CRITICAL SITE PERCOLATION ON THE TRIANGULAR LATTICE

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Abstract. This paper is an exposition of several results regarding site percolation on the triangular lattice. We begin by proving the critical probability of the lattice to be $p_c = \frac{1}{2}$ through the work of Kesten. Then we follow the work of Smirnov to demonstrate conformal invariance of crossing probabilities in critical percolation. Lastly, we follow the work of Lawler, Schramm and Werner to use this to compute the one arm exponent of critical site percolation.

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1. Introduction and Basic Background

The study of percolation begins with the following idea. Given a lattice, independently declare each vertex of the lattice to be open with probability p, or closed with probability $1 - p$. Note that one may instead choose to declare edges open or closed. This is referred to as *bond* percolation. In this paper we restrict ourselves to *site* percolation, where the vertices are declared open or closed as described above. For more information on bond percolation which contains plenty of beautiful results, the reader is encouraged to consult Grimmett's *Percolation* [4].

Once we have our percolation configuration, we wish to know more about the configuration of open clusters. An open cluster is a connected subset containing only open vertices. More specifically, we wish to know at what values of p does there exist an infinite open cluster beginning from an arbitrary point we declared the origin (this may be referred to as 'having percolation at 0'). We define the critical probability to be $p_c = \sup\{p | P_p[\text{An infinite open cluster contains } 0]\}$. In his work, Kesten [6] showed the critical probability of site percolation on the triangular lattice to be $\frac{1}{2}$, the proof of which we review below. Before we may do this, we have to first define the triangular lattice.

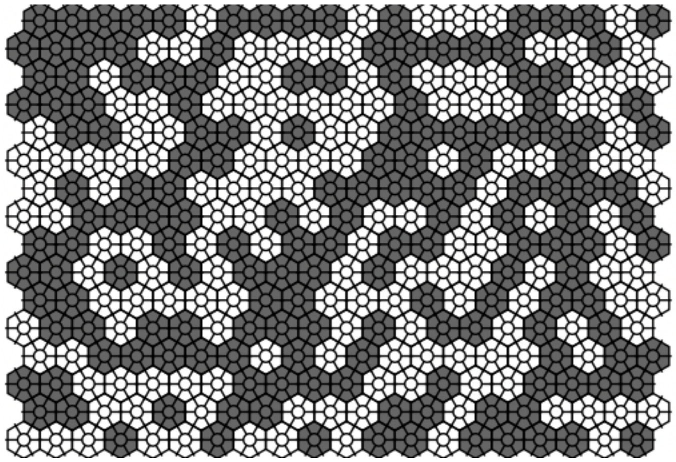

FIGURE 1. A sample percolation configuration taken from [12].

Definition 1.1. The triangular lattice ($\mathbb{T}$) has vertices on the points $(Re(\alpha + \beta e^{\frac{i\pi}{3}}), Im(\alpha + \beta e^{\frac{i\pi}{3}}))$ for $\alpha, \beta \in \mathbb{Z}$. We call a vertex $v = (v_\alpha, v_\beta)$ in addition to the usual x, y coordinates. Edges are lines connecting points of distance 1 away from each other.

Theorem 1.2. *For site percolation on the triangular lattice* $p_c = \frac{1}{2}$

After the critical probability has been found, a natural question to ask is what the percolation configuration looks like at the critical probability. A specific line of inquiry which has yielded a rich theory is describing the scaling limit of percolation interfaces. Given a simply connected subset of the plane (one where any loop can be contracted to a point such that the loop is always entirely in the domain), we define the exploration path as follows

Definition 1.3. Given a simply connected domain Ω and two points a, b on $\partial\Omega$, the **exploration path** denoted $\gamma^{e.p.}$ of a percolation configuration is the unique path separating all closed clusters connected to the arc ab from the open clusters connected to the arc ba.

We may also formulate such a path algorithmically. Allowing the arc ab to be entirely closed, and the arc ba to be entirely open, repeat the following process. Starting from the triangle with a as a vertex, choose the edge opposite a. Now, always move to an edge with an open vertex on its left, and a closed vertex on its right. This precisely yields the exploration path.

The next thing we wish to show is a property known as **conformal invariance**.

Definition 1.4. Given any two simply connected domains V, U with boundary points v_1, v_2 and u_1, u_2. Denote their lattice approximations by V_δ and U_δ. Then consider an exploration path $\gamma^{e.p.}$ from v_1 to v_2. Under any conformal mapping (bijective and holomorphic) $\psi : V \to U$ for which either $\psi(v_i) = \psi(u_i)$ then the distribution of $\gamma^{e.p.}$ is identical to the distribution of $\psi(\gamma^{e.p.})$.

We prove conformal invariance of the exploration path as well as **Cardy's Formula** (see **??**). We follow the work of Smirnov and provide the proof below [9].

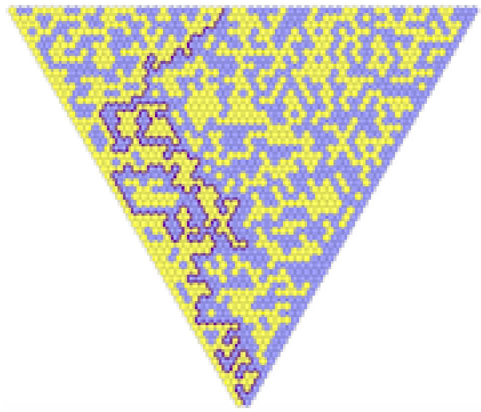

FIGURE 2. An exploration path pictured by Hugo Duminil-Copin and Vincent Beffara [1]

Lastly, conformal invariance is a sufficient condition to prove the convergence of the exploration path (as well as other interfaces we do not mention) to a random curve known as SLE_6. More information on the subject can be found here [2]. This opens up a wealth of new information about critical percolation, of notable mention being the study of *critical exponents*. The most elementary such exponent is the one arm exponent, or the probability that the open cluster from the origin intersects the disc of radius n. It was shown by Lawler, Schramm, and Werner [7] that the probability decays like $n^{-\frac{5}{48}}$. We shall review (some of) the techniques used to derive such an exponent below.

2. There is no Percolation at $p = \frac{1}{2}$

In this section, we prove the following fairly simple but necessary result for showing the critical probability is $\frac{1}{2}$.

Theorem 2.1. $P_{\frac{1}{2}}[\textit{There exists an infinite open path from the origin}] = 0$

Here, we very closely follow [11]. The approach below is simply an adaptation of this method to the triangular lattice, as some steps utilized will not be useful for site percolation. In order to produce any meaningful results, it is helpful to reformulate the percolation configuration as follows.

Definition 2.2. Given a lattice, a **percolation configuration** is an assignment to all vertices v of iid random variables $p(v)$ uniformly distributed from $[0, 1]$. Then given our probability p, we declare v open iff $p \geq p(v)$.

Since the $p(v)$'s are each distributed uniformly the probability that v is open is simply p. Note that we denote the probability of an event X on such a configuration with probability p by $P_p[X]$. Now, we also get the additional property of 'monotonicity' in p for certain events. As an example, suppose X was the event 'path W consists of entirely open vertices'. Suppose we have two probabilities p, p'. If $p' > p$, then $p > p(v)$ implies $p' > p(v)$ so we can only open vertices by increasing our value of p. Thus we conclude $P_p[X] \leq P_{p'}[X]$. This type of event is an **increasing event**, which we define using the notes of Steif [11].

Definition 2.3. Letting $\mathcal{V}$ be the set of vertices of the triangular lattice. Then define a partial order $\preceq$ on $\{0,1\}^{\mathcal{V}}$, where $x \preceq y$ iff entrywise it always holds that $y_i = 0$ implies $x_i = 0$. A function f on this space to the real numbers is increasing if $x \preceq y$ implies that $f(x) \leq f(y)$. An event is increasing if its indicator function is increasing.

Increasing events are an extremely powerful tool because they lend themselves quite well to certain inequalities. An essential such inequality is the FKG inequality.

Proposition 2.4. ***FKG Inequality:*** *Given two increasing events A, B, we have $P_p(A \cap B) \geq P_p(A)P_p(B)$. Equivalently $E[I_A I_B] \geq E[I_A]E[E_B]$.*

For a proof see [11] or [6]. Intuitively, increasing events will happen if 'many' vertices are open. If one such event has already happened, this suggests lots of vertices are already open so it is more likely that the other event will happen. This suggests increasing events are in a sense 'positively correlated'. Notice the same concept holds for decreasing events, but such a result will not be used here.

Corollary 2.5. *For increasing events $A_1, \ldots A_n$ $P[\bigcap_{i=1}^n A_i] \geq \prod_{i=1}^n P[A_i]$.*

Proof. Using induction and the fact that intersections of increasing events remain increasing, this is immediate. □

Definition 2.6. Allow $L_{n,m}$ to denote the event of an open crossing from the left boundary to the right boundary of $[0,n] \times [0,m]$ that travels strictly within the rectangle. Similarly let $L^*_{n,m}$ denote the closed analogue.

Remark 2.7. Notice that by the inherent symmetries of the lattice, the probability of an open crossing of an arbitrary $n \times m$ rectangle with a vertex in it's bottom left corner and a side parallel to the x-axis is identical to the probability of $L_{n,m}$, so we sometimes use $L_{n,m}$ as shorthand.

Proposition 2.8. ***Russo-Seymour-Welsh Theorem (RSW):*** *$P_{\frac{1}{2}}[L_{kn,n}] \geq c_k > 0$, where c_k depends only on k.*

For a proof of this theorem, see [11] section 7.2. When coupled with FKG, RSW estimates are extremely powerful. We use them as follows.

Definition 2.9. Given vertex v, a **Parallelogram Set** $\mathcal{P}(v,l)$ is the set of vertices $\{v : |v_\alpha|, |v_\beta| \leq l\}$.

Definition 2.10. An annulus about point v of length l, $A(v,l)$, is the set $\mathcal{P}(v,3l) \setminus \mathcal{P}(v,l)$. For convenience, an annulus centered about the origin will be denoted $A(l)$.

Definition 2.11. A traversing open circuit of an annulus is a loop of open vertices separating the inner boundary of the annulus from its outer boundary.

One immediately can tell this is useful, because if such an open circuit were to occur, we can apply symmetry to show the probability of a closed circuit is the same. If such a closed circuit were to occur, then we see that no open path can leave the annulus.

Lemma 2.12. *For any annulus, there exists a constant c such that for any v, l $P_{\frac{1}{2}}[$there exists a closed circuit traversing $A(v,l)] \geq c$.*

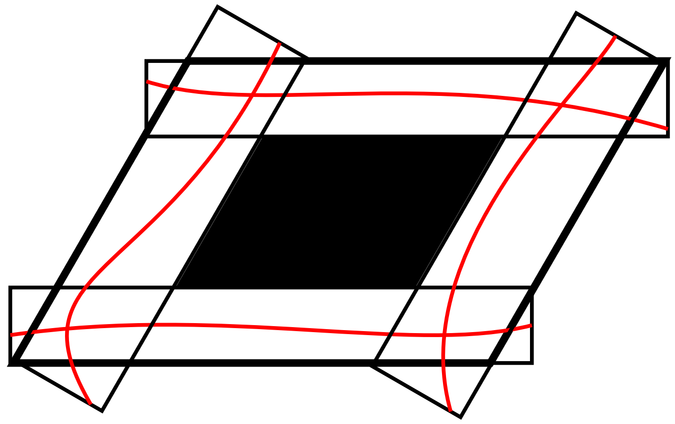

FIGURE 3. Using RSW to construct a traversing of an annulus

Proof. Note that we may cover each 'side' of the annulus $A(v,l)$ by a $6l \times \frac{\sqrt{3}l}{2}$ rectangle, and a crossing of all four of these rectangles implies a circuit traversing the annulus (see Figure 3). Note that $P_{\frac{1}{2}}[L_{6l,\frac{\sqrt{3}l}{2}}] \geq c_{12}$, and by the $\frac{2\pi}{3}$ rotational symmetry of the lattice we also observe that the probability of an open crossing of each of these rectangles is identical. Now by virtue of the fact that these open crossings are increasing events, by Proposition 2.4 we see
$P_{\frac{1}{2}}$[there exists a open circuit traversing A(v,l)] $\geq c_{12}^4 = c$. As $p = \frac{1}{2}$, we have symmetry implying that this inequality also applies to closed circuits. □

Remark 2.13. It is important to note that a similar upper bound may be derived, which we will denote q. One may do this by putting a rectangle over a side of the annulus, but we do not rigorously prove this statement. Moreover, this can be done for effectively any aspect ratio (the one from Definition 2.10 has an aspect ratio of 3), as well as other kinds of annuli. Most notably, this result will be used later for the hexagonal annulus. However, all such proofs are effectively the same.

Theorem 2.14. $P_{\frac{1}{2}}$[*There exists an infinite open path from the origin*] $= 0$

Proof. Consider the annuli $A_k = A(3^k l)$. Should any one such annulus contain a traversing closed circuit (denote the event in which A_k has such a circuit as C_k), we do not have an infinite open path from the origin. Additionally we see that these annuli are all disjoint, and thus events occurring within different annuli are independent. Since we have $P_{\frac{1}{2}}[C_k] \geq c$, the sum $\sum_{k=1}^{\infty} P_{\frac{1}{2}}[C_k]$ diverges and thus by the second Borel-Cantelli lemma we have that infinitely many such closed crossings occur, rendering percolation impossible. □

3. Crossing Probabilities for $p \geq \frac{1}{2}$

We begin this section with the essential topological tool about loops:

Proposition 3.1. *Consider a simple closed loop. Given four arcs A_1, A_2, A_3, A_4 appearing in counterclockwise order on the loops domain, there is an open (resp. closed) crossing from A_1 to A_3 if there is no closed (resp. open) crossing from A_2 to A_4.*

The proof of this statement is rather difficult and doesn't necessarily provide any stronger intuition behind this already intuitive statement. Thus we do not include it here, however curious readers are encouraged to check out the proof in [6].

From here we see that $U_{m,n}$ and $L_{m,n}$ are in fact not complements, as the vertical edges of a rectangle contain no edge of $\mathbb{T}$, and as such we briefly define two new events.

Definition 3.2. Allow $R_{m,n}$ (resp. $R^*_{m,n}$) and $D_{m,n}$ (resp. $D^*_{m,n}$) to denote open (resp. closed) crossings from the left edge to the right edge or from the bottom edge to the top edge respectively of the parallelogram $[0, m] \times [0, n]$ in α, β coordinates. By our previous proposition we have that these are complementary events.

Remark 3.3. Similarly to Proposition 2.8, we have that shifting our parallelogram by integer values in the α or β direction does not change these probabilities by symmetry of the lattice.

Now we want to show that these events become very unlikely for large parallelograms. The critical tool for this is Russo's formula, which goes as follows.

Definition 3.4. A vertex v is **pivotal** for an event A if changing the state of v changes whether or not the entire percolation configuration ω is contained in A. The event where v is pivotal for A is denoted $Piv_v(A)$.

Definition 3.5. The **influence** of a vertex is $P_p(Piv_v(A))$, and this is denoted by $I^p_v(A)$.

Proposition 3.6. ***Russo's Formula:*** *Suppose we have an increasing event A and a finite vertex set V. Given a percolation configuration, allow the set of pivotal vertices to be denoted by U. Then $\frac{d}{dp}P_p(A) = \sum_{v \in V} I^p_v(A) = E[\#U]$.*

The idea here is that as this event is increasing, it is clear only opening vertices can increase the probability of this event. Therefore if many vertices can be influential, then slightly increasing p can significantly increase the probability that some such vertices are open, which in turn greatly increases the likelihood that A is realized. For a proof, see [8]. Should we show the expected number of pivotal edges is infinite, we may use this to bound crossing probabilities as follows.

Proposition 3.7. *For $p > \frac{1}{2}$, $P_p[R^*_{2n,6n}] \to 0$ as $n \to \infty$.*

Lemma 3.8. *Allow $p(t) = \frac{1-t}{2} + tp$. Then $E_{p(t)}[\#U | R^*_{2n,6n}] \to \infty$ uniformly over t, then Proposition 3.7 holds.*

Proof. By the chain rule and Russo's formula, we have that $\frac{d}{dt}P_{p(t)}[R^*_{2n,6n}] = (\frac{1}{2} - p)E_{p(t)}[\#U]$, or equivalently $\frac{1}{P_{p(t)}[(R^*_{2n,6n})]}\frac{d}{dt}P_{p(t)}[R^*_{2n,6n}] = (\frac{1}{2} - p)E_{p(t)}[\#U | R^*_{2n,6n}]$. This is a fairly simple differential equation, which we integrate from $t = 0$ to $t = 1$ to get $ln(P_p[(R^*_{2n,6n})]) - ln(P_{\frac{1}{2}}[(R^*_{2n,6n})]) = (\frac{1}{2} - p)\int_0^1 E[\#U | R^*_{2n,6n}]dt$. Thus we see $P_p[(R^*_{2n,6n})] \leq P_{\frac{1}{2}}[(R^*_{2n,6n})]e^{-\int_0^1 E_{p(t)}[\#U | R^*_{2n,6n}]dt}$. From here the lemma is done. ∎

Remark 3.9. In order to continue the proof of Proposition 3.7, we need to define the notion of a 'lowest closed crossing'. To briefly justify why such a crossing can be chosen, fix our parallelogram P. Then for a closed crossing r, denote all the vertices below and including the crossing within P by the set $J^-(r)$. If $\mathcal{C}$ is the set of all closed crossings of P, then $\cap_{c\in\mathcal{C}} J^-(c)$ clearly has the lowest closed crossing as its upper boundary. Furthermore, a lowest closed crossing has what is referred to as a 'strong Markov property'. This means that it is independent of anything depending only on vertices above it.

Lemma 3.10. *$E_{p(t)}[\#U|R^*_{2n,6n}] \to \infty$ uniformly over t as $n \to \infty$.*

Proof. We first denote lowest closed horizontal crossing by $\mathfrak{R}$. We first note that any vertex in $\mathfrak{R}$ which has a vertical open crossing from the upper boundary of the parallelogram to just above that vertex is pivotal (this can be empty should our vertex be contained in the upper boundary). If not, this contradicts the fact that we chose the lowest crossing. We denote such a crossing as an open connection. Thus it is sufficient to estimate the number of vertices in the lowest closed crossing $\mathfrak{R}$ with an open connection, denoted by $N_{\mathfrak{R}}$. Now taking the expectation over all possible crossings r yields

$$E_{p(t)}[N_{\mathfrak{R}}] = \sum_r E_{p(t)}[N_r|\mathfrak{R} = r]P_{p(t)}[\mathfrak{R} = r]. \tag{1}$$

Now we define the events C_v for $v \in r$, where C_v is the event that v has an open connection. As this is a increasing event, we see that

$$E_{p(t)}[N_r|\mathfrak{R} = r] = E_{p(t)}[N_r] = \sum_{v\in r} P_{p(t)}[C_v] \geq \sum_{v\in r} P_{\frac{1}{2}}[C_v] = E_{\frac{1}{2}}[N_r] \tag{2}$$

where the first equality holds by the strong Markov property. Now plugging (2) into (1) gives

$$E_{p(t)}[N_{\mathfrak{R}}] \geq \min_r E_{\frac{1}{2}}[N_r] \sum_r P_{p(t)}[\mathfrak{R} = r]. \tag{3}$$

We then see that by the occurrence of $R^*_{2n,6n}$,

$$\sum_r P_{p(t)}[\mathfrak{R} = r] = 1$$

and we may thus simplify (3) to

$$E_{p(t)}[N_{\mathfrak{R}}] \geq \min_r E_{\frac{1}{2}}[N_r].$$

Thus our proof is complete if we show N_r becomes large in n regardless of our choice of r.

Now suppose there is an open crossing from r to the upper boundary of the parallelogram contained to the left of the line $v_\alpha = n$ (i.e. a crossing in the left half of the parallelogram). Again by Proposition 2.8 and rotational symmetry of the lattice we have $P_{\frac{1}{2}}[U_{n,6n}] \geq c_6$. Thus we may consider a leftmost vertical open connection from r, denote this crossing as s. Thus $E_{\frac{1}{2}}[N_r] \geq \min_s c_6 E_{\frac{1}{2}}[N_r|s]$. We show regardless of our choice of r and s, we are guaranteed an unbounded number of pivotal edges.

Allow v to denote the vertex where r and s meet. Then consider the annuli $A(v, 3^k)$. More specifically, consider the component of such an annulus to the right of s and above r (should the annulus never intersect s, then we say our annulus section intersects the upper boundary of the parallelogram and ends there. See Figure 4 for more). Such a section has probability of at least c of having a closed traversal, as it clearly is more likely than the entire annulus having a closed

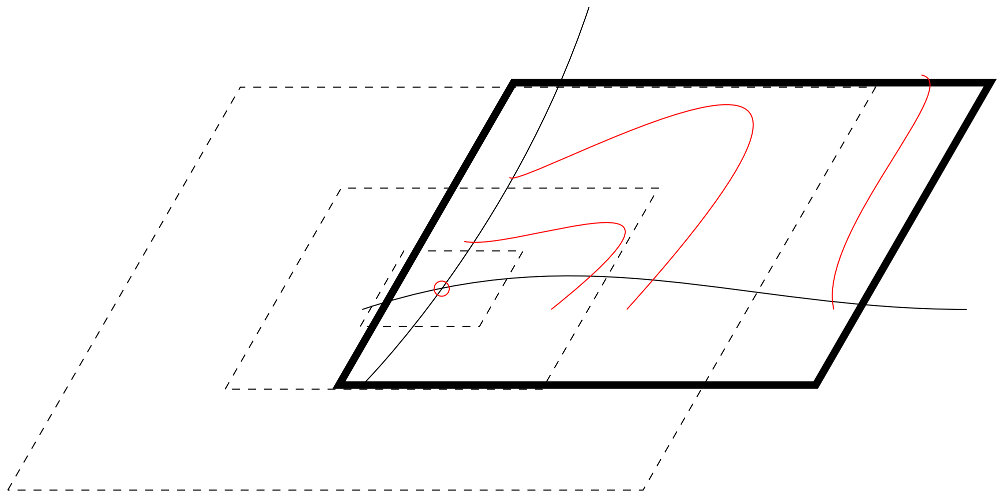

FIGURE 4. The construction of infinitely many pivotal vertices

traversal. By the strong Markov property this event is also independent of both s and r. Moreover, each annulus is independent of each other as they are disjoint. Lastly as s is contained in the left half of the parallelogram, we may include at least $\lfloor \log_3(n) \rfloor$ such annuli sections.

Thus the sum of probabilities of the annuli sections having a closed traversal is at least $c\lfloor \log_3(n) \rfloor$, where c is from Lemma 2.12. Notice each traversal in fact corresponds to a distinct pivotal vertex. Since $c\lfloor \log_3(n) \rfloor \to \infty$ as $n \to \infty$, then by the second Borel-Cantelli lemma we have infinitely many such traversals and $E_{\frac{1}{2}}[N_r|s] \to \infty$. □

Proposition 3.7 follows from Lemma 3.10 and Lemma 3.8.

Remark 3.11. The identical claim holds for $P_p[U^*_{6n,2n}]$, by a 90 degree rotation of the exact argument.

Corollary 3.12. *By the definition of a limit, there exists an $N = 2M$ such that $P_p[R^*_{N,3N}], P_p[U^*_{3N,N}] \leq \frac{1}{4}(50e)^{-49}$.*

4. Expected Cluster Size for $p > \frac{1}{2}$

Definition 4.1. Allow $C(v)$ to denote the closed cluster containing v on a fixed percolation configuration.

We may use the previous section's results to demonstrate the following:

Theorem 4.2. *$E_p[|C(0)|] < \infty$.*

The proof of this theorem is rather convoluted, but begins with the following fact. When keeping the configuration variable, the set of possible closed clusters with n vertices containing a specific vertex grows exponentially.

Definition 4.3. Denote the set of distinct connected sets of n vertices containing v by $\Omega_n(v)$.

Thus if we find a way to relate crossing probabilities with the size of a cluster, we can use the previous corollary to have the probability of the origin belonging to a size n cluster existing decay exponentially. This immediately implies the expected closed cluster size about the origin is finite. We do so by introducing the auxiliary graph. Effectively, the auxiliary graph declares specific vertices closed should such a crossing occur in a specific location. This is defined in more detail as follows.

Definition 4.4. The **auxiliary graph** with parameter $n \in \mathbb{N}$ has vertices on the points $(Re(n\alpha + n\beta e^{\frac{i\pi}{3}}), Im(n\alpha + n\beta e^{\frac{i\pi}{3}}))$ for $\alpha, \beta \in \mathbb{Z}$. Vertices will be referred to identically as on the triangular lattice, except with the following notational change $v \to \tilde{v}$ (this includes $0 \to \tilde{0}$). Additionally, the $n \times n$ parallelogram with $\tilde{v}$ in its bottom left corner will be referred to as $B(\tilde{v})$. Note that every $B(\tilde{v})$ has a (topologically) open top and right side, but closed bottom and left side. Edges are lines connecting vertices $\tilde{v}, \tilde{u}$ such that $|\tilde{v}_\alpha - \tilde{u}_\alpha|, |\tilde{v}_\beta - \tilde{u}_\beta| \leq 2n$. This graph will be referred to by $\mathcal{L}(n)$.

Remark 4.5. For all future uses of $\mathcal{L}$ we will assume $n = N$ from Corollary 3.12 and use $\mathcal{L}$ as a shorthand.

Now we define a percolation configuration on such a graph in a way that relates crossings to closed clusters.

Definition 4.6. An auxiliary percolation configuration on $\mathcal{L}$ goes as follows. A vertex $\tilde{v}$ is closed if there is a closed path which starts in its corresponding parallelogram and escapes the 3×3 grid of parallelograms surrounding it. Formally this is defined as follows. For a given $\tilde{v}$, we first define the four parallelograms $H^{0,\pm 1}(\tilde{v})$ and $H^{\pm 1,0}$. $H^{0,\pm 1}(\tilde{v}) = \cup_{i=1}^{3}(\tilde{v}_{\alpha\pm 1}, \tilde{v}_{\beta-2+i})$ and $H^{\pm 1,0}(\tilde{v}) = \cup_{i=1}^{3}(\tilde{v}_{\alpha-2+i}, \tilde{v}_{\beta\pm 1})$. Let $B'(\tilde{v}) = H^{0,1}(\tilde{v}) \cup H^{0,-1}(\tilde{v}) \cup H^{1,0}(\tilde{v}) \cup H^{-1,0}(\tilde{v})$. We say $\tilde{v}$ is closed if there exists a closed path on $\mathbb{T}$ with an end vertex in $B(\tilde{v})$ and one outside or on the boundary of $B'(\tilde{v})$.

Remark 4.7. Should $\tilde{v}$ be closed, by definition we either have a closed vertical crossing of one of $H^{\pm 1,0}(\tilde{v})$ or a closed horizontal crossing of one of $H^{0,\pm 1}(\tilde{v})$. Notice however the first scenario has the same probability as $U^*_{3N,N}$, and the second scenario has the same probability as $R^*_{N,3N}$. Thus by the union bound for any given $\tilde{v}$ we have $P_p[\tilde{v} \text{ closed}] \leq 2(P_p[R^*_{N,3N}] + P_p[U^*_{3N,N}])$ which helps us relate cluster sizes on $\mathcal{L}$ to crossing probabilities on $\mathbb{T}$. By Corollary 3.12 we may further strengthen this bound to $P_p[\tilde{v} \text{ closed}] \leq (50e)^{-49}$.

We therefore seek to relate cluster sizes on $\mathcal{L}$ and $\mathbb{T}$. In order to do so, we declare V to be the set of vertices $\tilde{v}$ with $|\tilde{v}_\alpha|, |\tilde{v}_\beta| \leq 3N$, which is a notion of 'close to the origin' on the auxiliary graph.

Proposition 4.8.

$$\max_{\tilde{v} \in V} |C(\tilde{v})| \geq \frac{|C(0)| - 16N^2}{49N^2}$$

The idea here goes as follows. Should the cluster at the origin be large, then there exists points very far from the origin which have a path back to the origin. This equates to a large path in $\mathcal{L}$ which returns relatively close to the origin, which is more rigorously stated in the following manner.

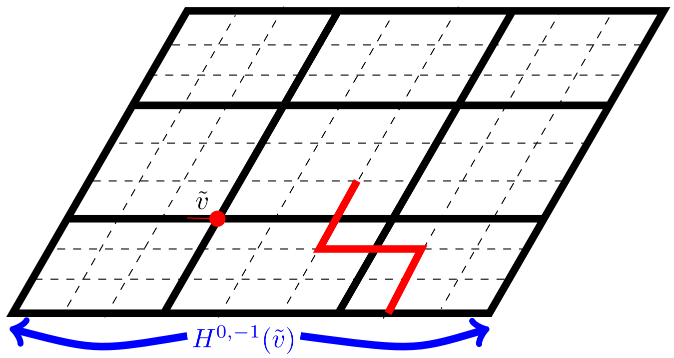

FIGURE 5. The auxiliary graph with parameter $N = 3$. Note $\tilde{v}$ is closed as there is a closed path starting in $B(\tilde{v})$ and leaving $B'(\tilde{v})$. Furthermore, observe the vertical crossing of $H^{0,-1}(\tilde{v})$

Lemma 4.9. *If we have $w^0 \in C(0)$ such that for some $x \geq 2$ either $xN \leq w^0_\alpha \leq (x+1)N$ or $xN \leq w^0_\beta \leq (x+1)N$ then there is a closed path on $\mathcal{L}$ ending on some vertex $\tilde{v} \in V$.*

Proof. First allow $B(\tilde{v}^0)$ to denote the unique parallelogram containing w^0. As w is in $C^*(0)$, then there is a closed path connecting w^0 to 0 which shall be called r. Moreover, r is a closed path escaping $B'(\tilde{v}^0)$ by the condition on w^0 (and thus $\tilde{v}^0$ is closed). This implies there is a closed vertex on $r \cap \partial B'(\tilde{v}^0)$ in $\mathbb{T}$. Denote this vertex as w^1 in parallelogram $B(\tilde{v}^1)$. As $w^1 \in B'(\tilde{v}^0)$, we see that $\tilde{v}^0$ is adjacent to $\tilde{v}^1$. We may now repeat this process on w^1 and iterate until we reach some w^b with $|w^b_\alpha|, |w^b_\beta| \leq N$, which provides a path on $\mathcal{L}$ starting at $\tilde{v}$ leading to some vertex $\tilde{v}^b$ such that $v \in V$. □

These paths are large enough to be considered a reasonably large cluster and supply an adequate bound on the minimum guaranteed cluster size. This is because any vertex outside of the $4N \times 4N$ parallelogram about the origin, should it be in $C(0)$, is guaranteed to have a path leading close to the origin. Thus there are at least $|C(0)| - 16N^2$ such vertices. There are at most 49 auxiliary vertices $\tilde{v}$ such that $|\tilde{v}_\alpha|, |\tilde{v}_\beta| \leq 3N$. Thus for at least one $\tilde{v}'$ there are $\frac{|C(0)|-16N^2}{49}$ such vertices on $\mathbb{T}$ which correspond specifically to $\tilde{v}'$. As there are N^2 vertices per $B(\tilde{v})$, we can see that $|C(\tilde{v}')| \geq \frac{|C(0)|-16N^2}{49N^2}$, proving Proposition 4.8.

This immediately gives us the following corollary which relates closed cluster sizes on $\mathbb{T}$ and $\mathcal{L}$.

Corollary 4.10. $|C(0)| = n$ *implies for some vertex $\tilde{v}$ such that $|\tilde{v}_\alpha|, |\tilde{v}_\beta| \leq 3N$ that $|C(\tilde{v})| \geq \frac{n-16N^2}{49N^2} \geq \frac{n}{49N^2} - 1 = An - 1$, where $A = \frac{1}{49N^2}$.*

Using the observation that an infinite cluster on one graph implies an infinite cluster on another, we know that $|C(\tilde{v})|$ is finite. The reason for this is that an infinite closed cluster from the origin is a decreasing event. We have shown that at $p = \frac{1}{2}$ this cannot happen, but here $p > \frac{1}{2}$. Thus we only consider the set of finite clusters. Thus by the union bound we see that

$$P_p[|C(0)| \geq n] \leq \sum_{\tilde{v} \in V} P_p[|C(\tilde{v})| \geq An - 1].$$

Now notice that if $|C(\tilde{v})| = m \geq An - 1$ then the cluster takes the form of some cluster in $\Omega_m(\tilde{v})$. Thus we see that we may further strengthen this bound to

$$P_p[|C(0)| \geq n] \leq \sum_{m=An-1}^{\infty} \sum_{\tilde{v} \in V} |\Omega_m(\tilde{v})| \sup_{\omega \in \cup_{\tilde{v} \in V} \Omega_m(\tilde{v})} P_p[\omega \text{ is closed}].$$

By symmetry of $\mathbb{T}$, we see for any $\tilde{v}, |\Omega_m(\tilde{v})| = |\Omega_m(\tilde{0})|$. The next step is to estimate this quantity.

Proposition 4.11. $|\Omega_m(\tilde{0})| \leq (7e)^m$.

Proof. In order to prove this, we need to consider subsets of $\Omega_m(\tilde{0})$. We denote these by $\Omega_{m,l}(\tilde{0})$, the set of clusters of m vertices containing $\tilde{0}$ which have l boundary vertices. The probability of $C(\tilde{0})$ being a specific $\omega \in \Omega_{m,l}$ is $p^l(1-p)^m$. It follows immediately that $P_p[|C(\tilde{0})| < \infty] = P_p[C(\tilde{0}) \in \cup_{m=1}^{\infty} \Omega_m] = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} |\Omega_{m,l}(\tilde{0})| p^l (1-p)^m \leq 1$. Thus we conclude for any fixed m we see that $\sum_{l=1}^{\infty} |\Omega_{m,l}(\tilde{0})| p^l (1-p)^m \leq 1$. Additionally, as the maximum degree of any vertex is 24, we see that any cluster with m vertices has as most $24m$ boundary vertices. Thus $\sum_{l=1}^{\infty} |\Omega_{m,l}(\tilde{0})| p^l (1-p)^m = \sum_{l=1}^{24m} |\Omega_{m,l}(\tilde{0})| p^l (1-p)^m \geq (1-p)^m p^{24m} \sum_{l=1}^{24m} |\Omega_{m,l}(\tilde{0})|$. This implies that $\sum_{l=1}^{\infty} |\Omega_{m,l}| \leq p^{-24m}(1-p)^{-m}$ for any p. Allowing $p = \frac{24}{25}$, we see that $\sum_{l=1}^{24m} |\Omega_{m,l}(\tilde{0})| \leq (25)^m (1 + \frac{1}{24})^{24m} \leq (25e)^m$. □

Plugging this estimate means that no more terms in the second sum depend on $\tilde{v}$. As there are 49 such $\tilde{v}$, we see that

$$P_p[|C(0)| \geq n] \leq 49 \sum_{m=An-1}^{\infty} (25e)^m \sup_{\omega \in \cup_{\tilde{v} \in V} \Omega_m(\tilde{v})} P_p[\omega \text{ is closed}].$$

Thus we have a final step of bounding $P_p[\omega \text{ is closed}]$. The idea is that if $\overline{B'(\tilde{v})}$ and $\overline{B'(\tilde{l})}$ (we just include the upper and right hand sides of these parallelograms here) are disjoint then $\tilde{v}$ and $\tilde{l}$ are independent, which yields the following lemma

Lemma 4.12. *Given a cluster of m vertices on $\mathcal{L}$, we may choose an independent set of vertices of size at least $t_m = \lceil \frac{m}{49} \rceil$.*

Proof. More concretely, if for $\tilde{v}$ and $\tilde{l}$ either $|\tilde{v}_\alpha - \tilde{l}_\alpha| \leq 3N$ or $|\tilde{v}_\beta - \tilde{l}_\beta| \leq 3N$ then we have an intersection of $\overline{B'(\tilde{v})}$ and $\overline{B'(\tilde{l})}$. Thus some $\tilde{l}$ in $C(\tilde{v})$ can have at most 48 other vertices which are not independent from it. Thus if we have a cluster of m vertices, we may choose an independent set of vertices of size at least $\lceil \frac{m}{49} \rceil$. □

Thus we obtain the bound

$$\sup_{\omega \in \cup_{\tilde{v} \in V} \Omega_m(\tilde{v})} P_p[\omega \text{ is closed}] \leq P_p[\tilde{l}^1, \tilde{l}^2 \ldots \tilde{l}^{t_m} \text{ all closed}] = \prod_{i=1}^{t_m} P_p[\tilde{l}^i \text{ closed}]$$

and by Remark 4.7 we may further manipulate this bound to get the simple form $\prod_{i=1}^{t_m} P_p[\tilde{l}^i \text{ closed}] \leq (50e)^{-49t_m} \leq (50e)^{-m}$.

We are now ready to prove Theorem 4.2:

Proof. By Proposition 4.11 and Lemma 4.12 we may conclude that $P_p[|C(0)| \geq n] \leq c_0 e^{-c_1 n}$, where $c_0, c_1 > 0$ depend only on p. Thus we have

$$E[|C(0)|] = \sum_{k=0}^{\infty} P_p[|C(0)| \geq n] \leq c_0 \sum_{k=0}^{\infty} (e^{-c_1})^k = \frac{c_0}{1 - e^{-c_1}} < \infty.$$

□

5. The Critical Probability is $\frac{1}{2}$

This section is fairly simple as it follows quite quickly from Theorem 4.2. This is done rigorously as follows.

Lemma 5.1. $P_p[U^*_{2^{k+1},2^k}] \leq 2^k P_p[|C(0)| \geq 2^k]$.

Proof. Denote the bottom boundary of the parallelogram of the event $U^*_{2^{k+1},2^k}$ by δ_*, and the top boundary by δ^*. If $U^*_{2^{k+1},2^k}$ occurs then some $v \in \delta_*$ is contained in a closed path to δ^*. Thus by the union bound

$$P_p[U^*_{2^{k+1},2^k}] \leq \sum_{v \in \delta_*} P_p[v \text{ has a closed path to } \delta^*]$$

However this means that v is in a closed cluster of size at least 2^k. Thus

$$P_p[U^*_{2^{k+1},2^k}] \leq \sum_{v \in \delta_*} P_p[|C(v)| \geq 2^k].$$

By symmetry, $P_p[|C(v)| \geq 2^k]$ is in fact identical over any $v \in \delta_*$. Thus $P_p[U^*_{2^{k+1},2^k}] \leq |\delta_*| P_p[|C(0)| \geq 2^k] = 2^{k+1} P_p[|C(0)| \geq 2^k]$ as desired. □

Lemma 5.2. *All but finitely many events of the form $R_{2^{k+1},2^k}$ occur almost surely.*

Proof. By Proposition 3.1, $(R_{2^{k+1},2^k})^c$ is $U^*_{2^{k+1},2^k}$. By our previous lemma, we have that

$$\sum_{k=0}^{\infty} P_p[U^*_{2^{k+1},2^k}] \leq \sum_{k=0}^{\infty} 2^{k+1} P_p[|C(0)| \geq 2^k] \leq \sum_{k=0}^{\infty} 4 \sum_{m=2^{k-1}-1}^{2^k} P_p[|C(0)| \geq m]$$

$$= 4 \sum_{m=0}^{\infty} P_p[|C(0)| \geq m] = E_p[|C(0)|] < \infty.$$

Thus by the Borel-Cantelli lemma we are done with this lemma. □

Remark 5.3. A rotation of this argument similarly proves all but finitely many events of the form $U_{2^k,2^{k+1}}$ occur almost surely.

Proposition 5.4. $p_c = \frac{1}{2}$.

Proof. It follows immediately from Lemma 5.2 and Remark 5.3 that we can construct an infinite cluster $\mathcal{C}$ with probability 1. This immediately gives by symmetry that $P_p[|C(0)| = \infty] > 0$ for $p > \frac{1}{2}$ so $p_c \geq \frac{1}{2}$. Thus by Theorem 2.14 we have shown $p_c = \frac{1}{2}$. □

6. Conformal Invariance

In this section, we follow the work of Smirnov [9] (very closely, but hopefully with more exposition) to prove conformal invariance of $\gamma^{e.p.}$ and Cardy's formula. We first state the original form, which is quite nasty so we do not dive into any more detail then merely the statement.

Consider a simply connected domain Ω with four boundary points A, B, C, D. Then discretize Ω by a factor of δ using a triangular lattice where $v = (\alpha + \beta e^{\frac{i\pi}{3}}) = (v_\alpha, v_\beta)$ for $\alpha, \beta \in \delta\mathbb{Z}$. Note that here we have simply mapped the embedding from $\mathbb{R}^2$ to $\mathbb{C}$, but the embedding is otherwise identical. Edges connect vertices distance δ apart (this is the mesh size). We say the probability of an open crossing from arc AB to arc CD is $C_\delta(\Omega, A, B, C, D)$. As $\delta \to 0$, we have $C_\delta(\Omega, A, B, C, D) \to C(\Omega, A, B, C, D)$.

Theorem 6.1. ***Cardy's Formula, Carleson Form:*** *Take an equilateral triangle T with vertices A, B, C in counterclockwise order. Choose $D \in BC$. Then $C(T, A, B, C, D) = \frac{diam(CD)}{diam(AB)}$.*

This provides a rather beautiful simplification of Cardy's formula, but its true use lies in the following proposition.

Theorem 6.2. *Given a conformal map ψ,*
$C(\Omega, A, B, C, D) = C(\psi(\Omega), \psi(A), \psi(B), \psi(C), \psi(D))$.

In words, these crossing probabilities are conformally invariant. Note therefore we see that given conformal invariance of crossing probabilities in the scaling limit and Carleson's form, we acquire Cardy's formula by conformally mapping T to Ω. Note the Riemann Mapping Theorem ensures this is well defined, so the approach is to prove the above and receive Cardy's formula for free. We thus provide some brief setup for how to approach such a question.

Definition 6.3. For a unit vector η, and function f

$$\frac{\partial f}{\partial \eta}(x) = \lim_{h \in \mathbb{R}, h \to 0} \frac{f(x + \eta h) - f(x)}{h}.$$

Definition 6.4. We allow $\tau = e^{\frac{2\pi i}{3}}$. Consider a simply connected domain Ω with a smooth boundary. Denote three points on the boundary of Ω in counterclockwise order by $a(1), a(\tau), a(\tau^2)$. We denote arcs along the boundary between two such points by their concatenation, i.e. $a(1)a(\tau)$. We define ν to be the counterclockwise unit tangent to $\partial\Omega$. Then for $z \in \Omega$ we have a triple of harmonic functions $h_\alpha(z), \alpha \in \{1, \tau, \tau^2\}$ which satisfy the following differential equation along the boundary:

$$\begin{cases} h_\alpha = 1, \text{ for } z = a(\alpha) \\ h_\alpha = 0 \text{ for } z \in a(\tau\alpha)a(\tau^2\alpha) \\ \frac{\partial}{\partial(\tau\nu)} h_\alpha = 0 \text{ for } z \in a(\alpha)a(\tau\alpha) \\ \frac{\partial}{\partial(-\tau^2\nu)} h_\alpha = 0 \text{ for } z \in a(\tau^2\alpha)a(\alpha) \end{cases}$$

Note that for reasons we overlook here, we may actually declare these h_α's to be unique. The solution to such a differential equation is actually consistent under conformal mappings, and takes Carleson's familiar form when mapped to an equilateral triangle. The goal therefore is to discretize crossing probabilities, and show that their limit satisfy this differential equation.

Definition 6.5. Allow $Q_\alpha^{\Omega,\delta}(z)$ to be the event that on Ω there is a closed simple (no loops) crossing from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$ which separates z from the arc $a(\tau\alpha)a(\tau^2\alpha)$. We refer to such a path as a **separating path**. Let H_α be the probability of such an event. This is the function we want to show converges to h_α.

Again this proof relies on the idea of a derivative of a discrete structure. As we do not vary p, but instead δ, we want a derivative based on size and direction. The way we capture such a derivative is through the parameter η, which is a vector pointing from the center of a triangle on our lattice approximation to the center of an adjacent (connected by an edge) triangle. We define the event $P_\alpha(z,\eta)$ to be the probability that a translate from point z by η is pivotal for $Q_\alpha(z)$. In other words, we have $P_\alpha(z,\eta) = P[Q_\alpha(z+\eta) \setminus Q_\alpha(z)]$. This is sufficient to yield our derivative.

Definition 6.6. $\frac{\partial}{\partial\eta}H_\alpha(z) = H_\alpha(z+\eta) - H_\alpha(z) = P_\alpha(z,\eta) - P_\alpha(z+\eta,-\eta)$.

Note that the second inequality holds quite quickly through DeMorgan's laws and the principle of inclusion-exclusion.

Similarly to Russo's formula, the derivative is the probability that we gain a point which is separated by translating z by η minus the probability that we lose a point which is separated by translating z by η. This formula can then be 'rotated' to create certain useful symmetries. Such a strategy is the entire mechanism behind the entire proof. The first and perhaps most 'trivial' example goes as follows.

Lemma 6.7. $P_\alpha(z,\eta) = P_{\tau\alpha}(z,\tau\eta)$.

Proof. In the opposite direction of η should lie one of the vertices of the triangle which has z as its center. Starting with this vertex in a counterclockwise order label the vertices of this triangle X, Y, Z. If $Q_\alpha(z+\eta) \setminus Q_\alpha(z)$ occurs then there is a closed path s containing Y and Z from the arc $a(\alpha)a(\tau\alpha)$ to the arc $a(\tau^2\alpha)a(\alpha)$. Now we may take the half of Ω with s and $a(\tau\alpha)a(\tau^2\alpha)$ in its boundary. We denote X as A_1, the portion of s in the 'Y direction' and $a(\tau^2\alpha)a(\alpha)$ as A_2, $a(\tau\alpha)a(\tau^2\alpha)$ as A_3, and the portion of s in the 'Z direction' and $a(\alpha)a(\tau\alpha)$ as A_4. By the occurrence of $Q_\alpha(z+\eta) \setminus Q_\alpha(z)$, we actually know that there is no closed crossing from A_2 to A_4. Thus by Proposition 3.1 we conclude there is an open crossing from X to $a(\tau\alpha)a(\tau^2\alpha)$. Therefore we have shown that the event $Q_\alpha(z+\eta) \setminus Q_\alpha(z)$ is equivalent to the event that these three crossings exist and are disjoint. We may declare all paths disjoint as we may erase all loops in the crossing from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$. We illustrate this in Figure 6.

Using the identical logic from Remark 3.9 we may choose a counterclockwise-most open crossing r_X from X to $a(\tau\alpha)a(\tau^2\alpha)$ and a clockwise-most closed crossing r_Y from Y to $a(\tau^2\alpha)a(\alpha)$. Now any closed crossing from z and $a(\alpha)a(\tau\alpha)$ must be completely contained in the region between r_Y and r_X containing Z. Notice however that the crossing from z to $a(\alpha)a(\tau\alpha)$ is equally likely to be open as it is closed by virtue of the fact that $p_c = \frac{1}{2}$, so we may freely switch open and closed sites in the region between r_X and r_Y (not inclusive) while preserving probability. Assuming we now have two open paths and one closed one, since $p_c = \frac{1}{2}$ we may again preserve probability while swapping open and closed sites in all of Ω. This results in the event $Q_{\tau\alpha}(z+\tau\eta) \setminus Q_{\tau\alpha}(z)$. □

Lemma 6.8. *The functions $H_\alpha^\delta(z)$ are uniformly Hölder on Ω if Ω is an equilateral triangle. Equivalently, $|H_\alpha^\delta(z) - H_\alpha^\delta(z')| \leq C|z-z'|^\epsilon$ for C only dependent on the domain Ω.*

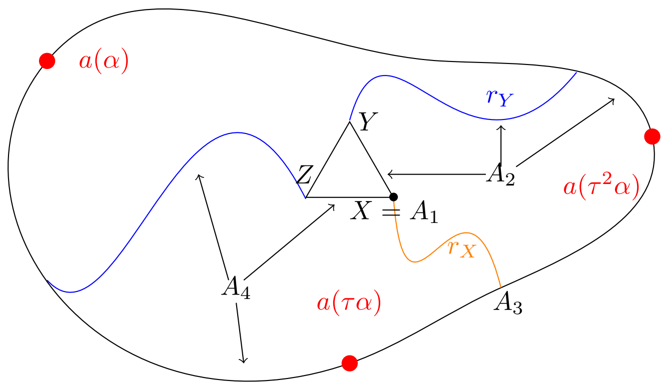

FIGURE 6. We equate blue with closed and orange with open. Assuming r_X and r_Y are the clockwise-most and counterclockwise-most such crossings, then we may freely swap the color of the third curve

Proof. We first remark $H_\alpha(z) - H_\alpha(z') = P[Q_\alpha(z) \setminus Q_\alpha(z')] - P[Q_\alpha(z') \setminus Q_\alpha(z)]$ and thus it suffices to estimate the first term on the right hand side and then apply the triangle inequality. We may identically bound the second term.

Next we note that given small enough r, every point in Ω is guaranteed to be a distance at least r from at least one boundary arc. This means the sets $\{z \in \Omega \mid \text{dist}(z, a(\alpha)a(\tau\alpha))\}$ for $\alpha \in \{1, \tau, \tau^2\}$ cover our triangle. Thus it suffices to prove this bound for points sufficiently far from a specific boundary arc.

Notice $Q_\alpha(z) \setminus Q_\alpha(z')$ can only occur if there is a closed path from $a(\alpha)a(\tau\alpha)$ to $a(\tau^2\alpha)a(\alpha)$ separating z from z', i.e. intersecting the interval $[zz']$. Thus by the identical reasoning of Lemma 6.7 we must have an open crossing from $[zz']$ to $a(\tau\alpha)a(\tau^2\alpha)$. We may assume this interval is at least some reasonable distance R away from the arc $a(\beta)a(\tau\beta)$. Then we may separate the interval with concentric annuli with size on the initial order of $r = |z - z'|$ and growing to order R. Given some fixed aspect ratio t, this results in $\log_t(R/r)$ disjoint annuli. However, as we have shown the existence of either an open or closed path from $[zz']$ to the boundary arc $a(\beta)a(\tau\beta)$, this implies all such annuli have either an open or closed crossing (This is not a traversing! In fact it is the complement) through them depending on our choice of β. Additionally, by RSW the probability of all such crossings of annuli is at most some fixed q. Thus the probability of this event occurring is at most $q^{\log_t(R/r)}$. Note this bound is actually identical on the other term as there was no special distinction between z and z', so by the triangle inequality we have $|P[Q_\alpha(z) \setminus Q_\alpha(z')] - P[Q_\alpha(z') \setminus Q_\alpha(z)]| \leq 2q^{\log_t(R/r)} = 2r^{|\log_t(q)| + \log_t(R)}$ as desired. □

Corollary 6.9. $P_\alpha(z) = O(\delta^\epsilon)$ *for some* $\epsilon > 0$.

Proof. Choose z' to be $z + \eta$ and we have proven our lemma as $||\eta|| = O(\delta)$. □

Definition 6.10. Take some equilateral triangle Δ with side length l, the bottom edge parallel to the x-axis, and with vertices $x(1), x(\tau), x(\tau^2)$ where $x(1)$ is on top. The **Discrete Contour Integral** is defined by

$$\int_\Delta H_\alpha(z)dz = \delta \sum_{z\in x(\tau)x(\tau^2)} H_\alpha(z) + \delta\tau \sum_{z\in x(\tau^2)x(1)} H_\alpha(z) + \delta\tau^2 \sum_{z\in x(1)x(\tau)} H_\alpha(z).$$

Lemma 6.11. $\int_\Delta H_\alpha(z)dz = \int_\Delta \frac{1}{\tau} H_{\tau\alpha}(z)dz + O(l\delta^\epsilon)$.

Proof. Color the lattice triangles in a chessboard fashion such that the triangles containing the vertices of Δ are colored black. This can be seen in Figure 7. Allow $\mathcal{B}$ to denote triangles with vertices colored black, and $\mathcal{W}$ to denote triangles with vertices colored white. Allow η to be $\frac{\delta e^{\frac{i\pi}{6}}}{\sqrt{3}}$ and $\eta' = \frac{\delta e^{\frac{i\pi}{2}}}{\sqrt{3}}$.

We see η is actually a vector from the center of a black triangle to the center of an adjacent white triangle, and η' is a vector from the center of a white triangle to the center of an adjacent black triangle.

The idea in vague terms then is that the sums in the black and white regions rotate and cancel appropriately to arrive at the desired lemma.

We first consider the sums over black triangles which 'point to a white triangle' in Ω, i.e. ones not on the arc $x(\tau^2\alpha)x(\alpha)$, as follows: For $\beta \in \{1, \tau, \tau^2\}$ we have

$$\sum_{z\in\mathcal{B}\setminus x(\tau^2\alpha)x(\alpha)} H_\beta(z+\eta) - H_\beta(z) = \sum_{z\in\mathcal{B}\setminus x(\tau^2\alpha)x(\alpha)} P_\beta(z,\eta) - P_\beta(z+\eta, -\eta).$$

Applying the 'rotation' from Lemma 6.7, this becomes

$$\sum_{z\in\mathcal{B}\setminus x(\tau^2\alpha)x(\alpha)} P_{\tau\beta}(z,\tau\eta) - P_{\tau\beta}(z+\eta,-\tau\eta) \tag{1}$$

$$= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} P_{\tau\beta}(z,\tau\eta) - P_{\tau\beta}(z+\tau\eta,-\tau\eta) + O(l\delta^{\epsilon-1}) \tag{2}$$

$$= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} H_{\tau\beta}(z+\tau\eta) - H_{\tau\beta}(z) + O(l\delta^{\epsilon-1}) \tag{3}$$

where (2) is justified as follows. Clearly the place where the (1) and (2) differ is in the second term. If some z is not on the boundary, then $z+\eta$ is a white triangle. As white triangles cannot be on the boundary, then we see that there $z+\eta-\tau\eta$ is in fact a different black triangle inside of Ω. In other words, we have shown a correspondence between $P_{\tau\beta}(z+\eta,-\tau\eta)$ and $P_{\tau\beta}((z+\eta-\tau\eta)+\tau\eta,-\tau\eta)$. Thus these two second terms can differ at most by the order of terms on the boundary arcs. There is an order of $\frac{l}{\delta}$ vertices along the boundary arcs, and each term differs by at most an order of δ^ϵ by Corollary 6.9. We may similarly get the equality

$$\sum_{z\in\mathcal{W}} H_\beta(z+\eta') - H_\beta(z) = \sum_{z\in\mathcal{W}} H_{\tau\beta}(z+\tau\eta') - H_{\tau\beta}(z) + O(l\delta^{\epsilon-1}). \tag{4}$$

We now wish to consider the following term:

$$\sum_{z\in\mathcal{B}\setminus x(\tau^2\alpha)x(\alpha)} H_\beta(z+\eta) - H_\beta(z) + \sum_{z\in\mathcal{W}} H_\beta(z+\eta') - H_\beta(z). \tag{5}$$

For any $z \in \mathcal{B} \setminus x(\tau^2\alpha)x(\alpha)$, we have $z+\eta \in \mathcal{W}$. Thus we may simplify (5) to

$$\sum_{z\in\mathcal{B}\setminus x(\tau^2\alpha)x(\alpha)} -H_\beta(z) + \sum_{z\in\mathcal{W}} H_\beta(z+\eta'). \tag{6}$$

Next we see that $z+\eta'$ can be any black triangle not on the boundary arc $x(\tau\alpha)x(\tau^2\alpha)$, thus (6) is equal to

$$\sum_{z\in x(\tau^2\alpha)x(\alpha)} H_\beta(z) - \sum_{z\in x(\tau\alpha)x(\tau^2\alpha)} H_\beta(z). \tag{7}$$

However, by (3) and (4) we also see that

$$\sum_{z\in\mathcal{B}\setminus x(\tau^2\alpha)x(\alpha)} H_\beta(z+\eta) - H_\beta(z) + \sum_{z\in\mathcal{W}} H_\beta(z+\eta') - H_\beta(z) \tag{8}$$

$$= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} H_{\tau\beta}(z+\tau\eta) - H_{\tau\beta}(z) + \sum_{z\in\mathcal{W}} H_{\tau\beta}(z+\tau\eta') - H_{\tau\beta}(z) + O(l\delta^{\epsilon-1}).$$

Then by repeating the exact same telescoping process that gave us (7), we find that

$$\sum_{z\in x(\tau^2\alpha)x(\alpha)} H_\beta(z) - \sum_{z\in x(\tau\alpha)x(\tau^2\alpha)} H_\beta(z) \tag{9}$$

$$= \sum_{z\in x(\alpha)x(\tau\alpha)} H_\beta(z) - \sum_{z\in x(\tau^2\alpha)x(\alpha)} H_\beta(z) + O(l\delta^{\epsilon-1}).$$

Taking appropriate linear combinations of (8) and (9) yields the following three equalities:

$$-\frac{\delta}{2}\Big(\sum_{z\in x(\tau^2)x(1)} H_\beta(z) - \sum_{z\in x(\tau)x(\tau^2)} H_\beta(z)\Big) \tag{10}$$

$$= -\frac{\delta}{2}\Big(\sum_{z\in x(1)x(\tau)} H_\beta(z) - \sum_{z\in x(\tau^2)x(1)} H_\beta(z)\Big) + O(l\delta^{\epsilon})$$

$$-i\frac{\sqrt{3}\delta}{2}\Big(\sum_{z\in x(1)x(\tau)} H_\beta(z) - \sum_{z\in x(\tau^2)x(1)} H_\beta(z)\Big) \tag{11}$$

$$= -i\frac{\sqrt{3}\delta}{2}\Big(\sum_{z\in x(\tau)x(\tau^2)} H_\beta(z) - \sum_{z\in x(1)x(\tau)} H_\beta(z)\Big) + O(l\delta^{\epsilon})$$

$$\frac{\delta}{2}\Big(\sum_{z\in x(\tau)x(\tau^2)} H_\beta(z) - \sum_{z\in x(1)x(\tau)} H_\beta(z)\Big) \tag{12}$$

$$= \frac{\delta}{2}\Big(\sum_{z\in x(\tau^2)x(1)} H_\beta(z) - \sum_{z\in x(\tau)x(\tau^2)} H_\beta(z)\Big) + O(l\delta^{\epsilon}).$$

Adding together (10), (11), and (12) immediately yields the lemma. ∎

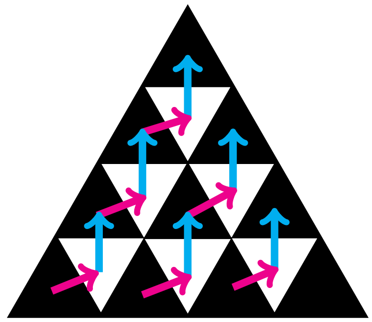

FIGURE 7. An illustration of the telescoping process used in Lemma 6.11, where the blue lines correspond to η' and the purple lines correspond to η. Notice the telescoping process occurs along a path.

Now we want to show that these H_α's converge uniformly to a function which satisfies the differential equation from Definition 6.4, and then invoke uniqueness of the solution. In order to do so, we first note that each H_α is a probability, so they are uniformly bounded by 1. Next, Lemma 6.8 immediately implies uniform equicontinuity. Thus by Arzela-Ascoli, we may choose a sequence $\{\delta_j\} \to 0$ such that $\{H_\alpha^{\delta_j}\}$ converges uniformly to some function f_α.

Lemma 6.12. $f_1(z) + f_\tau(z) + f_{\tau^2}(z) = 1.$

Proof. By the previous lemma and uniform convergence, we can clearly see that $\int_\Delta f_\beta(z)dz = \int_\Delta \frac{1}{\tau} f_\beta(z)dz$.

Therefore we see that $\int_\Delta f_\beta(z) - \tau^2 f_{\tau\beta}(z)dz = \int_\Delta \tau f_{\tau\beta}(z) - f_{\tau^2\beta}(z)dz$. Rearranging gives $\int_\Delta f_\beta(z) + f_{\tau\beta}(z) + f_{\tau^2\beta}(z)dz = 0$, implying the function inside this integral is holomorphic by Morera's theorem. As each f_β is the limiting funtion of $H_\beta^{\delta_i}$'s, which are real valued functions, f_β must also be real valued. Thus we see that $f_1 + f_\tau + f_{\tau^2}$ is some constant (this quite quickly comes from the Cauchy-Riemann equations). We thus evaluate the value of this function by evaluating the value of it at $x(1)$. Note clearly that f_τ and f_{τ^2} are 0 as each H_τ, H_{τ^2} is identically 0 on the arcs $x(\tau^2)x(1)$ and $x(1)x(\tau)$ respectively. Note that there are an order of $|\log_t(\frac{l}{\delta})|$ disjoint annuli with aspect ratio t separating $x(1)$ from $x(\tau)x(\tau^2)$. There is at least some positive probability c that there is a closed traversing of any given annulus. Thus there is probability at least on the order of $1-(1-c)^{|\log_t(\frac{R}{\delta})|}$ that at least one such traversing exists. As $\delta \to 0$ this is a lower bound approaching 1, so $f_1(x(1)) = 1$. Thus our lemma is proven. □

Lemma 6.13. $f_\alpha = h_\alpha$.

Proof. By essentially the same proof as in the previous lemma, we see that $\mathcal{G} = f_1 + \tau f_\tau + \tau^2 f_{\tau^2}$ is holomorphic. Thus we may use the following analogue of the Cauchy-Riemann equations: for any unit vector η we have $\frac{\partial \mathcal{G}}{\partial \eta} = \frac{\partial(\mathcal{G}/\tau)}{\partial(\tau\eta)}$ and as a consequence $\frac{\partial \mathrm{Re}[\mathcal{G}]}{\partial \eta} = \frac{\partial \mathrm{Re}[(\mathcal{G}/\tau)]}{\partial(\tau\eta)}$. Note that by Lemma 6.12 $\mathrm{Re}[\mathcal{G}] = f_1 - \frac{1}{2}(f_2 + f_3) = \frac{3f_1 - 1}{2}$, and similarly $\mathrm{Re}[(\mathcal{G}/\tau)] = \frac{3f_2 - 1}{2}$. Thus we see that $\frac{\partial f_1}{\partial \eta} = \frac{\partial f_\tau}{\partial(\tau\eta)} = \frac{\partial f_{\tau^2}}{\partial(\tau^2\eta)}$, were the second equality is found by observing that G/τ must also be holomorphic and repeating this process. Thus we see that on the arc $x(\alpha)x(\tau\alpha)$ we have $\frac{\partial f_\alpha}{\partial(\tau\nu)} = \frac{\partial f_{\tau^2\alpha}}{\partial\nu} = 0$ and on $x(\tau^2\alpha)x(\alpha)$ that $\frac{\partial f_\alpha}{\partial(-\tau^2\nu)} = \frac{\partial f_{\tau\alpha}}{\partial(-\nu)} = 0$ by the boundary conditions mentioned in Lemma 6.12. Thus we see that each f_α is a solution to the differential equation from Definition 6.4, which is unique. Thus our lemma is proven. □

As a corollary we get Theorem 6.1. Furthermore, Cardy's formula is sufficient to show not only conformal invariance of $\gamma^{e.p.}$, but actually shows convergence to a random curve known as SLE_6, which unlocks a significant amount of information about percolation on the triangular lattice. This is because $\gamma^{e.p.}$ actually follows Cardy's formula (and satisfies a specific locality property). For a complete proof of such a phenomenon, one is encouraged to see either [2] or [12].

7. The One Arm Exponent

The one arm exponent is the probability that the open cluster at the origin reaches some radius n (such an event will be denoted by $0 \leftrightarrow \Lambda_n$. This gets overshadowed by a broader class of events which we use later). Although nontrivial bounds can be found, convergence to SLE_6 gives a log asymptotic equality of $P[0 \leftrightarrow \Lambda_n] = n^{-\frac{5}{48}+o(1)}$, which we somewhat prove. However, as this requires by far the most advanced techniques covered in this paper, we neglect some major portions of the paper and take the results for granted. Instead, the goal here is taking SLE_6 as a blackbox and to demonstrate a flavor of what it can do.

We begin with a few definitions:

Definition 7.1. Let $\mathcal{H}(n)$ denote the set of all points which have a minimum path length to the origin of exactly n. Note that this will be a regular hexagon centered at the origin. Let $\mathbb{H}(n,m)$ be set of all points between $\mathcal{H}(n)$ and $\mathcal{H}(m)$, not including $\mathcal{H}(n)$. Let $\mathcal{T}(n,m)$ be the event that there is an open traversing of $\mathbb{H}(n,m)$.

Definition 7.2. Let $\mathcal{C}(r_1,r_2)$ denote an open crossing from $\mathcal{H}(r_1+1)$ to $\mathcal{H}(r_2)$, unless $r_1 = 0$. In which case this is the same event as Λ_{r_2}.

Then using the properties of SLE_6, it is possible to show the following theorem, which will be left unproven. For proofs, see [7] and [12]. These papers are the guide for this entire section.

Proposition 7.3. *$P[\mathcal{C}(n,Rn)]$ converges as $n \to \infty$ to some function $f(R)$ which satisfies for large R and positive constants c_1, c_2 s.t. $c_1 R^{-\frac{5}{48}} \le f(R) \le c_2 R^{-\frac{5}{48}}$.*

What we will show is that for any $\epsilon > 0$, $n^{-\frac{5}{48}-\epsilon} \le P(C[0,n]) \le n^{-\frac{5}{48}+\epsilon}$ as n becomes large. The critical idea is to create concentric annuli with a common aspect ratio t. As these annuli become large enough, they roughly follow the behavior of the model in the scaling limit, and therefore the probability of crossing these annuli converges to a crossing probability of $f(t)$. (In fact, [7] states Proposition 7.3 for $\mathcal{C}(r,1)$ in the scaling limit for small r. This however is equivalent to the formulation shown above, which is from [12]). Thus we combine this notion of convergence with Proposition 7.3 which comes from SLE_6 and we are done. We begin with the upper bound.

Lemma 7.4.
$$\limsup_{n\to\infty} \frac{\log(P[\mathcal{C}(0,n)])}{\log(n)} \le -\frac{5}{48}.$$

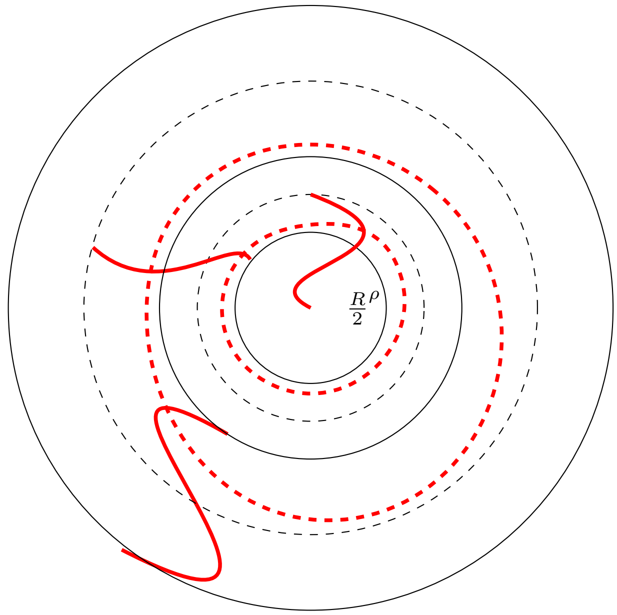

FIGURE 8. The chain construction used in Lemma 7.5

Proof. We begin by fixing an $\epsilon > 0$. Set R to be enough that $(c_2)^{\frac{2}{\epsilon}} \leq R$, and $R > 1$. Now by Proposition 7.3, we know that $\{P[\mathcal{C}(R^k, R^{k+1})]\}$ converges to $f(R)$, so we may choose ρ such that for all $k \geq \rho, P[\mathcal{C}(R^k, R^{k+1})] \leq R^{\frac{\epsilon}{2}} f(R) \leq c_2 R^{-\frac{5}{48}+\frac{\epsilon}{2}} \leq R^{-\frac{5}{48}+\epsilon}$.

Now we wish to consider very large n such that $R^{\rho+1} \leq n$. Allow ϑ to be the largest integer such that $R^{\vartheta} \leq n$. Then we see that $P[\mathcal{C}(0, n)] \leq P[\mathcal{C}(0, R^{\vartheta})]$.

The next key step is to observe that if $r_1 < r_2 < r_3$, then $\mathcal{C}(r_1, r_2)$ implies $\mathcal{C}(r_1, r_2) \cap \mathcal{C}(r_2, r_3)$ (moreover, the second and third event are independent). Thus we see $P[\mathcal{C}(0, R^{\vartheta})] \leq P[\mathcal{C}(0, R^{\rho})] \prod_{i=\rho}^{\vartheta-1} P[\mathcal{C}(R^i, R^{i+1})] \leq P[\mathcal{C}(0, R^{\rho})](R^{-\frac{5}{48}+\epsilon})^{\vartheta-\rho} \leq \frac{\mathcal{C}(0,R^{\rho})}{(R^{-\frac{5}{48}+\epsilon})^{\rho}} n^{-\frac{5}{48}+\epsilon}$. Now we may take $\frac{\mathcal{C}(0,R^{\rho})}{(R^{-\frac{5}{48}+\epsilon})^{\rho}}$ to be fixed due to its lack of dependence on n. Now take $n \to \infty$ and we have

$$\limsup_{n\to\infty} \frac{\log(P[\mathcal{C}(0, n)])}{\log(n)} \leq -\frac{5}{48} + \epsilon.$$

Taking $\epsilon \to 0$ we are done. □

Now we prove the upper bound. Here we do not have easy access to the independence used in the previous lemma. Thus we resort to connecting different smaller crossings with annuli, and conveniently lower bounding these probabilities using Proposition 2.8. Such constants then disappear in the limit.

Lemma 7.5.

$$\liminf_{n\to\infty} \frac{\log(P[\mathcal{C}(0, n)])}{\log(n)} \geq -\frac{5}{48}.$$

Proof. Fix $\epsilon > 0$. Allow $0 < c \leq P(\mathcal{T}(n, 2n))$ for any n (this can be done using Proposition 2.8). We choose $R > 2$ such that $R^{-\frac{\epsilon}{3}} \leq c_1$, $R^{-\frac{\epsilon}{3}} \leq c$. Now choose ρ such that for any $k \geq \rho$, $P[\mathcal{C}((\frac{R}{2})^k, 2(\frac{R}{2})^{k+1})] \geq R^{-\frac{\epsilon}{3}} f(R) \geq R^{-\frac{5}{48}-\frac{2\epsilon}{3}}$. Now we let ϑ denote the smallest integer such that $R^{\vartheta} \geq n$. Then we see $P[\mathcal{C}(0, n)] \geq P[\mathcal{C}(0, R^{\vartheta})]$.

Now, similarly to the last proof, we must split $\mathcal{C}(0, R^{\vartheta})$ into several different events (which have already been hinted at.) We do so as follows

$$\mathcal{C}(0, 2R^{\rho}) \cap \bigcap_{k=\rho}^{k=\vartheta-1} \mathcal{C}((\frac{R}{2})^k, 2(\frac{R}{2})^{k+1}) \cap \mathcal{T}((\frac{R}{2})^k, 2(\frac{R}{2})^k) \subset \mathcal{C}(0, R^{\vartheta}).$$

Notice that the crossings corresponding to both $\mathcal{C}((\frac{R}{2})^k, 2(\frac{R}{2})^{k+1})$ and $\mathcal{C}((\frac{R}{2})^{k-1}, 2(\frac{R}{2})^k)$ must intersect the traversing corresponding to $\mathcal{T}((\frac{R}{2})^k, 2(\frac{R}{2})^k)$ (see Figure 8). Thus this chain continues in an almost inductive fashion. Moreover, all such sub-events involved are *increasing*. Thus applying Proposition 2.4, we get

$$P[\mathcal{C}(0, R^{\vartheta})] \geq P[\mathcal{C}(0, 2R^{\rho})] \prod_{k=\rho}^{\vartheta-1} P[\mathcal{C}((\frac{R}{2})^k, 2(\frac{R}{2})^{k+1}] P[\mathcal{T}((\frac{R}{2})^k, 2(\frac{R}{2})^k)]$$

$$\geq P[\mathcal{C}(0, 2R^{\rho})](cR^{-\frac{5}{48}-\frac{2\epsilon}{3}})^{\vartheta-\rho} \geq \frac{P[\mathcal{C}(0, 2R^{\rho})]}{(R^{-\frac{5}{48}-\epsilon})^{\rho}} n^{-\frac{5}{48}-\epsilon}.$$

Again, consider $\frac{P[\mathcal{C}(0,2R^{\rho})]}{(R^{-\frac{5}{48}-\epsilon})^{\rho}}$ fixed and take $n \to \infty$. This yields

$$\liminf_{n\to\infty} \frac{\log(P[\mathcal{C}(0, n)])}{\log(n)} \geq -\frac{5}{48} - \epsilon.$$

Taking $\epsilon \to 0$, we are done. □

The previous two lemmas show that $P[0 \leftrightarrow \Lambda_n] = n^{-\frac{5}{48}+o(1)}$.

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