A GEOMETRIC VIEW OF BORDISM HOMOLOGY

ZHUO ZHANG

Abstract. Bordism homology is a generalized homology theory that is based on singular manifolds, i.e., maps from compact manifolds to a given space. Very few sources in the literature provide a complete proof of this fact. In this paper, we present a detailed description of bordism homology and give a geometric proof that it satisfies the four Eilenberg-Steenrod axioms. We will also discuss some of the consequences of these axioms.

Contents

1. Introduction

The notion of bordism first appeared in Thom's 1954 paper [9]. Bordism is an equivalence relation on manifolds: loosely speaking, two manifolds are *bordant* if their disjoint union is the boundary of some other manifold. We say that a manifold is *null bordant* if it is bordant to the empty set.

At first glance, this is a rather crude equivalence relation. For instance, S^n is null bordant since it is the boundary of D^{n+1}. However, it turns out that bordism is a fundamental concept in algebraic topology and other areas.

There are many motivations for this definition. For instance, classifying manifolds up to diffeomorphism or homeomorphism in dimensions 4 or higher is impossible because it is equivalent to the word problem in group theory, which is known to be unsolvable. Hence, it is natural to consider a weaker notion of equivalence between manifolds, namely bordism. Indeed, classifying manifolds up to bordism has been completely understood. This information is encoded in the bordism ring, as we will see later.

Another reason why people are interested in bordism is that it gives rise to a generalized homology theory, which is the main subject of this paper. Recall that the essential idea in defining homology groups is "kernel modulo image." In other words, the nth homology group is just n-dimensional objects without boundary

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modulo the boundary of $n+1$-dimensional objects. Therefore, it is natural to require the boundary of a manifold to be zero in our equivalence relation, and that is precisely what bordism guarantees, since the boundary of a manifold is, by definition, bordant to the empty set.

In this paper we shall follow Conner and Floyd's exposition [1] and give a self-contained introduction to bordism homology. We would assume that the readers are reasonably familiar with the basic concepts in algebraic topology, such as homology theories.

We still need quite a few technical tools from differential topology, so in section 2 we will review these relevant results for the readers. We shall follow Lee's [5] and tom Dieck's [10] books for this part.

In section 3, we define the bordism groups and the bordism ring, which will then be followed by a brief discussion of the various results on their structures.

In section 4, we define the relative bordism groups of a pair of spaces. This definition is slightly different from the one given in section 3, and the readers will see why. We will also discuss the functoriality of our constructions.

In section 5, we shall prove the four Eilenberg Steenrod axioms for bordism homology and point out why the dimension axiom fails. Thus, bordism homology is a generalized homology theory. It is the excision axiom that requires the most extensive use of the tools we developed in section 2.

In section 6, we discuss a few important consequences of the Eilenberg Steenrod axioms. We shall compute the bordism groups of spheres, discuss the Steenrod question, and finaly delve into the Mayer Vietoris sequence in the context of bordism homology.

2. Some Tools from Differential Topology

Most proofs related to bordism homology involve some technical tools from differential topology. We need to use some "big theorems," such as Sard's theorem and the collaring neighborhood theorem. These are all fundamental results in differential topology, whose proofs can be found in many standard textbooks such as Lee [5] and [10]. We also need some more subtle tools, such as giving a manifold an appropriate smooth structure. For this part, we mainly follow [1] and [10].

2.1. **Gluing manifolds along boundaries.** Suppose M and N are two manifolds with diffeomorphic boundaries. Intuitively, it is clear that we can glue them together to obtain a manifold without boundary. The following theorem makes precise and generalizes this intuition.

Theorem 2.1 (15.10.1 from from [10])**.** *Let M_0 and M_1 be smooth manifolds with boundary. Let $N_i \subseteq M_i$ be a union of components of ∂M_i, and let $\varphi : N_0 \to N_1$ be a diffeomorphism. Let M be the space obtained from $M_0 \sqcup M_1$, with N_i identified under φ. Then there is a smooth structure on M such that $M_i \backslash N_i \to M$ are smooth embeddings.*

Proof. The central ingredient of the proof is the existence of collaring neighborhoods. For a smooth manifold M with boundary, there always exists a neighborhood U of ∂M and a diffeomorphism $f : U \to \partial M \times [0, 1)$ so that $f(x) = (x, 0)$ for all $x \in \partial M$. The neighborhood U is called the collaring neighborhood.

Hence, we let $M = M_1 \cup_\varphi M_2$, which is seen to be a topological manifold of the same dimension. Let $N \subseteq M$ be the image of N_1 or N_2 in M under the identification. We give M a smooth structure as follows. For $(M_1 \setminus N_1) \cup (M_2 \setminus N_2)$ in M, we have an evident smooth structures, as they are open subsets of M_1 and M_2. For $N \subseteq M$, there is an open neighborhood U of N and a homeomorphism $f : U \to N \times (-1, 1)$, which restricts to diffeomorphisms onto $N \times (-1, 0]$ and $N \times [0, 1)$. We give U the smooth structure inherited from that of $N \times (-1, 1)$. Since the two smooth structures clearly agree on their overlap, we see that M is a smooth manifold. ∎

In particular, we see that the boundary of this new manifold M is just the union of $\partial M_i - N_i$, $i = 1, 2$.

2.2. **Smoothing the corner.** The second tool we need is called *smoothing the corner*, which is arguably the most subtle technique used in this paper. In Conner and Floyd's book [1], this technique was referred to as *straightening the angle*. Another excellent reference for this part is the short article [7] by Milnor, in particular the section entitled "pasting and straightening".

Suppose we are given two manifolds M and N with boundary. If we use products of charts, then the resulting structure on $M \times N$ is not smooth, since the upper half plane $\mathbb{R} \times [0, \infty)$ is homeomorphic but not diffeomorphic to the closed first quadrant $[0, \infty)^2$. In other words, there will be corners at $\partial M \times \partial N$. This can be a problem, since we would often need to work with spaces of the form $M \times I$, where M is a manifold with boundary.

All is not lost. One way to get around with this issue is to use "manifolds with corners" throughout the discussion, which however complicates the situation quite a bit. The other way is to use the technique of smoothing the corner.

Theorem 2.2 (15.10.2 from from [10]). *Let M and N be smooth manifolds with boundary. There exists a smooth structure on $M \times N$ such that $M \times N \backslash (\partial M \times \partial N) \subset M \times N$ and $\lambda : \mathbb{R}^2_+ \times \partial M \times \partial N \to M \times N$ are diffeomorphisms onto open parts of $M \times N$. Here, $\mathbb{R}_+ = [0, \infty)$, and $\partial M \times \mathbb{R}_+$, or equivalently $\partial M \times [0, 1)$, is the collaring neighborhood of ∂M.*

Proof. Fix a homeomorphism $\tau : \mathbb{R}^2_+ \to \mathbb{R}_+ \times \mathbb{R}$ that restricts to a diffeomorphism of $\mathbb{R}^2_+ \setminus (0, 0)$ onto $\mathbb{R}_+ \times \mathbb{R} \setminus (0, 0)$. For instance, we can take $\tau(r, \theta) = (r, 2\theta)$, written in polar coordinates. We give $(M \times N) \setminus (\partial M \times \partial N)$ the canonical smooth structure obtained from the product charts on M and N.

For $\partial M \times \partial N$, the collaring neighborhood theorem implies that there is a neighborhood U and a homeomorphism $f : U \to \mathbb{R}^2_+ \times \partial M \times \partial N$ that restricts to a diffeomorphism onto $(\mathbb{R}^2_+ \setminus (0, 0)) \times \partial M \times \partial N$. Composing f with the map τ we chose (or $\tau \times \mathrm{id}$, to be precise), we have a homeomorphism $\tau f : U \to \mathbb{R}_+ \times \mathbb{R} \times \partial M \times \partial N$. We now give the open set U the smooth structure inherited via τf. The two smooth structures on U and $(M \times N) \setminus (\partial M \times \partial N)$ are seen to agree, whence $M \times N$ now has a smooth structure with the desired properties. ∎

It follows from the proof of this theorem that the boundary of $M \times N$ is the union of $\partial M \times N$ and $M \times \partial N$. Hence, hereafter in this paper, we shall treat spaces of the form $M \times I$ as smooth manifolds without worrying about their corners.

2.3. **Smooth Urysohn's lemma.** The Urysohn's lemma states that for normal (i.e., T_4) topological spaces, closed sets can be separated by continuous functions. In other words, if $P, Q \subseteq X$ are disjoint closed subsets of a normal space X, then there is a continuous function from X to $[0,1]$ that restricts to 0 on P and 1 on Q. This theorem can be generalized to smooth manifolds, which can be proved easily if we assume the existence of partitions of unity.

Theorem 2.3. *Let P and Q be disjoint closed subsets of a smooth manifold M with or without boundary. Then there is a smooth function $\alpha : M \to [0,1]$ such that $\alpha|P = 0$ and $\alpha|Q = 1$.*

Proof. Let $\{\rho_1, \rho_2\}$ be a partition of unity subordinate to the open cover $\{P^c, Q^c\}$, where $\operatorname{supp} \rho_1 \subseteq P^c$. Then ρ_1 is the desired smooth function. ■

In fact, even more is true. One can choose α so that $\alpha^{-1}(0) = P$ and $\alpha^{-1}(1) = Q$. This is exercise 2-14 from Lee [5], but we will not need this fact.

2.4. **Sard's Theorem.** The following tools are central to the proofs in sections 5 and 6. They allow us to build submanifolds from smooth functions.

Definition 2.4. Let $f : M \to N$ be a smooth map. A point $c \in N$ is said to be a regular value of f if either

(1) $f^{-1}(c) = \varnothing$, or,
(2) for every $p \in f^{-1}(c)$, the map $df_p : T_pM \to T_cN$ is surjective.

If $c \in N$ is not regular, then it is said to be critical.

Regular values play an important role in differential topology. The well-known Sard's theorem asserts that for a smooth map, almost all points are regular values.

Theorem 2.5 (Sard's Theorem, 6.10 from [5])**.** *Suppose M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.*

Recall that a map is called proper if preimages of compact sets are compact. A map between manifolds is an immersion (resp. a submersion) if all induced maps on tangent spaces are injective (resp. surjective). A map between manifolds is a (smooth) embedding if it is both an immersion and a topological embedding.

Definition 2.6. If M is a smooth manifold with or without boundary, a regular domain in M is a properly embedded codimension-0 submanifold with boundary.

As we mentioned earlier, regular values allow us to construct submanifolds (in fact, regular domains). This idea is important in the proof of the excision axiom later on.

Theorem 2.7 (5.47 from [5])**.** *Suppose M is a smooth manifold without boundary and $f \in C^\infty(M)$. For each regular value r of f, the sublevel set $f^{-1}((-\infty, r])$ is a regular domain in M.*

This concludes our short discussion of the tools we need from differential topology. Hereafter in this paper, we shall make the following conventions.

Convention. *The word "manifold" means a compact smooth manifold with or without boundary, unless otherwise stated.*

Convention. *The empty set $\varnothing$ is a manifold of all dimensions.*

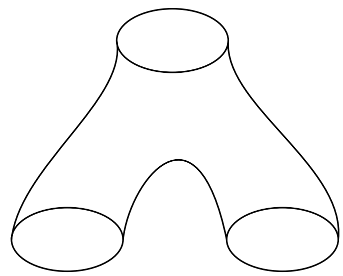

FIGURE 1. A "pair of pants". Here, M_0 is the union of the lower two circles, and M_1 is the upper circle, and $M_0 \sqcup M_1$ is the boundary of this pair of pants. It follows that $M_0 \sim M_1$

3. Definition of Bordism

Now we begin our formal discussion of bordism.

3.1. **Bordism groups.**

Definition 3.1. Two closed n-manifolds M_0 and M_1 are bordant if there exists an $n+1$-manifold N with boundary such that ∂N is diffeomorphic to $M_0 \sqcup M_1$. In this case, we write $M_0 \sim M_1$.

Definition 3.2. A closed n-manifold M is said to be null bordant if it is bordant to the empty set, i.e., there is an $n+1$-manifold N with boundary such that ∂N is diffeomorphic to M.

Figure 1 is a canonical example of bordism, which is commonly referred to as a "pair of pants."

The bordism we have just defined is actually *unoriented* bordism, and there is also an *oriented* version. Instead of considering all closed manifolds, we consider only oriented closed manifolds.

Definition 3.3. Two oriented closed manifolds are said to be bordant if $M_0 \sqcup -M_1$ is the boundary of some oriented $n+1$-manifold N, where $-M_1$ denotes the opposite orientation.

In this paper, we shall focus on the unoriented case, but the constructions for the oriented case are mostly similar. We will point out their distinctions when necessary.

Convention. *In this paper, the word "bordism" means unoriented bordism unless otherwise stated.*

The first thing to check is that bordism is indeed an equivalence relation. To this end we shall need the tools from the previous section, namely gluing manifolds along their boundaries.

Theorem 3.4. *Bordism is an equivalence relation.*

Proof. Symmetry is trivial. Suppose M is a closed n-manifold. Then $M \times I$ is a compact manifold with boundary equal to $M \sqcup M$, so reflexivity holds.

For transitivity, suppose $M_0 \sim M_1$ and $M_1 \sim M_2$. Then there exist $n+1$-manifolds N and N' such that $\partial N = M_0 \sqcup M_1$ and $\partial N' = M_1 \sqcup M_2$. By theorem

2.1, we may glue N and N' along M_1. The resulting set $W = N \cup_{M_1} N'$ can be given a smooth structure so that the inclusions $M_0, M_2 \hookrightarrow W$ are smooth embeddings, whence $\partial W = M_0 \sqcup M_2$. Thus, transitivity holds. ■

We write the equivalence class of M as $[M]$.

Definition 3.5. The nth unoriented bordism group, denoted by $\mathfrak{N}_n$, is defined as the set of all closed n-manifolds up to bordism, where addition is defined as disjoint union: $[M_0] + [M_1] = [M_0 \sqcup M_1]$. The identity element is the empty manifold. It is not hard to check that this operation is well defined. From the proof of theorem 3.4 it is immediate that $M \sqcup M$ is null bordant, so every element in $\mathfrak{N}_n$ is its own inverse.

Example 3.6. Since closed 0-manifolds are just finite sets of discrete points, and compact 1-manifolds with boundary are just finite disjoint unions of $[0, 1]$, we conclude that a closed 0-manifold is null bordant if and only if it has an even number of points. Hence, $\mathfrak{N}_0 = \mathbb{Z}/2$.

Example 3.7. Since closed 1-manifolds are just disjoint unions of S^1, and S^1 is null bordant, we conclude that $\mathfrak{N}_1 = 0$.

Example 3.8. Orientable closed 2-manifolds are exactly spheres and connected sums of tori, so they are all null bordant. On the other hand, one can show that $\mathbb{R}P^2$ is not null bordant, say using Euler characteristic. Hence, $\mathfrak{N}_2 \neq 0$. In fact, $\mathfrak{N}_2 = \mathbb{Z}/2$. For the proof of this fact and the calculation of $\mathfrak{N}_n$ in higher dimensions, readers may look at Thom's original paper [9].

3.2. **Bordism ring.** In fact, bordism groups together form a graded ring.

Definition 3.9. The unoriented bordism ring is defined as $\mathfrak{N}_* = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n$, where the multiplication is induced by products of manifolds: $[M_0] \cdot [M_1] = [M_0 \times M_1]$. We leave it to the readers to check that this is well defined.

Since every nonzero element in $\mathfrak{N}_*$ has order 2, $\mathfrak{N}_*$ is actually a graded algebra over $\mathbb{Z}/2$. The oriented bordism groups and the oriented bordism ring are defined entirely analogously and are denoted by Ω_n and Ω_*, respectively. But in the oriented case, elements may no longer have order 2, since $\partial(M \times I) = M \sqcup -M$ if one takes orientation into account.

Thom in his remarkable 1954 paper [9] first determined the structure of the ring $\mathfrak{N}_*$: it is isomorphic to the graded polynomial ring cover $\mathbb{Z}/2$, with one generator in each dimension not of the form $2^k - 1$, i.e.,

$$\mathfrak{N}_* \cong \mathbb{Z}/2\left[x_i \mid i \neq 2^k - 1\right].$$

This gives us a complete and computable description of all bordism groups.

(1) When $n = 0$, there is one generator "1", so $\mathfrak{N}_0 = \mathfrak{N}_2 = \mathbb{Z}/2$.
(2) When $n = 1, 3$, there is no polynomial in these degrees, so $\mathfrak{N}_1 = \mathfrak{N}_3 = 0$.
(3) When $n = 2$, there is one generator x_2, so $\mathfrak{N}_0 = \mathfrak{N}_2 = \mathbb{Z}/2$.
(4) When $n = 4$, there is one generator x_4, as well as an element x_2^2 in degree 4, so $\mathfrak{N}_4 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Following this pattern, one can easily compute $\mathfrak{N}_n$ for arbitrary n. One might wonder: what manifolds can be taken as the representatives of these generators? Thom showed in [9] that the generators in even dimensions can be taken as $\mathbb{R}P^{2n}$. The explicit generators in odd dimensions were described by Dold in [2].

As the readers may expect, the oriented bordism ring Ω_* has a much more complicated structure. In Thom's paper [9], it was shown that Ω_k are finitely generated abelian groups. Furthermore, Thom proved that $\Omega_* \otimes \mathbb{Q}$ is isomorphic to the polynomial algebra whose generators are $\mathbb{C}P^{2n}$ for $n \geq 0$. Milnor showed in his 1960 paper [8] that Ω has no odd torsion; furthermore, $\Omega/\Omega_{\text{tor}}$ is a polynomial algebra with a generator in each dimension $4k$. A complete algebraic description of Ω_* was given by Wall in [11].

In this paper we shall not focus on the bordism groups and ring themselves, but rather on the homology theory they give rise to. In particular, we will study in the next section the bordism homology groups of a space, just like in the theory of singular homologies.

4. Bordism Homology of a Space

Now we work towards constructing our (generalized) homology theory from bordism. Recall that a homology theory assigns, to each pair of spaces (X, A), a sequence of abelian groups. Singular homology is defined in terms of singular chain complexes. Our definition of bordism homology will not involve such chain complexes, since bordism is entirely geometric in nature.

4.1. **Relative bordism groups.**

Instead of working with singular simplices, we now consider singular manifolds.

Convention. *A pair (X, A) of topological spaces is just a space X together with a subspace $A \subseteq X$.*

Definition 4.1. Let (X, A) be a pair of topological spaces. A singular n-manifold on (X, A) is a pair (M, f) where M is a compact n-manifold and $f : M \to X$ is a continuous map, with $f(\partial M) \subseteq A$.

Now we need to define the bordism of such singular manifolds. One might be tempted to write: two singular manifolds (M_i, f_i) are bordant if there is another manifold W and a map $g : W \to X$ with $\partial W = M_0 \sqcup M_1$ and $g|W_i = f_i$. We may then define the bordism group $\mathfrak{N}_n(X, A)$ to be the set of all such singular n-manifolds up to bordism, with the group operation being disjoint union. This seems to be a natural generalization of definition 3.1.

However, this definition does not work well. The most obvious reason is that ∂W should not have boundaries, but M_0 and M_1 might. Furthermore, this is not even an equivalence relation, as it is not reflexive. Suppose (M, f) is a singular manifold on (X, A). Then $\partial(M \times I)$ is not just the disjoint union of M and itself, but has the additional piece $\partial M \times I$.

The exactness axiom also fails miserably. Consider $\mathfrak{N}_n(A) \to \mathfrak{N}_n(X) \to \mathfrak{N}_n(X, A)$. The composition of these two arrows is not necessarily zero. Indeed, if we have a singular manifold on A, a priori there is no reason to believe that it is the boundary of some other manifold, so it can fail to be null bordant in $\mathfrak{N}_n(X, A)$.

How can we get around with these issues? Recall how we define the relative singular homology groups of a pair. We first define the relative chains $C_k(X, A) = C_k(X)/C_k(A)$, and then take its homology groups. The subset A is in the quotient, so its information is meant to be "ignored." With this in mind, we have the following definitions.

Definition 4.2. Two singular n-manifolds (M_i, f_i) on (X, A) are said to be bordant if there exist a compact $n+1$-manifold W and a map $g : W \to X$ such that

(1) $M_0 \sqcup M_1$ is an embedded submanifold of ∂W, and $g|M_i = f_i$,
(2) $g(\partial W \setminus M_0 \sqcup M_1) \subseteq A$.

This makes precise the idea that A is in the "quotient:" two singular manifolds M_i are bordant if their disjoint union forms the boundary of another manifold, up to an additional piece whose image lies in A. In other words, we are allowed to freely add pieces that lie in A without changing the equivalence class of a manifold.

In particular, this new definition of relative bordism is reflexive: $\partial(M \times I) = (M \sqcup M) \cup (\partial M \times I)$, and we have by assumption that $g(\partial M \times I) = f(\partial M) \subseteq A$, where g is the natural map $g(p,t) = g(p)$. We leave it to the readers to verify that relative bordism is an equivalence relation. (Theorem 15.10.3 from [10] might be helpful.)

Definition 4.3. The nth (relative) bordism group of a pair (X, A), denoted by $\mathfrak{N}_n(X, A)$, is the set of all singular n-manifolds on (X, A) up to bordism, with the group operation being disjoint union. We denote the equivalence class of (M, f) by $[M, f]$.

Definition 4.4. The nth (absolute) bordism group of a space X is, just like in singular homology, defined as $\mathfrak{N}_n(X) := \mathfrak{N}_n(X, \varnothing)$.

Remark 4.5. *In particular, if $A = \varnothing$, then we have the following.*

(1) All singular manifolds are closed, i.e., without boundary.
(2) We are no longer allowed to put in the "additional piece" that lies in A.

Hence, relative bordism reduces to the case described at the beginning of this section.

Example 4.6. If X is taken to be the one point space pt, then its bordism groups are just the usual bordism groups, i.e., $\mathfrak{N}_n(pt) = \mathfrak{N}_n$, since there is only one map into a one point space.

4.2. **Functoriality of $\mathfrak{N}$.** If $\phi : (X, A) \to (Y, B)$ is a map of pairs, then there is an induced map $\phi_* : \mathfrak{N}_n(X, A) \to \mathfrak{N}_n(Y, B)$, given by

$$\phi_*[M, f] = [M, \phi f].$$

It is easy to see that $\mathfrak{N}_n$ are functors from the category of pairs of topological spaces to the category of abelian groups.

We still need the boundary map $\partial : \mathfrak{N}_n(X, A) \to \mathfrak{N}_{n-1}(A)$, which is defined in the obvious way:

$$\partial[M, f] = [\partial M, f|\partial M].$$

This is well defined since we imposed the condition that $f(\partial M) \subseteq A$. It is not hard to check that ∂ is a natural transformation, i.e., we need to check that the following square commutes:

$$\begin{tikzcd}
\mathfrak{N}_n(X, A) \arrow[r, "\partial"] \arrow[d, "\phi_*"'] & \mathfrak{N}_n(A) \arrow[d, "\phi_*"] \\
\mathfrak{N}_n(Y, B) \arrow[r, "\partial"] & \mathfrak{N}_n(B)
\end{tikzcd}$$

and this holds since

$$\partial \circ \phi_*[M, f] = \partial[M, \phi f] = [\partial M, (\phi f)|\partial M] = [\partial M, \phi(f|\partial M)]$$

$$= \phi_*[\partial M, f|\partial M] = \phi_* \circ \partial[M, f].$$

Finally, note that $\partial^2 = 0$ by construction.

Remark 4.7. *Note that there is a slight abuse of notation here: we are using the symbol $\mathfrak{N}_n$ to denote both the nth bordism group, as in definition 3.5, and to denote the functor from the category of pairs of topological spaces to the category of abelian groups, but it should be clear from the context which one is being used.*

4.3. **Modules over the bordism ring.** We conclude this section with a brief remark that the direct sum $\mathfrak{N}_*(X, A) = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n(X, A)$ is a module over the bordism ring $\mathfrak{N}_*$ we constructed in section 3.

Indeed, given $[M, f] \in \mathfrak{N}_m(X, A)$ and $[W] \in \mathfrak{N}_n$, we have $[M \times W, f \circ \pi] \in \mathfrak{N}_{m+n}(X, A)$, where $\pi : M \times W \to M$ is the projection. It is easy to check that the operation thus defined satisfies the associative and distributive laws. Notice that this is a right module. We could have put $[W]$ on the left. However, in this case of oriented bordism, a left module structure would introduce a sign in the multiplication. It turns out that the sign is always positive, but to prove it requires nontrivial results about the structure of the oriented bordism ring Ω_*, such as those listed at the end of section 3.

5. The Eilenberg-Steenrod Axioms

Now we have all the tools necessary for proving that bordism defines a generalized homology theory. The proofs presented in this section can be easily generalized to the oriented case.

The following lemma is going to be extremely useful, so we state it early here.

Lemma 5.1. *Let $[M, f] \in \mathfrak{N}_n(X, A)$ and V be an embedded submanifold of M of codimension 0. Suppose $[V, f|V] \in \mathfrak{N}_n(X, A)$ and $f(M \setminus V) \subseteq A$. Then $[M, f] = [V, f|V]$ in $\mathfrak{N}_n(X, A)$.*

Proof. Consider the manifold $W = M \times I$. Its boundary consists of two copies of M, call them M_0 and M_1, together with $\partial M \times I$. Let g be the map $g(p, t) = f(p)$. We may break down M_1 as $M_1 = (M_1 \setminus V) \cup V$, so $M \times I$ is a bordism between M and V, since

$$g(\partial W - M \sqcup V) = g((M_1 \setminus V) \cup (\partial M \times I)) = f(M \setminus V) \cup f(\partial M) \subseteq A,$$

so the claim holds by the definition of relative bordism. ∎

5.1. **Dimension axiom.** We first note that the dimension axiom fails for bordism homology. As pointed out in Example 4.6, the nth bordism group of a point is just $\mathfrak{N}_n$ itself. We have seen that $\mathfrak{N}_2 \neq 0$. In fact, it is well known that $\mathbb{R}P^{2n}$ is not the boundary of any manifold, so $\mathfrak{N}_{2n} \neq 0$ for all $n \geq 1$.

5.2. **Homotopy axiom.**

Theorem 5.2. *Suppose $\phi \simeq \psi : (X, A) \to (Y, B)$ are homotopic maps of pairs. Then $\mathfrak{N}_n(\phi) = \mathfrak{N}_n(\psi)$ for all n.*

Proof. By definition, there exists a continuous map $H : X \times I \to Y$ with $H(x, 0) = \phi(x)$, $H(x, 1) = \psi(x)$, and $H(A, t) \subseteq B$ for all t.

Let $[M, f] \in \mathfrak{N}_n(X, A)$. We need to show that $[M, \phi \circ f] = [M, \psi \circ f]$ in $\mathfrak{N}_n(Y, B)$. Let $\widetilde{H} : M \times I \to Y$ be defined as

$$\widetilde{H}(p, t) = H(f(p), t).$$

Then $\widetilde{H}$ is continuous. Furthermore, $\widetilde{H}|(M\times 0)=\phi\circ f$ and $\widetilde{H}|(M\times 1)=\psi\circ f$. It remains to show that $\widetilde{H}$ maps $\partial(M\times I)-M\times\{0,1\}$ into B. This is true since

$$\widetilde{H}(\partial M\times I)=\bigcup_{t\in[0,1]}\widetilde{H}(\partial M\times t)=\bigcup_{t\in[0,1]}H(f(\partial M),t)\subseteq\bigcup_{t\in[0,1]}H(A,t)\subseteq B.$$

Therefore, $(M\times I,\widetilde{H})$ is a bordism between $(M,\phi\circ f)$ and $(M,\psi\circ f)$. ∎

5.3. **Sum axiom.**

Theorem 5.3. *If $X=\bigsqcup_\alpha X_\alpha$ and $i_\alpha: X_\alpha\to X$ are the inclusions, then $\oplus_\alpha i_{\alpha*}:\bigoplus_\alpha\mathfrak{N}_n(X_\alpha)\to\mathfrak{N}_n(X)$ is an isomorphism.*

Proof. The map $\iota=\oplus_\alpha i_{\alpha*}$ is given by the following

$$\iota\left(\oplus_\alpha[M_\alpha,f_\alpha]\right)=\left[\bigsqcup M_\alpha,\bigsqcup f_\alpha\right].$$

Since $[M_\alpha,f_\alpha]=0$ for all but finitely many α, we may assume that $\bigsqcup M_\alpha$ is compact, so ι is a well defined homomorphism.

To show that ι is injective, suppose $\iota(\oplus_\alpha[M_\alpha,f_\alpha])=0$ in $\mathfrak{N}_n(X)$. Then there exists a $n+1$-manifold W and a map $g:W\to\bigsqcup X_\alpha$ such that $\partial W=\bigsqcup M_\alpha$ and $g|M_\alpha=f_\alpha$. Note that W is the disjoint union (as a space) of $W_\alpha:=g^{-1}(X_\alpha)$, each of which is open and closed in W. Hence, W_α is compact, and it is a manifold since it is a union of the components in W. Furthermore, $\partial W_\alpha=\partial W\cap W_\alpha=M_\alpha$ and $g|\partial W_\alpha=f_\alpha$. It follows that (M_α,f_α) are all null bordant in X_α via $(W_\alpha,g|W_\alpha)$. Hence, ι is injective.

To show that ι is surjective, suppose $[M,f]\in\mathfrak{N}_n(X)$. By the same argument, M is the disjoint union of $M_\alpha:=f^{-1}(X_\alpha)$, and each M_α is a compact manifold. It follows that $\oplus_\alpha[M_\alpha,f|M_\alpha]$ is sent to $[M,f]$ under ι. ∎

5.4. **Long exact sequence axiom.**

Before proving the long exact sequence axiom, we need a preliminary lemma.

Lemma 5.4. *Suppose (M,f) has no boundary and is null bordant in $\mathfrak{N}_n(X,A)$ via W. Then $\partial W-M$ is a compact n-manifold without boundary.*

Proof. By definition ∂W contains M as an embedded submanifold. Since M is compact and W is Hausdorff, M is closed in ∂W. Recall that embedded submanifolds without boundary of codimension 0 in a manifold without boundary are precisely the open submanifolds. For a proof, see proposition 5.1 in Lee [5]. It follows that M is also open in ∂W. Hence, M is a union of components in ∂W, which implies that $\partial W-M$ is a compact n-manifold without boundary, since ∂W itself has no boundary. ∎

Theorem 5.5. *For a pair (X,A), let $i:A\to X$ and $j:(X,\varnothing)\to(X,A)$ be the inclusions. Then the following sequence is exact:*

$$\cdots\longrightarrow\mathfrak{N}_n(A)\xrightarrow{i_*}\mathfrak{N}_n(X)\xrightarrow{j_*}\mathfrak{N}_n(X,A)\xrightarrow{\partial}\mathfrak{N}_{n-1}(A)\longrightarrow\cdots$$

Proof. There are six inclusions to verify.

(1) $\operatorname{Im} i_*\subseteq\ker j_*$. Suppose $[M,f]\in\mathfrak{N}_n(A)$. Take $V=\varnothing$ in lemma 5.1, we see that $[M,f]=0$ in $\mathfrak{N}_n(X,A)$, so $j_*i_*=0$.

(2) $\ker j_* \subseteq \operatorname{Im} i_*$. Suppose $[M, f] \in \mathfrak{N}_n(X)$ is such that $[M, f] = 0$ in $\mathfrak{N}_n(X, A)$. Then there exists a pair (W, g) satisfying the conditions in definition 4.2. Note that M has no boundary. By lemma 5.4, $N := \partial W - M$ is a compact n-manifold without boundary. Furthermore, $g(N) \subseteq A$ by assumption, so $[N, g|N] \in \mathfrak{N}_n(A)$. It follows that $i_*[N, g|N] = [M, f]$ in $\mathfrak{N}_n(X)$, since $\partial W = M \sqcup N$.

(3) $\operatorname{Im} j_* \subseteq \ker \partial$. Suppose $[M, f] \in \mathfrak{N}_n(X)$. Since M has no boundary, $\partial j_*[M, f] = \partial[M, f] = [\varnothing, \varnothing]$, so the result holds trivially.

(4) $\ker \partial \subseteq \operatorname{Im} j_*$. Suppose $[M, f] \in \mathfrak{N}_n(X, A)$ and $\partial[M, f] = [\partial M, f|\partial M] = 0$ in $\mathfrak{N}_{n-1}(A)$. By definition, there exists an n-manifold W on A and a map $g : W \to A$ with $\partial W = \partial M$ and $g|\partial W = f|\partial M$. By theorem 2.1, we may glue together M and W along their common boundary. We now obtain us a manifold B without boundary and a map $h = f \cup_{\partial M} g$, $h : B \to X$. Then $[B, h] \in \mathfrak{N}_n(X)$. Furthermore $[M, f]$ is bordant to $[B, h]$ by lemma 5.1, since $h(B \setminus M) \subseteq g(W) \subseteq A$. Hence, $j_*[B, h] = [M, f]$.

(5) $\operatorname{Im} \partial \subseteq \ker i_*$. Suppose $[M, f] \in \mathfrak{N}_{n+1}(X, A)$. Then $[\partial M, f|\partial M] \in \mathfrak{N}_n(A)$. When considered as an element in $\mathfrak{N}_n(X)$, $[\partial M, f|\partial M]$ becomes trivial because it is null bordant via M itself, so $i_*\partial = 0$.

(6) $\ker i_* \subseteq \operatorname{Im} \partial$. Suppose $[M, f] \in \mathfrak{N}_n(A)$ and $[M, f] = 0$ in $\mathfrak{N}_n(X)$. By definition, there exists an $n + 1$-manifold W and a map $g : W \to X$ with $\partial W = M$ and $g|\partial W = f$. It follows that $g(\partial W) = f(M) \subseteq A$, whence $[W, g] \in \mathfrak{N}_{n+1}(X, A)$, so we have $\partial[W, g] = [M, f]$. ∎

5.5. **Excision axiom.** The final axiom we need to verify is the excision axiom. In addition to all the tools we have developed in this section and section 2, we will need the following crucial lemma.

Lemma 5.6. *Let P and Q be disjoint closed subsets of a compact smooth manifold M. There exists a closed topological submanifold $B \subseteq M$ with boundary that contains P and is disjoint from Q. Furthermore, B can be given a smooth structure.*

Proof. We shall only prove this lemma in the special case that M has no boundary, and this version will be also useful in section 6.3. We use the smooth Urysohn's lemma (2.3). Take a smooth function $\alpha : M \to [0, 1]$ such that $\alpha|P = 0$ and $\alpha|Q = 1$. By Sard's theorem (2.5), there exists a regular value $r \in (0, 1)$. Consider $B = \alpha^{-1}([0, r]) = \alpha^{-1}((-\infty, r])$. Then B is closed, nonempty, and is a regular domain in M by theorem 2.7. ∎

Remark 5.7. *The proof of the general case can be found in theorem 3.1 of [1], which is slightly beyond the scope of this paper. The proof is similar to the one given above, in the sense that the submanifold B is also constructed as a regular sublevel set $\alpha^{-1}([0, r])$ of some smooth function. However, theorem 2.7 fails when M has boundary, and one needs to smooth certain corners, using additional techniques.*

The following lemma tells us that the boundary of the submanifold B built this way only falls into a certain part of the big manifold.

Lemma 5.8. *Under the same hypothesis and notation as in lemma 5.6, we have that*

$$\partial B \subseteq \partial M \cup \alpha^{-1}(r) \subseteq \partial M \cup (P^c \cap Q^c).$$

Proof. Suppose $p \in \partial B \setminus \partial M$. We will show that $\alpha(p) = r$. Since $p \notin \partial M$, p is an interior point of M, so there is a chart (φ, U) in M about p.

Since $p \in B$, we know that $\alpha(p) \leq r$. Suppose $\alpha(p) = r_0 < r$. Take $r_1 \in (r_0, r)$, and consider $V = \alpha^{-1}([0, r_1))$. Then $p \in V$ and $V \subseteq B$. Note that V is open in M, and hence also in B. Then $U \cap V$ is an open neighborhood of p in B that is diffeomorphic to an open subset of $\mathbb{R}^n$. Hence, p is an interior point of B, a contradiction. Finally, $\alpha^{-1}(r) \subseteq P^c \cup Q^c$ because $\alpha|P = 0$ and $\alpha|Q = 1$. ∎

Remark 5.9. *Lemma 5.8 is a purely topological consequence of our construction. Hence, it works for the general case as well.*

Now we can prove the excision axiom. The argument is a bit convoluted, and the readers are encouraged to draw pictures themselves.

Theorem 5.10. *Suppose X is a space with subspaces Z and A satisfying $\overline{Z} \subseteq \operatorname{Int} A$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism $i_* : \mathfrak{N}_n(X \setminus Z, A \setminus Z) \to \mathfrak{N}_n(X, A)$ on the bordism homology groups.*

Proof. We first show surjectivity. Suppose $[M, f] \in \mathfrak{N}_n(X, A)$. Then $P = f^{-1}(X \setminus \operatorname{Int} A)$ and $Q = f^{-1}(\overline{Z})$ are disjoint closed subsets of M. Apply lemma 5.6 to obtain a closed submanifold $B \subseteq M$ with boundary such that $P \subseteq B$ and $B \cap Q = \varnothing$.

Since $B \cap Q = \varnothing$, we know that $f(B) \subseteq X \setminus \overline{Z}$. By lemma 5.8, $\partial B \subseteq \partial M \cup (P^c \cap Q^c)$. Thus, for a point $p \in \partial B$, either $p \in \partial M$, in which case $f(p) \in A$, or $p \in P^c \cap Q^c$, in which case $p \in P^c$ implies $f(p) \subseteq \operatorname{Int} A$. In either case, we have $f(\partial B) \subseteq A \setminus \overline{Z}$. Therefore, $[B, f|B] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$. Furthermore, $f(M \setminus B) \subseteq \operatorname{Int} A$ because B contains $f^{-1}(X \setminus \operatorname{Int} A)$. By lemma 5.1, $i_*[B, f|B] = [M, f]$.

Now we show injectivity. Suppose $[M, f] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$ and $i_*[M, f] = 0$ in $\mathfrak{N}_n(X, A)$. Then there exists an $n+1$-manifold W and a map $g : W \to X$ such that M is an embedded submanifold of ∂W, $g(\partial W \setminus M) \subseteq A$, and $g|M = f$. Let us use the same trick: let $P = g^{-1}(X \setminus \operatorname{Int} A)$ and $Q = g^{-1}(\overline{Z})$. Find an embedded submanifold $B \subseteq W$ such that $P \subseteq B$ and $B \cap Q = \varnothing$. Clearly $[\partial B, g|\partial B] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$ and is null bordant.

We claim that $M \cap \partial B = M \cap B$. One inclusion is clear. For the other inclusion, suppose $p \in M \cap B$. Then $p \in \partial W$, so there is a chart $\varphi : U \to \mathbb{H}^{n+1}$ about p in W such that $\varphi(p) \in \mathbb{R}^n \times 0$. It follows that $p \in \partial B$, so the equality holds.

This implies that $M \cap \partial B$ is an embedded submanifold in M, since

$$M \cap \partial B = M \cap B = (\alpha|M)^{-1}([0, r]), \tag{$*$}$$

where, by changing r if necessary, we may assume that r is also a regular value of $\alpha|M$, since the set of critical values has measure zero.

To simplify our notation, let $L = M \cap \partial B = M \cap B$. Observe that $[L, f|L] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$. To see this, first note that $f(L) \subseteq f(M) \subseteq X \setminus Z$. It remains to show that f maps ∂L into $A \setminus Z$. This follows from ($*$) and lemma 5.8, since $\partial L \subseteq \partial M \cup (P^c \cap Q^c)$.

We now claim that $[M, f] = [L, f|L]$ in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$. By lemma 5.1, it suffices to show that $f(M \setminus L) = f(M \setminus \partial B) \subseteq A \setminus Z$. If $p \in M \setminus \partial B = M \setminus B$, then $f(p) \in X \setminus Z$ because $p \in M$, and $f(p) \in \operatorname{Int} A$ because $p \notin B$. The claim then follows.

We can show by the same argument that $[L, f|L] = [\partial B, g|\partial B]$ in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$. Since L is an embedded submanifold of M, it is also an embedded submanifold of ∂B. Now, it suffices to show that $g(\partial B \setminus L) = g(\partial B \setminus M) \subseteq A \setminus Z$. Recall that $\partial B \subseteq \partial W \cup (P^c \cap Q^c)$. If $p \in \partial B \setminus M$, then either $p \in (P^c \cap Q^c) \setminus M$ or $p \in \partial W \setminus M$. In the first case, we are obviously done. In the second case, we know that $p \notin Q$ since B and Q are disjoint, so $g(p) \notin \overline{Z}$. Furthermore, we have by assumption that $g(\partial W \setminus M) \subseteq A$, so $g(p) \in A$. Hence, in either case, $g(p) \in A \setminus Z$. Therefore, (M, f), $(L, f|L)$ and $(\partial B, g|\partial B)$ are bordant, as desired, but since the last one is obviously null bordant in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$, we conclude that $[M, f] = 0$ in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$. ■

6. Consequences of the Axioms

Having proved the Eilenberg Steenrod axioms for bordism homology, we now have a number of new tools and definitions.

For instance, we may define the reduced bordism homology groups $\widetilde{\mathfrak{N}}_n(X)$ as the kernel of the map $\varepsilon : \mathfrak{N}_n(X) \to \mathfrak{N}_n(pt)$. For a *good pair* (X, A), i.e., where A has a neighborhood that deformation retracts onto A, we can prove using excision that

$$\mathfrak{N}_n(X, A) \cong \widetilde{\mathfrak{N}}_n(X/A).$$

Note that $\widetilde{\mathfrak{N}}_n\,(pt) = 0$ for all n. Furthermore, we have

$$\mathfrak{N}_n(X) = \widetilde{\mathfrak{N}}_n(X) \oplus \mathfrak{N}_n(pt) = \widetilde{\mathfrak{N}}_n(X) \oplus \mathfrak{N}_n.$$

This can be proved using the fact that the long exact sequence of the pair (X, pt) splits via the retraction of X onto pt.

6.1. **Computation of $\mathfrak{N}_n(S^k)$.** For any homology theory, one often is interested in computing the homology groups of spheres.

Proposition 6.1. *There are isomorphisms*

$$\widetilde{\mathfrak{N}}_n(S^k) = \mathfrak{N}_{n-k} \quad \textit{and} \quad \mathfrak{N}_n(S^k) = \mathfrak{N}_{n-k} \oplus \mathfrak{N}_n.$$

Proof. We may use the suspension isomorphisms $\widetilde{\mathfrak{N}}_n(X) \cong \widetilde{\mathfrak{N}}_{n+1}(SX)$, where S denotes the suspension functor. This holds for any generalized homology theory. Hence, $\widetilde{\mathfrak{N}}_n(S^k) \cong \widetilde{\mathfrak{N}}_{n-k}(S^0)$, so it boils down to calculating $\widetilde{\mathfrak{N}}_n(S^0)$. We claim that $\widetilde{\mathfrak{N}}_n(S^0) = \mathfrak{N}_n$, which is defined to be the kernel of $h_* : \mathfrak{N}_n(S^0) \to \mathfrak{N}_n(pt)$, where $h : S^0 \to pt$ is the collapse map.

By example 4.6, we may identify $\mathfrak{N}_n(pt)$ with $\mathfrak{N}_n$. Since S^0 is the disjoint union of two discrete points, the sum axiom implies that $\iota : \mathfrak{N}_n \oplus \mathfrak{N}_n \to \mathfrak{N}_n(S^0)$ is an isomorphism. Here ι sends a pair $[M] \oplus [M']$ to $[M \sqcup M', i \sqcup i']$, where i and i' send M and M' to the two distinct points of S^0, respectively. It suffices to calculate $\ker(h_* \circ \iota)$. Note that

$$h_* \circ \iota([M] \oplus [M']) = [M \sqcup M'] = [M] + [M'].$$

Therefore, $[M] \oplus [M'] \in \ker(h_* \circ \iota)$ iff $[M] = [M']$, so $\ker h_* = \mathfrak{N}_n$. ■

6.2. **Steenrod problem.** There is a close connection between bordism homology and singular homology. Consider $[M, f] \in \mathfrak{N}_n(X)$. M contains a fundamental class $[M] \in H_n(M; \mathbb{F}_2)$ if we use $\mathbb{F}_2$ coefficients. Hence, we have $f_*[M] \in H_n(X; \mathbb{F}_2)$. This gives us a map μ from $\mathfrak{N}_n(X)$ to $H_n(X; \mathbb{F}_2)$.

One needs to show that this map is well defined. It suffices to consider the case where (M, f) is null bordant, i.e., $(M, f) = (\partial W, g|\partial W)$ for some W, g. Then

W contains a fundamental class $[W] \in H_{n+1}(W, \partial W; \mathbb{F}_2)$, which is sent to the fundamental class $[M]$ under the boundary map ∂ in the long exact sequence:

$$\begin{tikzcd}
H_{n+1}(W, M; \mathbb{F}_2) \arrow[d, "\partial"'] & \\
H_n(M; \mathbb{F}_2) \arrow[d, "i_*"'] \arrow[dr, "f_*"] & \\
& H_n(X; \mathbb{F}_2) \\
H_n(W; \mathbb{F}_2) \arrow[ur, "g_*"'] &
\end{tikzcd}$$

Hence, $f_*[M] = g_* i_* \partial[W] = 0$ because $i \circ \partial = 0$. Therefore, μ is well defined. For each map of spaces $\phi : X \to Y$, we have the following diagram:

$$\begin{tikzcd}
\mathfrak{N}_n(X) \arrow[r, "\mu"] \arrow[d, "\phi_*"'] & H_n(X; \mathbb{F}_2) \arrow[d, "\phi_*"] \\
\mathfrak{N}_n(Y) \arrow[r, "\mu"] & H_n(Y; \mathbb{F}_2)
\end{tikzcd}$$

One sees that this square commutes since

$$\phi_* \mu[M, f] = \phi_* f_*[M] = \mu[M, \phi f] = \mu \phi_*[M, f].$$

Hence, μ is a natural transformation from bordism homology to ordinary homology. Similarly, one can define a natural transformation from oriented bordism homology to ordinary homology, with coefficients in $\mathbb{Z}$ rather than $\mathbb{F}_2$.

Steenrod raised the following question, which appeared in Eilenberg's 1949 paper [3], problem 25. It was formulated as follows.

Question 6.2 (Steenrod Problem)**.** *Given a space X and a homology class $\alpha \in H_n(X; R)$, does there always exist a closed singular manifold (M, f) such that $f_*[M] = \alpha$? Here, $R = \mathbb{F}_2$ in the unoriented case, and $R = \mathbb{Z}$ in the oriented case.*

In other words, is the natural transformation μ defined above always surjective? This problem was first solved by Thom in [9], where he showed that the answer to the unoriented case is affirmative, and that the statement does not hold in the oriented case. The proofs are quite involved. Interested readers may look at Thom's original paper or Miller's notes [6] on this subject.

6.3. **Mayer Vietoris sequence in bordism homology.** One of the most useful tools built from the Eilenberg Steenrod axioms is the Mayer Vietoris sequence. To derive this fact from the axioms, one needs to invoke the following algebraic version of the theorem.

Lemma 6.3. *Suppose the following diagram commutes, where the the rows are exact and every third vertical arrow is an isomorphism (as indicated in the diagram).*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "k_{n+1}", "\cong"'] & A_n \arrow[r, "f_n"] \arrow[d, "i_n"] & B_n \arrow[r, "g_n"] \arrow[d, "j_n"] & C_n \arrow[r, "\delta_n"] \arrow[d, "k_n", "\cong"'] & A_{n-1} \arrow[r] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\delta'_{n+1}"] & A'_n \arrow[r, "f'_n"] & B'_n \arrow[r, "g'_n"] & C'_n \arrow[r, "\delta'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Then the following sequence is exact.

$$\cdots \longrightarrow A_n \xrightarrow{i_n \oplus f_n} A'_n \oplus B_n \xrightarrow{f'_n - j_n} B'_n \xrightarrow{\partial_n} A_{n-1} \longrightarrow \cdots$$

where $\partial_n = \delta_n \circ k_n^{-1} \circ g'_n$.

The proof of this lemma is a standard exercise in homological algebra, which involves lots of diagram chasing.

Now suppose X is covered by the interiors of A and B. We have an inclusion of pairs $(B, A \cap B) \hookrightarrow (X, A)$, and hence following maps between the two long exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathfrak{N}_n(A \cap B) \arrow[r, "i_*"] \arrow[d] & \mathfrak{N}_n(B) \arrow[r, "j_*"] \arrow[d] & \mathfrak{N}_n(B, A \cap B) \arrow[r, "\partial"] \arrow[d, "k_*"] & \mathfrak{N}_{n-1}(A \cap B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathfrak{N}_n(A) \arrow[r, "i_*"] & \mathfrak{N}_n(X) \arrow[r, "j_*"] & \mathfrak{N}_n(X, A) \arrow[r, "\partial"] & \mathfrak{N}_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

The maps k_* are isomorphisms because they there is a homeomorphism $B/(A \cap B) \cong X/A$. Hence, we have a long exact sequence

$$\cdots \longrightarrow \mathfrak{N}_n(A \cap B) \longrightarrow \mathfrak{N}_n(A) \oplus \mathfrak{N}_n(B) \longrightarrow \mathfrak{N}_n(X) \xrightarrow{\partial_n} \mathfrak{N}_{n-1}(A \cap B) \longrightarrow \cdots$$

In our case, the boundary homomorphism $\partial_n : \mathfrak{N}_n(X) \to \mathfrak{N}_{n-1}(A \cap B)$ has a nice geometric formula. To construct it, we follow the recipe in lemma 6.3.

Given a singular n-manifold $[M, f] \in \mathfrak{N}_n(X)$, we need to construct a singular $n-1$-manifold on $A \cap B$. We follow the diagram.

(1) The first map j_* does nothing but viewing $[M, f]$ as a singular manifold on (X, A). Note that M has no boundary.
(2) Next, we need to find an element in $\mathfrak{N}_n(B, A \cap B)$ whose image under k_* is $[M, f]$. To this end, we consult the proof of the excision axiom (5.10). Note that the set that is being excised is $Z = X \setminus B$, where $\overline{Z} = X \setminus \operatorname{Int} B \subseteq \operatorname{Int} A$. Hence, let $P = f^{-1}(X \setminus \operatorname{Int} A)$ and $Q = f^{-1}(X \setminus \operatorname{Int} B)$. Since M has no boundary, the special case of lemma 5.6 we proved applies. Thus, pick a smooth function $\alpha : M \to [0, 1]$ such that $\alpha|P = 0$ and $\alpha|Q = 1$. Choose any regular value $0 < r < 1$. Then $L = \alpha^{-1}([0, r])$ satisfies $k_*[L, f|L] = [M, f]$.
(3) Finally, we apply the boundary map to L. In this situation, we actually have $\partial L = \alpha^{-1}(r)$. This follows from the elementary fact that the boundary of a regular domain B, inside a manifold M without boundary, coincides with its topological boundary, as proposition 5.46 in Lee [5] shows. We refer the readers to [4] for more details

In summary, the boundary map does the following to M: first choose a smooth function $\alpha : M \to [0, 1]$ satisfying $\alpha|P = 0$ and $\alpha|Q = 1$, then take the preimage

$\alpha^{-1}(r)$ for any regular value r of α. It is not obvious that this formula is well defined (i.e., independent of the choice of α or r), but since we are following the diagram and the proof of excision, well-definedness is automatic.

The formula above is precisely the recipe described by tom Dieck in theorem 21.1.7 of [10]. However, the author presents this formula before proving the excision axiom and then shows that this is well defined. What we just did is to motivate the construction of this formula. Alternatively, since the Mayer-Vietoris sequence and the excision axiom are, in some sense, equivalent, one could also proceed in the opposite direction as tom Dieck did.

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