopy groups in dimension less than n and an epimorphism of homotopy groups in dimension n .

PROBLEM 21.0.4. *Let X be a minimal finite space. Give a descriptive interpretation of what this says about $|\mathcal{K}(X)|$.*

ADDENDUM 21.0.2. There is a nice paper of Osaki [55] that interprets Stong's process of passing from an F -space to its core Y . He shows that $\mathcal{K}(Y)$ is obtained from $\mathcal{K}(X)$ by a sequence of elementary simplicial collapses, so that $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ have the same "simple" homotopy type. It follows that if X and Y are homotopy equivalent F -spaces, then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. If K is not collapsible, then $\mathcal{X}(K)$ is a minimal finite space. As Osaki points out and is clear from Example 3.4.15, there are non-collapsible triangulations K_1 and K_2 of S^1 such that $\mathcal{X}(K_1)$ and $\mathcal{X}(K_2)$ are not homeomorphic and therefore, being minimal, not homotopy equivalent. Barmak and Minian [10] went further and proved that two finite spaces X and Y are homotopy equivalent if and only if $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ have the same simple homotopy type.

reference to Barmak's book already here?

Finite spaces can be weak homotopy equivalent but not homotopy equivalent, as we have seen in Example 3.4.14 and Example 3.4.15. The following problems

Not written yet

are far more difficult than their analogues for homotopy equivalence, which we have treated in §Section 2.5, following the REU paper of Fix and Patrias. Note that the work of Fix and Patrias implicitly addresses the problem of finding a computationally effective algorithm for enumerating the homotopy types of finite spaces.

PROBLEM 21.0.5. *Are there computationally effective algorithms for enumerating the weak homotopy types of finite spaces for small n ? What is the asymptotic behavior of the number of weak homotopy types of spaces with at most n elements?*

ADDENDUM 21.0.3. Osaki [55] has given two theorems that describe when one can shrink an F -space, possibly minimal, to a smaller weakly homotopy equivalent F -space. He asks whether all weak homotopy equivalences are generated by the simple kinds that he describes. The question has since been answered in the negative, by Barmak and Minian [8]. Barmak's thesis, which was inspired by my 2003 REU notes and has now become the book [7], goes a good deal further. There is much more to be done on this problem, which is still not well understood.

PROBLEM 21.0.6. *Is there a combinatorial way of determining when a weak homotopy equivalence of finite spaces is a homotopy equivalence?*

PROBLEM 21.0.7. *Rather than restricting to finite simplicial complexes, can we model the world of finite CW complexes, or at least the world of finite regular CW complexes, in the world of finite spaces. The discussion of spheres and cones in §Section 3.4 gives a possible starting point. This is related to the combinatorially interesting question of relating finite topological spaces to discrete Morse theory.*

Bibliography

- [1] D. Achlioptas. Database-friendly random projections: Johnson-Lindenstrauss with binary coins. *Journal of Computer and System Sciences* 66(4)(2003), 671–687.
- [2] P.S. Alexandroff. Diskrete Räume. *Matematicheskii Sbornik (N.S.)* 2(1937), 501–518.
- [3] P.S. Alexandroff and H. Hopf. *Topologie I*. Berlin, 1935.
- [4] F.G. Arenas. Alexandroff spaces. *Acta Math. Univ. Comenianae* 68(1999), 17–25.
- [5] M. Arkowitz. *Co-H-Spaces*. Handbook of Algebraic Topology, Elsevier Science B.V. 1995.
- [6] J. Anusiak and K.P., Shum. Remarks on finite topological spaces. *Colloquium Mathematicum* 23(1971), 217–223.
- [7] J.A. Barmak. Algebraic topology of finite spaces and applications. *Lecture Notes in Mathematics* 2032. Springer 2011.
- [8] J.A. Barmak and E.G. Minian. Minimal finite models. *J. Homotopy Relat. Struct.* 2(2007), 127–140.
- [9] J.A. Barmak and E.G. Minian. One-point reductions of finite spaces, h-regular CW-complexes and collapsibility. *Alg. Geom. and Topology* 8 (2008), 1763–1780.
- [10] J.A. Barmak and E.G. Minian. Simple homotopy types and finite spaces. *Adv. Math.* 218(2008), 87–104.
- [11] A. Björner. Posets, Regular CW Complexes and Bruhat Order. *European Journal of Combinatorics* 5(1)(1984), 7–16.
- [12] J. Bourgain, T. Figiel and V. Milman. On Hilbertian Subsets of Finite Metric Spaces. *Israel Journal of Math.* 55(1986), 147–152.
- [13] G. Brinkmann and B. McKay. Posets on up to 16 points. *Order* 19(2002), 147–149.
- [14] R. Bumbly, R. Fisher, H. Levinson and R. Silverman. Topologies on Finite Sets. *Proc. 9th S-E Conf. Combinatorics, Graph Theory, and Computing* (1978), 163–170.
- [15] N. Cianci and M. Ottina. Poset splitting and minimality of finite models. *J. Comb. Theory, Ser. A* 157(2018), 120–161.
- [16] N. Cianci and M. Ottina. Poset splitting and minimality of finite models. *J. Comb. Theory, Ser. A* 157(2018), 120–161.
- [17] N. Cianci and M. Ottina. Smallest homotopically trivial non-contractible spaces. 1608.05307(2016).
- [18] E. Clader. Inverse Limits of Finite Topological Spaces. *Homology, Homotopy and Applications*, vol. 11(2), 2009, 223–227.
- [19] Cohen, M.M.. *A Course in Simple Homotopy Theory*. Springer-Verlag New York, Heidelberg, Berlin (1970).
- [20] J.C. Culberson and G.J.E. Rawlins. New Results from an Algorithm for Counting Posets. *Order* 7 (90/91), no 4, 361–374.
- [21] M. M. Deza. *Geometry of Cuts and Metrics*. Springer-Verlag Berlin Heidelberg. 1997.
- [22] A. Dold and R. Thom. Quasifaserungen und unendliche symmetrische Produkte. *Annals of Math.* 67(1958), 239–281.
- [23] M. Erne and K. Stege. Counting Finite Posets and Topologies. *Order* 8 (91), p 247-265.
- [24] J.W. Evans, F. Harary, and M.S. Lynn. On the computer enumeration of finite topologies. *Communications of the ACM* 10(1967), 295–297, 313.
- [25] R. H. Fox. On topologies for function spaces. *Bull. Amer. Math. Soc.* 51(1945), 429-432.
- [26] R. Fritsch and R.A. Piccinini. *Cellular structures in algebraic topology*. Cambridge University Press. 1990.
- [27] J.E. Goodman, J. O’Rourke and C.D. Tóth. *Handbook of Discrete and Computational Geometry* (3rd ed.). Chapman and Hall/CRC. 2017.
- [28] D. Gorenstein. *Finite Groups*. Harper and Row. 1968.

- [29] K.A. Hardie and J.J.C. Vermeulen. Homotopy theory of finite and locally finite T_0 -spaces. *Expositiones Math.* 11(1993), 331–341.
- [30] K.A. Hardie, J.J.C. Vermeulen, and P.J. Witbooi. A nontrivial pairing of finite T_0 -spaces. *Topology and its Applications* 125(2002), 533–542.
- [31] A. Hatcher. *Algebraic topology*. Cambridge University Press. 2002.
- [32] R. D. Helmstutler and R.S. Higginbottom. Finite topological spaces as a pedagogical tool. *PRIMUS*, 22(1)(2012), 64–72.
- [33] R. Helmstutler and Ryan Vaughn. Finite Co-H-spaces are Contractible: A Dual to a Theorem of Stong. <http://files.umwblogs.org/blogs.dir/4710/files/2010/10/RDH-RMV-co-H.pdf>
- [34] P.J. Hilton and S. Wylie. *Homology theory*. Cambridge University Press. 1960.
- [35] P. Indyk and R. Motwani. Approximate Nearest Neighbors: Towards Removing the Curse of Dimensionality. *STOC '98: Proceedings of the 30th ACM Symposium on Theory of Computing*(1998), 604–613.
- [36] W. Johnson and J. Lindenstrauss. Extensions of the Lipschitz Maps into a Hilbert space. *Contemporary Math.* 26(1984), 189–206.
- [37] D. Kleitman and B. Rothschild. The number of finite topologies. *Proc. AMS*, 25, 1970, 276–282.
- [38] D. Kleitman and B. Rothschild. Asymptotic enumeration of partial orders on a finite set. *Trans. Amer. Math. Soc.* 205 (1975), 205–220.
- [39] Y. Koda. The numbers of finite lattices and finite topologies. *Bulletin of the Institute of Combinatorics and its applications* 10 (1994), 83–89.
- [40] T.Y. Kong and E. Khalimsky. Polyhedral analogs of locally finite topological spaces. *Lecture Notes in Pure and Applied Mathematics Vol 123, General Topology and Applications*, 153–164. Marcel Dekker. 1990.
- [41] V. Krishnamurthy. On the number of topologies on a finite set. *Amer. Math. Monthly* 73(1966), 154–157.
- [42] M.J. Kukiela. On homotopy types of Alexandroff spaces. *Order* 27(2010), 9–21.
- [43] N. Linial. Finite Metric Spaces. *Proceedings of the ICM* 3(2002), 573–586
- [44] J. Matousek. On the Distortion Required for Embedding Finite Metric Spaces into Normed Spaces. *Israel J. Math.* 93, 333–344 (1996).
- [45] J.P. May. *Simplicial objects in algebraic topology*. The University of Chicago Press. 1967; reprinted 1982.
- [46] J.P. May. *Classifying spaces and fibrations*. *Memoirs Amer. Math. Soc.* 155. 1975.
- [47] J.P. May. Weak equivalences and quasifibrations. In: R. Piccinini, editor, *Groups of self-equivalences and related topics*. *Lecture Notes in Mathematics Vol. 1425*. Springer-Verlag, 1990, 91–101.
- [48] J.P. May. *A concise course in algebraic topology*. The University of Chicago Press. 1999.
- [49] J.P. May and K. Ponto. *More concise algebraic topology: localization, completion, and model categories*. University of Chicago Press. 2012.
- [50] M.C. McCord. Singular homology groups and homotopy groups of finite topological spaces. *Duke Mathematical Journal* 33(1966), 465–474.
- [51] B.D. McKay and A. Piperno. Practical Graph Isomorphism, II, *J. Symbolic Computation* 60(2013), 94–112.
- [52] Milnor, J. Whitehead Torsion, II, *Bull. Amer. Math. Soc.*(1966), 358–426.
- [53] C.L. Morgan. Embedding metric spaces in Euclidean space. *J. Geom.* 5, 101–107 (1974).
- [54] J.R. Munkres. *Topology*. Second edition. Prentice-Hall. 1975.
- [55] T. Osaki. Reduction of finite topological spaces. *Interdisciplinary Information Sciences* 5(1999), 149–155.
- [56] H. Prömel. Counting Unlabeled structures. *Journal of Combinatorial Theory, Series A.* 44(1987), 83–93.
- [57] D. Quillen. Homotopy properties of the poset of nontrivial p -subgroups of a group. *Advances in Mathematics* 28(1978), 101–128.
- [58] H. Räcke. CMCS 39600: Theory of Metric Embeddings. *TTIC Lecture notes*. 2006.
- [59] P. Renteln. On the enumeration of finite topologies. *Journal of Combinatorics, Information, and System Sciences* 19(1994), 201–206.
- [60] P. Renteln. Geometrical Approaches to the Enumeration of Finite Posets: An Introductory Survey. *Nieuw Archief voor Wiskunde* 14 (1996), 349–371.

- [61] I.J. Schoenberg. Metric Spaces and Positive Definite Functions. *Trans. Amer. Math. Soc.* 44 (1938), 522–536.
- [62] H. Seifert and W. Threlfall. *A textbook of topology.*
- [63] E.H. Spanier. *Algebraic topology.* McGraw-Hill. 1966.
- [64] R.P. Stanley. *Enumerative Combinatorics Vol.1.* Cambridge University Press. 1997.
- [65] R.E. Stong. Finite topological spaces. *Trans. Amer. Math. Soc.* 123(1966), 325–340.
- [66] R.E. Stong. Group actions on finite spaces. *Discrete Mathematics* 49(1984), 95–100.
- [67] M. J. Thibault. *Homotopy Theory of Combinatorial Categories.* Dissertation at the University of Chicago.
- [68] J.H.C. Whitehead. Combinatorial homotopy I, II. *Bulletin Amer. Math. Soc.* 55(1948), 213–245 and 453–496.
- [69] P.J. Witbooi. Globalizing weak homotopy equivalences. *Topology and its Applications* 100(2000), 229–240.