# INTRODUCTION TO GAUSSIAN FREE FIELD AND LIOUVILLE QUANTUM GRAVITY

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Abstract. Liouville Quantum Gravity (LQG) surface is a natural, canonical model of describing a random two-dimensional Riemannian manifold. LQG surface is defined using the Gaussian Free Field (GFF), a multi-dimensional-time analog of Brownian motion. The GFF has the Markov property and its circle average is a Brownian motion. $\gamma-$LQG surface is constructed using the GFF, and the coefficient $\gamma$ determines the strength of the singular points, on which the Liouville area measure is supported. This paper introduces the GFF, the construction of LQG surfaces, and their properties.

Contents



## 1. Introduction

*Liouville quantum gravity* (LQG) is a class of canonical two-dimensional random surfaces. LQG surfaces can be seen as "random Riemannian manifolds," though their singular points prevent them from being smooth, and the description is not exactly accurate. However, LQG surfaces are equipped with a measure, a metric, and a conformal structure, similar to those of a Riemann manifold.

LQG surfaces were first introduced in the physics literature in the 1980s, and have important applications in stat. LQG can be defined on various orientable surfaces such as disks, spheres, and torii. LQG is also proved to be the limit of several random planar maps under their corresponding embedding. [DDG21] introduces recent developments on the convergence of random planar maps to LQG.

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## 2. Gaussian Free Field

2.1. **Definition.** LQG is defined using the *Gaussian Free Field* (GFF), a centered Gaussian process defined in the following way. Consider the space $H_s(D)$ of smooth, real-valued functions on $R^d$ that are supported on a compact subset of a domain $D \subset R^d$. The *Dirichlet inner product* on this space is defined by

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx$$

where $\nabla$ denotes the gradient and $\cdot$ denotes the dot product. Let $H(D)$ be the Hilbert space completion of $H_s(D)$ with the the Dirichlet inner product. $H(D)$ is in fact the Sobolev space of index 1 with distribution functions and their gradients in $L^2(D)$.

Roughly speaking, GFF is a standard Gaussian variable $h$ on $H(D)$. A standard Gaussian random variable $v$ is defined on a finite-dimensional vector space.

**Definition 2.1.** (Standard Gaussian variable) $v$ is a standard Gaussian variable on $V = \mathbb{R}^d$ if it is one of the following [Jan97]:

a) $v : (V, \mathcal{F}, \mu_V) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ has law $\mu_V$, where

$$\mu_V := e^{-\frac{\langle v,v\rangle}{2}} Z^{-1} d\nu$$

is the probability measure on $V$. $d\nu$ is the Lebesgue measure and $Z$ is a normalizing constant.

b) $v$ has the same law as $\sum_{j=1}^d \alpha_j v_j$ where $v_1, \cdots, v_d$ are a deterministic orthonormal basis for $V$ and the $\alpha_j$ are i.i.d. Gaussian random variables with mean zero and variance one.

c) The characteristic function of $v$ is given by

$$\mathbb{E}(e^{i(z,v)}) = e^{-\frac{||z||^2}{2}} \tag{2.2}$$

for any $z \in \mathbb{R}^d$.

d) For each fixed $w \in V$ , the inner product $\langle v, w\rangle$ is a zero mean Gaussian random variable with variance $\langle w, w\rangle$.

The GFF is defined as an analog of a standard Gaussian variable on a infinite-dimensional Hilbert space completion $H(D)$ mentioned earlier. We use the second definition of a standard Gaussian variable as the analog, extending the orthonormal basis to infinite dimensions. Note that the definition below is under the assumption of zero boundary conditions.

**Definition 2.3.** (Gaussian Free Field) Let $\{f_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $H(D)$. The GFF $h$ is defined by

$$h := \sum_{j\in\mathbb{N}} \alpha_j f_j$$

where $\alpha_j$ are i.i.d. Gaussian random variables with mean zero and variance one. This is an analogue of the discrete GFF on a finite graph.

In terms of the Green's Function, the GFF can also be defined as the centered Gaussian process $h$ with covariances [DDG21]:

$$Cov(h(z), h(w)) = G(z, w) := \log \frac{\max\{|z|, 1\} \max\{|w|, 1\}}{|z - w|} \tag{2.4}$$

for $z, w \in \mathbb{C}$.

Since $G(z,w) \to \infty$ as $z \to w$, $h$ does not converge pointwise. However, the GFF is well-defined as a random distribution on $H(D)$. Fix an orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$, for any $f \in H(D)$, the Dirichlet inner product $(h,f)_\nabla = \int_D (\nabla h \cdot \nabla f) dz$ is a random variable that can be almost surely expressed as the limit of the partial sums. Let $f = \sum_{j\in\mathbb{N}} \beta_j f_j$, we have $(h,f)_\nabla = \lim_{k\to\infty} \sum_{j=1}^k \alpha_j \beta_j$. Moreover, the $L^2$ inner product is well-defined as a random variable. For $\phi \in H(D)$, the $L^2$ inner product

$$\langle h, \phi \rangle = \int_{\mathbb{C}} h(z)\phi(z) d^2 z \tag{2.5}$$

is well-defined as a random variable. For $f, g \in H(D)$, the random variables $(h,f)_\nabla$ and $(h,g)_\nabla$ have covariance

$$Cov((h,f)_\nabla, (h,g)_\nabla) = (f,g)_\nabla \tag{2.6}$$

which makes $H(D)$ a Sobolev space. The definition of GFF $h$, though not a function in a rigorous sense since it is not defined pointwise, can instead be viewed as a distribution on $H(D)$. We can formulate an alternative definition for the GFF using its property of preserving the inner product [She03]:

**Proposition 2.7.** *A Gaussian Free Field is any Gaussian Hilbert space of random variables denoted by $(h,f)_\nabla$ for each $f \in H(D)$ that inherits the same inner product structure of $H(D)$, i.e.,*

$$\mathbf{E}[(h,a)_\nabla (h,b)_\nabla] = \mathbf{E}(a,b)_\nabla$$

*In other words, for any $f \in H(D)$, $(h,f)_\nabla$ is linear in $f$ and each $(h,f)_\nabla$ is a centered Gaussian variable with variance $(f,f)_\nabla$.*

Now we introduce some properties of the GFF.

**Theorem 2.8.** *(Conformal Invariance) If $\psi : D \to D'$ is a conformal map, and $h \in H(D)$ is a GFF on $D$, then $h \circ \psi^{-1}$ is a GFF on $D'$.*

*Proof.* The Dirichlet inner product is conformal invariant:

$$\int_{D'} \nabla(f_1 \circ \psi^{-1}) \nabla(f_2 \circ \psi^{-1}) dx = \int_D (\nabla f_1 \nabla f_2) dx$$

Then, if $\{f_i\}$ is an orthonormal basis of $H(D)$, then $\{f_i \circ \psi^{-1}\}$ is an orthonormal basis of $H(D')$. Then from 2.3, $h \circ \psi^{-1} = \sum_j a_j f_j \circ \psi^{-1}$ is then a GFF on $D'$. □

2.2. **Markov Property.** The GFF satisfies the (domain) Markov property. This property states that conditioned on the value of $h$ outside some subset $U \subset D$, the value of the GFF inside $U$ can be obtained by adding an independent GFF to the harmonic extension of the functions defined outside of $U$.

**Theorem 2.9.** *(Markov Property) Fix open $U \subset D$, and let $h$ be a GFF without zero boundary condition on $D$. Then $h$ can be written as*

$$h = h_0 + \varphi$$

*where $h_0$ is a zero boundary condition GFF in $U$ and vanishes outside of $U$, $\varphi$ is harmonic in $U$, and $h_0$ and $U$ are independent.*

*Proof.* Note that the space of harmonic functions in $U$ is orthogonal to $H(D)$ in $U$. Let $h_0 \in H_U(D)$ and $\varphi$ be a harmonic function in $U$. Then using integration by parts, we have

$$(h_0, \varphi)_\nabla = \int_D \nabla h_0(z) \nabla \varphi(z) dx = -\int_D \nabla h_0(z) \Delta \varphi(z) dx = 0$$

Hence $H_U^\perp(D)$ is the space of harmonic functions in $U$. We want to show that the two orthogonal spaces span $H(D)$.

If $\partial D$ is regular, then the Dirichlet problem has a unique solution $\phi$, which is a continuous function which agrees with $h$ on $D \setminus U$ and is harmonic inside $U$. We can then take $h_0 = h - \phi$ and obtain the decomposition. However, when $\partial D$ has irregular points, the harmonic solution will not be continuous at those points. Hence we make an approximation. For any $x \in U$, let $U_\delta := \{y \in U : dist(y, U^c) = \delta\}$ be the closed subset of $U$ whose points are $\delta$ distance away from the complement of $U$. Let $\tau = \min\{t > 0 : B_t \in U_\delta\}$ be the first time a Brownian motion starting at $x$ hits $U_\delta$. Then let

$$\varphi_\delta = \mathbf{E}^x[h(B_\tau)]$$

be the expectation of $h$ at the first hitting point. Then $\varphi_\delta$ is the Dirichlet problem to the domain enclosed by $U_\delta$. Then we let $h_\delta = h - \varphi_\delta$, which is compactly supported on the domain enclosed by $U_\delta$ and is smooth. Now we have $H_U(D) = \bigcup \overline{H_{U_\delta}}$. As $\delta \to 0$, we obtain an increasing sequence of functions $h_\delta$ that converges pointwise to some function $h_0 \in H_U(D)$. Then $\varphi_\delta$ converges to some $\varphi$, and since the limit of harmonic functions is a harmonic function, $\varphi$ is also harmonic in $U$. Hence we have found a unique decomposition $h = h_0 + \varphi$ belonging to $H_U(D)$ and $H_U^\perp(D)$. □

2.3. **Circle Average.** Let $D$ be a bounded domain of $\mathbb{C}$. Let $0 < \epsilon < dist(z, \partial D)$. Let $\rho_\epsilon(w, z)$ denote the uniform distribution on the circle of radius $\epsilon$ around $z$.

**Definition 2.10.** (Circle Average) The circle average $h_\epsilon(z)$ is defined as the following:

$$h_\epsilon(z) = (h, \rho_\epsilon) = \int_D h(w) \rho_\epsilon(w, z) d^2 w$$

where $\rho_\epsilon(w, z)$ is the uniform measure on the circle around $z$.

The circle average of $h$ at a given point is in fact a Brownian motion [BP21]:

**Theorem 2.11.** *Fix $z \in D$ and let $0 < \epsilon_0 < dist(z, \partial D)$. For $t \geq t_0 = \log 1/\epsilon_0$, set $B_t = h_{e^{-t}}(z)$, then $(B_t, t \geq t_0)$ has the law of a Brownian motion started from $B_{t_0}$.*

*Proof.* Recall from 2.6 that $Cov((h, f)_\nabla, (h, g)_\nabla) = (f, g)_\nabla$. By 2.5, the circle average $h_\epsilon = (h, \rho_\epsilon)$ is a Gaussian process with covariance

$$Cov(h_{\epsilon_1}(z_1), h_{\epsilon_2}(z_2)) = \int_{D^2} G(x, y) \rho_{\epsilon_1}(x, z_1) \rho_{\epsilon_2}(y, z_2) dx dy$$

where $\rho_\epsilon(w, z)$ is the uniform measure on the circle around $z$ as defined above. Then we have

$$Var(h_\epsilon(z)) = -\log \epsilon + \log C(z; D) \tag{2.12}$$

With $\epsilon$ replaced by $e^{-t}$, we can see that the variance of $B_t$ is equal to $t - t_0$. Since the $h_\epsilon(z)$ are jointly Gaussian random variables by definition, $B_t$ is a Brownian motion starting at $B_{t_0}$. □

*Proof.*

- Independent increments follow from the Markov property. Let $B_t = h_{e^{-t}}(z)$. Let $0 < s < t$, and let $\epsilon_1 = e^{-s}$ and $\epsilon_2 = e^{-t}$, then $0 < \epsilon_2 < \epsilon_1$. Let $U = B_{\epsilon_1}(z)$, from 2.2, $h$ can be written as $h^0 + \phi$, where $h^0$ is a GFF in $U$ and $\phi$ is harmonic. Then we have
$$h_{\epsilon_2}(z) = \int_{x \in B_{\epsilon_2}(z)} h(x)\rho_\epsilon(x, z) d^2x$$
where $\int_{x \in B_{\epsilon_2}(z)} d\mu = 1$. Then we plug in the decomposition:
$$h_{\epsilon_2}(z) = \int_{x \in B_{\epsilon_2}(z)} h^0(x)\rho_\epsilon(x, z) d^2x + \int_{x \in B_{\epsilon_2}(z)} \phi(x)\rho_\epsilon(x, z) d^2x$$
Since $\phi$ is harmonic, the circle average of radius $\epsilon_1$ is the same as that of radius $\epsilon_2$.
$$h_{\epsilon_2}(z) = h_{\epsilon_2}(z)^0 + \phi_{\epsilon_1}(z)$$
For $x \in \partial U$, $h(x) = \phi(x)$, hence the circle average $\phi_{\epsilon_1}(z) = h_{\epsilon_1}(z)$. Therefore,
$$B_t - B_s = h_{\epsilon_2}(z) - h_{\epsilon_1}(z) = h_{\epsilon_2}(z)^0$$
Since $h^0$ and $\phi$ are independent, $B_t - B_s$ is independent of $B_s = \phi_{\epsilon_1}(z)$. Hence $B_t$ has independent increments.
- I.i.d Gaussian distribution comes from the definition of the GFF. From above we have $B_t - B_s = h_{\epsilon_2}(z)^0$, which is the circle average of the zero boundary condition GFF in $U$. By definition, we know that $h_{\epsilon_2}(z)^0$ has mean 0, and it suffices to prove that $Var(B_t - B_s) = t - s$. To see this, we use the covariance definition definition of the GFF. Recall from 2.6 that $Cov(h(z), h(w)) = G(z, w)$. Then for any $f, g \in H(D)$, we have
$$Cov((h(z), f(z)), (h(w), g(w))) = \int_{D \times D} f(z)g(w)G(z, w)dzdw$$
Then circle average $h_\epsilon = (h, \rho_\epsilon)$ is a Gaussian process with covariance
$$Cov(h_\epsilon(z_1), h_\epsilon(z_2)) = \int_{U^2} G(x, y)\rho_\epsilon(x, z_1)\rho_\epsilon(y, z_2)dxdy$$
where $\rho_\epsilon(w, z)$ is the uniform measure on the circle of radius $\epsilon$ around $z$ as defined above. Then we have
$$Var(h_\epsilon) = Cov((h, \rho_\epsilon), (h, \rho_\epsilon)) = \int_U G(x, y)\rho(x)\rho(y)dxdy$$
Since the Green's function is harmonic, with respect to $y$, we have
$$G(x, z) = \int_{|y|=\epsilon} G(x, y)\rho_\epsilon(y)dy$$
Hence
$$Var(h_\epsilon) = \int_U G(x, z)\rho_\epsilon(x)dx$$
$$= \int_{B_\epsilon} -\log x\rho_\epsilon(x)dx = -\log \epsilon$$

Now we have $B_t - B_s = h_{\epsilon_2}(z) - h_{\epsilon_1}(z) = h_{\epsilon_2}(z)^0$. We already know that $\mathbf{E}(B_t - B_s) = \mathbf{E}(h_{\epsilon_2}(z)^0) = 0$, and $Var(h_{\epsilon_2}(z)) = -\log e^{-t} = t$ and $Var(h_{\epsilon_1}(z)) = s$. Then

$$Var(B_t - B_s) = \mathbf{E}((B_t - B_s)^2) - (\mathbf{E}(B_t - B_s))^2$$

$$= \mathbf{E}(h_{\epsilon_1}(z)^2 + h_{\epsilon_2}(z)^2 - 2h_{\epsilon_1}(z)h_{\epsilon_2}(z))$$

$$= \mathbf{E}(h_{\epsilon_2}(z)^2) - \mathbf{E}(h_{\epsilon_1}(z)^2) - 2\mathbf{E}(h^0_{\epsilon_2}(z))\mathbf{E}(h_{\epsilon_1}(z))$$

Since $h^0_{\epsilon_2}$ is independent of $h_{\epsilon_1}$, we have

$$Var(B_t - B_s) = \mathbf{E}(h_{\epsilon_2}(z)^2) - \mathbf{E}(h_{\epsilon_1}(z)^2)$$

$$= Var(h_{\epsilon_2}(z)) - Var(h_{\epsilon_1}(z)) + \mathbf{E}(h_{\epsilon_2}(z) - h_{\epsilon_1}(z)) = t - s$$

- Continuity follows from the continuity of the GFF $h^0 \in H(U)$.

□

## 3. Liouville Quantum Gravity

3.1. **Definition.** With the definition of the GFF, we are able to construct the LQG surface with a parameter $\gamma \in (0, 2)$.

**Definition 3.1.** (Liouville Quantum Gravity) The LQG is defined using isothermal coordinates. A $\gamma$-LQG surface parametrized by $\mathbb{C}$ is the random two-dimensional Riemannian manifold with Riemannian metric tensor [G21]:

$$e^{\gamma h(z)}(d^2x + d^2y), \quad \text{for } z = x + iy \tag{3.2}$$

where $d^2x + d^2y$ denotes the Euclidean metric tensor of $\mathbb{C}$.

Since the GFF is not defined pointwise, the definition above is not rigorous. To define the LQG in a rigorous way, we approximate $h$ by a collection of $\{h_\epsilon\}_{\epsilon>0}$ and send $\epsilon \to 0$. Among the several choices of $\{h_\epsilon\}$, we discuss two methods: convolution with the heat kernel and the circle average.

**Definition 3.3.** (Convolution with the Heat kernel) $h^*_\epsilon$ can be defined as the convolution of $h$ with the heat kernel of $\mathbb{C}$:

$$g_\epsilon(w, z) = \frac{1}{\pi\epsilon^2} e^{-|z-w|^2/\epsilon^2}$$

$$h^*_\epsilon(z) = \int_{\mathbb{C}} h(z) g_\epsilon(w, z) d^2w \tag{3.4}$$

As $\epsilon \to 0$, $g_\epsilon(z, w)$ approximates the point mass at point $z$, which means that $h^*_\epsilon$ approximates $h$ in the distributional sense when $\epsilon$ is small.

Another possible choice for the approximation of $h$ is the average value over a small circle around a given point $z$.

$$h_\epsilon(z) = (h(z), \rho_\epsilon(z, x)) = \int_{|x|=\epsilon} h(z)\rho(z, x)dx$$

where $\rho(z, x)$ denotes the uniform measure on the ball of radius $\epsilon$ around $z$, as defined in 2.10.

### 3.2. Liouville Area Measure.

**Definition 3.5.** (LQG Area Measure) The $\gamma-$LQG area measure is defined as the following:

$$\mu_h = \lim_{\epsilon\to 0} \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} d^2 z \tag{3.6}$$

The leading normalizing factor $\epsilon^{\gamma^2/2}$ comes from the fact that $\mathbb{E}(e^{\gamma h_\epsilon(z)}) \simeq -\epsilon^{\gamma^2/2}$. This approximation goes according to the following. For a Gaussian random variable $N$ with mean $a$ and variance $b$, we have $\mathbb{E}(e^N) = e^{a+\frac{b}{2}}$. Since $h_\epsilon$ is defined with mean 0 on the whole plane, we have

$$\mathbb{E}(e^{\gamma h_\epsilon(z)}) = e^{\frac{1}{2} Var(\gamma h_\epsilon(z))}$$

By 2.12, we have

$$Var(h_\epsilon(z)) = G_\epsilon(z,z) = \log C(z;D) - \log\epsilon \tag{3.7}$$

where $C(z;D)$ is a constant depending on $z$ and $D$. Then we have the following:

$$\mathbb{E}(e^{\gamma h_\epsilon(z)}) = \exp[\frac{\gamma^2}{2}(-\log\epsilon + \log C(z;D))] = \left(\frac{C(z;D)}{\epsilon}\right)^{\gamma^2/2}$$

Hence, $\mu_h$ normalized with the leading $\epsilon^{\gamma^2/2}$. We can also compute the expectation of $\mu_\epsilon(S)$, where $S \subset D$:

$$\mathbb{E}(\mu_\epsilon(S)) = \int_S C(z;D)^{\frac{\gamma^2}{2}} dz \tag{3.8}$$

Now we want to show that the limit in 3.6 exists and the area measure is well-defined. The proof is given in [BP21]. First, we want to show convergence of the sequence of measures for a fixed bounded Borel subset $S \subset D$.

Let

$$\overline{h}_\epsilon(z) = \gamma h_\epsilon(z) - \frac{1}{2} Var(\gamma h_\epsilon(z))$$

Note that

$$\mathbb{E}(e^{\overline{h}_\epsilon(z)}) = \mathbb{E}(e^{\gamma h_\epsilon(z)})(\frac{\epsilon}{C(z;D)})^{\gamma^2/2} = 1$$

and

$$\mathbb{E}(\mu_\epsilon(z)) = \mathbb{E}(e^{\overline{h}_\epsilon(z)}) \cdot C(z;D)^{\frac{\gamma^2}{2}} \tag{3.9}$$

Let $S$ be fixed and let $I_\epsilon = \mu_\epsilon(S) = \int_S e^{\gamma h_\epsilon(z)} \epsilon^{\frac{\gamma^2}{2}} dz$. We want to show $I_\epsilon$ converges. First, we show that along the sequence of $\epsilon = 2^{-k}$, $I_\epsilon$ is Cauchy.

**Proposition 3.10.** *If $\gamma \in [0,\sqrt{2})$ and $\epsilon > 0$, $\delta = \frac{\epsilon}{2}$, then*

$$\mathbb{E}((I_\epsilon - I_\delta)^2) \le C\epsilon^{2-\gamma^2}$$

*Proof.* We consider two Brownian motions $h_\epsilon(x)$ and $h_\epsilon(y)$ and their normalized versions. By 3.9, we have

$$\mathbb{E}((I_\epsilon - I_\delta)^2) = \int_{S^2} \mathbb{E}\left((e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)})(e^{\overline{h}_\epsilon(y)} - e^{\overline{h}_\delta(y)})(C(x;D)C(y;D))^{\frac{\gamma^2}{2}} dxdy\right)$$

$$= \int_{S^2} \mathbb{E}\left((e^{\overline{h}_\epsilon(x)+\overline{h}_\delta(x)})(1 - e^{\overline{h}_\epsilon(x)-\overline{h}_\delta(x)})(1 - e^{\overline{h}_\epsilon(y)-\overline{h}_\delta(y)})(C(x;D)C(y;D))^{\frac{\gamma^2}{2}} dxdy\right)$$

For $|x-y| \geq 2\epsilon$, by the Markov property, $h_\epsilon(x) - h_\delta(x)$ and $h_\epsilon(y) - h_\delta(y)$ are independent. We can write

$$h = \tilde{h} + \psi$$

where $\psi$ is harmonic in the disjoint union of two balls $B(x,\epsilon) \cup B(y,\epsilon)$. Note that the first term in the product inside the integral, $e^{\overline{h}_\epsilon(x)+\overline{h}_\delta(x)}$, depends on $\psi$. But the second term, $e^{\overline{h}_\epsilon(x)-\overline{h}_\delta(x)}$, a Brownian motion independent of $\psi$, only depends on $\tilde{h}$ restricted to $B(x,\epsilon)$. The third term $e^{\overline{h}_\epsilon(y)-\overline{h}_\delta(y)}$ only depends on $\tilde{h}$ restricted to $B(y,\epsilon)$. In fact, the three terms in the product inside the integral are independent and we can write the expectation of them separately. For a fixed point $x$, by the Martingale property of Brownian motion, $\mathbb{E}(e^{\overline{h_\delta(x)}}|h_\epsilon(x)) = e^{\overline{h_\epsilon(x)}}$, and therefore,

$$\mathbb{E}[e^{\overline{h}_\epsilon(x)-\overline{h}_\delta(x)}|\overline{h}_\epsilon] = 1$$

Then the second and third terms $e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)}$ and $e^{\overline{h}_\epsilon(y)} - e^{\overline{h}_\delta(y)}$ both have expectation 1, the above integral is valued 0 when $|x-y| \geq 2\epsilon$.

In the case where $|x-y| \leq 2\epsilon$, we have

$$\mathbb{E}((e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)})^2) = \mathbb{E}(e^{2\overline{h}_\epsilon(x)} + e^{2\overline{h}_\delta(y)} - 2e^{\overline{h}_\epsilon(x)+\overline{h}_\delta(x)})$$

Using the martingale property again, we have

$$= \mathbb{E}(e^{2\overline{h}_\delta(x)} - e^{2\overline{h}_\epsilon(x)}) \leq \mathbb{E}(e^{2\overline{h}_\delta(x)}) = C\mathbb{E}(e^{2\overline{h}_\epsilon(x)})$$

by Cauchy-Schwarz, we have

$$\mathbb{E}((I_\epsilon - I_\delta)^2) \leq$$

$$\int_{|x-y|\leq 2\epsilon} \sqrt{\mathbb{E}((e^{\overline{h}_\epsilon(x)} - e^{\overline{h}_\delta(x)})^2)\mathbb{E}((e^{\overline{h}_\epsilon(y)} - e^{\overline{h}_\delta(y)})^2)}C(x;D)^{\gamma^2/2}C(y;D)^{\gamma^2/2}dxdy$$

$$\leq \int_{|x-y|\leq 2\epsilon} \sqrt{\mathbb{E}(e^{2\overline{h}_\epsilon(x)})\mathbb{E}(e^{2\overline{h}_\epsilon(y)})}C(x;D)^{\gamma^2/2}C(y;D)^{\gamma^2/2}dxdy$$

$$\leq \int_{|x-y|\leq 2\epsilon} C\epsilon^{\gamma^2} e^{\frac{1}{2}(2\gamma^2)\log(\frac{1}{\epsilon})}dxdy$$

$$\leq C\epsilon^{2+\gamma^2-2\gamma^2} = C\epsilon^{2-\gamma^2}$$

Since $\gamma^2 < 2$, the above goes to 0 as $\epsilon \to 0$. □

In fact, the convergence of $I_\epsilon$ is almost surely strictly positive. Since

$$\mathbb{E}(\lim_{\epsilon\to 0} I_\epsilon) = \lim_{\epsilon\to 0} \mathbb{E}(I_\epsilon) > 0$$

we know that $\mathbb{P}(\lim_{\epsilon\to 0} I_\epsilon > 0) > 0$. By definition of the GFF, $\{f_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $H(D)$, and $h_\epsilon(x)$ is a function of the seuqnece of coefficients, which are independent standard Gaussian variables. Along the sequence of $\epsilon = 2^{-k}$, $\lim_{\epsilon\to 0} I_\epsilon > 0$ is a tail event of the $\sigma-$algebra generated by the sequence of coefficients. By Kolmogorov's $0-1$ law, since it the probability of the event is positive,

$$\mathbb{P}(\lim_{\epsilon\to 0} I_\epsilon > 0) = 1$$

Hence the limit is almost surely strictly positive.

In the case where $\gamma > \sqrt{2}$, the convergence is not immediate. In fact, we use to the fact that points sampled according to the area measure is no more than $\gamma-$thick (3.14). The proof of convergence for $\gamma \in [\sqrt{2}, 2)$ given in [DS11] uses the fact that the contribution of the rare points that are strictly larger than $\gamma-$thick is exponentially small with respect to $\epsilon$. Therefore, we can remove these points and consider only the good event $G_\epsilon(x) = \{h_\epsilon(x) \leq \alpha \log \frac{1}{\epsilon}\}$ for some $\alpha > \gamma$.

Now we are ready to proof that the Liousville area measure $\mu_h$ in 3.6 is well-defined as a random measure.

**Theorem 3.11.** *For $\gamma \in [0, 2)$, $\mu_\epsilon$ converges weakly a.s. to a random measure $\mu_h$, along the subsequence $\epsilon = 2^{-k}$.*

*Proof.* Since $\mu_\epsilon(D)$ converges a.s., the measure $\mu_\epsilon$ is a.s. tight in the space of Borel measure on $D$, with weak convergence along the sequence $\epsilon = 2^{-k}$ as shown in 3.10. Let $\tilde{\mu}$ be any weak limit of the the sequence $\{\mu_\epsilon\}$.

Let $\mathcal{A}$ be the collection of subsets of the form $A = [x_1, y_1) \times [x_2, y_2)$ where $x_i, y_i \in \mathbb{Q}$ and $\overline{A} \subset D$. From 3.10, $\mu_\epsilon(A)$ converges to some limit $\mu(A)$ for any $A \in \mathcal{A}$. Since $\mathcal{A}$ is a countable collection of boxes, the limit exists simultaneously for all $A \in \mathcal{A}$. Denote the limit by $\mu(A)$. We want to show that $\tilde{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

For $A = [x_1, y_1) \times [x_2, y_2)$, we have

$$\mu(A) = \sup_{x'_i, y'_i \in \mathbb{Q}} \{\mu([x'_1, y'_1] \times [x'_2, y'_2])\} \tag{3.12}$$

where $x'_i > x_i$ and $y'_i < y_i$. Since the smaller boxes are contained in $A$, the left-hand side is larger than or equal to the right-hand side. From 3.8, we have

$$\mathbb{E}(\mu(A)) = \int_A C(z, D)^{\gamma^2/2} dz$$

Find a sequence of boxes approaching $A$, denoted by $\{A_i\}$, then

$$\mathbb{E}(\mu(A_i)) = \int_A \mathbf{1}_{z \in A_i} C(z, D)^{\gamma^2/2} dz$$

By the monotone convergence theorem,

$$\mathbb{E}(\mu(A)) = \mathbb{E}(\lim_{i \to \infty} \mu(A_i))$$

Hence we have equality. For $x'_i > x_i$, $y'_i < y_i$, by the Portmanteau lemma, we have

$$\tilde{\mu}(A) \geq \tilde{\mu}([x'_1, y'_1] \times [x'_2, y'_2]) \geq \limsup_{\epsilon \to 0} \mu_\epsilon([x'_1, y'_1] \times [x'_2, y'_2])\} = \mu([x'_1, y'_1] \times [x'_2, y'_2])$$

By 3.12, taking the supremum over the boxes, we have

$$\tilde{\mu}(A) \geq \mu(A)$$

Likewise, using the identity

$$\mu(A) = \inf_{x''_i, y''_i \in \mathbb{Q}} \{\mu([x''_1, y''_1] \times [x''_2, y''_2])\}$$

where $x''_i < x_i$ and $y''_i > y_i$, we have

$$\tilde{\mu}(A) \leq \mu(A)$$

Hence $\tilde{\mu}(A) = \mu(A)$ for any $A \in \mathcal{A}$ and the Liouville measure $\mu_h$ is well-defined. □

3.3. **Thick Points.** For a fixed $z \in D$, $e^{\gamma h_\epsilon(z)}\epsilon^{\gamma^2/2}$ converges to 0 a.s. Therefore, there are some atypical points that support the measure $\mu_h$. One can sample a point $z$ randomly according to the normalized LQG measure, and one natural question is what $h$ looks like near this point $z$. To understand the distribution of the GFF, we can look at *thick points*, i.e. points that have atypical values.

**Definition 3.13.** (Thick Points) Let $h$ be a GFF in $D$ and let $\alpha > 0$. A point $z \in D$ is $\alpha-$thick if

$$\liminf_{\epsilon\to 0} \frac{h_\epsilon(z)}{\log 1/\epsilon} = \alpha$$

Note that $z \in D$ is typically not thick. since $h_{e^{-t}}(z)$ is a Brownian motion with variance 1, $h_\epsilon$ is a Brownian motion at scale $\log 1/\epsilon$. Therefore, the typical value of $h_\epsilon(z)$ is of order $\sqrt{\log 1/\epsilon}$, a result of a standard Brownian motion. This implies $\lim\limits_{\epsilon\to 0} \frac{h_\epsilon(z)}{\log 1/\epsilon} \simeq \frac{1}{\sqrt{\log 1/\epsilon}} \to 0$ a.s., so thick points are in fact atypical. However, when sampled according to the $\gamma-$LQG area measure, $z \in D$ behaves otherwise.

**Theorem 3.14.** *Let $D$ be a bounded domain. Let $z$ be a point sample according to the LQG area measure $\mu_h$, normalized to be a probability distribution. Then, $z$ is a.s. a $\gamma-$thick point, i.e.,*

$$\lim_{\epsilon\to 0} \frac{h_\epsilon(z)}{\log(1/\epsilon)} = \gamma$$

*Proof.* Let $\mathbb{P}(dh)$ denote the distribution of $h$ according to the law of the GFF. Let $Q_\epsilon$ denote the joint law of $(h, z)$, where $z$ is sampled according to the approximating LQG measure $\mu_\epsilon$:

$$Q_\epsilon = \frac{1}{Z}\epsilon^{\gamma^2/2}e^{\gamma h_\epsilon(z)}dz\mathbb{P}(dh)$$

We want to show that the $z$ sampled this way is almost surely a $\gamma-$thick point. We show this by considering both the marginal and conditional distribution of $h$ and $z$.

The marginal distribution of $h$ over the domain $D$ is:

$$Q_\epsilon(dh) = \frac{1}{Z}\mathbb{E}(\mu_\epsilon(D))\mathbb{P}dh = \frac{1}{Z}\int_D C(z; D)^{\gamma^2/2}dz\mathbb{P}(dh)$$

Since $\mu(h)$ is normalized to be a probability measure, we can deduce the normalizing constant $Z = \int_D C(z; D)^{\gamma^2/2}dz$, where $C(z; D)$ is given in 3.7.

The marginal distribution of $z$ is:

$$Q_\epsilon(dz) = \frac{1}{Z}\mathbb{E}(\epsilon^{\gamma^2/2}e^{\gamma h_\epsilon(z)})dz = \frac{1}{Z}C(z; D)^{\gamma^2/2}dz$$

Note that $Q_\epsilon(dz)$ is not dependent on $\epsilon$ and is absolutely continuous with respect to the Lebesgue measure.

Next, we consider the conditional distribution of $h$ given $z$, also called the *rooted random measure* in [DS11]:

$$Q_\epsilon(dh|z) = \frac{1}{Z(z)}\epsilon^{\gamma^2/2}e^{\gamma h_\epsilon(z)}\mathbb{P}(dh)$$

Recall from 2.10 that $h_\epsilon(z) = (h, \rho_\epsilon)$ is the $L^2$ inner product of $h$ and $\rho_\epsilon$. $Q_\epsilon(dh|z)$ is then $\mathbb{P}(dh)$ re-weighted by a exponential linear functional over $h$. To see how this affects the distribution, we need the following lemma:

**Lemma 3.15.** *(Girsanov Theorem.) Let $X = (X_1, \cdots, X_n)$ be a Gaussian vector under the law $\mathbb{P}$, with mean 0 and covariance matrix $V$. Let $\alpha \in \mathbb{R}^n$ and define a new probability measure by the Radon-Nikodym derivative*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{\langle \alpha, X\rangle}}{\mathbb{E}(e^{\langle \alpha, X\rangle})}$$

*Then under the new measure $\mathbb{Q}$, $X$ is still a Gaussian vector, with covariance matrix $V$ and mean $V\alpha$.*

The theorem is saying that if we tilt the measure by linear functional, then the process remains Gaussian with the same variance, but the mean is shifted by the amount of the covariance of the function we used to tilt the measure. Various proofs can be given to the theorem, and here we use the Laplace transform.

*Proof.* When the Gaussian vector is scaled by $\alpha$, the variance of $e^{\langle \alpha, X\rangle}$ is scaled by $e^{\alpha^2/2}$ while the mean remains 0, so the expectation

$$\mathbb{E}(e^{\langle \alpha, X\rangle}) = e^{\frac{1}{2}\langle \alpha, V\alpha\rangle}$$

Now suppose $\lambda \in \mathbb{R}^n$, then we have

$$\mathbb{Q}(e^{\langle \lambda, X\rangle}) = \frac{\mathbb{E}(e^{\langle \alpha, X\rangle} e^{\langle \lambda, X\rangle})}{\mathbb{E}(e^{\langle \alpha, X\rangle})}$$

$$= \frac{\mathbb{E}(e^{\langle \lambda+\alpha, X\rangle})}{e^{\frac{1}{2}\langle \alpha, V\alpha\rangle}} = \frac{e^{\frac{1}{2}\langle \lambda+\alpha, V(\lambda+\alpha)\rangle}}{e^{\frac{1}{2}\langle \alpha, V\alpha\rangle}} = e^{\frac{1}{2}\langle \lambda, V\lambda\rangle + \langle \lambda, V\alpha\rangle}$$

where $\langle \lambda, V\lambda\rangle$ is the Gaussian term and $\langle \lambda, \alpha V\rangle$ is the drift term, which means that under the new measure $\mathbb{Q}$, $X$ has variance $V$ and mean $V\alpha$. ☐

Since $Q_\epsilon(dh|z) = \frac{1}{Z(z)}\epsilon^{\gamma^2/2}e^{\gamma h_\epsilon(z)}\mathbb{P}(dh)$ is $\mathbb{P}(dh)$ re-weighted by an exponential $(h, \gamma\rho_\epsilon)$ and normalized, from Girsanov theorem above, $h$ under $Q_\epsilon(dh|z)$ has the same variance with a drift in the mean given by:

$$Cov(h(w), \gamma h_\epsilon(z)) = \gamma \log\frac{1}{|w-z|} + C$$

by 3.7. Now define $Q(dz, dh)$ by $\mu_h(dz)\mathbb{P}(dh)$, where $\mu_h$ is defined as the a.s. limit. Therefore, as $\epsilon \to 0$, under the measure measure $Q$,

$$\lim_{\epsilon\to 0}\frac{h_\epsilon(z)}{\log 1/\epsilon} = \gamma$$

almost surely. This concludes the proof that $z$ is a.s. a $\gamma-$thick point. ☐

The circle average introduces a logarithmic singularity of strength $\gamma$ at the point $z$. This implies that $\gamma$ actually defines how "rough" the LQG surface is. In fact, different values of $\gamma$ correspond to several random planar maps. Results on the convergence of random planar maps to LQG can be found in [GMS20].

The result in [HMP10] shows that $\gamma-$thick points under $\mu_h$ has Hausdorff dimension $2 - \frac{\gamma^2}{2}$ almost surely.

**Theorem 3.16.** $\forall\ 0 \le a \le 2$, *let*

$$T(a; D) = \{z \in D : \lim_{r\to 0}\frac{h_r(z)}{\log 1/r} = \sqrt{2a}\}$$

*where $h_r(z)$ denotes the circle average of the GFF at point $z$ of radius $r$. Then almost surely,*

$$dim_{\mathcal{H}} T(a; D) = 2 - a$$

The theorem is proved by estimating the upper and lower bounds of the dimension of the set. For the upper bound limit, we will need the following lemma, the proof of which in [HMP10] uses the Kolmogorov–Chensov theorem.

**Lemma 3.17.** *Suppose $D$ is bounded with smooth boundary. For every $0 < \gamma < \frac{1}{2}$ and $\epsilon, \xi > 0$, there exists a constant $M$ dependent on $\gamma, \epsilon, \xi$ such that the circle averages*

$$|h_r(z) - h_s(w)| \le M(\log \frac{1}{r})^{\xi} \frac{(|z-w| + |r-s|)^{\gamma}}{r^{\gamma+\epsilon}}$$

*for all $z, w \in D$ and $r, s \in (0, 1]$ with $r/s \in [\frac{1}{2}, 2]$.*

*Proof.* (The upper bound in 3.16) Let $\epsilon > 0$ be arbitrary. Let $\xi \in (0, 1)$ and $\gamma \in (0, \frac{1}{2})$. be fixed. Let $\{r_n\} = n^{-\epsilon^{-1}}$, $t \in [\log \frac{1}{r_n}, \log \frac{1}{r_{n+1}}]$. By lemma 3.17, there exists $M(\gamma, \xi, \gamma\epsilon)$ such that

$$|h_{e^{-t}}(z) - h_{r_n}(z)| \le M\epsilon^{-\xi}(\log n)^{\xi} \frac{(r_{n+1} - r_n)^{\gamma}}{r_{n+1}^{\gamma+\gamma\epsilon}} = O((\log n)^{\xi} n^{\frac{\gamma}{\epsilon}} - (\frac{\epsilon+1}{\epsilon})\gamma) = O((\log n)^{\xi})$$

Next, let $\{z_{nj}\}$ be a set of discrete points spaced by $r_n^{1+\epsilon}$ within $D$. If $z \in h_{r_n}(z_{nj})$, then by lemma 3.17 again,

$$|h_{r_n}(z) - h_{r_n}(z_{nj})| \le M'(\frac{\log n}{\epsilon})^{\xi} \frac{|r_n|^{\gamma}}{r_n^{\gamma+\epsilon}} = O((\log n)^{\xi})$$

Let $\delta(n) = C(\log n)^{\xi-1}$. Let $I_n = \left\{ j : |h_{r_n}(z_{nj})| \ge \sqrt{2}(\sqrt{a} - \delta(n)) \log \frac{1}{r_n} \right\}$ denote the indexes of points $z_{nj}$ such that the the Gaussian variable is larger than the above value.

We will use the following estimate of Gaussian variables to proceed:

Let $Z$ be a Gaussian random variable, then

$$\mathbb{P}(Z > \lambda) = \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \le \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\lambda x/2} dx$$
$$= \frac{2}{\sqrt{2\pi}} \frac{1}{\lambda} e^{-\lambda^2/2}$$

As $\lambda \to \infty$, we have the estimate

$$\mathbb{P}(Z > \lambda) = O(\sqrt{\frac{2}{\pi}} \lambda^{-1} e^{-\lambda^2/2})$$

Using this, we have

$$\mathbb{P}(j \in I_n) = \mathbb{P}\left( \frac{|h_{r_n}(z_{nj})|}{\sqrt{\log \frac{1}{r_n}}} \ge (\sqrt{a} - \delta(n)) \sqrt{2 \log \frac{1}{r_n}} \right) = O(r_n^{a - O(1)})$$

Since there are approximately $O(\frac{1}{r_n^{2(1+\epsilon)}})$ points in $\{z_{nj}\}$, the expectation

$$\mathbb{E}(|I_n|) \le O\left( \frac{r_n^{a-O(1)}}{r_n^{2(1+\epsilon)}} \right) = O(r_n^{a-O(1)-2(1+\epsilon)})$$

Let $\alpha = 2 - a + \frac{2+a}{1+\epsilon}\epsilon$, we have

$$\mathbb{E}(\sum_{n\geq N}\sum_{j\in I_n} h_{r_n^{1+\epsilon}}(z_{nj})^{\alpha}) \leq \sum_{n\geq N} r_n^{(1+\epsilon)\alpha + a - O(1) - 2(1+\epsilon)}$$

$$= \sum_{n\geq N} O(r_n^{2\epsilon - O(1)}) = \sum_{n\geq N} O(n^{-2+O(1)}) < +\infty$$

The above shows that the Hausdorff-$\alpha$ measure of $T(a;D)$ is 0. Therefore,

$$dim_{\mathcal{H}}(T(a;D)) \leq 2 - a + \frac{2+a}{1+\epsilon}\epsilon \to 2 - a$$

since $\epsilon > 0$ is arbitrary. Hence, the Hausdorff dimension of $T(a;D)$ is no more than $2-a$. □

The proof for the lower bound of the Hausdorff dimension of the thick points can be found in [HMP10].

**Corollary 3.18.** *The $\gamma-$LQG area measure $\mu_h$ is supported on a subset of $\mathbb{C}$ with Hausdorff dimension $2 - \gamma^2/2$. In particular, for an unbounded domain $D$, $\mu_h$ is mutually singular with respect to the Lebesgue measure.*

*Proof.* From the definition of thick points and the theorem above, we have that the Hausdorff dimension of the set of $\gamma-$thick points is indeed $2 - \gamma^2/2$. From 3.14 we have seen that the points sampled according to $\mu_h$ is a.s. a $\gamma-$thick point. Denote the set of $\gamma-$thick points by $\mathcal{T}_\gamma$, then we have $\mu_h(T_\gamma^c) = 0$, which means $\mu_h \perp \nu$ where $\nu$ denotes the Lebesgue measure.

□

3.4. **LQG Metric.** In addition to the LQG measure, we also introduce the LQG metric. On a surface $\mathcal{S}$, the distance between two points $z, w$ is given as the following:

$$D_h(z,w) = \inf_{P:z\to w} \int_a^b e^{h(P(t))/2} |P'(t)| dt \tag{3.19}$$

where the inf is over all continuously differentiable path $P$. The above is a only a formal definition, since $h$ is still not defined pointwise. Under the isothermal coordinates, the distance between two points on the surface is calculated with $h$ replaced by $\frac{2\gamma}{d_\gamma}h_\epsilon$, where $d_\gamma$ is the distance exponent, also the Hausdorff dimension of the metric space $(D, D_h)$.

**Definition 3.20.** (LQG Metric) The $\gamma-$LQG metric is defined as the following:

$$D_h^\epsilon(z,w) = \inf_{P:z\to w} \int_a^b e^{\frac{\gamma}{d_\gamma} h_\epsilon(P(t))} |P'(t)| dt$$

The reason to include $d_\gamma$ instead of simplifying it to be 2 is because we want $D_h^\epsilon$ to be scaled by $C^{\frac{1}{d_\gamma}}$ when the $\gamma-$LQG area measure is scaled by $C$.

3.5. **Conformal Coordinate Change.** The LQG surface can be parameterized in different ways. Under conformal coordinate changes, the covariance of the LQG area measure will change accordingly. Let $h$ be a GFF on $D$ and let $\phi\colon \tilde{D} \to D$ be a conformal map. The pullback of $h$ can be defined as a distribution on the new domain $\tilde{D}$ by the following inner product:

$$(h \circ \phi, \tilde{\rho}) = (h, \rho)$$

where $\rho \in C_0^\infty$ and $\tilde{\rho} = |\phi'|^2 \rho \circ \phi$. It was shown in [DS11] that the a new LQG area measure defined this way is the image of the old one under the conformal map:

**Theorem 3.21.** *Define $\tilde{h}$ on the new domain $\tilde{D}$ by*

$$\tilde{h} = h \circ \phi + Q \log |\phi'|, \quad \textit{where } Q = \tfrac{\gamma}{2} + \tfrac{2}{\gamma}$$

*Then a.s., the LQG measure of $\tilde{h}$ is the image of the LQG measure of $h$:*

$$\mu_{\tilde{h}}(X) = \mu_h(\phi(X)), \quad \textit{for all Borel set } X \subset \tilde{D}$$

The theorem implies that a conformal coordinate change produces a covariant (i.e., the image of the original measure) LQG area measure on the new domain, and as shown in [GM19], also a conformally covariant LQG metric. Pulling back $\mu_h$ by $\phi$, we have a new measure absolutely continuous with respect to $\mu_{\tilde{h}}$ on $\tilde{D}$ with density $e^{\gamma Q \log |\phi'(z)|}$, where $z \in \tilde{D}$. If we set rescale $\tilde{h}$ by adding a factpr $Q \log |\phi'|$, the new measure will be adjusted to a.s. equal the $\gamma-$LQG measure $\mu_{\tilde{h}}$.

*Proof.* We now prove Theorem 3.21. We consider the first $N$ vectors of the orthonormal basis of the Hilbert space $H(D)$ $\{f_1, \cdots, f_N\}$ and let $h^N = \sum_1^N \alpha_j f_j$ be a GFF projected on the space of the first $N$ orthonormal vectors. Define

$$\mu^N(S) = \int_S e^{\gamma h^N(z) - \frac{\gamma^2}{2} Var(h^N(z))} C(z; D)^{\gamma^2/2} dz \tag{3.22}$$

where $C(z; D)$ is shown in 3.7. The factor $-\frac{\gamma^2}{2} Var(h^N(z))$ on the exponent normalizes the expectation of $e^{\gamma h_\epsilon^N(z)}$ as a Gaussian variable, and we are left with the constant term $C(z; D)^{\gamma^2/2} dz$.

$$\mathbb{E}(\mu^N(S)) = \int_S e^{\gamma^2/2 Var(h^N(z)) - \gamma^2/2 Var(h^N(z))} C(z; D)^{\gamma^2/2} dz = \int_S C(z; D)^{\gamma^2/2} dz$$

This is equal to the expectation of $\mu_h(S)$, as shown in 3.8. In fact, $\mu^N(S)$ is a martingale with respect to the filtration $\mathcal{F}_N$ generate by $\{X_j\}_1^N$. By Doob's martingale convergence theorem, we know that the limit of $\mu^N$ exists. we want to show that the limit actually converges to $\mu_h$.

**Lemma 3.23.** *Let $\mu^N$ be defined as above. For a measurable $S \subset D$, almost surely,*

$$\lim_{N \to \infty} \mu^N(S) = \mu_h(S)$$

*Proof.* Recall that the LQG measure is defined using the limit of the circle average $h_\epsilon$. Now suppose $h_\epsilon = h'_\epsilon + h_\epsilon^N$. By orthogonality, $h_\epsilon^N$ and $h'_\epsilon$ are independent. Then we have

$$\epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} = \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon^N(z)} e^{\gamma h'_\epsilon(z)}$$

Introducing the circle average of $h$ will not change the martingale property of $\mu^N$, hence the conditional expectation of $\mu_\epsilon(S)$ given $\mathcal{F}_N$ is equal to the value of $\mu_\epsilon^N(S)$:

$$\mathbb{E}(\mu_\epsilon(S)|\mathcal{F}_N) = \mu_\epsilon^N(S) := \int_S e^{\gamma h_\epsilon^N(z) - \frac{\gamma^2}{2} Var(h_\epsilon^N(z))} C(z;D)^{\gamma^2/2} dz$$

The second half of the equation is a definition of $\mu_\epsilon^N$ using the circle average $h_\epsilon^N$. Taking $\epsilon \to 0$, we know that $\mu_\epsilon^N$ converges to $\mu^N$, hence

$$\mu^N(S) = \lim_{\epsilon\to 0} \mathbb{E}(\mu_\epsilon(S)|\mathcal{F}_N)$$

By Fatou's lemma, we have

$$\lim_{\epsilon\to 0} \mathbb{E}(\mu_\epsilon(S)|\mathcal{F}_N) \geq \mathbb{E}(\lim_{\epsilon\to 0} \mu_\epsilon(S)|\mathcal{F}_N)$$

Therefore

$$\mu^N(S) \geq \mathbb{E}(\lim_{\epsilon\to 0} \mu_\epsilon(S)|\mathcal{F}_N) = \mathbb{E}(\mu_h(S)|\mathcal{F}_N)$$

When taking $N \to \infty$, from Doob's martingale convergence theorem again, we have the following convergence:

$$\lim_{N\to\infty} \mu^N(S) \geq \mu_h(S) \tag{3.24}$$

This is direction of the inequalities that we will use to derive the equality. The other one comes again from Fatou's Lemma:

$$\mathbb{E}(\lim_{N\to\infty} \mu^N(S)) \leq \lim_{N\to\infty} \mathbb{E}(\mu^N(S)) = \lim_{N\to\infty} \int_S C(z;D)^{\gamma^2/2} dz \tag{3.25}$$

From 3.8, we have

$$\mathbb{E}(\mu_h(S)) = \int_S C(z;D)^{\gamma^2/2} dz$$

which is equal to the right-hand side of 3.25. Hence

$$\mathbb{E}(\lim_{N\to\infty} \mu^N(S)) \leq \mathbb{E}(\mu_h(S)) \tag{3.26}$$

From 3.24 and 3.26 we know that $\mathbb{E}(\mu_h(S)) = \mathbb{E}(\lim_{N\to\infty} \mu^N(S))$. By the martingale property,

$$\mu_h(S) = \lim_{N\to\infty} \mu^N(S)$$

which shows that $\mu^N$ indeed converges to $\mu_h$. □

Since $\{f_1, \cdots, f_N\}$ is an orthonormal basis of $H^N(D)$, $f_n \circ \phi$ will still give an orthonormal basis. Therefore, $\tilde{h}$ truncated can be written as $\tilde{h}_N = h_N \circ \phi + Q \log |\phi'|$. With the lemma above, it suffices to show the conformal property for $\mu^N$ and $\tilde{\mu}^N$. When applying the conformal map, we scale the small circle $\epsilon$ roughly by a factor of $\frac{1}{|\phi'|}$. Therefore the constant $C(z;D)$ dependent on $z$ and $D$ will also change by this factor. Hence $C(z';\tilde{D}) = C(z;D)|\phi'|$. Applying the conformal map to $\mu^N$ and by change of variable, we have

$$\mu^N(\phi(\tilde{D})) =$$

$$\int_{\phi^{-1}(D)} exp\left(\gamma h^N(\phi(z)) + \gamma Q \log|\phi'| - \frac{\gamma^2}{2} Var(h^N(\phi(z)))\right) \frac{(C(z;D))^{\gamma^2/2}}{|\phi'|^2} dz'$$

$$= \int_{\tilde{D}} exp\left(\gamma h^N(\phi(z')) - \frac{\gamma^2}{2} Var(h^N(\phi(z')))\right) |\phi'|^{\gamma^2+2} \frac{C(z';\tilde{D})^{\gamma^2/2}}{|\phi'|^{2+\frac{\gamma^2}{2}}} dz'$$

$$= \int_{\tilde{D}} exp\left(\gamma h^N \circ \phi(z') - \frac{\gamma^2}{2} Var(h^N \circ \phi)\right) C(z'; \tilde{D})^{\frac{\gamma^2}{2}} dz'$$

$$= \int_{\tilde{D}} d\tilde{\mu}^N(z') = \tilde{\mu}^N(\tilde{D})$$

which shows the conformal covariant property for $\mu^N$. From the lemma above, we know that the measure $d\mu'_N \to d\mu_{h'}$ and $\mu^N(D) \to \mu_h$. Hence

$$\tilde{\mu}_h(\tilde{D}) = \mu_h(\phi(\tilde{D}))$$

which concludes the proof for 3.21. □

Using the above theorem, an equivalence relation determined by a conformal map can be defined. Denote the set of pairs of open domains and distributions by $\mathcal{DH} = \{(D, h) \colon D \subset \mathbb{C}$ is an open set, $h$ is a distribution on $D\}$. An equivalence relation on $\mathcal{DH}$ can be defined as the following:

**Definition 3.27.** $(D, h) \sim_\gamma (\tilde{D}, \tilde{h})$ if there exists a conformal map $\phi \colon \tilde{D} \to D$ st. $\tilde{h} = h \circ \phi + Q|\phi'|$, where $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

$(D, h)$ and $(\tilde{D}, \tilde{h})$ represent the same LQG surface under different parametrizations. An *embedding* of the LQG surface is a choice of representation $(D, h)$ from the equivalence class in $\mathcal{DH}/\sim_\gamma$.

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