NASH-MOSER ITERATION

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ABSTRACT. In this paper, we explain the connection between two theorems. The first proves existence of C^k isometric embeddings of Riemannian manifolds into $\mathbb{R}^N$. The second finds stable solutions of reversible mechanical systems. These theorems are proved using a common technique based on Newton's method.

CONTENTS

1. INTRODUCTION

In 1956, Nash proved that a Riemannian manifold with C^k metric $(k > 2)$ can be realized as a C^k submanifold of $\mathbb{R}^N$ for large enough N. The idea behind his proof was to define an ODE with smoothing and "feedback".

This idea has been discretized into an iterative technique and is used to prove the Nash-Moser inverse function theorem, which is used nowadays to solve differential equations where inverting the linearization loses derivatives. Nash-Moser iteration is also used in KAM theory, which studies when solutions of reversible (non-dissipative) mechanical systems maintain predictable trajectories under small perturbations of the system.

In Section 2 we give the proof of the Nash-Moser inverse function theorem. The proof of the embedding theorem is discussed in Section 3. In Section 4 we apply

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Nash-Moser iteration to a model problem for the KAM theorem. Proofs in Section 4 demonstrate ideas used in Section 5, where we introduce the KAM theorem.

2. Inverse Function Theorems

Let X, Y be Banach spaces and $U \subseteq X$ be open. A map $F : U \to Y$ is differentiable at $x \in U$ if there is a linear map $T : X \to Y$ such that

$$F(x+h) - F(x) = T(h) + o(\|h\|)$$

for all h in some neighborhood of 0. Then $DF = T$ is the Fréchet derivative of F at x.

On $C^k(\mathbb{R}^n)$, for $r \leq k$ let

$$\|u\|_r = \max_{|\alpha| \leq r} \sup_{x \in \mathbb{R}^n} |D^\alpha u(x)|.$$

The Nash-Moser inverse function theorem was formulated by Schwartz in [10]. Its application to solving PDEs was studied by Moser [4] [5]. We mostly follow the presentation in [3]. The theorem states

Theorem 2.1. *Let $B_1(0) \subset C^k(\mathbb{R}^n)$ and $T : B_1(0) \to C^{k-m}(\mathbb{R}^n)$ for $0 \leq m \leq k$. Suppose*

(1) T has two continuous Fréchet derivatives bounded by $M \geq 1$
(2) There exists $L : B_1(0) \to \mathcal{L}(C^\ell(\mathbb{R}^N), C^{\ell-m}(\mathbb{R}^N))$ for every $\ell > 0$ such that
   (a) $\|L(u)h\|_{k-m} \leq M\|h\|_k$ for any $u \in B_1(0), h \in C^k$
   (b) $DT(u) \circ L(u)h = h$ for any $u \in B_1(0), h \in C^{k+m}$
   (c) $\|L(u)T(u)\|_{k+9m} \leq M(1 + \|u\|_{k+10m})$ for any $u \in C^{k+10m}$.

Let $P = 61$. If

$$\|T(0)\|_{k+9m} \leq 2^{-P-1}M^{-5P-2},$$

then $0 \in T(B_1(0))$.

For each $u \in B_1(0)$ the map $L(u)$ is a linear operator which loses m derivatives and inverts $DT(u)$ on $C^{k+m} \subset C^k$. Condition (2)(b) replaces the invertibility condition on DT required by the classical inverse function theorem. The constant $P = 61$ ensures $T(0)$ is small enough for a Newton-type iteration to converge (we will point out where it is used in the proof). The iterative technique used to prove Theorem 2.1 is modeled on the following.

Proposition 2.2. *Let X be a Banach space and let $B_1(0) \subset X$. Suppose $T : B_1(0) \to X$ such that*

(1) T has two continuous Fréchet derivatives on $B_1(0)$ bounded by $M > 2$
(2) There exists $L : B_1(0) \to \mathcal{L}(X, X)$ such that
   (a) $\|L(u)h\| \leq M\|h\|$ for any $h \in X, u \in B_1(0)$
   (b) $DT(u) \circ L(u)h = h$ for any $h \in X, u \in B_1(0)$.

If $\|T(0)\| < M^{-5}$, then $0 \in T(B_1(0))$.

Applying Proposition 2.2 to $F = T - T(0) - \tilde{h}$ where $|\tilde{h}| < M^{-5}$ gives $h \in B_1(0)$ such that $T(h) = T(0) + \tilde{h}$. Then a neighborhood of $T(0)$ has a pre-image under T. The proof uses Newton method, which we first describe.

Proposition 2.3. *(Newton's method) Let $f \in C^2(\mathbb{R})$. We look for a root of f. Let $x_0 \in \mathbb{R}$ and inductively define x_n by linearizing f at x_n, to get $f(x_n)+f'(x_n)(x-x_n)$, and taking the root*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $f(a) = 0$ and $f'(a) \neq 0$ there exists $\delta > 0$ such that if $x_0 \in B_\delta(a)$ then $\lim_{n\to\infty} x_n = a$, with $|x_{n+1} - a| = O(|x_n - a|^2)$.

The last statement is shown using the second order Taylor expansion of f. In the following, inductive quantities are defined as in Newton's method with T in place of f, and L inverting DT.

Proof. (of Proposition 2.2)

Let $\lambda = \frac{3}{2}$ and $\beta = \frac{8}{3}\log(M)$. Let $u_0 = 0$, and

$$u_{n+1} = u_n - L(u_n)T(u_n). \tag{2.4}$$

Suppose for $n \geq 0$

$(P1)$ $u_n \in B_1(0)$

$(P2)$ $\|u_{n+1} - u_n\| \leq e^{-\beta\lambda^n}$.

Then $u_n \to u \in B_1(0)$ and by $(P1)$ and condition $(2)(b)$,

$$T(u_n) = DT(u_n)u_n - DT(u_n)(u_n - L(u_n)T(u_n)) = DT(u_n)(u_n - u_{n+1})$$

which by $(P2)$ means $T(u) = 0$.

For $n = 0$, $(P1)$ is immediate, and

$$\|u_1 - u_0\| = \|L(0)T(0)\| \leq M\|T(0)\| \leq M^{-4} \leq e^{-\beta}$$

which is $(P2)$.

Suppose $(P1)$ and $(P2)$ hold for $k \leq n$. Since $(\lambda - 1)k \leq \lambda^k$,

$$\begin{aligned}\|u_{n+1}\| &\leq \sum_{k=0}^{n} \|u_{k+1} - u_k\| \leq \sum_{k=0}^{n} e^{-\beta\lambda^k} \\ &\leq e^{-\beta} + \sum_{k=1}^{\infty} e^{-\beta(\lambda-1)k} \leq e^{-\beta} + \frac{e^{-\beta(\lambda-1)}}{1 - e^{-\beta(\lambda-1)}} < 1.\end{aligned}$$

For $(P2)$, note the Lagrange remainder term

$$T(u+h) = T(u) + DT(u)h + \int_0^1 (1-t)D^2T(u+th)(h,h)dt$$

and $\| \int_0^1 (1-t)D^2T(u+th)(h,h)dt\| \leq M\|h\|^2$. Set

$$u = u_n, \qquad h = u_{n+1} - u_n = -L(u_n)T(u_n).$$

Recall $u_{n+1}, u_n \in B_1(0)$. Then

$$\begin{aligned}\|T(u_{n+1})\| &\leq \|T(u_n) - DT(u_n)L(u_n)T(u_n)\| + M\|u_{n+1} - u_n\|^2 \\ &= M\|u_{n+1} - u_n\|^2\end{aligned}$$

by condition $(2)(b)$. Thus

$$\begin{aligned}\|u_{n+2} - u_{n+1}\| &= \|L(u_{n+1})T(u_{n+1})\| \leq M\|T(u_{n+1})\| \\ &\leq M^2\|u_{n+1} - u_n\|^2 \leq M^2 e^{-2\beta\lambda^n} \leq e^{-\beta\lambda^{n+1}}\end{aligned}$$

by choice of β and λ. ∎

We cannot directly use Newton iteration to prove Theorem 2.1. To see this, suppose we define

$$u_{n+1} = u_n - L(u_n)T(u_n)$$

where $u_0 \in C^k$ and L as in Theorem 2.1. Then u_{n+1} has m less derivatives than u_n and iteration is defined only for finite steps. An idea of Nash is to apply smoothing to u_n during Newton iteration. As $n \to \infty$ the amount of smoothing is rapidly decreased to ensure iteration converges to a correct solution.

Define a family of smoothing operators

$$S(t) : \mathbb{R}^+ \to \{\text{operators from } C^{k-m} \to C^{k+10m}\}$$

such that for $k - m \le r \le \rho \le k + 10m$,

$$\|S(t)u\|_\rho \le Mt^{\rho-r}\|u\|_r, \; u \in C^r \tag{2.5}$$

$$\|(I - S(t))u\|_r \le Mt^{r-\rho}\|u\|_\rho, \; u \in C^\rho. \tag{2.6}$$

We send $t \to \infty$ during the iteration process. An example of such an operator will be constructed in the next section.

The proof of Theorem 2.1 closely follows Proposition 2.2, with extra steps involving the smoothing operator. We provide the proof as it demonstrates Nash-Moser iteration.

Proof. (of Theorem 2.1) Let $\lambda = \frac{3}{2}$, $\mu = \frac{9}{4}$, and $\beta = \frac{8}{m}\log(2M^5)$. Recall m is the number of derivatives which L loses. Define

$$S_n = S(e^{\beta\lambda^n})$$

$$u_{n+1} = u_n - S_n L(u_n)T(u_n)$$

where $u_0 = 0$. Note $\{u_n\}$ are defined as in Newton iteration except quantities are smoothed. As $n \to \infty$, the amount of smoothing decreases to zero.

Similar to proposition 2.1, the result follows if

$(P1)$ $u_n \in B_1(0)$

$(P2)$ $\|u_n - u_{n-1}\|_k \le e^{-\mu m\beta\lambda^n}$

$(P3)$ $1 + \|u_n\|_{k+10m} \le e^{\mu m\beta\lambda^n}$ for all $n \ge 1$.

Condition $(P3)$ is new. As $n \to \infty$, $(P3)$ bounds the increase of "non-smooth" behavior of u_n.

Beginning induction at $n = 1$, $(P1)$ follows by the choice of β and property (2.5). Letting $r = k - m$ and $\rho = k$ in (2.5) gives

$$\|u_1\|_k = \|S_0 L(0)T(0)\|_k \le M(e^\beta)^m\|L(0)T(0)\|_{k-m}$$
$$\le M^2 e^{m\beta}\|T(0)\|_{k+9m} \le \exp(m\beta - P\log(2M^5)) < \exp(-\mu m\beta\lambda),$$

since

$$m\beta - P\log(2M^5) = 8\log(2M^5) - 61\log(2M^5) \le -27\log(2M^5) = -\mu m\beta\lambda.$$

Note we used the exponential decrease of smoothing. This gives $(P2)$. Below, the constant $P = 61$ is used similarly to cancel terms with λ, μ, β.

Letting $r = k-m$ and $\rho = k+10m$ in (2.5), and using hypotheses (2)(a), (3)(a), and the bound on $\|T(0)\|_{k+9m}$ gives

$$1 + \|u_1\|_{k+10m} = 1 + \|S_0 L(0)T(0)\|_{k+10m} \le 1 + M^2 e^{11m\beta}\|T(0)\|_k$$

$$\leq 1 + M^2 e^{11m\beta}\|T(0)\|_{k+9m} \leq 2^{-P}M^{-5P}e^{11m\beta} = e^{\mu m\beta\lambda}$$

which is $(P3)$.

Assume $(P1)-(P3)$ for $k \leq n$. We proceed as in Proposition 2.1.

Using $\lambda^j \geq (\lambda-1)j$ and $\mu m\beta(\lambda-1) = 9\log(2M^5) \geq \log 2$ will give $(P1)$ for u_{n+1}. For $(P2)$, the smoothing operator does not allow cancelling in the Lagrange remainder term:

$$\begin{aligned}\|T(u_n)\|_k &\leq \|T(u_{n-1}) - DT(u_{n-1})S_{n-1}L(u_{n-1})T(u_{n-1})\|_k + M\|u_n - u_{n-1}\|_k^2 \\ &\leq M\left[\|(I-S_{n-1})L(u_{n-1})T(u_{n-1})\|_k + e^{-2\mu m\beta\lambda^n}\right].\end{aligned}$$

Then

$$\begin{aligned}\|u_{n+1}-u_n\|_k &= \|S_nL(u_n)T(u_n)\|_k \\ &\leq M^2e^{m\beta\lambda^n}\|T(u_n)\|_k \\ &\leq M^3e^{m\beta\lambda^n}\left[\|(I-S_{n-1})L(u_{n-1})T(u_{n-1})\|_k + e^{-2\mu m\beta\lambda^n}\right].\end{aligned}$$

Using $r = k$ and $\rho = k+9m$ in (2.6), 2(c), and $(P3)$ for $k \leq n$ in order gives

$$\|u_{n+1}-u_n\|_k \leq M^3(M^2+1)e^{-\frac{21}{4}m\beta\lambda^{n-1}} \leq e^{-\mu m\beta\lambda^{n+1}}$$

since $M^3(M^2+1) \leq 2M^5$ and $\frac{21}{4}\log(2M^5) \geq \mu\lambda^2$.

For (P3), using (2.5) with $\rho = k+10m, r = k+9m$, condition (2)(c), and (P3) for $k \leq n$ in order,

$$\begin{aligned}1+\|u_{n+1}\|_{k+10m} &\leq 1+\sum_{j=0}^n \|S_jL(u_j)T(u_j)\|_{k+10m} \leq 1+\sum_{j=0}^n Me^{m\beta\lambda^j}\|L(u_j)T(u_j)\|_{k+9m} \\ &\leq 1+M^2\sum_{j=0}^n e^{m\beta\lambda^j}(1+\|u_j\|_{k+10m}) \leq 1+M^2\sum_{j=0}^n e^{m(1+\mu)\beta\lambda^j}.\end{aligned}$$

Computations using the above, which can be found in [3], then show

$$(1+\|u_{n+1}\|_{k+10m})e^{-\mu m\beta\lambda^{n+1}} \leq 1.$$

This gives $(P3)$. □

The statements of Proposition 2.2 and Theorem 2.1 have similar structure; hypotheses and the conclusions are of the same type. Theorem 2.1 is significantly stronger than Proposition 2.2 as weaker conditions are required on the operator L. Theorem 2.1 can also be stated using a general decreasing family of Banach spaces. In applications, such as to solving systems of differential equations, we usually work with spaces C^k.

3. The Isometric Embedding Theorem

In this section we describe Nash's proof of the isometric embedding theorem for compact Riemannian manifolds. Solving the linearization of a natural ODE to this problem loses space derivatives. Here the inverse of the linearization is like "L" in Theorem 2.1 where $m = 2$. The embedding is instead found as the limit of a perturbation process. Nash prevents "loss of derivative" during perturbation by continually smoothing the embedding and metric.

We present Nash's original proof as it contains interesting ideas, though nowadays the embedding can be found using the Nash-Moser inverse function theorem

[11], Appendix II. Key ideas and proof steps will be described, and details are in [2] ch.3 and [8].

Let (Σ, g) be a compact n-manifold. On Σ fix a smooth atlas $\{U_\ell\}$.

Suppose $f : \Sigma \to \mathbb{R}^m$ and $h = h_{ij} dx_i \otimes dx_j$ is a $(0,2)$ tensor on Σ. (At each point, a (p,q) tensor is a $(p+q)-$linear function on p dual vectors and q vectors.)

Let $\|\cdot\|$ denote the Frobenius norm when applied to a matrix, and

$$\|Df\|_{0,U_\ell} := \sup_{p \in U_\ell} \|Df(p)\|, \qquad \|Df\|_0 := \sup_\ell \|Df\|_{0,U_\ell}$$

$$\|h\|_{0,U_\ell} := \sup_{p \in U_\ell} \sup_{ij} \|h_{ij}(p)\|, \qquad \|h\|_0 := \sup_\ell \|h\|_{0,U_\ell}.$$

Let

$$\|D^k f\|_0 = \sup_{|\alpha|=k} \|\partial^\alpha f\|_0,$$

where $\partial^\alpha f$ is defined with respect to coordinates on each U_ℓ, and let $\|f\|_k$ be the C^k norm of f.

If $z : \Sigma \to \mathbb{R}^N$ an embedding, let $z^\sharp e$ denote the pullback of the Euclidean metric under z. In coordinates,

$$(z^\sharp e)_{ij} = \frac{\partial z}{\partial x_i} \cdot \frac{\partial z}{\partial x_j}. \tag{3.1}$$

Then z is isometric if $z^\sharp e = g$.

We say z is *free* if for every U_ℓ with local coordinates x and $p \in U_\ell$, the vectors

$$\frac{\partial z}{\partial x_i}(p), \;\; \frac{\partial^2 z}{\partial x_j x_k}(p)$$

are linearly independent for $i, j, k = 1, ...N$ and $k \leq j$. The independence of these vectors is preserved under coordinate change.

In [8], Nash proves

Theorem 3.2. *Let (Σ, g) be a compact $n-$manifold with C^k metric g. Then there exists a C^k isometric embedding $z : \Sigma \to \mathbb{R}^N$ where $N \geq \frac{3n^2+11n}{2}$.*

The proof proceeds as follows. If z_0 is the initial embedding, we find a family of embeddings $z(t)$ such that $z(0) = z_0$ and $\lim_{t\to\infty} z(t) := z_\infty$ realizes the metric g. The embeddings $z(t)$ induce a path of metrics $g(t)$ on Σ defined by

$$z(t)^\sharp e = g(t).$$

Nash ensures $\lim_{t\to\infty} g(t) = g$.

3.1. **Initial Setup and Loss of Derivative.** Given a rate of metric change $\dot{g_{ij}}$, $1 \leq i \leq j \leq n$, we find $\dot{z_\alpha}, \alpha = 1, ..., N$ which induce the metric change. The $\dot{g_{ij}}$ will specify directions to "flow" and decrease metric error. Let $\dot{}$ denote differentiation with respect to time. By (3.1) we are finding $\dot{z_\alpha}$ which satisfies

$$\dot{g_{ij}} = \sum_\alpha \left[\frac{\partial z_\alpha}{\partial x_i}\frac{\partial \dot{z_\alpha}}{\partial x_j} + \frac{\partial \dot{z_\alpha}}{\partial x_i}\frac{\partial z_\alpha}{\partial x_j}\right]. \tag{3.3}$$

We add constraints

$$\sum_\alpha \frac{\partial z_\alpha}{\partial x_i}\dot{z_\alpha} = 0 \qquad \text{for all } i,$$

$$\sum_\alpha (\dot{z_\alpha})^2 \text{ is minimized subject to the above}$$

i.e. we require the perturbation to be normal to the embedding and to have minimal norm. By the first constraint,

$$\sum_\alpha \frac{\partial^2 z_\alpha}{\partial x_i x_j}\dot{z_\alpha} + \frac{\partial z_\alpha}{\partial x_i}\frac{\partial \dot{z_\alpha}}{\partial x_j} = 0,$$

which allows us to write the system as

$$\begin{aligned} \dot{g_{ij}} &= -2\sum_\alpha \frac{\partial^2 z_\alpha}{\partial x_i x_j}\dot{z_\alpha} \\ 0 &= \sum_\alpha \frac{\partial z_\alpha}{\partial x_i}\dot{z_\alpha} \qquad \text{for all } i. \end{aligned} \tag{3.4}$$

The first constraint thus converts (3.3) to a linear system of equations. Assuming z is free (which happens if $N \geq \frac{n(n+3)}{2}$), the unique solution to (3.4) can be written:

$$\dot{z_\alpha} = \sum_{i \leq j} \dot{g_{ij}} F_{aij}(Dz, D^2 z).$$

Lemma 3.5. *$F_{\alpha ij}$ depends analytically on $Dz, D^2 z$ and is coordinate independent.*

A proof is in ([2], Lemma 3.5.1). Thus we can denote the solution above by

$$\dot{z} = L(Dz, D^2 z)\dot{g}, \tag{3.6}$$

where L is a linear operator which loses two space derivatives.

Let $g_0 = z_0^{\#} e$. Suppose

$$h_{ij} = (g - g_0)_{ij}$$

is the initial metric error, and $h(t)$ is a path of metrics such that $h(0) = 0$ and $\lim_{t\to\infty} h(t) = h$. Letting $\dot{h_{ij}} = \dot{g_{ij}}$ in (3.6) formally gives an ODE whose solution $z(t)$ solves

$$\frac{\partial z(t)}{\partial x_i} \cdot \frac{\partial z(t)}{\partial x_j} = \frac{\partial z_0}{\partial x_i} \cdot \frac{\partial z_0}{\partial x_j} + h_{ij}(t). \tag{3.7}$$

However we cannot solve (3.6) using standard approximation techniques such as in Proposition 2.2, which doesn't use smoothing, as $\dot{z}$ depends on Dz and $D^2 z$. A proof using the Nash-Moser Inverse Function Theorem to solve (3.6) is in [11], Appendix II. We will describe Nash's original proof which constructs and solves a new ODE. Similar to the proof of Theorem 2.1, a smoothing operator will sustain an approximation process.

Note the path of metrics will not be pre-determined. We will solve for the path of the metric and perturbation of the embedding simultaneously. The idea is ensure the $\dot{g_{ij}}$ always point toward decreasing error. Smoothed quantities are substituted into (3.6) which give perturbations that approximately induce $\dot{g_{ij}}$. The $\dot{g_{ij}}$ must account for accumulated error. This is the feedback.

3.2. **Smoothing Operator.** A family of smoothing operators satisfying (2.5) and (2.6) is constructed for functions on $\mathbb{R}^n$ using standard techniques in analysis. We then describe how to smooth functions and tensors on manifolds, and embeddings. Since the smoothness of a function is linked with the decay of its Fourier transform, we can smooth out a function by multiplying its Fourier transform with a cut-off function. Let $\psi(x) \in C^\infty(\mathbb{R})$ such that $\psi(x) = 1$ for $x \leq 1$, is decreasing for $x \in [1,2]$, and is 0 for $x \geq 2$. Define

$$\widehat{S_t f}(\xi) = \widehat{f}(\xi) \cdot \psi(\frac{|\xi|}{t}).$$

As $t \to \infty$, the amount of smoothing decreases. Note Σ is compact, thus we will only have to consider $\operatorname{supp}(f)$ compact. If $\widehat{K_t} = \psi(\frac{|\xi|}{t})$ then

$$S_t f = K_t * f.$$

Note

$$K_t(x) = \int \psi(\frac{|\xi|}{t}) e^{i\xi x} d\xi = t^n \int \psi(|\xi|) e^{i\xi t x} d\xi = t^n K_1(tx).$$

For $\beta < \alpha$ with $|\beta| = b, |\alpha| = a$, we have

$$\begin{aligned} \partial^\alpha(K_t * f) &= (\partial^\beta K_t) * (\partial^{\alpha-\beta} f) \\ &= t^{n+b} \partial^\beta K_1(tx) * \partial^{\alpha-\beta} f = t^b \int \partial^\beta K_1(x) \partial^{\alpha-\beta} f(y - \frac{x}{t}) dx \\ &\leq C_b t^b \|f\|_{a-b}. \end{aligned} \tag{3.8}$$

We will also need bounds on $\partial^\alpha(\dot{K}_t * f)$ in terms of derivatives of f. Note

$$\widehat{\dot{K}_t}(|\xi|) = \dot{\widehat{K_t}}(|\xi|) = \dot{\psi}(\frac{|\xi|}{t}) = -\frac{|\xi|}{t^2} \psi'(\frac{|\xi|}{t}).$$

Let $L = \dot{K}_t$ at $t = 1$. Then

$$\dot{K}_t(|\xi|) = -\int \frac{|\xi|}{t^2} \psi'(\frac{|\xi|}{t}) e^{i\xi x} d\xi = -t^{n-1} \int y \psi'(|y|) e^{it\xi y} dy = t^{n-1} L(t\xi).$$

Similar to above we obtain

$$\partial^\alpha(\dot{K}_t * f) \leq D_b t^{b-1} \|f\|_{a-b}.$$

Thus

$$\begin{aligned} \|D^a(S_t f)\|_0 &\lesssim t^b \|f\|_{a-b} \\ \|D^a(\dot{S}_t f)\|_0 &\lesssim t^{b-1} \|f\|_{a-b}. \end{aligned} \tag{3.9}$$

We now describe how to smooth functions on a manifold. Then embeddings and tensors are smoothed component-wise. We use the following lemma from [7].

Lemma 3.10. *If $z : \mathcal{M} \to \mathbb{R}^m$, $z(\mathcal{M}) = R$, is a smooth embedding, there exists a neighborhood $\mathcal{N}$ of R such that for every $x \in \mathcal{N}$, there is a unique $y(x) \in R$ such that $dist(x, y(x)) = dist(x, R)$, and $x \to y(x)$ is smooth.*

Let $z : \mathcal{M} \to \mathbb{R}^m$ be a smooth embedding, $R = z(\mathcal{M})$, and $\mathcal{N}$ a neighborhood of R as in Lemma 3.10 with y the projection function. Define ϕ on $\mathcal{N}$ by

$$\phi(x) = \psi(\frac{\text{dist}(x, y(x))}{\varepsilon})$$

with ε small so that $\phi(x) = 0$ in a neighborhood of $\partial\mathcal{N}$. Let $f(y)$ be a function on R. Extend $f(y)$ to $\mathbb{R}^m$ by $f(x) = 0$ if $x \notin \mathcal{N}$, and $f(x) = \phi(x)f(x)$ if $x \in \mathcal{N}$.

Then $S_t f(y)$ is defined by
1. $f(y) \to f(x)$, extending f from R to $\mathbb{R}^m$
2. $f(x) \to K_t * f(x) := g(x)$
3. $g(x) \to g(y) = g|_R$.

Then $S_t f(y) := g(y)$. Fix a smooth embedding z. To smooth a tensor, identify $\mathcal{M}$ with $z(\mathcal{M})$ and smooth component-wise. Embeddings are also smoothed component-wise. We obtain bounds identical to those in (3.9), where f is replaced by an embedding or a tensor.

3.3. **System with Feedback.** We assume z_0 is an analytic initial embedding that is free and $z_0^\sharp e - g$ is small enough to ensure that $z(t)$ is free for all t. Refer to [8] section C for the construction of such a z_0 (during which the condition $N \geq \frac{3n^2+11n}{2}$ is used).

In Nash's notation, let

$$T \leq K \left[\, t \,\middle|\, \begin{smallmatrix} p & q \\ r & s \end{smallmatrix} \right]$$

denote the set of bounds

$$\|T\|_0 \leq Kt^p$$
$$\vdots$$
$$\|D^r T\|_0 \leq Kt^p$$
$$\|D^{r+1} T\|_0 \leq Kt^{p+1}$$
$$\vdots$$
$$\|D^s T\|_0 \leq Kt^q.$$

Here $q - p = s - r$. If the context is clear we write $\left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix} \right]$ instead of $\left[\, t \,\middle|\, \begin{smallmatrix} p & q \\ r & s \end{smallmatrix} \right]$, and we write $\left[\begin{smallmatrix} 0 \\ r \end{smallmatrix} \right]$ in place of $\left[\begin{smallmatrix} 0 & 0 \\ r & r \end{smallmatrix} \right]$.

The new ODE is defined as follows. Let g be the given metric, $H = g - z_0^\# e$ the metric error at $t = t_0$, and $\zeta(t) = S_t z(t)$. Recall L is the linear operator defined by (3.6). We solve

$$\begin{aligned} z(t_0) &= z_0 \\ \dot{z}(t) &= L(D\zeta(t), D^2\zeta(t))\dot{h}(t) \\ \dot{h}(t) &= \dot{u}(t - t_0)S_t H + u(t - t_0)\dot{S}_t H + \dot{S}_t E(t) + S_t \dot{E}(t) \end{aligned} \tag{3.11}$$

where $u(t) = \psi(2 - t)$. The expression for $\dot{h}(t)$ is obtained by differentiating

$$h(t) := u(t - t_0)S_t H + S_t E(t)$$

where $E(t)$ is accumulated error

$$E(t) = \int_{t_0}^{t} e(\tau)u(t-\tau)d\tau$$

$$e(t) = \dot{h}(t) - \dot{g}(t),$$

$\dot{g}(t)$ is given by (3.3) and $e(t)$ is the difference at time t between $\dot{g}$ and the desired rate of metric change. This error is caused by substituting smoothed quantities in for exact quantities in L. The system (3.11) can be regarded as an ODE with $g(t)$ and $h(t)$ as unknowns. Here $h(t)$ is a "guide" path of metrics which continuously adjusts for accumulated error, while $g(t)$ is the path we actually traverse. Note $E(t)$ is defined with lag, so that $\dot{h}(t)$ is defined by quantities from strictly earlier times.

Suppose (3.11) has a solution for all $t \geq t_0$ and $\lim_{t\to\infty} z(t)$, $\lim_{t\to\infty} h(t)$ exist. The total metric change accomplished would be

$$\int_{t_0}^{\infty} \dot{g} = \int_{t_0}^{\infty} \dot{h} - \int_{t_0}^{\infty} e(t)$$

$$= h(\infty) - E(\infty) = H + E(\infty) - E(\infty) = H.$$

which means $z(\infty)$ is an isometric embedding.

Beginning with the case $g \in C^3$, Nash shows (3.11) has a solution $z(t)$ for all $t \geq t_0$ by:

1. Let (†) denote a set of (apriori) bounds on quantities in (3.11).
2. A solution to (3.11) on an interval $[t_0, t]$, satisfying (†), satisfies strictly stronger bounds (†′) given t_0 large enough and the initial metric error small enough.
3. Given that (3.8) satisfies (†′) at some t, there exists $h > 0$ depending only on (†′) such that a solution exists on $[t, t+h]$. The solution satisfies (†).

Iterating (2) and (3) gives a solution to (3.11) for all $t \geq t_0$. This is an example of the method of continuity.

The first part of (3) follows from Picard's theorem: Since $S_\theta, S'_\theta, ..., \dot{S}_\theta, \dot{S}'_\theta, ...$ are smoothing operators, we can rewrite quantities in terms of $z, Dz, E, \dot{E}$ and time derivatives. Suppose initial conditions are bounded. On a compact set, initial conditions will be Lipschitz, thus for a time h we may solve (3.11) using Picard's theorem.

The existence of limits $\lim_{t\to\infty} z(t)$, $\lim_{t\to\infty} h(t)$, and their regularity will follow from (†).

We discuss bounds $(\dagger), (\dagger')$.

First, (3.11) requires $L(\zeta'(t), \zeta''(t))$ to be non-singular for all $t \geq t_0$, which happens if

$$\zeta - z_0 \leq \varepsilon \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$$

for small enough $\varepsilon > 0$ (then $F(\zeta', \zeta'')$ is close to $F(z_0', z_0'')$ which is non-singular). This is given by including

$$z - z_0 \le \beta \left[\begin{smallmatrix} 0 & 1 \\ 3 & 4 \end{smallmatrix}\right],$$

in (†) and choosing t_0 large enough so that $S_t z_0 - z_0 \le \frac{\varepsilon}{2} \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$ for all $t \ge t_0$.

Given $z(t)$ exists on $[t_0, t]$, we have (†) be the following apriori bounds. First,

$$z_0 \le \alpha \left[\begin{smallmatrix} 0 & 1 \\ 3 & 4 \end{smallmatrix}\right]$$

is immediate. By construction,

$$H \le \delta \left[\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right]$$

where $\delta > 0$ can be chosen. As above

$$\zeta - z_0 \le \varepsilon \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$$

$$z - z_0 \le \beta \left[\begin{smallmatrix} 0 & 1 \\ 3 & 4 \end{smallmatrix}\right]$$

$$S_\theta z_0 - z_0 \le \frac{\varepsilon}{2} \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right],$$

which means

$$z \le \xi \left[\begin{smallmatrix} 0 & 1 \\ 3 & 4 \end{smallmatrix}\right].$$

The rest of the bounds are

$$E \le \lambda \left[\begin{smallmatrix} 0 \\ 3 \end{smallmatrix}\right]$$

$$\dot{h} \le \mu \left[\begin{smallmatrix} -4 & 0 \\ 0 & 4 \end{smallmatrix}\right]$$

$$\dot{z} \le \gamma \left[\begin{smallmatrix} -4 & 0 \\ 0 & 4 \end{smallmatrix}\right]$$

$$e \le \eta \left[\begin{smallmatrix} -5 & -2 \\ 0 & 3 \end{smallmatrix}\right]$$

Stronger bounds (†′) are derived from and have the same form as (†), with smaller constants in place of $\alpha, \delta, \varepsilon$ etc. Note that third derivatives are included in bounds. This controls space derivatives of $z(t)$ to order three, and is used to show the limit embedding is C^3 (see [8]). The C^k case is proven by induction.

If stars denote corresponding constants in (†′), the result is

$$\lambda^* = \frac{1}{t_0}\eta$$

$$\mu^* = C_1(t_0+1)^4\delta + C_2\delta + C_3\lambda^* + C_4\frac{1}{t_0}\eta$$

$$\gamma^* = P_1(\xi)\mu^*$$

$$\eta^* = C_5\xi\gamma^*$$

$$\beta^* = P_2(\xi)(1+\xi+\gamma)(\mu^*+\delta+\lambda^*)$$

$$\alpha^* = \alpha + \beta^*$$

$$\varepsilon^* = \frac{\varepsilon}{2} + C_6\beta^*.$$

If δ is chosen small enough and t_0 large enough, starred constants (except α^*) will be smaller than the original constants. The presence of smoothing is key in allowing (†), (†′) to include bounds on higher order derivatives using (3.9). When deriving (†′), positive powers of t are cancelled by negative powers of t from lower derivatives. This paradigm was seen in the proof of Theorem 2.1.

4. Diffeomorphisms of a Circle

In this section we apply Nash-Moser iteration to a model problem for the KAM theorem. The presentation is adapted from [12].

Let ϕ lift an analytic diffeomorphism of $\mathcal{S}^1$. Assume

$$\phi(x) = x + \rho + \eta(x)$$

where $|\eta|$ is a small perturbation and

$$\rho = \lim_{n\to\infty} \frac{\phi^{(n)}(x) - x}{n}$$

is the rotation number of ϕ. A result is that the rotation number exists, is independent of x, and is invariant under conjugation.

Assume ϕ, η are real analytic ([1] does $\varphi \in C^\infty$). Our goal is to show there exists an analytic coordinate change $H(x) = y$ such that

$$H^{-1} \circ \phi \circ H(x) = x + \rho.$$

If we only require H to be a homeomorphism, Denjoy's theorem gives H for irrational ρ. Additional smoothness of H allows us to obtain more detailed information on the dynamics of ϕ, for example the distributions of orbits.

We will see that $|\eta|$ small implies

$$H(x) = x + h(x)$$

where $|h|$ is small. The coordinate change H must satisfy

$$\phi \circ H(x) = H(x + \rho)$$

or

$$x + h(x) + \rho + \eta(x + h(x)) = x + \rho + h(x + \rho),$$

thus

$$h(x + \rho) - h(x) = \eta(x + h(x)).$$

We prove the existence of a solution as the limit of approximate solutions. Linearizing gives

$$h(x + \rho) - h(x) = \eta(x). \tag{4.1}$$

The first step is to solve (4.1) and show a solution gives $H(x) = x + h(x)$ which conjugates ϕ to a diffeomorphism that differs from $x + \rho$ by an error that is second order in $|\eta|, |h|$. We then can iterate the process as in Theorem 2.1. This type of iteration is used to prove the KAM theorem.

4.1. The Linearization.

To solve (4.1), write

$$\sum_k \hat{h}(k)[e^{2\pi i k\rho} - 1]e^{2\pi i kx} = \sum_k \hat{\eta}(k)e^{2\pi i kx}.$$

For simplicity assume $\hat{\eta}(0) = 0$. Later we discuss the case $\hat{\eta}(0) \neq 0$. Formally,

$$h(x) = \sum_{k\neq 0} \frac{\hat{\eta}(k)}{e^{2\pi i k\rho} - 1} e^{2\pi i kx} \tag{4.2}$$

is a solution to (4.1). If ρ is rational the series is not defined. If ρ is well approximated by rationals, we do not have control over the smallness of denominators $|e^{2\pi i k\rho} - 1|$ and the series may diverge. We thus need a diophantine condition on ρ.

Definition 4.3. If $\alpha \in \mathbb{R}$ and $K > 0, \nu \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$,

$$|\alpha - \frac{m}{n}| > \frac{K}{n^\nu},$$

then α is type (K, ν).

In general ρ will be type (K, ν) for some $K > 0$, $\nu \geq 2$.

Lemma 4.4. *For $\nu > 2$, almost every $\alpha \in \mathbb{R}$ is type (K_α, ν) for some $K_\alpha > 0$.*

Proof. Without loss consider $\alpha \in (0, 1)$. Let $\nu > 2$. Let

$$E_{q,K} = (-\frac{K}{q^\nu}, \frac{K}{q^\nu}) \cup (\frac{1}{q} - \frac{K}{q^\nu}, \frac{1}{q} + \frac{K}{q^\nu}) \cup ... \cup (\frac{q-1}{q} - \frac{K}{q^\nu}, \frac{q-1}{q} + \frac{K}{q^\nu}).$$

If α not type (K, ν) then $\alpha \in E_{q,K}$ for infinitely many q. Since $\sum_{q \in \mathbb{N}} |E_{q,K}| = \sum_{q \in \mathbb{N}} \frac{2}{q^{\nu-1}} < +\infty$ we have $|\limsup E_{q,K}| = 0$. Take the union over $K \in \mathbb{Q}$ and the result follows. □

The following two Lemmas help prove the convergence of (4.2) if ρ is type (K, ν).

Lemma 4.5. *If ρ is type (K, ν), then*

$$|e^{2\pi i k \rho} - 1| > 2K|k|^{-(\nu-1)}$$

for $k \neq 0$.

Proof. We have $|k\rho - n| > K|k|^{-(\nu-1)}$ and thus $|e^{2\pi i k\rho} - 1| > 2K|k|^{-(\nu-1)}$ (bound chord of unit circle below by $\frac{len(arc)}{\pi}$). □

Define

$$S_\sigma = \{z \in \mathbb{C} : |\text{Im } z| < \sigma\}$$
$$\|f\|_\sigma = \sup_{x \in S_\sigma} |f(x)|$$
$$B_\sigma = \{f : f \text{ analytic on } S_\sigma, \|f\|_\sigma < \infty\}.$$

Lemma 4.6. *Suppose $\eta \in B_\sigma$. Then*

$$|\hat{\eta}(n)| \leq \|\eta\|_\sigma e^{-2\pi|n|\sigma}$$

Proof. Note η has period 1. By Cauchy's theorem the path integral of η over the rectangle with one side $[0, 1]$ and height σ is zero. For $n > 0$, this means

$$|\hat{\eta}(n)| = |\int_0^1 \eta(x) e^{-2\pi i n x} dx| = e^{-2\pi n\sigma} |\int_0^1 \eta(x - i\sigma) e^{-2\pi i n x} dx| \leq \|\eta\|_\sigma e^{-2\pi n \sigma}$$

and similarly for $n < 0$. □

Now suppose ρ is type (K, ν). For $|\text{Im } z| < \sigma - \delta$,

$$|h(z)| = |\sum_{n \neq 0} \frac{\hat{\eta}(n)}{e^{2\pi i n \rho} - 1} e^{2\pi i n z}| \leq \sum_{n \neq 0} \frac{|n|^{\nu-1}}{2K} \|\eta\|_\sigma e^{-2\pi\sigma|n|} e^{2\pi(\sigma-\delta)|n|}$$
$$\leq \frac{C_\nu}{K(2\pi\delta)^\nu} \|\eta\|_\sigma.$$

We used

$$\sum_{n \neq 0} |n|^{\nu-1} e^{-2\pi\delta|n|} \lesssim \int_0^\infty x^{\nu-1} e^{-2\pi\delta x} dx = \frac{\Gamma(\nu)}{(2\pi\delta)^\nu}.$$

Thus

$$\|h\|_{\sigma-\delta} \leq \frac{C_\nu}{K(2\pi\delta)^\nu}\|\eta\|_\sigma. \tag{4.7}$$

The condition $|Imz| < \sigma - \delta$ provides exponential decay to balance terms $|n|^{\nu-1}$ and is necessary for convergence. Thus we can only bound h on a strictly smaller domain of analyticity than that of ϕ. Note we have freedom in choosing the amount δ which the domain shrinks. For H to be a valid coordinate change we need H^{-1} to be analytic. The following is discussed in [12].

Lemma 4.8. *If $2\pi C_\nu\|\eta\|_\sigma < K(2\pi\delta)^{\nu+1}$ and $\delta \in (0, \min(\sigma, \frac{1}{4\pi}))$ then $H(z) = z + h(z)$ has analytic inverse on $H(S_{\sigma-2\delta})$.*

Solving the linearization (4.1) is analogous to applying the operator L in Theorem 2.1. Instead of derivatives lost, domain of analyticity (on which solutions are bounded) is lost. In both cases, the linear operator involved is degenerate, and extra steps are needed to carry out Newton iteration. Theorem 2.1 uses a smoothing operator. Here, we control the loss of domain at each step, so that in total, the domain shrinks less than a specified amount. We will see that choosing the amount to shrink at each step is delicate; the less we shrink the domain, the worse the bound on the solution of (4.1). A balance will be found.

4.2. **Iteration to a Solution.** We are proving

Theorem 4.9. *(Arnold's theorem [12], Theorem 2.1) Let ρ, the rotation number of ϕ, be type (K,ν) and $\sigma > 0$. Then there exists $\varepsilon(K,\nu,\sigma) > 0$ such that if $\phi(x) = x + \rho + \eta(x)$ with $\|\eta\|_\sigma < \varepsilon(K,\nu,\sigma)$, then there exists an analytic change of coordinates $H(x) = y$ such that $H^{-1} \circ \phi \circ H(x) = x + \rho$.*

We will prove H_k defined in the following way gives $H = \lim_{k\to\infty} H_0 \circ H_1 \circ \cdots H_k$ as the desired coordinate change. Define

$$\begin{gathered}
\phi_0(x) = \phi(x) = x + \rho + \eta_0(x) \\
H_0(x) = x + h_0(x) \text{ where } h_0(x+\rho) - h_0(x) = \eta_0(x) - \widehat{\eta_0}(0), \\
\vdots \\
\phi_k(x) = H_{k-1}^{-1} \circ \phi_{k-1} \circ H_{k-1} = x + \rho + \eta_k(x) \\
H_k(x) = x + h_k(x) \text{ where } h_k(x+\rho) - h_k(x) = \eta_k(x) - \widehat{\eta_k}(0)
\end{gathered}$$

for $k \geq 0$. Here ϕ_k is a diffeomorphism conjugated from ϕ_{k-1} by H_{k-1}, a coordinate change solving (4.1) with $\eta_{k-1} - \widehat{\eta_{k-1}}(0)$ in place of η. We will show (in section 4.2.1) that subtracting the zeroth Fourier mode of the right hand side does not affect the convergence of ϕ_k to a pure rotation. The key is the existence of a point on $\mathcal{S}^1$ which ϕ_k rotates exactly by ρ. Note all ϕ_k are conjugate thus have the same rotation number.

The plan now is to specify how much domain is lost at each step. Note in (4.7):
- if δ increases, more domain is lost but H is closer to id
- if δ decreases, less domain is lost but H is farther from id

In order for coordinate changes $H_0 \circ H_1 \circ \cdots \circ H_n$ to converge, we must have $H_n \to id$. For the limit to be analytic we must ensure the domains of analyticity of h_k do not shrink to zero. A balance for choosing $\delta > 0$ at each step is given in the following.

Theorem 4.10. *Let*

$$\delta_n = \frac{\sigma}{36(1+n^2)},\ n \geq 0$$
$$\sigma_0 = \sigma,\ \sigma_{n+1} = \sigma_n - 6\delta_n,\ n \geq 0$$
$$\varepsilon_0 = \|\eta\|_\sigma,\ \varepsilon_n = \varepsilon_0^{(\frac{3}{2})^n},\ n \geq 0.$$

We can assume $\sigma, K < 1$. *If*

$$\varepsilon_0 < (\frac{K}{16\pi C_\nu}(\frac{\sigma}{36})^{\nu+1})^8,$$

then $\eta_{n+1} \in B_{\sigma_{n+1}}$ *and*

$$\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \varepsilon_{n+1},$$
$$\|h_n\|_{\sigma_n - \delta_n} \leq \frac{C_\nu \varepsilon_n}{K(2\pi\delta_n)^\nu}.$$

Write $H_n^{-1}(x) = x - h_n(x) + g_n(x)$, *then*

$$\|g_n\|_{\sigma - 4\delta_n} \leq \frac{2\pi C_\nu \varepsilon_n^2}{K^2(2\pi\delta_n)^{2\nu+1}}.$$

For Theorem 4.10 to hold we will need $|\eta_{k+1}|$ being second order in $|\eta_k|$ i.e. fast convergence of $\eta_k \to 0$. We first show Theorem 4.10 implies Arnold's theorem.

Proof. (of Theorem 4.9) By Theorem 4.10, $\lim_{n\to\infty} \phi_n(x) = x + \rho$. Let

$$\mathcal{H}_n(x) = H_0 \circ H_1 \circ \cdots \circ H_n(x) = x_n$$

where

$$x_0 = x + h_n(x),\ x_k = x_{k-1} + h_{n-k}(x_{k-1}).$$

First we show $\lim_{n\to\infty} \mathcal{H}_n$ exists on S_{σ^*} where $\sigma^* > 0$. Since $|H_k(x) - x| = |h_k(x)|$, the composition $\mathcal{H}_n$ moves $x \in S_{\sigma_n}$ by $h_n(x) + \sum_{k=1}^n h_{n-k}(x_{k-1})$. Then using inductive bounds in Theorem 4.10,

$$|\mathcal{H}_n(x) - x| \leq \sum_{k\geq 0} |h_k(x)| \leq \sum_{k\geq 0} \frac{C_\nu \varepsilon_k}{K(2\pi\delta_k)^\nu} = A < \infty$$

since ε_n decays exponentially and δ_n like $\frac{1}{n^2}$. We also have

$$|\mathcal{H}_n'| \leq \prod_{k=1}^n |1 + h_k'| \leq \prod_{k=1}^n (1 + \frac{2\pi C_\nu \varepsilon_k}{K(2\pi\delta_k)^{\nu+1}}) \leq 1 + \frac{4}{\delta_n} A.$$

since cross terms are very small.

Then

$$\mathcal{H}_{n+1} - \mathcal{H}_n = \mathcal{H}_n \circ H_{n+1} - \mathcal{H}_n$$
$$= \mathcal{H}_n(x + h_{n+1}(x)) - \mathcal{H}_n(x) = \int_0^1 \mathcal{H}_n'(x + th_{n+1}(x))h_{n+1}(x)dx$$
$$\leq (1 + \frac{4A}{\delta_{n+1}})\frac{C_\nu \varepsilon_{n+1}}{K(2\pi\delta_{n+1})^\nu}.$$

Thus since $\sum_{n\geq 0} |\mathcal{H}_{n+1} - \mathcal{H}_n| < \infty$ on S_{σ^*}, the $\mathcal{H}_n$ are uniformly Cauchy. Since

$$\sigma^* = \lim_{n\to\infty} \sigma_n = \sigma - 6\sum_{n\geq 0} \delta_n = \sigma - 6\sum_{n\geq 0} \frac{\sigma}{36(1+n^2)}$$
$$\geq \sigma - \frac{1}{6}(\frac{\pi^2}{6}) \geq \frac{\sigma}{2},$$

we have $\lim_{n\to\infty} \mathcal{H}_n = \mathcal{H}$ where $\mathcal{H}$ is analytic on S_{σ^*}. Write

$$\mathcal{H}(z) = z + \tilde{h}(z).$$

By above $\|\tilde{h}\|_{\sigma^*} < A$. Let $\delta^* = \frac{\sigma^*}{16}$. We bound $\|\tilde{h}'\|_{\sigma^*-\delta^*}$ to use the inverse function theorem. Using definition of ε_n, δ_n we find $A < (\delta^*)^2$. Since $\|\tilde{h}\|_{\sigma^*} < A$ by the Cauchy integral formula $\|\tilde{h}'\|_{\sigma*-\delta^*} \leq \frac{A}{\delta^*} < \delta^*$. Then $\mathcal{H}$ has inverse on $\mathcal{H}(S_{\sigma^*-\delta^*}) \supseteq S_{\sigma^*-2\delta^*}$. Thus $\mathcal{H}$ is an analytic coordinate change on $S_{\sigma*-2\delta*}$ such that

$$H^{-1} \circ \phi \circ H(x) = x + \rho.$$

□

We now prove Theorem 4.10 using (4.7) and the following two propositions.

Proposition 4.11. *If η, δ as in Lemma 4.8, then $H^{-1}(z) = z - h(z) + g(z)$ where*

$$\|g\|_{\sigma-4\delta} \leq \frac{2\pi C_\nu^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2.$$

Proof. Let $g(z) = H^{-1}(z) - z + h(z)$. We use $g(z + h(z)) = h(z + h(z)) - h(z)$ to write

$$g(z + h(z)) = \int_0^1 \frac{dh}{dt}(z + th(z))h(z)dt.$$

In (4.7) and Lemma 4.8 we have bounds on $\|h\|_{\sigma-\delta}, \|h'\|_{\sigma-2\delta}$. Now $z \in S_{\sigma-3\delta}$ implies $z + th(z) \in S_{\sigma-2\delta}$, thus restrict $z + h(z) \in S_{\sigma-4\delta}$. Then $\|g\|_{\sigma-4\delta} \leq \|h\|_{\sigma-\delta}\|h'\|_{\sigma-2\delta}$ gives the result. □

Proposition 4.12. *Let η, δ as in Lemma 4.8, and $\tilde{\phi}(x) = H^{-1}\circ\phi\circ H(x) = x+\rho+\tilde{\eta}$. Then*

$$\|\tilde{\eta}\|_{\sigma-6\delta} \leq \frac{16\pi C_\nu^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2.$$

The proof of Proposition 4.12 proceeds by using Proposition 4.11 to expand

$$\begin{aligned}\tilde{\phi}(x) &= H^{-1} \circ \phi \circ H(x)\\ &= x + h(x) + \rho + \eta(x + h(x)) - h(x + h(x) + \rho + \eta(x + h(x)))\\ &\quad + g(x + h(x) + \rho + \eta(x + h(x)))\\ &= x + \rho + \Big[h(x) - h(x + \rho) + \eta(x)\Big] + \Big[\eta(x + h(x)) - \eta(x)\Big]\\ &\quad + \Big[h(x + \rho) - h(x + h(x) + \rho + \eta(x + h(x)))\Big]\\ &\quad + g(x + h(x) + \eta(x + h(x))).\end{aligned} \tag{4.13}$$

The right hand side excluding $x + \rho$ is $\tilde{\eta}(x)$. Each bracketed term is bounded separately using Cauchy's estimate and previously derived bounds.

Proof. (of Theorem 4.10)

For $n = 0$, by (4.7) and $\varepsilon_0 = \|\eta_0\|_\sigma$, we have $\|h_0\|_{\sigma-\delta_0} \leq \frac{C_\nu\varepsilon_0}{K(2\pi\delta_0)^\nu}$. Similarly Proposition 4.11 gives bound on $\|g_0\|_{\sigma-4\delta_0}$. Now $\sigma_1 = \sigma - 6\delta_0$ and η_0, δ_0 satisfies hypotheses of Lemma 4.8, thus by Proposition 4.12

$$\|\eta_1\|_{\sigma_1} \leq \frac{16\pi C_\nu^2}{K^2(2\pi\delta_0)^{2\nu+1}}\varepsilon_0^2$$

and $\frac{16\pi C_\nu^2}{K^2(2\pi\delta_0)^{2\nu+1}}\varepsilon_0^{\frac{1}{2}} < 1$ by choice of ε_0.

Assume the Theorem holds for $k \leq n-1$. Note $\delta_n < \delta_{n-1}$ and $\|\eta_n\|_{\sigma_n} \leq \varepsilon_n$. Then (4.7) implies the bound on $\|h_n\|_{\sigma_n-\delta_n}$ and Proposition 4.11 the bound on $\|g_n\|_{\sigma_n-4\delta_n}$. Then since $\sigma_{n+1} = \sigma_n - 6\delta_n$, Proposition 4.12 and the definition of ε_n implies

$$\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \frac{16\pi C_\nu^2}{K^2(2\pi\delta_n)^{2\nu+1}}\varepsilon_n^2 \leq \varepsilon_0^{(\frac{3}{2})^{n+1}}$$

□

4.2.1. *The Zeroth Fourier Coefficient.* Consider the case $\hat{\eta}(0) \neq 0$. Then h as defined above solves

$$h(x+\rho) - h(x) = \eta(x) - \hat{\eta}(0).$$

Proposition 4.11 still holds and the coordinate changes $\mathcal{H}_n$ will converge as before. We show that Proposition 4.12 still holds using the following

Lemma 4.14. *If $\phi(x) = x + \rho + \eta(x)$ where ρ is the rotation number of ϕ then there is some x_0 such that $\eta(x_0) = 0$.*

Proof. If $\eta \neq 0$ on $\mathcal{S}^1$ then since η is continuous, $\min_{\mathcal{S}^1}|\eta| > \varepsilon > 0$. For $\eta > 0$ this would mean $\rho > \rho + \varepsilon$. The same argument applies for $\eta < 0$. □

Then in 4.13 we have $|\hat{\eta}(0)|$ equal to the last three terms evaluated at x_0. Since these terms are $O(\|\eta\|_\sigma^2)$, Proposition 4.12 holds and thus ϕ_n still converges to a pure rotation.

5. The KAM theorem

The goal now is to state the KAM theorem and present a reformulation which can be proven with Nash-Moser iteration. We focus on setting up the iteration process. The proof that iteration works is similar to proving Theorem 4.10 and is in [6] ch.5. We adopt (and simplify) the approach of [6] and consider systems

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned} \tag{5.1}$$

such that

$$\begin{aligned}f(-x,y) &= f(x,y)\\ g(-x,y) &= -g(x,y).\end{aligned}$$

Here $x, y \in \mathbb{R}^n$ and f, g are real analytic with period 2π in x_k for $k = 1, ..., n$. Such systems are reversible systems. A Hamiltonian system

$$\dot{p_k} = \frac{\partial H}{\partial q_k}$$

$$\dot{q_k} = -\frac{\partial H}{\partial p_k}$$

is an example of such a system.

Definition 5.2. The system (5.1) is in normal form if

$$f(x,y) = F(y)$$

$$g(x,y) = 0.$$

A system in normal form has solutions

$$x(t) = F(c)t + b \ (mod\ 2\pi), \qquad y(t) = c. \tag{5.3}$$

These are lines of constant slope on $n-$tori in $\mathbb{R}^{2n}$.

Definition 5.4. Suppose for $y = c$ there exists $z \in \mathbb{Z}^n$ such that

$$F(c) \cdot z = 0.$$

Then the components of $F(c)$ are rationally dependent and $\{x_k \in [0, 2\pi), y = c\}$ is called a resonant torus.

Definition 5.5. A trajectory (5.3) is quasi-periodic if components of $F(c)$ are rationally independent.

A system (5.1) in normal form has periodic solutions on resonant tori, and quasi-periodic solutions on non-resonant tori. Quasi-periodic trajectories are dense on tori. We will see that many non-resonant tori survive small perturbations.

We work with coordinate changes that preserve the reversibility of (5.1). Let $\mathcal{G}$ denote the set of coordinate changes

$$x = u(\xi, \eta), \qquad y = v(\xi, \eta)$$

with $(u, v) \approx (\xi, \eta)$ and

$$\begin{aligned} u(-\xi, \eta) &= u(\xi, \eta) \\ v(-\xi, \eta) &= -v(\xi, \eta). \end{aligned}$$

Then $(x, y) \to (\xi, \eta)$ is close to the identity. We require

$$u(\xi, \eta) - \xi, \ v(\xi, \eta)$$

be analytic and have period 2π in ξ to preserve properties of f, g.

5.1. **Statement of the Theorem.** Exposition is, beginning now, parallel to Section 4.1. We state the KAM theorem and describe how it is proved. Note KAM stands for Kolmogorov, Arnold, and Moser, whose work founded the theory.

Consider a perturbed reversible system

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \\ \dot{y} &= g(x, y, \mu) \end{aligned} \tag{5.6}$$

such that the system at $\mu = 0$ is in normal form. For all $\mu \geq 0$ we require

$$\begin{aligned} f(-x, y, \mu) &= f(x, y, \mu) \\ g(-x, y, \mu) &= -g(x, y, \mu). \end{aligned} \tag{5.7}$$

The KAM theorem tells us when a solution to (5.6) at $\mu > 0$ is a perturbation of a quasi-periodic solution at $\mu = 0$. We require a diophantine condition analogous to Definition 4.3. Its necessity will arise when inverting a linearized system, as before.

Definition 5.8. If $\omega \in \mathbb{R}^n$ and $\gamma > 0, \tau \in \mathbb{N}$ such that for any $j = (j_1, ..., j_n) \in \mathbb{Z}^n$, we have

$$|\sum_{k=1}^{n} j_k \omega_k| \geq \gamma |j|^{-\tau},$$

then ω is type (γ, τ).

By a similar proof as Lemma 4.4, for a.e. $\omega \in \mathbb{R}^n$ is type (γ, τ) for some γ, τ.

Theorem 5.9. *(KAM theorem) Consider (5.6). If for some c we have $F(c)$ type (γ, τ) and*

$$det(\frac{\partial F_k}{\partial y_\ell}) \neq 0$$

at $y = c$, then there exists solutions

$$x = \theta + u(\theta, \mu)$$

$$y = c + v(\theta, \mu)$$

$$\dot{\theta}_k = F_k(c)$$

where u, v are real analytic in θ, have period 2π in θ,

$$u(-\theta, \mu) = -u(\theta, \mu), \qquad v(-\theta, \mu) = v(\theta, \mu),$$

and $u(\theta, 0) = v(\theta, 0) = 0$.

At $\mu = 0$, we have the quasi-periodic solution

$$\dot{x} = F(c), \qquad y = c.$$

As μ varies the solution is only slightly perturbed and remains predictable. The perturbed solution is a dense flow on a deformed torus, and is "qualitatively quasi-periodic". These are stable solutions of (5.6).

5.2. **Reformulation.** To prove Theorem 5.9, we reformulate the problem as follows. Given a solution to (5.6), we find when there exists a coordinate change in $\mathcal{G}$ that transforms the solution to a quasi-periodic trajectory on a torus. Since the coordinate change is analytic and close to the identity, the solution would be a small perturbation of a quasi-periodic trajectory.

Theorem 5.10. *([6], Theorem 5.1) Let ω be type (γ, τ). Let $0 < \alpha < 1$ and $0 < \beta < min(\frac{\alpha}{4n+4\tau+1}, 1-\alpha)$. Then there exists $\delta_0 = \delta_0(\alpha, \beta, \gamma, \tau, n)$ such that if we have a reversible system*

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

with

$$f(-x, y) = f(x, y), \qquad g(-x, y) = -g(x, y)$$

such that for some $0 < \delta < \delta_0$,

$$|f - \omega - y| + |g| < \delta^{1+\alpha}$$

for any x, y such that $|Im x_k| < \delta^\beta$, $|y_k| < \delta^*$, then there exists a coordinate change

$$x = u(\xi, \eta), \qquad y = v(\xi, \eta)$$

in $\mathcal{G}$ which transforms the system into

$$\dot{\xi} = \phi(\xi, \eta), \qquad \dot{\eta} = \psi(\xi, \eta)$$

with

$$\phi = w + \eta + O(\eta^2), \qquad \psi = O(\eta^2).$$

In addition, we can choose u, v to be linear in η and satisfy

$$|u - \xi| + |v| < \delta$$

in the region $|Im \xi_k| < \frac{1}{2}\delta^\beta$, $|\eta_k| < \frac{\delta}{2}$.

The system under consideration is (5.6) at a fixed μ. For $\eta = 0$ we have the solution $\xi(t) = \omega t + \xi(0)$ which gives

$$x = u(\omega t + \xi(0), 0), \qquad y = v(\omega t + \xi(0), 0)$$

as a solution to the original system. Since $(u, v) \approx (x, y)$ this is approximately $\dot{x} = w, \dot{y} = 0$.

Lemma 5.11. *Theorem 5.10 implies the KAM theorem.*

Proof. In the KAM theorem, we begin with an unperturbed system $\dot{x} = F(y)$, $\dot{y} = 0$ defined on a complex domain $X \times Y$. By assumption $det(\frac{\partial F_\ell}{\partial y_k}) \neq 0$ for $y = c \in Y$, and $\omega = F(c)$ is type (γ, τ).

Since $DF(c)$ is non-singular, write

$$y = c + DF(c)^{-1}z, \ z \in X.$$

By Taylor's theorem,

$$\begin{aligned} F(c + DF(c)^{-1}z) &= F(c) + z + \tilde{G}(DF(c)^{-1}z) \\ &= F(c) + z + G(z) \end{aligned}$$

where $|G(z)| = O(|z|^2)$. Rewrite the unperturbed system as

$$\begin{aligned} \dot{x} &= F(c) + z + G(z) \\ \dot{z} &= 0. \end{aligned}$$

For some $\delta > 0$, restricting to $|z| < \delta$ gives $|G(z)| < \frac{1}{2}\delta^{1+\alpha}$,

$$\{c + DF(c)^{-1}z : |z| < \delta\} \subset Y,$$

and $|y - c| = |DF(c)^{-1}z| < \frac{1}{8}\delta^{1+\alpha}$. Since (5.6) depends analytically on μ there exists $\delta' > 0$ such that for $\mu < \delta'$,

$$|f - \omega - y| < \frac{1}{8}\delta^{1+\alpha}, \qquad |g| < \frac{1}{8}\delta^{1+\alpha}.$$

This system, for $\mu < \delta'$, then satisfies the hypotheses of Theorem 5.10, and has quasi-periodic solutions

$$x = u(\omega t + \xi(0), 0), \qquad y = v(\omega t + \xi(0), 0).$$

as required in the KAM theorem. $\square$

We conclude this section by deriving the system and its linearization for the proof of Theorem 5.10.

Suppose $x = u(\xi, \eta), y = v(\xi, \eta)$ satisfies the conclusions of Theorem 5.10. Write

$$f = \omega + y + \tilde{f}, \qquad g = \tilde{g}.$$

Now $\dot{\xi} = \phi, \dot{\eta} = \psi$ where

$$\begin{aligned} \phi &= \omega + \eta + o(\eta^2) \\ \psi &= o(\eta^2). \end{aligned} \tag{5.12}$$

We also have

$$\begin{aligned} \dot{x} &= u_\xi \phi + u_\eta \psi = f(u, v) \\ \dot{y} &= v_\xi \phi + v_\eta \psi = g(u, v). \end{aligned} \tag{5.13}$$

Since $(\xi, \eta) \mapsto (u, v)$ is close to the identity, $u - \xi$ has period 2π in ξ, and u, v are linear in η, write

$$
\begin{aligned}
u &= \xi + u_0(\xi) + u_1\eta \\
v &= \eta + v_0(\xi) + v_1\eta
\end{aligned} \tag{5.14}
$$

where u_0, v_0 are periodic in ξ and u_1, v_1 are matrices. Set $\eta = 0$. This is where we look for a quasi-periodic solution. After substituting (5.12) and (5.14) into (5.13), and equating coefficients of terms linear in η, an approximate system is

$$
\begin{aligned}
\frac{\partial u_0}{\partial \xi} \cdot \omega - v_0 &= \tilde{f}(\xi, 0) \\
\frac{\partial v_0}{\partial \xi} \cdot \omega &= \tilde{g}(\xi, 0) \\
\frac{\partial u_1}{\partial \xi} - v_1 = -\frac{\partial u_0}{\partial \xi} &+ \partial_2 \tilde{f}(\xi, 0) \\
\frac{\partial v_1}{\partial \xi} \cdot \omega = -\frac{\partial v_0}{\partial \xi} &+ \partial_2 \tilde{g}(\xi, 0).
\end{aligned} \tag{5.15}
$$

Solving (5.15) involves solving

$$
\sum_{k=1}^{n} \omega_k \frac{\partial v_0}{\partial \xi_k} = \tilde{g}(\xi, 0),
$$

or

$$
\sum_{k=1}^{n} \sum_{m} \omega_k i m_k \hat{v_0}(m) e^{2\pi i m \cdot \xi} = \sum_{m} i\omega \cdot m \hat{v_0}(m) e^{2\pi i m \cdot \xi} = \sum_{m \neq 0} \hat{g}(m) e^{2\pi i m \cdot \xi}
$$

since g is odd in ξ. Formally,

$$
v_0(\xi) = \sum_{m} \frac{\hat{g}(m)}{i\omega \cdot m} e^{2\pi i m \cdot \xi}.
$$

The condition ω type (γ, τ) ensures convergence of the series as in the 1D case. Bounds on $|(u_0, v_0)|$ are derived using Cauchy's estimate, thus hold only on strictly smaller domains. We iterate while ensuring the domain of analyticity does not shrink to zero as in diffeomorphisms of a circle.

5.3. **Completing the Picture.** In the previous section we showed that many quasi-periodic solutions to reversible systems are stable under small reversible perturbations. In the complement, numerical simulations indicate all but finitely many resonant tori break into unpredictable behavior [9]. This unpredictable behavior has only been proven for a measure zero cantor set of resonant tori (the proof is in [6], ch.3) and is being studied to this day.

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