TWO MODELS OF HYPERBOLIC GEOMETRY

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ABSTRACT. This paper explores hyperbolic geometry, with a particular focus on the interaction between two models: the Poincaré's Unit Disk Model and Upper-Half Plane Model. The paper first defines the concept of geometry and discusses Euclid's five postulates. The departure from Euclidean principles is then examined through a thorough investigation of these postulates under the context of hyperbolic geometry. The investigation culminates with the discussion of the isomorphic Möbius Transformation that maps between the two models. Through establishing the isomorphic map and analyzing its properties, this paper explores how isomorphism preserves geometric invariants.

CONTENTS

1. INTRODUCTION

The exploration of non-Euclidean geometries has beckoned mathematicians to scrutinize geometric systems that deviate from the classical Euclidean axioms. Hyperbolic geometry is particularly distinguished by its departure from Euclid's fifth postulate. This departure gives rise to an intricate and captivating geometry, constructed upon distinct axioms and properties that challenge conventional Euclidean notions.

Central to the understanding of hyperbolic geometry are the geometric models that provide tangible representations of its abstract concepts. Among these models, the Unit Disk Model and the Upper-Half Plane Model of the French mathematician

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Henri Poincaré's (1854-1912) stand prominently. One essential facet of this paper is the re-calibration of Euclid's five postulates under the hyperbolic context. Each postulate undergoes a rigorous reformulation and re-evaluation of validity in the model of hyperbolic geometry.

The Unit Disk Model, realized through the unit circle within the complex plane, and the Upper-Half Plane Model through the upper-half of the complex plane, each captures the essence of hyperbolic geometry from different perspectives. This paper first discusses the properties of the two models separately. To unite the understanding of the two models, the paper examines the concept of inversion as a foundational transformation in the Möbius transformation that provides the isomorphic map between them. The isomorphic map, which this paper will denote as "S," serves as the conduit between these models. The properties of this isomorphism and the consequential preservation of geometric invariants is analyzed. The paper derives and presents the preservation of length for smooth curves for the Upper-Half Plane Model from the Unit Disk Model, underscoring its congruence facilitated by the isomorphic map.

This paper is structured as follows. The beginning section introduces the definition of the concept of geometry as a pair of underlying set and transformation group. Specifically, it discusses the definition of Euclidean geometry, as well as Euclid's five postulates. The next section explores the foundations of hyperbolic geometry through the Unit Disk Model and the Upper-Half Plane Model respectively. It establishes rigorous definitions and theorems. It then examines each of Euclid's five postulates under this newly established context. The following section connects the two models by introducing isomorphic mapping. It reviews circle inversion as an essential Möbius transformation that help constructs the isomorphism between the models. The isomorphism is unveiled using the Fundamental Theorem of Möbius Geometry. Finally, the paper discusses the preservation of invariants under isomorphic transformations that unite different models of hyperbolic geometry.

2. Euclidean Geometry

This section discusses two different approaches to define Euclidean geometry. The first approach defines geometry as a pair of two sets (S, G) consisting of a nonempty set S and a transformation group G on S. The second approach defines geometry as an axiomatic system based on primitive terms and postulates. In constructing the models that follow the two definitions, the interplay between them will become evident.

2.1. **Euclidean Geometry defined by Transformation Groups.**

Before we define a geometry, we first define a transformation group.

Definition 2.1. Let S be a nonempty set. A *transformation group* G is a collection of transformations $T : S \to S$ such that
(a) G contains identity Id_s
(b) the transformations in G are invertible, and their inverses are in G
(c) G is closed under composition.

Based on this definition of transformation group, the following defines a geometry.

Definition 2.2. A *geometry* is a pair (S, G) consisting of a nonempty set S and a transformation group G on S.

By this definition, the set S is the underlying set of the geometry, and the set G is the transformation group of the geometry. We now have the tools to define Euclidean geometry. The underlying set S of Euclidean geometry is the complex plane $\mathbb{C}$. The transformation group G is s set E of transformations of the form

$$Tz = e^{i\theta}z + b, (\theta \in \mathbb{R}, b \in \mathbb{C}) \tag{2.3}$$

We may check that E satisfies the definition of transformation group.
(a) When $\theta = 0$, $b = 0$, T is the identity transformation.
(b) If transformation $Tz = e^{i\theta}z + b$, then T is invertible, and the inverse of T denoted as $T^{-1}z = e^{i(-\theta)}z + (-e^{-i\theta}b)$ is in E.
(c) Let T_1 and T_2 be two transformations. If $T_1z = e^{i\theta_1}z + b_1$, $T_2z = e^{i\theta_2}z + b_2$, then

$$(T_1 \circ T_2)z = e^{i(\theta_1+\theta_2)}z + (b_2e^{i\theta_1} + b_1).$$

This concludes that E is a transformation group. [1]

Euclidean geometry is modeled by the pair $(\mathbb{C}, E)$, where $\mathbb{C}$ is the complex plane, and E is the transformation group on $\mathbb{C}$.

2.2. **Euclidean Geometry defined by Euclid's 5 Postulates.**

Euclid defined geometry as an axiomatic system in the *Elements* [3]. This section discusses his five postulates.

Postulate 2.1. It is possible to draw one and only one straight line from any point to another point.

Postulate 2.2. From each end of a finite straight line, it is possible to produce it continuously in a straight line by an amount greater than any assigned length.

Postulate 2.3. It is possible to describe one and only one circle with any center and radius.

Postulate 2.4. All right angles are equal to one another.

Postulate 2.5. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The fifth postulate can be rephrased as the following. Through a point not on a line, there is a unique line parallel to the given line. Unlike the first four postulates, the fifth cannot be proven within the system of Euclidean geometry without introducing a different postulate. The attempt to prove the fifth postulate started with Poseidonios (c. 135-c. 51 B.C.) and was followed by mathematicians throughout history, including Wallis, Lambert, and Gauss [2]. However, none of them succeeded in deducing it without replacing it with another postulate. It was finally revealed that the fifth postulate is independent of other Euclid's axioms when N.I.

Lobachevskii (1826) created a consistent geometry where all of Euclid's postulates hold except the fifth [2]. This is the initial construction of hyperbolic geometry

3. Hyperbolic Geometry

Proof of the consistency of a geometry as an axiomatic system is attained by the existence of a model. The model gives meaning to the primitive definitions and turn the axioms into true statements. This section discusses two models of hyperbolic geometry from Poincaré-the Unit Disk Model and the Upper-Half Plane Model. Specifically, it investigates how the two models satisfy Euclid's first four postulates but not the fifth.

3.1. Poincaré's Unit Disk Model.

We have previously defined geometry as a pair consisting of an underlying set and a transformation group acting on the set. In this case, let $\mathbb{D}$ be the unit disk in the complex plane. Let H be a set of transformations on $\mathbb{D}$ of the form

$$Tz = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}z}. \tag{3.1}$$

By this definition, the Unit Disk Model can be denoted as $(\mathbb{D}, H)$. It models hyperbolic geometry with a group of transformations H consisting of all Möbius transformations mapping the unit disk $\mathbb{D}$ onto itself. The set D is called the hyperbolic plane, and the group H is called the hyperbolic group.

We now provide primitive definitions for the model.

Definition 3.2. A hyperbolic *straight line* is a Euclidean circle or Euclidean straight line in $\mathbb{D}$ that intersects the unit circle at a right angle.

Definition 3.3. In the hyperbolic plane, the *length* of a smooth curve γ with parametrization $z(t) = x(t) + iy(t)$, where $a \leq t \leq b$, is given by

$$l(\gamma) = 2\int_a^b \frac{|z'(t)|}{1 - |z(t)|^2}, \tag{3.4}$$

where $z'(t) = x'(t) + iy'(t)$. [1]

Definition 3.5. Hyperbolic *angle* is the same as Euclidean angle.

Definition 3.6. A hyperbolic *circle* is a portion of a Euclidean circle or straight line inside the unit disk that is entirely contained in $\mathbb{D}$.

In any model of a geometry, an invariant is a property that remains unchanged after any transformation of its object from its transformation group. It is defined as the following.

Definition 3.7. Let (S, G) be a geometry. Let D be a set whose elements are subsets of S.

- The set D is called an *invariant* in the geometry (S, G) if for any $B \in D$ and $T \in G$, then $T(B) \in D$.

• A function f defined on D is called an *invariant* in the geometry (S, G) if for any $B \in D$ and $T \in G$, then $f(T(B)) = f(B)$.

To gain a better understanding of this concept, we look at an example in Euclidean geometry.

Lemma 3.8. *Euclidean angle is an invariant in Euclidean geometry.*

Proof. We want to show that for any transformation of the form $Tz = e^{i\theta}z + b, (\theta \in \mathbb{R}, b \in \mathbb{C})$, if x, y are elements of $\mathbb{C}$, then

$$\frac{< Tx,\ Ty >}{\|Tx\| \cdot \|Ty\|} = \frac{< x,\ y >}{\|x\| \cdot \|y\|}.$$

The angle between Tx and Ty after transformation is the same as the angle between the two lines formed by Tx and $T(0)$, and Ty and $T(0)$. We know that $T(0) = b$. This gives us the following equality:

$$\frac{< Tx - T(0),\ Ty - T(0) >}{\big\|Tx - T(0)\big\| \cdot \big\|Ty - T(0)\big\|} = \frac{< e^{i\theta}x,\ e^{i\theta}y >}{\big\|e^{i\theta}x\big\| \cdot \big\|e^{i\theta}y\big\|}.$$

Since $e^{i\theta}$ is an orthogonal transformation, we have that

$$\frac{< e^{i\theta}x,\ e^{i\theta}y >}{\big\|e^{i\theta}x\big\| \cdot \big\|e^{i\theta}y\big\|} = \frac{< x,\ y >}{\|x\| \cdot \|y\|}.$$

This concludes the proof. □

Sanity check. As is stated in the section of Euclidean Geometry, the transformation group E of a Euclidean geometry includes compositions of rotation and translation, that is, rigid motions. To see why rigid motions preserve Euclidean angles, we can use the cosine formula, $cos(x, y) = \frac{<x,\,y>}{\|x\| \cdot \|y\|}$. Since rigid motions preserve $\frac{<x,\,y>}{\|x\| \cdot \|y\|}$, $cos(x, y)$ is preserved.

Having proven that Euclidean angle is an invariant in Euclidean geometry, we may now check whether hyperbolic angle and hyperbolic length are invariants in hyperbolic geometry.

Theorem 3.9. *Hyperbolic angle is an invariant in hyperbolic geometry.*

Proof. By Lemma 3.8, Euclidean angle is an invariant in Euclidean geometry. Since hyperbolic angles are defined the same way as Euclidean angles are, hyperbolic angle is an invariant in hyperbolic geometry. □

Theorem 3.10. *Hyperbolic straight line is an invariant in hyperbolic geometry.*

Proof. By Theorem 3.9, hyperbolic angles are preserved during transformations. Thus, angles between a hyperbolic straight line and the unit circle are preserved during transformations. By Definition 3.2, a hyperbolic straight line intersects the unit circle at a right angle. This quality is preserved after transformations. Hence, hyperbolic straight line is also an invariant. □

Now, we check that hyperbolic length is also an invariant.

Theorem 3.11. *Let T be a transformation of hyperbolic group, and γ be a smooth curve. Then*

$$l(T(\gamma)) = l(\gamma).$$

Proof. Let $w = Tz = e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z}$, where $|z_0| < 1, \theta \in \mathbb{R}$, and $\gamma = z(t)$. Then

$$T(\gamma) = T(z(t)) = e^{i\theta}\frac{z(t)-z_0}{1-\overline{z_0}z(t)}.$$

And

$$w'(t) = e^{i\theta}\frac{1-|z_0|^2}{(1-\overline{z_0}z(t))^2}z'(t).$$

This yields the equality

$$\frac{|w'(t)|}{1-|w(t)|^2} = \frac{|z'(t)|}{1-|z(t)|^2}.$$

Hence, we have that

$$l(T(\gamma)) = 2\int_a^b \frac{|w'(t)|}{1-|w(t)|^2} = 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2} = l(\gamma)$$

This concludes the proof. □

3.1.1. *Why Unit Disk Model Models Hyperbolic Geometry.* Having established the basic definitions for the Unit Disk Model, we now proceed to checking whether this model of hyperbolic geometry is consistent with the first four postulates in Euclidean geometry. Before we begin with the validity of the first postulate, we define cross ratio and symmetric points.

Definition 3.12. The *cross ratio* is the following functions of four extended complex variables:

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3}\cdot\frac{z_1 - z_3}{z_1 - z_2}.$$

Definition 3.13. Let C be a cline (a circle or a straight line) passing through 3 distinct points z_1, z_2, and z_3. Two points z and z^* are called *symmetric with respect to* C, if

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$

Remark 3.14. Symmetry is an invariant in hyperbolic geometry. If z and z^* are symmetric with respect to C, then Tz and Tz^* are symmetric with respect to $T(C)$. We can denote it as $T(z^*) = (Tz)^*$.

Lemma 3.15. *Let C_1 be a cline with center $O1$. Let Z_1 and Z_2 be distinct symmetric points with respect to C. Any cline C_2 with center $O2$ that is orthogonal to C_1 and passing through Z_1 must also be passing through Z_2.*

Lemma 3.15 is visualized by Figure 1.

Lemma 3.15 implies that if Z_1 coincides with $O1$, cline C_2 will be the diameter of cline C_1. In other words, cline C_2 will be a hyperbolic straight line that passes through 0 and ∞. One can imagine it to be a circle of infinite radius.

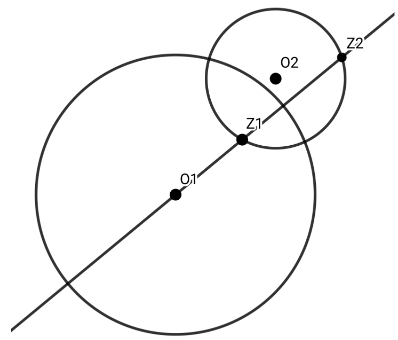

FIGURE 1.

Now, we can prove Euclid's first postulate under the Unit Disk Model.

Postulate 3.1. Given two points in $\mathbb{D}$, it is possible to draw one and only one hyperbolic straight line through the two points.

Proof. Let z_1, z_2 be 2 distinct points in $\mathbb{D}$. We define

$$Tz = e^{i\theta}\frac{z - z_1}{1 - \overline{z_1}z}.$$

Then, Tz takes z_1 to 0. Choose $\theta = -arg(\frac{z_2 - z_1}{1 - \overline{z_1}z2})$. Then

$$Tz_2 = e^{i\theta}\frac{z_2 - z_1}{1 - \overline{z_1}z_2} = |\frac{z_2 - z_1}{1 - \overline{z_1}z_2}|.$$

We have that Tz_2 lies on the x-axis. By Lemma 3.15, a hyperbolic straight line passing through 0 must also pass through ∞. Thus, this line that passes through 0 and Tz_2 is a diameter of the unit disk. Since $Tz_2 \in \mathbb{R}$, the diameter must be the x-axis. This concludes the proof that there exists a unique hyperbolic straight line passing through z_1 and z_2, and it is the x-axis. □

Before delving into the second postulate, we first acquaint ourselves with the distance formula for the Unit Disk Model. We will only introduce the less complicated formula of the distance between 0 and another point in $\mathbb{D}$.

Corollary 3.16. *The distance $d(0, z)$ between 0 and z can be attained through the formula*

$$d(0, z) = ln(\frac{1 + |z|}{1 - |z|}).$$

Proof. We have stated previously that a hyperbolic straight line passing through 0 is a segment of a Euclidean straight line. Thus, we have that $d(0, z) = l(\gamma)$, where $\gamma = z(t) = tz, 0 \leq t \leq 1$. This implies that

$$d(0, z) = 2\int_0^1 \frac{|z'(t)}{1 - |z(t)|^2}dt = 2\int_0^{|z|} \frac{d\gamma}{1 - \gamma^2}, \ \gamma = t|z|,$$

$$= ln(\frac{1 + |z|}{1 - |z|}).$$

□

Additionally, we need to prove an essential lemma in Möbius transformation.

Lemma 3.17. *Every hyperbolic straight line can be mapped to the x-axis by a Möbius transformation.*

Proof. Let C be a hyperbolic straight line. Let z_0 be a point on C. By Lemma 3.15, the symmetry of z_0, z_0^* also lies on C. Denote Möbius transformation T as

$$Tz = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}z},$$

where we choose θ in later steps. Then T is a transformation in the hyperbolic group $\mathbb{H}$ that takes z_0 to 0 and takes $z_0^* = \frac{1}{\overline{z_0}}$ to ∞.

Hence, $T(C)$ is a cline that passes through 0 and ∞, and is orthogonal to $\partial\mathbb{D}$. This implies that $T(C)$ is a diameter of the unit circle. Then, we can choose θ so that the diameter corresponds to the x-axis. □

Now, we have acquired enough tools to prove the second postulate.

Postulate 3.2. From each end of a hyperbolic straight line in $\mathbb{D}$, it is possible to produce it continuously by infinite length.

Proof. By Lemma 3.17, every hyperbolic straight line can be mapped to the x-axis through Möbius transformation. Consequently, it is sufficient to exclusively consider the distance between the center of $\mathbb{D}$ and the other point after Möbius transformation. Let $d(0, r)$ denote the distance. By Corollary 3.16, we have that

$$d(0, r) = ln\frac{1 + r}{1 - r}$$

From this formula, we may see that as r approaches 1, $d(0, r)$ approaches ∞. This means that for any $N > 0$, there exists $r' < 1$, such that

$$d(0, r') > d(0, r) + N.$$

Hence, every hyperbolic straight line is extendable continuously to infinity. □

Postulate 3.3. A circle can be described with any center and hyperbolic radius.

Proof. By Lemma 3.17, it is sufficient for us to consider the case when the center of the circle coincides with the center of $\mathbb{D}$. Let the radius of the circle be denoted as R. For any R greater than 0, there exists $0 < r < 1$, such that

$$R = ln\frac{1 + r}{1 - r}.$$

This shows that for any given R, there exists a hyperbolic circle of radius R. □

Postulate 3.4. All right angles are congruent.

Proof. By Theorem 3.9, angle is an invariant in hyperbolic geometry. This is equivalent to saying that hyperbolic transformations are conformal. Therefore, the truth of the fourth postulate in Euclidean geometry extends to its truth in hyperbolic geometry. □

Before we proceed to the fifth postulate, we need to define parallel lines in the Unit Disk Model.

Definition 3.18. Two hyperbolic lines are called *parallel* if they do not intersect in the interior of $\mathbb{D}$ but share one point on the boundary of $\mathbb{D}$.

The following figure helps visualize parallel lines in the Unit Disk Model. In Figure 2, the cline with center O_2 is orthogonal to the cline with center O_1. The hyperbolic straight line BD has two parallel lines, O_1B and O_1D.

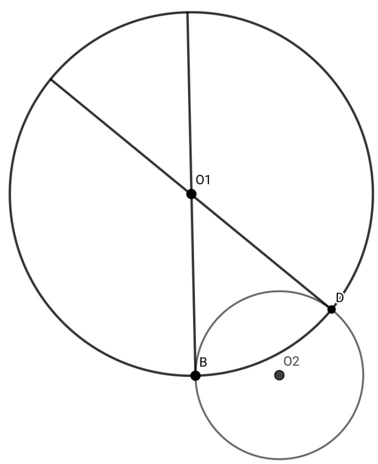

FIGURE 2.

Next, we show that Euclid's fifth postulate is inconsistent in hyperbolic geometry.

Postulate 3.5. Through a given point not on a given line interior to $\mathbb{D}$, there exists a unique line that is parallel to to the given line.

Proof. By Lemma 3.17, for any hyperbolic straight line C, there exists a transformation T in $\mathbb{H}$ such that $T(C)$ is the x-axis. This implies that if $z \in \mathbb{D}$ is a point not on C, then it must be true that Tz is a point not on the x-axis.

By Euclidean geometry, there exists a circle C_1 such that C_1 is a hyperbolic straight line passing through Tz and 1. C_1 has no other intersection with the x-axis. This means that C_1 is a hyperbolic straight line passing through Tz and parallel to the x-axis. Thus, $T^{-1}(C_1)$ is a hyperbolic straight line passing through z and parallel to C.

By the same construction, there exists a hyperbolic straight line C_{-1} passing through Tz and -1. C_{-1} is parallel to $T(C)$. Thus, $T^{-1}(C_{-1})$ is another hyperbolic straight line passing through z and parallel to C.

Since $T^{-1}(C_{-1}) \neq T^{-1}(C_1)$, we have constructed 2 hyperbolic straight lines passing through z and parallel to C.

This concludes the proof. □

3.2. **Poincaré's Upper-Half Plane Model.** We now briefly introduce the second model of Poincaré, the Upper-Half Plane Model. It models hyperbolic geometry with the underlying set $\mathbb{U}$ and the group of transformations $\overline{\overline{\mathbb{H}}}$. The set $\mathbb{U}$ is a subset of the complex plane $\mathbb{C}$, where

$$\mathbb{U} = z : Im(z) > 0.$$

The group of transformation $\overline{\overline{\mathbb{H}}}$ on $\mathbb{U}$ is denoted as

$$w = Tz = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{R},\ ad-bc > 0.$$

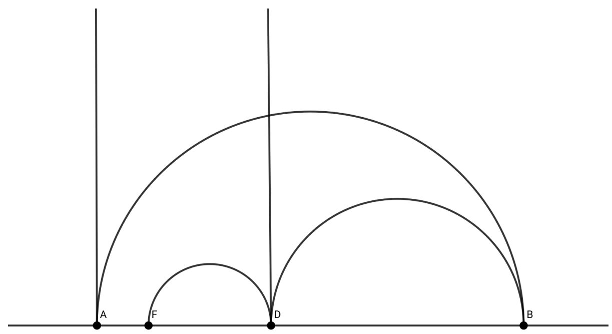

FIGURE 3. Straight lines in upper-half plane

By this definition, the Upper-Half Plane Model $(\mathbb{U}, \overline{\overline{\mathbb{H}}})$ models hyperbolic geometry. Next, we define the basic terms for hyperbolic geometry in the Upper-Half Plane Model.

Definition 3.19. A hyperbolic *straight line* is a Euclidean circular arc perpendicular to the x-axis or a vertical Euclidean straight line perpendicular to the x-axis.

Definition 3.20. In the upper-half plane, the *length* of a smooth curve γ with parametrization $z(t) = x(t) + iy(t)$ is given by

$$l(\gamma) = \int_a^b \frac{|z'(t)|}{y(t)} dt, \tag{3.21}$$

where $z'(t) = x'(t) + iy'(t)$.[1]

Definition 3.22. Hyperbolic *angle* is the same as Euclidean angle.

Definition 3.23. A hyperbolic *circle* is a portion of a Euclidean circle or straight line inside the unit disk that is entirely contained in $\mathbb{D}$.

4. Mapping Between Two Models

4.1. **Inversion.** Before we dive into the mapping between the two models, we introduce inversion as a specific form of transformation. We first define inversion of a point.

Definition 4.1. Given a circle C with center O and radius r. Let P be a point other than O. P' is called the *inversion* of P if it satisfies the following conditions:
1) $|OP| \cdot |OP'| = r^2$.
2) P' lies on the same side of O as P does.

Figure 4 depicts Q as the inversion of the point P with respect to the circle with center O. In particular, $|OP| \cdot |OQ| = |OD|^2$.

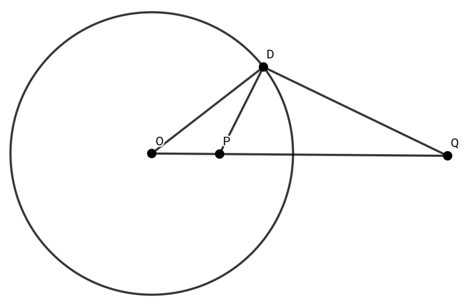

Figure 4.

Notice that when P coincides with the center of the circle O, Q will be sent to infinitely far away from P by this definition. Because of this exception, we include the point at infinity as the inversion of the center of the circle. Meanwhile, the inversion of the point at infinity is the center of the circle.

Here are some basic properties of inversion. Inversion is a transformation that maps:
a) A straight line containing center O into itself,
b) A straight line not containing center O into a circle through O,
c) A circle through center O onto a straight line not containing O,
d) A circle not through center O onto a circle not through O.

This mapping is essential to understanding the isomorphism between the Unit Disk Model and the Upper-Half Plane Model.

4.2. **Isomorphism.** This section discusses the isomorphism between the Unit Disk Model and the Upper-Half Plane Model. To find the isomorphic map between the two models, we first introduce the fundamental theorem of Möbius geometry.

Theorem 4.2. *There is a unique Möbius transformation that takes any 3 distinct extended complex numbers, z_1, z_2, and z_3 to any other 3 distinct extended complex numbers w_1, w_2, and w_3.*

To prove this theorem, we introduce the definition of fixed points of Möbius transformations.

Definition 4.3. A *fixed point* of a transformation T is a point z such that $Tz = z$.

Lemma 4.4. *A Möbius transformation with 3 or more fixed points must be the identity transformation.*

Proof. Let T be a Möbius transformation of the form $Tz = \frac{az+b}{cz+d}, (ad - bc \neq 0)$. Then, $Tz = z$ indicates that

$$cz^2 + (d-a)z - b = 0. \tag{4.5}$$

If $c \neq 0$, then equation (4.5) has 1 or 2 roots. This implies that transformation T has 1 or 2 fixed points in $\mathbb{C}$.

If $c = 0$ and $a \neq d$, then equation (4.5) has 1 solution, namely $z = \frac{b}{d-a}$. In this case, T has 2 fixed points, namely $\frac{b}{d-a}$ and ∞.

If $c = 0$ and $a = d \neq 0$, then $Tz = z + \frac{b}{d}$. If $b = 0$, then T has infinitely many fixed points. If $b \neq 0$, then T has a unique fixed point at infinity.

Therefore, the only case when T has more than 2 fixed points is when $a = d \neq 0$ and $b = c = 0$. In this case, T is the identity transformation. □

We may now proceed to the proof of Theorem 4.2.

Proof. For any 3 distinct points z_1, z_2, and z_3, there exists a Möbius transformation T, such that

$$Tz_1 = 1,\ Tz_2 = 0,\ Tz_3 = \infty. \tag{4.6}$$

The Möbius transformation that satisfies this condition is denoted of the form

$$Tz = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}.$$

For any 3 distinct points z_1, z_2, and z_3 and 3 distinct points w_1, w_2, and w_3, there exists a Möbius transformation U, such that

$$Uz_1 = w_1,\ Uz_2 = w_2,\ Uz_3 = w_3.$$

By equation (4.6), we know that there must exist transformations T and S, such that

$$Tz_1 = 1,\ Tz_2 = 0,\ Tz_3 = \infty,$$

and

$$Sw_1 = 1,\ Sw_2 = 0,\ Sw_3 = \infty.$$

Then, the Möbius transformation U of the form $U = S^{-1} \circ T$ satisfies the following:

$$Uz_1 = S^{-1} \circ Tz_1 = S^{-1}1 = w_1$$
$$Uz_2 = S^{-1} \circ Tz_2 = S^{-1}0 = w_2$$
$$Uz_3 = S^{-1} \circ Tz_3 = S^{-1}\infty = w_3$$

This shows that U is the required transformation.

Finally, we want to prove the uniqueness of U. To achieve that, we need to show that if U_1 and U_2 are Möbius transformations such that

$$U_i(z_j) = w_j,\ j = 1, 2, 3,\ i = 1, 2,$$

then $U_1 = U_2$.

To show this, consider the Möbius transformation $U_2^{-1} \circ U_1$. This transformation satisfies

$$U_2^{-1} \circ U_1 = U_2^{-1}(w_i) = z_i,\ i = 1, 2, 3.$$

This implies that z_1, z_2, z_3 are fixed points of the transformation $U_2^{-1} \circ U_1$. By Lemma 4.4, since $U_2^{-1} \circ U_1$ has at least 3 fixed points, it must be the identity transformation. This implies that $U_1 = U_2$. □

With the Fundamental Theorem of Möbius Geometry, we can now establish the isomorphism between the Unit Disk Model and the Upper-Half Plane Model of hyperbolic geometry.

Consider the transformation

$$w = Sz = i\,\frac{1+z}{1-z}. \tag{4.7}$$

Then S maps 3 distinct points on the Unit Disk $\mathbb{D}$ to other 3 distinct points on the Upper-Half Plane $\mathbb{H}$, namely

$$S(-1) = 0,\ S(0) = i,\ S(1) = \infty.$$

Figure 5 depicts the transformation S that maps the 3 points on the Unit Disk to the other 3 points on the Upper-Half Plane.

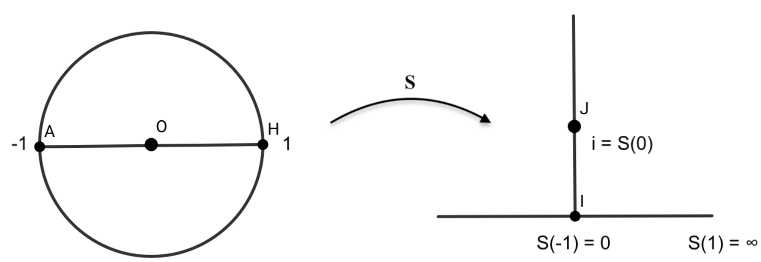

FIGURE 5. Map between Unit Disk $\mathbb{D}$ and Upper-Half Plane $\mathbb{H}$ [1]

By Theorem 4.2, S is the unique transformation that maps the Unit Disk $\mathbb{D}$ to the Upper-Half Plane $\mathbb{U}$. On the other hand, the inverse of S maps $\mathbb{U}$ to $\mathbb{D}$:

$$S^{-1}w = \frac{iw+1}{iw-1} \tag{4.8}$$

Thus, we have found the isomorphism between the two models of hyperbolic geometry.

To better understand the isomorphic map S, we analyze it through breaking down its individual steps. It is a Möbius transformation that combines translation, inversion, and dilation.

The first step entails translation. The term *1+z* in the numerator translates points in the unit disk by adding 1 to each point. This translation moves the center of the unit disk to the point -1 in the complex plane, which is the center of inversion.

The second step entails inversion. The term *1-z* in the denominator introduces inversion about the unit circle. Inversion in the Unit Disk Model is equivalent to reflection with respect to the boundary of $\mathbb{D}$. This step reflects the translated points across the unit circle according to the properties we have established about inversion in the previous section.

The third step entails dilation and rotation. The multiplication by i rotates the points obtained from inversion by 90 degrees counterclockwise and at the same time scales the points.

Remark 4.9. Both $(\mathbb{D}, \mathbb{H})$ and $(\mathbb{U}, \overline{\mathbb{H}})$ are models of the same abstract geometry, namely hyperbolic geometry.

4.3. **Why Isomorphism Preserves Invariant.** In the final section, we look at why the isomorphic map S is able to preserve the property of invariant between the Unit Disk Model to the Upper-Half Plane Model. More specifically, we will look at the invariant *length* for both models.

By Definition 3.3, we know that the length of a smooth curve γ in the Unit Disk Model is defined as

$$l(\gamma) = 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}, \tag{4.10}$$

where $z'(t) = x'(t) + iy'(t)$.

By Definition 3.20, we know that the length of a smooth curve γ in the Upper-Half Plane Model is

$$l(\gamma) = \int_a^b \frac{|z'(t)|}{y(t)} dt, \tag{4.11}$$

where $z'(t) = x'(t) + iy'(t)$.

In the remaining part of this paper, we derive equation (4.11) from equation (4.10) with isomorphism S. Then, we explain why isomorphism necessarily preserves invariant.

Proof of Equation (4.11) by (4.10)

Proof. Let $\gamma : z(t) = x(t) + iy(t)$, $(a \le t \le b)$ be a smooth curve in the upper-half plane. Then,

$$\hat{\gamma} : \hat{z}(t) = S^{-1}(z(t)) = \frac{iz(t)+1}{iz(t)-1}.$$

We have that $\hat{\gamma}$ is a smooth curve in the unit disk.

This implies that

$$|\hat{z}'(t)| = \frac{|(iz(t)-1)iz'(t) - (iz(t)+1)iz'(t)|}{|iz(t)-1|^2} = \frac{2|z'|}{|iz-1|^2}.$$

Thus, we have

$$l(\gamma) = 2\int_a^b \frac{|\hat{z}'|}{1-|\hat{z}|^2}\,dt = 2\int_a^b \frac{2|z'|}{|iz-1|^2 - |iz+1|^2}\,dt$$

$$= \int_a^b \frac{4|z'|}{(1+y)^2 + x^2 - (1-y)^2 - x^2}\,dt$$

$$= \int_a^b \frac{|z'|}{y} dt.$$

□

We have shown that the length formula of the Upper-Half Plane Model can be derived from the Unit Disk Model using the isomorphic map. This explores its significance in how isomorphism maintains the property of invariant.

Theorem 3.11 shows that length is an invariant in the Unit Disk Model. This can be denoted as $l(T(\gamma)) = l(\gamma)$, for any smooth curve γ in the unit disk and any transformation T in the hyperbolic group. We also know that the length of curve γ in the unit disk can be translated to a different length of curve γ' in the upper-half plane by the isomorphic map S. The parametrization $z'(t)$ of γ' can be expressed with respect to the parametrization $z(t)$ of γ as the following relationship:

$$z'(t) = S(z(t)),\ a \le t \le b.$$

Therefore, we have that $T(\gamma) = T(z(t)) = T(S^{-1}z'(t)) = (T \circ S^{-1})(z'(t))$.

Let transformation U denote the composition of transformations T and S^{-1}: $U = T \circ S^{-1}$. Transformation T is in the transformation group $\mathbb{H}$ denoted by equation (3.1), while transformation S^{-1} is the isomorphic map from the upper-half plane to the unit disk: $S^{-1} = \frac{iw+1}{iw-1}$. The resulting composition U can be denoted of the form

$$Uz = \frac{az+b}{cz+d},\ a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

This implies that U is an element of the transformation group $\overline{\mathbb{H}}$ on the upper-half plane $\mathbb{U}$.

By Theorem 3.11, since $l(T(\gamma)) = l(\gamma)$, then

$$l(U(\gamma')) = l(\gamma').$$

This concludes that isomorphism preserves length as invariant in different models of geometry.

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