AN INTRODUCTION TO DE RHAM COHOMOLOGY

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ABSTRACT. In this paper, we provide an introduction of the de Rham Cohomology to readers with a background in algebraic topology. We first investigate the notion of differential forms and singular homology. We then introduce de Rham cohomology and present a proof of de Rham's theorem. Finally, we cover a few applications of the de Rham theorem.

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1. INTRODUCTION

We will introduce the de Rham Cohomology with two motivating examples from vector calculus. Let us consider the following vector field on $\mathbb{R}^2$.

$$F(x,y) = \langle 2y^2 + y, 4xy + x\rangle$$

F has two important properties. The first property is that F is the result of the gradient operation applied to some scalar function, f (in this case, $f = 2xy^2 + xy$). A physicist would call any vector field that adheres to this property a "conservative" field, as a system with a conservative force field always conserves energy. This is a result of the path-independence of all line integrals of F, which follows from the fundamental theorem of line integrals.

The second property is that the curl of F is 0 (i.e. $\nabla \times F = 0$). An engineer might call this field "irrotational," as, if F described the motion of some fluid in the euclidean plane, this property would indicate that elements of the fluid do not undergo rotation.

We will now examine the relationship between conservative and irrotational fields. We first notice that all conservative fields are irrotational–a result that one may obtain by computing the curl of the gradient of an arbitrary function, h, by hand. However, we cannot say that all irrotational fields are conservative. Let

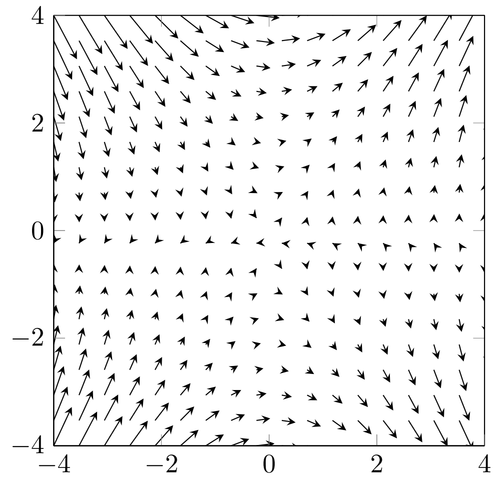

FIGURE 1. A graph of the vector field associated with F on $\mathbb{R}^2$.

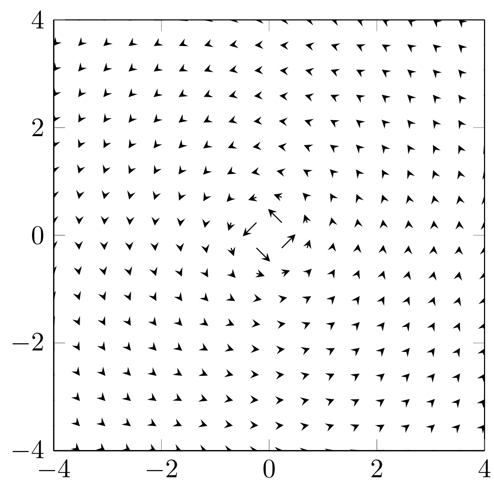

FIGURE 2. A graph of the vector field associated with G on $\mathbb{R}^2 \sim \{0,0\}$.

us consider the following vector field on $\mathbb{R}^2 \sim \{0,0\}$.

$$G(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$$

We notice that this field is irrotational, as $\nabla \times G = 0$. Now, taking some point p in Figure 2, we see that any integral over a counter-clockwise path from p to some point directly across the origin, q, works in the direction of the arrows in the figure and yields a positive result. On the other hand, a clockwise path to the same point works against the arrows of the vector field and yields a negative result. Since G is not path-independent, we conclude that G is not conservative.

We shall investigate, for the entirety of this paper, how topology determines the extent to which irrotational fields on a certain manifold are conservative or non-conservative. It might surprise the reader, for example, that all irrotational vector fields on $\mathbb{R}^2$, like F, are conservative, but many irrotational vector fields on $\mathbb{R}^2 \sim \{0, 0\}$, like G, are nonconservative.

Our first steps will be to generalize to differential forms and exterior derivatives. For instance, a function can be considered an object called a 0-form, which we will define shortly. Likewise, one can *associate* something called a 1-form with a vector field. We can also generalize conservative fields to exact forms and irrotational fields to closed forms. And, finally, the exterior derivative, which we will also soon introduce, can be thought of as a generalization of the gradient and curl.

Now, generalizing our mission statement, our goal will be to probe how topology impacts the "exact-ness" of closed forms on manifolds. We will do this by introducing the de Rham cohomology, which, simply defined, is the space of closed forms on a smooth manifold quotiented by its exact forms. Then, after covering the de Rham Cohomology, we will prove the de Rham theorem, which demonstrates a natural isomorphism between de Rham cohomology and singular cohomology groups.

2. The Exterior Derivative

We will first take a moment to briefly explain what a differential form is, following the definition provided in Chapter 6 of [1]. We begin by defining a k-tensor.

Definition 2.1. A k-tensor, $f : S^k \to \mathbb{R}$, is a *multilinear* function that takes k-tuples of vectors in a vector space, S, to real numbers. By *multilinear*, we mean that f is linear with respect to each of its input vectors, $s_1, \ldots, s_k$.

Definition 2.2. We call any k-tensor, f, an *alternating k-tensor* if the following property holds for all i:

$$f(s_1, \ldots, s_i, s_{i+1}, \ldots, s_k) = -f(s_1, \ldots, s_{i+1}, s_i, \ldots, s_k)$$

Definition 2.3. Given a manifold, $M \subset \mathbb{R}^n$, we define a *differential k-form on* M to be a function that assigns an alternating k-tensor, $f_p : T_p(M) \to \mathbb{R}$, to each point $p \in M$. Here, $T_p(M)$ represents the tangent space of M at p.

It follows from the above definition that a 0-form is a function, as 0-forms do not rely on any vector input and therefore simply assign a scalar to each point on a manifold. We will now focus our attention on the exterior derivative.

Definition 2.4. We define the exterior derivative, $df : S \to \mathbb{R}$, of some 0-form, f, to be the directional derivative of f in the direction of S.

Example 2.5. One can write a k-form, η, uniquely as the sum of k-forms:

$$\sum_J f_J dx_J$$

where f_J is a function and $dx_J = dx_{j_1} \wedge \ldots \wedge dx_{j_k}$ for each J. Here, $\wedge$ denotes the wedge product and each dx_{j_i} is the exterior derivative of the projection function, x_{j_i}. See Chapter 6 of [1] for a proof.

Definition 2.6. We now define the exterior derivative of a differential k-form, η, as

$$d\eta = \sum_{1 \leq j_1 < \ldots < j_k \leq n} dg_{j_1,\ldots,j_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_k}$$

where η is written in the form of the above example:

$$\eta = \sum_{1 \leq j_1 < \ldots < j_k \leq n} g_{j_1,\ldots,j_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_k}$$

We will now define closed and exact forms.

Definition 2.7. We call a differential k-form, η, *closed* if $d\eta = 0$.

Definition 2.8. We call a differential k-form, θ, *exact* if $\theta = d\omega$ for some $(k-1)$-form ω.

Given an open subset, F, of $\mathbb{R}^n$, there exists an isomorphism between the gradient of scalar fields in F and the exterior derivative of 0-forms in F. Similarly, there are isomorphisms between the curl of vector fields in F and the exterior derivative of 1-forms in F. This connection between exterior differentiation and the gradient and curl operators, proven in Theorem 31.2 of [1], shows us that closed and exact forms are the counterparts to irrotational and conservative vector fields, respectively. In the next part of this section, we will focus on showing that all exact forms are closed.

Lemma 2.9. *If g is a 0-form, $d(dg) = 0$.*

Proof. Applying our definition of the exterior derivative, we have,

$$\begin{aligned} d(dg) &= d\left(\sum_i \frac{\partial g}{\partial x_i} dx_i\right) \\ &= \sum_j \sum_i \frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial x_i}\right) dx_j \wedge dx_i \end{aligned}$$

We eliminate all parts of the sum in which $j = i$ and utilize the alternating property of the wedge product to yield the following.

$$d(dg) = \sum_{j<i} \left(\frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial g}{\partial x_j}\right)\right) dx_j \wedge dx_i$$

The partial derivatives portion of the expression evaluates to zero for each j, i pair. We conclude that $d(dg) = 0$.

□

Lemma 2.10. *Given some k-form, θ, and l-form, ω,*

$$d(\theta \wedge \omega) = d(\theta) \wedge \omega + (-1)^k \theta \wedge d\omega$$

Proof. We will show this with some computation. Let $\theta = g dx_J$ and $\omega = f dx_I$. Utilizing our definition of the exterior derivative, we have,

$$\begin{aligned}
d(\theta \wedge \omega) &= d(g \cdot f dx_J \wedge dx_I) \\
&= d(g \cdot f) \wedge dx_J \wedge dx_I \\
&= \left(\sum_m \left(\frac{\partial (g \cdot f)}{\partial x_m} \right) dx_m \right) \wedge dx_J \wedge dx_I \\
&= ((dg) \wedge f + g \wedge (df)) \wedge dx_J \wedge dx_I \\
&= (dg \wedge dx_J) \wedge (f \wedge dx_I) + (-1)^k (g \wedge dx_J) \wedge (df \wedge dx_I) \\
&= d\theta \wedge \omega + (-1)^k \theta \wedge d\omega
\end{aligned}$$

□

Theorem 2.11. *Every exact k-form is closed.*

Proof. Let θ be an exact $(k+1)$-form. Then $\theta = d\omega$ for some k-form $\omega = \sum_{1 \leq i \leq \binom{n}{k}} g \wedge dx_{J_i}$. Thus,

$$\begin{aligned}
d(\theta) &= d(d\omega) \\
&= d\left(d\left(\sum_i g \wedge dx_{J_i} \right) \right) \\
&= \sum_i d(dg \wedge dx_{J_i})
\end{aligned}$$

Using Lemmas 2.9 and 2.10, we get the following.

$$\begin{aligned}
d(\theta) &= \sum_i d(dg) \wedge dx_{J_i} - dg \wedge d(dx_{J_i}) \\
&= \sum_i d(dg)0 - dg \wedge d(1) \wedge dx_{J_i} \\
&= 0
\end{aligned}$$

□

We need to prove one final property of the exterior derivative to use in Section 4.

Theorem 2.12. *Let $\phi : A \to W$ be a smooth map where A and W are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Take ϕ^* to be the corresponding pullback of differential forms (see section 32 of [1], for reference). Take ω to be some k-form on W. Then*

$$\phi^*(d\omega) = d(\phi^* \omega)$$

Proof. We will prove this for ω as a zero form. Commutivity of the pullback for differential forms of greater order follows from this, the linearity of the dual transformation (see Theorem 26.5 of [1] for further explication), and the definition of the exterior derivative.

$$\begin{aligned} d(\phi^*\omega) &= d(\omega \circ \phi) \\ &= \sum_{i=1}^{m} \frac{\partial(\omega \circ \phi)}{\partial x_i} dx_i \end{aligned}$$

After applying the chain rule and some simplification, we get the following.

$$\begin{aligned} d(\phi^*\omega) &= \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \phi_j} \circ \phi \right) \frac{\partial \phi_j}{\partial x_i} dx_i \\ &= \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \phi_j} \circ \phi \right) d\phi_j \\ &= \phi^*(d\omega) \end{aligned}$$

□

We omit proof of the uniqueness of the exterior derivative and a few other properties in this paper. The reader is encouraged to review proofs of such properties in [1] or [2].

3. Singular Homology and Cohomology

We cover important concepts of Algebraic Topology in this section before introducing the de Rham Cohomology. We follow a similar approach to [3].

Definition 3.1. We define the standard k-simplex, Δ^k, as follows.

$$\{\Delta^k = \{x_0, \dots, x_k\} \in \mathbb{R}^{k+1} : \sum_i x_i = 1 \text{ and } x_i \geq 0 \text{ for all i}\}$$

Definition 3.2. We define a singular k-simplex as a smooth map, $\sigma : \Delta^k \to M$. For our purposes, this is simply a map from the standard k-simplex, Δ^k, to a smooth manifold M.

Definition 3.3. We let $C_k(M)$ denote the free abelian group with basis comprised by the set of singular k-simplices in M.

Definition 3.4. Singular k-chains are elements of $C_k(M)$. We write k-chains as formal linear combinations of k-simplices.

Definition 3.5. Here, we will define integration over chains for later use. Let c be some k-chain such that $c \in C_k(M)$ for a smooth manifold, M. Using the definition of k-chains, we write $c = \sum_{i=1} a_i \sigma_i$, where each σ_i is a k-simplex. Furthermore, let ω be a smooth differential k-form on M. We define the *integral of* ω *over* c as

$$\int_c \omega = \sum_i a_i \int_{\sigma_i} \omega = \sum_i a_i \int_{\Delta_k} \sigma_i^* \omega$$

Definition 3.6. The boundary operator, ∂, takes $C_k(M)$ to $C_{k-1}(M)$. We define ∂ on a k-simplex, σ, as follows.

$$\partial(\sigma) = \sum_i (-1)^i \sigma_i$$

In the above, σ_i is the restriction of σ to the simplex with its ith vertex removed.

Theorem 3.7. $\partial \circ \partial = 0$

Proof. A proof of this fact is relatively straightforward. Using the definition of the boundary operator, we see that

$$\begin{aligned}(\partial \circ \partial)(\sigma) &= \partial(\sum_i (-1)^i \sigma_i) \\ &= \sum_{j<i} (-1)^j (-1)^i \sigma_{j,i} + \sum_{j>i} (-1)^{j-1} (-1)^i \sigma_{i,j}\end{aligned}$$

where $\sigma_{i,j}$ is σ restricted to the simplex with the ith and jth vertices removed. Switching i and j in the second sum, we yield

$$\begin{aligned}(\partial \circ \partial)(\sigma) &= \sum_{j<i} (-1)^j (-1)^i \sigma_{j,i} - \sum_{j<i} (-1)^i (-1)^j \sigma_{j,i} \\ &= 0\end{aligned}$$

□

Definition 3.8. We define the singular homology group, $H_k(M)$, by

$$H_k(M) = \frac{\mathrm{Ker}(\partial_k)}{\mathrm{Im}(\partial_{k+1})}$$

where $\mathrm{Ker}(\partial_k)$ and $\mathrm{Im}(\partial_{k+1})$ denote the kernel of $\partial : C_k(M) \to C_{k-1}(M)$ and image of $\partial : C_{k+1}(M) \to C_k(M)$, respectively.

It is often helpful to visualize homology groups and their boundary operators with a *chain complex*. A chain complex is an algebraic structure that represents a series of abelian groups connected by homomorphisms. We will include a few diagrams of these and related complexes, called *cochain complexes*, in this paper.

$$\cdots \xrightarrow{\partial} C_{k+1}(M) \xrightarrow{\partial} C_k(M) \xrightarrow{\partial} C_{k-1}(M) \xrightarrow{\partial} \cdots$$

FIGURE 3. A chain complex of singular k-simplex free abelian groups, $C_k(M)$, and the boundary operator, ∂, on M.

Cochain complexes, on the other hand, can be thought of as dual to chain complexes. A *cochain*, C^k, is defined by $\mathrm{Hom}(C_k, G)$ where C_k is the corresponding chain group and G is an arbitrary abelian group. A *coboundary map* is a mapping between adjacent cochains. Cohomology is simply the homology of a cochain complex.

Definition 3.9. The k-th singular cohomology group, $H^k(M)$, is given by

$$H^k(M) = \frac{\mathrm{Ker}(\partial^k)}{\mathrm{Im}(\partial^{k-1})}$$

where $\mathrm{Ker}(\partial^k)$ and $\mathrm{Im}(\partial^{k-1})$ denote the kernel of $\partial^* : C^k(M) \to C^{k+1}(M)$ and image of $\partial^* : C^{k-1}(M) \to C^k(M)$, respectively. Here, we define $C^k(M)$ as $\mathrm{Hom}(C_k, \mathbb{R})$. In this way, ∂^* is the corresponding dual homomorphism of the boundary operator of the singular chain complex.

$$\cdots \xrightarrow{\partial^*} C^{k-1}(M) \xrightarrow{\partial^*} C^k(M) \xrightarrow{\partial^*} C^{k+1}(M) \xrightarrow{\partial^*} \cdots$$

FIGURE 4. A cochain complex with abelian groups, $C^k(M)$, and the dual function, ∂^*, on M.

4. The De Rham Cohomology and Chain Maps

Definition 4.1. We define the k-th de Rham group, $H^k_{DR}(M)$, as

$$H^k_{DR}(M) = \frac{\mathrm{Ker}(d^k)}{\mathrm{Im}(d^{k-1})}$$

where $\mathrm{Ker}(d^k)$ and $\mathrm{Im}(d^{k-1})$ denote the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ and image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$, respectively. Here, we define $\Omega^k(M)$ as the space of k-forms on M and d as the exterior derivative. We have already shown that $d \circ d = 0$ (Theorem 2.11).

$$\cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots$$

FIGURE 5. The corresponding cochain complex for the sequence of differential form spaces, $\{\Omega^0(M), \Omega^1(M), \Omega^2(M), \ldots\}$.

We will now introduce *chain maps*, which are sets of morphisms between chain complexes. Let us consider the following chain map.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(M) \arrow[r, "d"] \arrow[d, "p^*_{k-1}"] & \Omega^k(M) \arrow[r, "d"] \arrow[d, "p^*_k"] & \Omega^{k+1}(M) \arrow[r, "d"] \arrow[d, "p^*_{k+1}"] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1}(W) \arrow[r, "d"] & \Omega^k(W) \arrow[r, "d"] & \Omega^{k+1}(W) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

FIGURE 6. A chain map of the set of morphisms, $\{p^*_i : \Omega^i(M) \to \Omega^i(W)\}_i$.

By definition, chain maps commute. We have already shown that the pullback associated with a smooth function, $p : M \to W$, commutes with the exterior derivative. It follows that the above chain map is valid. This chain map induces morphisms, $\{g^*_i : H^i_{DR}(M) \to H^i_{DR}(W)\}$, between the de Rham cohomologies of M and W.

It should be noted that commutivity of the boundary operator gives rise to chain maps between k-simplices on M and W. Such a chain map induces mappings between the singular homology groups. Likewise, mappings are induced between singular cohomologies, which results from the commutivity of the dual of the boundary operator.

5. Stokes' and Additional Theorems

In this section, we will follow an approach similar to that provided in [4]. We begin with Stokes' theorem. The theorem plays a key role in our proof of de Rham's theorem. Specifically, it prompts a map between the singular and de Rham cohomologies. For our purposes, we will present a version of Stokes' theorem applied to k-chains.

Theorem 5.1 (Stokes)**.** *Let M be a smooth manifold containing a k-chain, c. Let α be a smooth $(k-1)$-form defined on M. Then,*

$$\int_{\partial c} \alpha = \int_c d\alpha$$

Proof. We omit the proof of this theorem. Readers unfamiliar with Stokes' are encouraged to review a proof of the theorem in the seventh chapter of [1]. □

We define a set of homomorphisms between $\Omega^k(M)$ and $C^k(M)$ by $\alpha \to \int \alpha$. With Stokes' Theorem, we see that these homomorphisms commute with the boundary operator and exterior derivative. Thus, we have the following chain map.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle h_{k-1}} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_{k+1}} & & \\ \cdots & \xrightarrow{\partial^*} & C^{k-1}(M) & \xrightarrow{\partial^*} & C^k(M) & \xrightarrow{\partial^*} & C^{k+1}(M) & \xrightarrow{\partial^*} & \cdots \end{array}$$

Figure 7. A chain map of the set of homomorphisms, $\{h_i : \Omega^i(M) \to C^i(M)\}_i$.

This chain map therefore induces a set of homomorphisms between the de Rham cohomologies and singular cohomologies.

Definition 5.2. We define the *de Rham homomorphism* as the map between de Rham and singular cohomologies. We denote this set of induced homomorphisms by $D_k(M) : H^k_{DR}(M) \to H^k(M)$. We omit proof that the de Rham homomorphism is well defined (see, for instance, [4]). We will now prove two key theorems regarding the homomorphism.

Theorem 5.3. *Let M and N be smooth manifolds. Let $f : M \to N$ be a smooth mapping. Let p^* denote the two pullbacks (for the de Rham and singular cohomologies). Then the following diagram commutes.*

$$\begin{array}{ccc} H^k_{DR}(N) & \xrightarrow{p^*} & H^k_{DR}(M) \\ \downarrow{\scriptstyle D_k(N)} & & \downarrow{\scriptstyle D_k(M)} \\ H^k(N) & \xrightarrow{p^*} & H^k(M) \end{array}$$

Proof. We can show this from the definition of a k-simplex. Let α be a k-form and σ a smooth k-simplex. We have,

$$\int_\sigma p^*\omega = \int_{\Delta^k} \sigma^* p^* \omega = \int_{\Delta^k} (p \circ \sigma)^* \omega = \int_{p \circ \sigma} \omega$$

□

Theorem 5.4. *Let U and V be open subsets of the smooth manifold M such that $U \cup V = M$. We use β and γ to denote the connecting homomorphisms (see Chapter 3 of [3] for a definition) of the de Rham and singular cohomologies, respectively. Then the following diagram commutes.*

$$\begin{array}{ccc} H^{k-1}_{DR}(U \cap V) & \xrightarrow{\beta} & H^k_{DR}(M) \\ \big\downarrow{\scriptstyle D_{k-1}(U\cap V)} & & \big\downarrow{\scriptstyle D_k(M)} \\ H^{k-1}(U \cap V) & \xrightarrow{\gamma} & H^k(M) \end{array}$$

Proof. Let $[\alpha] \in H^{k-1}_{DR}(U \cap V)$ and $[c] \in H_k(M)$. Furthermore, we choose δ_U and δ_V in $H^k_{DR}(U)$ and $H^k_{DR}(V)$, respectively, so that they satisfy the following property. We must have that $\alpha = (\delta_U - \delta_V)|_{U\cap V}$ and $\beta([\alpha]) = d\delta_U$ on U while $\beta([\alpha]) = d\delta_V$ on V.

Also, we specify the choice of chain c to satisfy a few properties. Let $c_U \in C_k(U)$ and $c_V \in C_k(V)$ be smooth. We choose c such that $c = c_U + c_V$ is smooth and, letting ∂^i be the induced map of the boundary map from $C_k(M)$ to $C_k(U \cap V)$, $\partial^i([c]) = [\partial c_U] = -[\partial c_V]$.

We want to show that $D_k(\beta[\alpha]) = \gamma(D_k([\alpha]))$:

$$D_k([\alpha])(\partial^i([c])) = \int_{\partial c_U} \alpha - \int_{\partial c_V} \alpha = \int_{\partial c_U} \delta_U + \int_{\partial c_V} \delta_V$$

Furthermore,

$$D_k(\beta([\alpha]))([c]) = \int_{c_U} d\delta_U + \int_{c_V} d\delta_V$$

The desired result follows immediately from Stokes'. □

6. De Rham's Theorem

We will prove De Rham's theorem following a method presented in [5]. We begin by proving a few supporting lemmas.

Definition 6.1. A smooth manifold M is *de Rham* if $D_k(M)$ is an isomorphism for each $k \geq 0$.

Lemma 6.2. *Let U be a convex subset of $\mathbb{R}^n$. U is de Rham.*

Proof. We use the result that the de Rham and singular cohomologies are topological invariants. We will omit proof of this fact (see chapter 17 of [6]). Due to topological invariance, U must have the same de Rham and singular cohomology groups as a point, $\{p\}$. We now must compute the cohomologies for $\{p\}$.

We begin with the de Rham cohomologies. When $k = 0$, $H^k_{DR}(\{p\}) \cong \mathbb{R}$, as we are looking only at functions from $\{p\}$ to $\{\mathbb{R}\}$. Furthermore, when $k > 0$, $H^k_{DR}(\{p\})$ is trivial, as $\{p\}$ has dimension 0.

Now we compute the singular cohomologies. That $H_0(\{p\}) \cong \mathbb{Z}$ and $H_k(\{p\}) \cong 0$ for $k > 0$ is a straightforward calculation of singular homology (which we omit here). It follows trivially that $H^0(\{p\}) \cong \mathbb{R}$ and $H^k(\{p\}) \cong 0$ (for $k > 0$), as

$H^i(\{p\})$ is naturally isomorphic with $\mathrm{Hom}(H_i(\{p\}), \mathbb{R})$.

All that is left to show is that the de Rham homomorphism is not trivial for $k = 0$. We know that $H^0_{DR}(U)$ is the space of constant functions. The constant function $g = 1$ is therefore in $H^0_{DR}(U)$, and we see that $D_k([g])([s]) = (g \circ s)(0) = 1$. □

Lemma 6.3. *Let M be a smooth manifold and S a basis of M. There is an open cover $\{D_i\}$ of M which is countable so that D_i is the finite union of elements of the basis for each i, and, if $D_i \cap D_j = \emptyset$, then $i \neq j \pm 1$.*

Proof. We omit this proof here. See Lemma 3.4 of [9]. □

Lemma 6.4. *Let $M = \bigcup_{i=1}^q D_i$ be a smooth manifold where $q \in \mathbb{Z}$ and D_i is open for each i. If the de Rham homomorphism is an isomorphism on each D_i and each finite intersection of D_i's, then the de Rham homomorphism on all of M is an isomorphism.*

Proof. We shall show this for $q = 2$. The full proof follows by induction. We write the following diagram from the Mayer-Vietoris sequences for de Rham and singular cohomology.

$$\begin{array}{ccccc}
H^{k-1}_{DR}(D_1) \bigoplus H^{k-1}_{DR}(D_2) & \longrightarrow & H^{k-1}_{DR}(D_1 \cap D_2) & \longrightarrow & H^k_{DR}(M) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
H^{k-1}(D_1) \bigoplus H^{k-1}(D_2) & \longrightarrow & H^{k-1}(D_1 \cap D_2) & \longrightarrow & H^k(M)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^k_{DR}(D_1) \bigoplus H^k_{DR}(D_2) & \longrightarrow & H^k_{DR}(D_1 \cap D_2) \\
 & \downarrow \delta & & \downarrow \epsilon \\
\longrightarrow & H^k(D_1) \bigoplus H^k(D_2) & \longrightarrow & H^k(D_1 \cap D_2)
\end{array}$$

Theorems 5.3 and 5.4 show commutivity of the diagram. The Five Lemma (see [3] chapter 2) tells us that the γ is an isomorphism due to our hypothesis that the first, second, fourth, and fifth vertical homomorphisms are isomorphisms. We conclude that M is de Rham.

□

Theorem 6.5 (de Rham)**.** *If M is a smooth manifold, $D_k(M)$ is an isomorphism for each k.*

Proof. Let us say that $M = \bigcup_j M_j$ is the disjoint union of manifolds $\{M_j\}$. We will show that if each M_j is de Rham, M is de Rham. We introduce inclusion maps $i_j : M_j \to M$. The inclusion maps induce isomorphisms between the direct products of the de Rham cohomology groups of M_j and the singular cohomology groups of $\bigcup_j M_j$. It follows from Theorem 5.3 that $D_k(M)$ is an isomorphism.

We will now choose an open cover, $\{U_i\}$, of M as we did in Lemma 6.3. We know that $M = U^{\mathrm{odd}} \cup U^{\mathrm{even}}$ where we define $U^{\mathrm{odd}} = U_{2q+1}$ and $U^{\mathrm{even}} = U_{2q}$ for all natural q. We know that only adjacent U_i's have nonempty intersections from the Lemma. Thus, we may write the set of intersections of the open cover, U^{t}, as $U^{\mathrm{t}} = \cup(U_{q+1} \cap U_q)$.

Using Lemma 6.4, if we show that the de Rham homomorphism is isomorphic on U^{even}, U^{odd}, and U^{t}, then M must be de Rham. We can simplify this even further as all three of these sets are the disjoint union of U_q and $U_q \cap U_{q+1}$.

Using Lemma 6.4 again, we see that we simply need to show the existence of a basis for M such that D_k is an isomorphism for each element in the basis. This is because we can write U_q as a finite union and $U_q \cap U_{q+1}$ as a finite intersection of elements of the basis.

If we assume that M has dimension n, we take the basis of M to be the domain charts, which are diffeomorphic to open sets contained in $\mathbb{R}^n$.

We need now only show that open subsets of $\mathbb{R}^n$ are de Rham. We may take a basis of balls of an open set $O \subset \mathbb{R}^n$. Balls and the intersection of balls are convex and, by Lemma 6.2, are de Rham. Thus, O is de Rham. $\square$

7. Applications

Returning to the punctured plane, one immediately sees that there exist differential forms on $\mathbb{R}^2 \sim \{0,0\}$ that are closed but not exact. Take, for example, the 1-forms on $\mathbb{R}^2 \sim \{0,0\}$. We know, from the de Rham theorem, that there exists an isomorphism $\alpha : H^1_{DR}(\mathbb{R}^2 \sim \{0,0\}) \to H^1(\mathbb{R}^2 \sim \{0,0\})$. Given that $H^1(\mathbb{R}^2 \sim \{0,0\}) \cong \mathbb{R}$, we see that $H^1_{DR}(\mathbb{R}^2 \sim \{0,0\}) \cong \mathbb{R}$. The de Rham cohomology is not trivial, and thus we have proven the existence of non-exact closed forms in the punctured plane.

Let us consider the n-sphere, S^n. Once again, we may use de Rham's theorem to calculate the de Rham cohomology of S^n. Calculation of the singular cohomology reveals that $H^k(S^n)$ is isomorphic to $\mathbb{R}$ for $k = 0, n$ and is trivial for $0 < k < n$. Thus, we see that all closed k-forms are exact on S^n for $0 < k < n$.

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