NOTES ON CONFORMAL WELDING OF LIOUVILLE QUANTUM GRAVITY SURFACES

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Abstract. As a fundamental work in the field of random geometry, Sheffield [40] introduced the theory on conformal welding of Liouville quantum gravity surfaces, establishing the first rigorous connection between two canonical random fractal objects: SLE and LQG. This paper aims to be a self-contained notes on [40], covering discussions from discrete models and necessary preliminaries to general results about conformal welding of LQG surfaces in [7].

Contents

1. Introduction

In the past few decades, Liouville quantum gravity (LQG) and Schramm-Loewner Evolution (SLE) have played vital roles in physics and mathematics. Liouville quantum gravity, first introduced by Polyakov in 1981 [34] in the context of bosonic string theory, is a canonical model of a two-dimensional random surface. Schramm-Loewner evolution, first introduced by Schramm in 1999 [36], is a canonical two-dimensional random curve that does not cross itself. Both LQG and SLE enjoy a

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conformal structure, and have deep relationships with physics. In this article, we will review some fundamental theorems about conformal welding of LQG surfaces due to [40], which establishes the first rigorous connection between LQG and SLE. In particular, it justifies that the "right" relationships between γ (parameter of LQG) and κ (parameter of SLE) is $\kappa \in \{\gamma^2, 16/\gamma^2\}$.

Roughly speaking, an LQG surface with parameter γ (which we simply write as γ-LQG in the remaining part of this paper) is a random two-dimensional Riemannian manifold parameterized by a domain $D \subseteq \mathbb{C}$ whose Riemannian metric tensor is $e^{\gamma h}(dx^2 + dy^2)$. Here $dx^2 + dy^2$ is the Euclidean metric tensor, and h is some variant of the Gaussian free field (GFF), which can be heuristically thought of as an infinite Gaussian vector parameterized by D, with covariance given by the Green's function on D. Note that the above definition does not make literal sense since h is a distribution, and certain regularization techniques are required for a rigorous definition. Under the formal definition, an LQG surface would become an equivalence class of measure space endowed with a conformal structure, where two surfaces are in the same equivalence class if they differ by a conformal mapping. Note that certain LQG surfaces can also be equipped with a metric space structure or be identified with a mating of two continuum random trees, but these processes would be hard and involved and would not be the focal point of this article. Still, it's worth mentioning that (generalized versions) of the theorems in [40] are important contributors to [7], which justifies that some specific LQG surfaces are equivalent to the *peanosphere*, a mating of two continuum random trees.

SLE_κ is a one-parameter family of random fractal curves, and is uniquely characterized by its two fundamental properties, *conformal invariance* and *domain Markov property*. Heuristically speaking, the parameter κ characterizes the "speed" of the curve, and as κ grows larger, the curve becomes "windier" (the author learned this word from [42]) and the range of curve tends to be "denser". Note that the theory on conformal welding of LQG surfaces provides another perspective of certain SLE's, that is, the interface obtained from "gluing" two independent LQG surfaces.

The results in [40] can be interpreted in the following four aspects, where we always fix $\gamma \in (0, 2)$ and $\kappa = \gamma^2$. For simplicity, we assume that all the LQG surfaces in the following four aspects are parameterized by the upper half plane $\mathbb{H}$.

(1) **Stationarity property of SLE-decorated LQG (Theorem 4.1)**.

Suppose we start with a specific type of γ-LQG surface, decorated with an independent SLE_κ curve η_0 parameterized by half-plane capacity (a natural paramerization of SLE). The theorem says that, after we have explored the surface along $\eta_0([0, t])$, that is, some initial segment of the curve, the remaining surface decorated with the undiscovered curve has the same law (modulo a conformal mapping $\tilde{g}_t : \mathbb{H} \setminus \eta_0([0, t]) \mapsto \mathbb{H}$) as the original one. In other words, the curve-decorated surface has a stationarity property.

(2) **Zipping up one LQG surface along the SLE curve (Theorem 4.16 and Theorem 4.17); see Figure 3 for some visualization.**

In aspect (1), the conformal mapping $\tilde{g}_t^{-1} : \mathbb{H} \mapsto \mathbb{H} \setminus \eta_0([0, t])$ can almost surely be continuously extended to a homeomorphism on the martin boundary. Under this extended homeomorphism, boundary arcs of $\mathbb{H}$ are matched together, and the identified arc is $\eta_0([0, t])$. Therefore, as t changes, the dynamic exploration process can be intepreted as "zipping up" one γ-LQG surface along the SLE_κ curve, or in other words, welding the boundary arcs of one γ-LQG surface together into an

SLE_κ curve. Both Theorem 4.16 and Theorem 4.17 give a stationary dynamic zipping process of SLE_κ-decorated γ-LQG; see Figure 3 for some visualization. However, the underlying γ-LQG surfaces and the time parameterizations of the zipping process are not the same. In particular, one time paramerization is the natural parameterization of SLE_κ curve, that is, the capacity parameterization, and the other is the so called "quantum length parameterization" (which will be explained in aspect (3)). In light of this, the two stationary zippers in Theorem 4.16 and Theorem 4.17 will be respectively called the capacity zipper and the quantum zipper.

(3) **A natural random length measure of SLE (Theorem 4.25).**

A γ-LQG surface with metric tensor $e^{\gamma h}(dx^2 + dy^2)$ and parameterized by $\mathbb{H}$ can be endowed with a canonical quantum boundary length measure that can be heuristically denoted by $e^{\gamma h/2}d\lambda$, where $d\lambda$ is the Euclidean measure on the real line. During the zipping procedure in aspect (2), boundary segments of $\mathbb{H}$ are welded together into an SLE_κ curve. Theorem 4.25 and its corollary, Corollary 4.26, say that, almost surely, any two segments that are identified together have the same quantum boundary length. Therefore, the quantum boundary length measure $e^{\gamma h/2}d\lambda$ gives rise to a random length measure of the SLE_κ curve, and the "quantum length parameterization" in aspect (2) is actually obtained by measuring the units of quantum length being zipped.

(4) **Conformal welding of two LQG surfaces (Theorem 4.29); see Figure 6 for some visualization.**

We can also conformally weld two γ-LQG surfaces together, which may be viewed as an infinite version of the zipping process in aspect (2). Theorem 4.29 says that, if we glue together two specific independent γ-LQG surfaces $(\mathcal{W}_1, \mathcal{W}_2)$ along boundary segments in a quantum-boundary-length-preserving way and conformally map the glued surface into $\mathbb{H}$, then we will get a new γ-LQG surface $\mathcal{W}$ parameterized by $\mathbb{H}$, and the interface ζ will be an SLE_κ independent with $\mathcal{W}$. The independence between $\mathcal{W}$ and ζ means that the geometric structure of the combined surface yields no information of the conformal structure of the interface, which may be quite surprising at first glance. Another translation of the theorem is to slice $\mathcal{W}$ by ζ, and then get an independent pair of $(\mathcal{W}_1, \mathcal{W}_2)$. It is worth mentioning that $(\mathcal{W}_1, \mathcal{W}_2)$ and $(\mathcal{W}, \zeta)$ determine each other.

Although all the above contents, including the four aspects of our main theory, are presented in a continuum manner, LQG and SLE are intimately related with certain discrete models. We will give a brief explanation here; see Section 2 and Appendix A for details.

A planar map is a graph embedded in the sphere $\mathbb{C} \cup \{\infty\}$, viewed modulo orientation preserving homeomorphisms from the sphere to itself. Roughly speaking, LQG surfaces can be seen as continuum random surfaces weighted by the partition functions of statistical physics models in two-dimensional conformal field theory, and hence should be the scaling limits of random planar maps weighted by partition functions of discrete statistical physics models. In addition, because of its conformal invariance property, SLE also arises as a natural candidate of the scaling limit for interfaces of critical statistical physics model. Therefore, as a collection of the above two viewpoints, conformal welding of LQG surfaces can be interpreted as a scaling limit of the discrete results with respect to random planar maps decorated by interfaces of critical statistical physics models. This point of view also provides motivations and intuitions for our main theorems from the discrete side. In particular, it makes necessary the constructions of “scale-invariant” LQG surfaces that serve as natural candidates for the scaling limits. Moreover, the relation $\kappa = \gamma^2$ is also in line with the following coincidence: the critical statistical physics model that generates the interfaces on random planar maps is exactly the model that us used to reweight the planar maps.

We also mention here that there are generalizations and many related works of [40]. An example is the seminal paper [7] (see [9] for a survey on main results and applications of [7]). Indeed, [7] proves some generalized versions of the theorems in [40]. These generalized theorems play central roles in the mating-of-trees theorem, which lays foundations for the first rigorous convergence results for random planar maps to LQG surfaces. We note here that part of these results can be seen as the same results of [40], except that they are weighted on a different measure. In addition, [40] is also closely related with imaginary geometry introduced in the striking series of paper [27, 28, 29, 30].

This paper is organized as follows. Section 2 gives some motivations and intuitions of our main theorems from the discrete side. Section 3 gives the definitions and some basic propertied of GFF, LQG and SLE, and also introduces a scale-invariant LQG surface, the quantum wedge. Section 4 provides the statements, proofs and ideas of the main results, following the order of the four aspects above. Section 5 reviews some results of [7], which are generalizations of the main theorems in this paper. Appendix A is about the relationships between the continuum models with the discrete models, and can be seen as a complement of Section 2.

2. Motivations and intuitions from the discrete side

This section is devoted to giving some motivations and intuitions for some of our main theorems in Section 4. For more relationships between the continuum models and discrete models, see Appendix A.

We will study the discrete analog of the conformal welding theorem (Theorem 4.29), that is, gluing two random planar maps decorated by some critical statistical physics model; see Figure 1 for an illustration. We will specify the setting under the case $\kappa = 3/8$ and $\gamma = \sqrt{3/8}$. We remark here that $\mathrm{SLE}_{3/8}$ is believed to be the scaling limit of self-avoiding walk; see [20], while $\sqrt{8/3}$-LQG is believed to be (and proved in certain cases, see [14, 9]) the scaling limit of random planar maps decorated by a critical Bernoulli site percolation configuration. Note that in the latter case, if we fix the number of vertices of the planar map, then the conditional law of the underlying map is uniform. For more about scaling limit results with regard to self avoiding walks on random planar maps, see [5, 12].

Fix an integer n. Let D be a sampl of uniform triangulation with simple boundary, a marked point a on the outer face and exactly n edges, except that each edge on the outer face is counted as half an edge (since after gluing two boundary half edges will be identified with one interior edge). Denote by ℓ the boundary length of D. By the bijection in [4] we know that ℓ is of order $\sqrt{n}$. We further fix $\ell \in \{[\sqrt{n}], [\sqrt{n}] + 1, [\sqrt{n}] + 2\}$ such that $n + \ell$ is a multiple of 3, and by Euler's formula the number of vertices in D is then fixed to be $(n + \ell)/3 + 1$.

We can embed D into a two-dimensional manifold by endowing each triangle with the metric of an equilateral triangle with side length one. The manifold can further be endowed with counting measure on vertices of D normalized by the number of

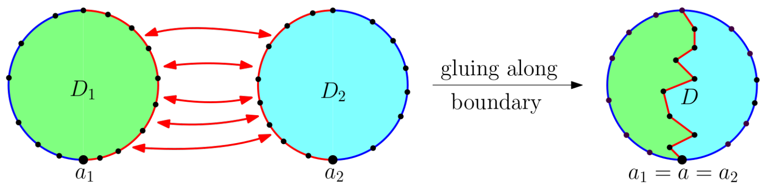

FIGURE 1. An illustration of gluing triangulations. **Left:** two independent uniform triangulations (D_1, a_1) and (D_2, a_2) with n edges and boundary length ℓ, which will be glued together along the boundary. **Right:** a uniform triangulation (D, a) with $2n$ edges, boundary length $2(\ell - r)$ and decorated with a self-avoiding walk started from a, obtained after gluing.

vertices. We then choose a conformal map ϕ that maps the above manifold to the upper half plane $\mathbb{H}$, under the requirement that $\phi(a) = 0$ and the scaling be fixed so that the area (under the pushforward measure) of unit (half) disk is a fixed integer $k < n$. We can now consider the limit distribution (in the sense of weak topology) of uniform distribution on (D, a) as k, n approach infinity in such a way that n/k tends to infinity, and this limit should be a scale-invariant $\sqrt{8/3}$-LQG surface.

Now consider two independent uniform triangulations (D_1, a_1) and (D_2, a_2) with n edges and boundary length ℓ. We uniformly choose r from $\{1, 2, \dots, \ell - 1\}$ and glue these two triangulations together along a boundary segment of length r, under the additional requirement that a_1 and a_2 is identified. Then we get a new triangulation D with $2n$ edges, a simple boundary with length $2(\ell - r)$, one marked point $a = a_1 = a_2$ on the outer face, and one interface η obtained by identifying two boundary arcs; see Figure 1. In addition, conditioned on (D, a), the conditional distribution of η should be that of a self-avoiding walk conditioned to connect a with another boundary point and have no more intersections with the outer face. We then consider the scaling limit of (D, a, η) (in the same sense as in the last paragraph), which (if exists) should again be a scale-invariant LQG surface decorated by a continuous non-crossing curve. Note that there is no reason to believe that the parameter of the new LQG surface is $\sqrt{8/3}$ since the marginal distribution of D is not uniform. That being said, due to a universality conjecture (see Appendix A for detail), the limit curve should still be an $\mathrm{SLE}_{8/3}$ independent with the LQG surface. Furthermore, the two parts sliced from the limit surface by the limit curve should be the scaling limit of (D_1, a_1) and (D_2, a_2), that is, two independent scale-invariant $\sqrt{8/3}$-LQG surfaces.

The flavor of the quantum zipper theorem (Theorem 4.17) is like the following. Since r is uniform in $\{1, 2, \dots, \ell - 1\}$, for any (random) integer m, the law of r and $r + m$ are approximately the same as long as $m/\sqrt{n}$ is (with high probability) small. Therefore, in the welding procedure of last paragraph, if we zip up (or zip down when $m < 0$) m more steps (if possible), then the resulting surface-interface pair should approximately maintain the same law. We expect a similar property to hold in the continuum case as in Theorem 4.17.

3. GFF, LQG and SLE overview

In this section we introduce the necessary background for our main theorems, including GFF, LQG and SLE. We will give the precise definitions and some important properties related with these random objects. We may provide some heuristic ideas or key observations, but detailed proof of the properties would not be given. See [39], [2] and [3] for more detailed discussions.

3.1. **Gaussian Free Field.**

3.1.1. *Zero-boundary GFF.* We say a domain $D \subseteq \mathbb{C}$ is **regular** if Brownian motion starting from any $x \in \partial D$ a.s. hits D^c instantaneously. In this subsection, we always assume that D is a proper regular domain of $\mathbb{C}$. Roughly speaking, we want to define the zero-boundary GFF on D as an infinite Gaussian vector $(h(z))_{z\in D}$, with its covariance structure given by

$$\mathbb{E}[h(z)h(w)] = G_0^D(z,w), \tag{3.1}$$

where $G_0^D(x,y)$ is the Green's function on D with zero boundary conditions. However, since $G_0^D(z,w)$ has a log-singularity on the diagonal, the zero-boundary GFF can only be understood as a random generalized function on D.

For generalized functions (which can be equivalently seen as measures) ρ_1, ρ_2, we set

$$\Gamma_0(\rho_1,\rho_2) := \frac{1}{2\pi}\iint_{D^2} G_0^D(z,w)\rho_1(z)\rho_2(w)\, d^2z\, d^2w.$$

Denote by $\mathcal{M}_0(D)$ the set of generalized functions ρ such that $\Gamma_0(\rho,\rho)$ is finite. Indeed, one can prove that $\mathcal{M}_0(D)$ is the Sobolev space $\mathcal{H}_0^{-1}(D)$ (the definition of it would be explained later). In light of (3.1), it is then natural to have the following definition (which is also a theorem) of zero-boundary GFF, that is, a Gaussian process indexed by $\mathcal{M}_0(D)$ with covariance sturcture given by Γ_0.

Definition 3.2 (Zero boundary GFF)**.** There exists a *unique* stochastic process $(h,\rho)_{\rho\in\mathcal{M}_0(D)}$, which we call **zero-boundary Gaussian free field** or **Dirichlet-boundary Gaussian free field**, such that for every $\rho_1,\dots,\rho_n \in \mathcal{M}_0(D)$, the random vector $((h,\rho_1),\dots,(h,\rho_n))$ is a centered Gaussian vector with covariance structure given by

$$\mathbb{E}[(h,\rho_1)(h,\rho_2)] = \Gamma_0(\rho_1,\rho_2) = \frac{1}{2\pi}\iint_{D^2} G_0^D(z,w)\rho_1(z)\rho_2(w)\, d^2z\, d^2w. \tag{3.3}$$

We can show existence by Komolgorov's extension theorem and uniqueness by the fact that a stochastic process is uniquely characterized by its finite-dimensional marginals.

Note that Definition 3.2 does not give much information about zero-boundary GFF; it just claim that such a stochastic process exists and is unique. We then try to seek a explicit and more doable form of (an instance of) zero-boundary GFF.

Let Sobolev space $\mathcal{H}_0^1(D)$ be the Hilbert space completion of the set of smooth, compactly supported functions on D with respect to the **Dirichlet inner product**,

$$(\varphi,\psi)_\nabla = \frac{1}{2\pi}\int_D \nabla\varphi(z)\cdot\nabla\psi(z)\, d^2z. \tag{3.4}$$

The Sobolev space $\mathcal{H}_0^{-1}(D)$ is both the image of $\mathcal{H}_0^1(D)$ under Laplacian and the dual space (or space of continuous linear functionals) on $\mathcal{H}_0^1(D)$. For any $\rho_1, \rho_2 \in$

$\mathcal{D}_0(D)$, we can find $\varphi_1, \varphi_2 \in \mathcal{H}_0^1(D)$ such that $-\Delta\varphi_i = 2\pi\rho_i$ for $i = 1, 2$. Then by Gauss-Green formula, or essentially integration by parts, we have

$$\frac{1}{2\pi}\iint_{D^2} G_0^D(z,w)\rho_1(z)\rho_2(w)\, d^2z\, d^2w = (\varphi_1, \varphi_2)_\nabla.$$

Therefore, if we set for $i = 1, 2$

$$(h, \varphi_i)_\nabla := (h, \rho_i),$$

then (3.3) becomes

$$\mathbb{E}[(h,\varphi_1)_\nabla (h,\varphi_2)_\nabla] = (\varphi_1, \varphi_2)_\nabla, \tag{3.5}$$

which strongly suggests the following proposition.

Proposition 3.6 (Zero-GFF as a random Fourier series)**.** *Let $X_n, n \geq 1$ be i.i.d standard one-dimensional Gaussian random variables and $\{e_n\}_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}_0^1(D)$. Set for $N \geq 1$*

$$h_{(N)} := \sum_{n=1}^{N} X_n e_n.$$

Then for any $\rho \in \mathcal{H}_0^{-1}(D)$, $(h_{(N)}, \rho)$ converges almost surely and in $L_2(\mathbb{P})$. Denote the limit by (h, ρ) (where we abuse $(\cdot,\cdot)$ for L^2 inner product). Then (h, ρ) is a centered one-dimensional Gaussian random variable with variance $\Gamma_0(\rho, \rho)$. We can thus construct and view (an instance of) zero-boundary GFF as a formal sum

$$h := \lim_{N\to\infty} h_{(N)} = \sum_{n=1}^{\infty} X_n e_n. \tag{3.7}$$

Proposition 3.6 can be proved by martingale convergence theorem. Note that for any $\rho \in \mathcal{H}_0^{-1}$,the law of (h, ρ) does not depend on the choice of the orthonormal basis $\{e_n\}_{n\geq 1}$.

Remark 3.8. Indeed, for any $\varepsilon > 0$, one can show that the random Fourier series (3.7) converges almost surely in the Sobolev space $\mathcal{H}_0^{-\varepsilon}(D)$. Moreover, one can show that the law of the limit does not depend on the choice the orthonormal basis $\{e_n\}_{n\geq 1}$. We can then try to define the zero-boundary GFF as this limit, which has the following benefits compared with Definition 3.2:

(1) one obtain an explicit form of zero-boundary GFF;
(2) one can make use of properties of Sobolev spaces, including separability, decomposition property and certain amount of regularity of its elements.

However, it is worth noting this new definition requires more regularity in ρ. In particular, by law of large numbers, the random series (3.7) almost surely does *not* converge in $\mathcal{H}_0^1$, so we cannot treat all $\rho \in \mathcal{H}_0^{-1}$ at the same time. That being said, Definition 3.2 and the definition above are still consistent on the intersection of their domains (which can be taken as $\mathcal{H}_0^\epsilon$ for any $\epsilon > 0$), and we can also take advantage of (3.7) (at least as a formal sum) when using Definition 3.2 as our definition. The only caveat is that, if we want to treat zero-boundary GFF as an element in a Sobolev space with negative order but also consider its L^2 product with functions in $\mathcal{H}_0^{-1}$ at the same time, then some justification is needed. Such nontrivial justification will be manageable in every case we encounter, and in this article we will omit their proofs. From now on, the readers can forget about the subtlety and exploit the merits of two definitions above at the same time.

Remark 3.9. We choose the term "zero-boundary GFF" because of the choice of Green's function with zero boundary conditions in Definition 3.2 and because one can choose the orthonormal basis $\{e_n\}_{n\geq 1}$ as eigenfunctions of the $-\Delta$ on D, with zero or Dirichlet boundary conditions.

We than provide some main properties of zero-boundary GFF. One important property is the conformal invariance.

Proposition 3.10 (Conformal invariance)**.** *The law of zero-boundary GFF is conformally invariant, meaning that if h is the zero-boundary GFF on D and $\phi : D' \to D$ is a conformal map, then $h \circ \phi$ is the zero-boundary GFF on D'.*

Proof. Proposition 3.10 follows directly from the conformal invariance of Green's function with zero boundary condition or Dirichlet inner product. □

Another important property about zero-boundary GFF is related with circle averages. In the remaining part of this paper, we always write $B(z, r)$ as the disk of radius r around z. Let $\rho_{z,r}$ be the uniform measure on $\partial B(z, r)$. Then $\rho_{z,r} \in \mathcal{H}_0^{-1}$ (here we also see $\rho_{z,r}$ as a generalized function). Therefore, one can also define for $r < \mathrm{dist}(z, \partial D)$

$$h^{(r)}(z) := (h, \rho_{z,r}).$$

We then have the following proposition.

Proposition 3.11 (Circle averages)**.** *Fix $z \in D$. Let $0 < r_0 < dist(z, \partial D)$ and $t_0 = -\log r_0$. For $t \geq t_0$, set*

$$B_t := h^{(e^{-t})}(z).$$

Then $(B_t)_{t\geq t_0}$ has the law of Brownian motion started from B_{t_0}.

Proof. Proposition 3.11 follows from computing the covariance structure by (3.3). The key intuition here is that the speed of Brownian motion should match the order of log-singularity at the diagonal of "covariance matrix" of h, which is one in this case due to property of Green's function with zero boundary conditions. □

It is shown in [8] that there exists a modification of h such that $h^{(r)}(z)$ is a.s. jointly Hölder continuous of order $\frac{1}{2}$-, which is the same order as that of a standard Brownian motion. In the remaining part of this paper, we always assume that h has been replaced by such a modification, making the B_t in Proposition 3.11 truly a Brownian motion.

The last important property is the decomposition of zero-boundary GFF. We explain the rough idea here, and will omit the technical proof. In view of (3.7), if we decompose the Sobolev space $\mathcal{H}_0^1(D)$ as the direct sum of two orthogonal subspaces $\mathcal{H}^1$ and $\mathcal{H}^2$, then in Proposition 3.6 we can choose an orthonormal basis of $\mathcal{H}_0^1(D)$ as the the disjoint union of an orthonormal basis of $\mathcal{H}^1$ and an orthonormal basis of $\mathcal{H}^2$. By doing this, we can view h as sum of two independent formal sums, which are "projections" of h onto $\mathcal{H}^1$ and $\mathcal{H}^2$.

Proposition 3.12 (Markov Property)**.** *Suppose h is a zero-boundary GFF on D. Fix $U \subseteq D$ open. Then we can write $h = h_{supp} + h_{harm}$, where h_{supp} is zero outside U and a zero-boundary GFF in U, h_{harm} is harmonic in U, and h_{supp} and h_{harm} are independent.*

Proof. Proposition 3.12 follows from the decomposition

$$\mathcal{H}_0^1(D) = \mathcal{H}_0^1(U) \oplus \mathrm{Harm}(U),$$

where $\mathcal{H}_0^1(U)$ is Sobolev space on U (whose definition is similar to that of $\mathcal{H}_0^1(D)$), and $\mathrm{Harm}(U)$ consists of those generalized functions in $\mathcal{H}_0^1(D)$ that are harmonic on U. □

3.1.2. *Free-boundary GFF and whole-plane GFF.* Free-boundary GFF and whole-plane GFF share many similarities with zero-boundary GFF. However, they are both defined modulo constants, while zero-boundary GFF is not. For simplicity, we only state results about free-boundary GFF case since it is our focus in the main theorems, but one can also adapt them to the whole-plane GFF case. In this subsection, we still assume that D is a proper regular domain of $\mathbb{C}$. Note that the whole-plane GFF case actually corresponds to $D = \mathbb{C}$

Roughly speaking, the free boundary GFF on D is an infinite Gaussian vector $(h(z))_{z\in D}$, with its covariance structure given by

$$\mathbb{E}[h(z)h(w)] = G^D(z,w), \tag{3.13}$$

where $G^D(z,w)$ a (choice of) Green function with Neumann boundary conditions on D (or in the whole-plane GFF case, $-2\pi \log|z-w| + C$ with C an arbitrary constant). However, since G^D is defined only modulo constants, we can a priori only define free-boundary GFF (or whole-plane GFF) as random generealized funtions **modulo constants**. To be precise, let

$$\Gamma(\rho_1,\rho_2) := \frac{1}{2\pi}\iint_{D^2} G^D(z,w)\rho_1(z)\rho_2(w)\, d^2z\, d^2w. \tag{3.14}$$

Denote by $\bar{\mathcal{M}}(D)$ the set of generalized functions $\bar\rho$ such that $\Gamma(\bar\rho,\bar\rho)$ is finite and $\int_D \bar\rho(z)\, d^2z = 0$. Also define

$$\mathcal{M}(D) := \{\rho = \bar\rho + \varphi : \bar\rho \in \bar{\mathcal{M}}(D) \text{ and } \varphi \text{ is smooth, compactly supported in } D\}.$$

A priori, we can only define a free-boundary GFF as a Gaussian process indexed by $\bar{\mathcal{M}}(D)$. To solve this problem, we need to fix $\rho_0 \in \mathcal{M}(D) \setminus \bar{\mathcal{M}}(D)$, declare $(h,\rho_0) = 0$, and then use linearity to give definitions for other $\rho \in \mathcal{M}$.

Definition 3.15 (Free-boundary GFF)**.** There exists a *unique* stochastic process $(h,\bar\rho)_{\bar\rho\in\bar{\mathcal{M}}(D)}$, which we call **free-boundary Gaussian free field** or **Neumann-boundary Gaussian free field**, such that for every $\bar\rho_1,\dots,\bar\rho_n \in \mathcal{M}_0(D)$, the random vector $((h,\bar\rho_1),\dots,(h,\bar\rho_n))$ is a centered Gaussian vector with covariance structure given by

$$\mathbb{E}[(h,\bar\rho_1)(h,\bar\rho_2)] = \Gamma(\bar\rho_1,\bar\rho_2) = \frac{1}{2\pi}\iint_{D^2} G^D(z,w)\bar\rho_1(z)\bar\rho_2(w)\, d^2z\, d^2w. \tag{3.16}$$

Suppose we declare $(h,\rho_0) = 0$ for some $\rho_0 \in \mathcal{M}(D) \setminus \bar{\mathcal{M}}(D)$, then we can define

$$(h,\rho) = (h, \rho - \frac{\int_D \rho(z)\, d^2z}{\int_D \rho_0(z)\, d^2z}\rho_0)$$

for each $\rho \in \mathcal{M}(D)$.

With the help of Gauss-Green formula, we can also get the counterpart of Definition 3.2. We say two smooth functions are **equivalent modulo constants** if their difference is a constant function. Let Sobolev space $\bar{\mathcal{H}}^1(D)$ be the Hilbert space completion of the set of smooth (not necessarily compactly supported, compared with $\mathcal{H}_0^1(D)$ in Section 3.1.1) functions on D with respect to the Dirichlet inner product (3.4), defined modulo constants. Note that we define $\bar{\mathcal{H}}^1(D)$ modulo constants to make $(\cdot,\cdot)_\nabla$ really be an inner product. In one word, (an instance of) free-boundary GFF is a random Fourier series as in (3.7), except that $\{e_n\}_{n\geq 1}$ is now an orthonormal basis of $\bar{\mathcal{H}}^1(D)$.

Proposition 3.17 (Free-boundary GFF as a random Fourier series)**.** *Let $X_n, n \geq 1$ be i.i.d standard one-dimensional Gaussian random variables and $\{e_n\}_{n\geq 1}$ be an orthonormal basis of $\bar{\mathcal{H}}^1(D)$. Then (an instance of) free-boundary GFF can be expressed as*

$$h = \sum_{n=1}^{\infty} X_n e_n, \tag{3.18}$$

where we see (3.18) as a formal sum in a similar way as Proposition 3.6.

Remark 3.19. Similar to the zero-boundary GFF case, the formal sum (3.7) does converge almost surely in the Sobolev space $\bar{\mathcal{H}}^{-\varepsilon}(D)$ for any $\varepsilon > 0$, so we may think of free-boundary GFF as an element in $\bar{\mathcal{H}}^{-\varepsilon}$. Similar to Remark 3.9, this point of view will bring both benefits and concerns, and in this paper we will also forget about the necessary but subtle justification.

Remark 3.20. In defining $\bar{\mathcal{H}}(D)$, we can equally start with smooth funcions on D with Neumann-boundary conditions, and end up with the same space after taking the closure with respect to $(\cdot,\cdot)_\nabla$. In particular, the orthonormal basis $\{e_n\}_{n\geq 1}$ can be chosen as the eigenfunctions of Laplacian with Neumann boundary conditions.

In parallel with Proposition 3.10, Proposition 3.11 and Proposition 3.12, we can get properties of free-boundary GFF (and whole-plane GFF) with regard to conformal invariance, circle averages and domain Markov property. Note that we need to deal with different boundary conditions when giving the Markov property, and usually we make h_{supp} still a zero-boundary GFF on U and include all the boundary conditions on h_{harm}, which is defined modulo constants. We can also handle the boundary conditions via a "taking the even part" trick, which will be discussed in the next subsection.

We choose to offer two other fundamental properties here. Note that in all these cases the free-boundary GFF are viewed mololo constants, and one should proceed with extra caution when the constant is fixed; see Remark 3.23.

One important property of free-boundary GFF is the semicircle average property. Suppose h is a free-boundary GFF on D whose boundary has a linear segment L. Let $z \in L$. Suppose for $r \leq r_0$ we have $D \cap \partial B(z,r) \subseteq L$. For $r \leq r_0$, let $\rho_{z,r}$ be the uniform measure on the semicircle $D \cap \partial B(z,r)$. We can then define for $r \leq r_0$

$$h^{(r)}(z) := (h, \rho_{z,r} - \rho_{z,r_0}).$$

Similar to Section 3.1.1, we always assume that h has been replaced by a nice modification such that $h^{(r)}(z)$ is continuous in z and r. We then have the following proposition.

Proposition 3.21. *Fix $L \subseteq D$ and $z \in L$. Set $t_0 = -\log r_0$. For $t \geq t_0$, set*

$$B_t := h^{(e^{-t})}(z).$$

Then $(B_{t-t_0})_{t\geq t_0}$ is $\sqrt{2}$ times of a standard Brownian motion.

Proof. Proposition 3.21 follows from computing the covariance function by (3.16). Note that different with Proposition 3.11, the Green's function with Neumann boundary conditions now has a log-singularity of order 2 at (z, z), which explains the choice of the parameter $\sqrt{2}$ (Here we also note that the Green's function with zero boundary conditions does not have this property). □

The following proposition (which can also be generalized to the whole-plane GFF case) has a similar flavour as Proposition 3.12.

Proposition 3.22 (Radial decomposition for half plane)**.** *Let h be a free-boundary GFF (viewed modulo constants) on the half plane $\mathbb{H}$. Then we can write $h = h_{rad} + h_{circ}$, where h_{rad} (resp. h_{circ}) is constant (resp. has average zero) on each semicircle centered at zero, and h_{rad} and h_{circ} are independent. Note that h_{rad} is defined modulo constants, while h_{circ} has additive constants fixed.*

Proof. Proposition 3.22 follows from the decomposition

$$\bar{\mathcal{H}}(\mathbb{H}) = \bar{\mathcal{H}}^R(\mathbb{H}) \oplus \bar{\mathcal{H}}^C(\mathbb{H}),$$

where $\bar{\mathcal{H}}^R(\mathbb{H})$ (resp. $\bar{\mathcal{H}}^C(\mathbb{H})$) are space of functions in $\bar{\mathcal{H}}$ that are constant (resp. have average zero) on each semicircle centered at zero. □

Remark 3.23. Although it is sometimes helpful to specify the additive constant of the free-boundary GFF, one should take care with conformal invariance, circle averages and the ecomposition results. Let h be a free-boundary GFF on D and we declare $(h, \rho_0) = 0$ for some $\rho_0 \in \mathcal{M}(D) \setminus \bar{\mathcal{M}}(D)$. Then the following holds (Note that we state (3) and (4) in the setting of Proposition 3.22, and similar results also hold for other decompositions).

(1) If ϕ is a conformal map, then the additive constant of $h \circ \phi$ is also fixed. In particular, if ϕ is a conformal map from D to itself, then h and $h \circ \phi$ (viewed as true random distributions) may *not* be equal in law.
(2) For $z \in D$ (resp. $z \in \partial D$), the circle averages (resp. semicircle averages) $(h, \rho_{z,e^{-t}})$ evolves like a standard Brownian motion ($\sqrt{2}$ times a standard Brownian motion) *plus* a random constant $(h, \rho_{z,1})$, where $(h, \rho_{z,1})$ may *not* be independent with the Brownian motion.
(3) In the setting of Proposition 3.22, h_{rad} and h_{circ} may *not* be independent. To be more specific, the additive constant of h_{rad} is also fixed (according to ρ_0), but this constant may be random and may depend on h_{circ}.
(4) However, if ρ_0 is well-chosen, then h_{rad} and h_{circ} can be truly independent. For example, we can let ρ_0 be radially symmetric, that is, $\rho_0(z) = \rho(|z|)$ for some generalizaed function ρ on $\mathbb{R}$. In this case, the random additive constant of h_{rad} is independent with h_{circ} since $(h_{circ}, \rho_0) = 0$.

3.1.3. *Mixed boundary conditions and some technical results.* This part could be skipped on a first reading. Its main purpose is to introduce the notion of GFF with mixed boundary conditions, to derive relationships with free-boundary GFF, and to state some related technical results that will be useful in the proofs of our main theorems.

To be concise, the definition of GFF with mixed boundary conditions is given as a Fourier series (which is similar to (3.7) and (3.18)), and we omit the couterpart to Definition 3.2 and Definition 3.15 here.

Definition 3.24. Suppose $D \in \mathbb{C}$ is a regular domain and $\partial D = \partial^Z \cup \partial^F$ where $\partial^Z \neq \emptyset$ and $\partial^Z \cap \partial^F = \emptyset$. The GFF on D with zero boundary conditions on ∂^Z and free boundary conditions on ∂^F is constructed using a series expansion as in (3.18), except that the space $\mathcal{H}^1(D)$ is replaced with the Hilbert space closure with respect to $(\cdot,\cdot)_\nabla$ of the subspace of smooth functions on D which have an L^2 gradient and vanishes on ∂^Z.

Note that GFF with mixed boundary conditions is *not* defined modulo constants.

We now introduce the "taking the odd/even part" trick, which helps us to relate GFF with different boundary conditions together. Given an instance of whole-plane GFF h (defined modulo constants), if we define (in the sense of distribution)

$$h^O(z) := \frac{1}{\sqrt{2}}(h(z) - h(\bar{z})), \quad h^E(z) := \frac{1}{\sqrt{2}}(h(z) + h(\bar{z})), \tag{3.25}$$

then h^O and h^E are independent projections of h onto complementary orthogonal spaces complemetary orthogonal spaces. Recall that the covariance structure of a zero-boundary GFF on $\mathbb{H}$, an (instance of) free-boundary GFF on $\mathbb{H}$ and an (instance of) whole-plane GFF can be respectively given by (using $\bar{z}$ to denote the conjugate of z)

$$G_0^{\mathbb{H}}(z,w) = -\log|z-w| + \log|z-\bar{w}|, \quad (z,w) \in \mathbb{H}^2;$$
$$G^{\mathbb{H}}(z,w) = -\log|z-w| - \log|z-\bar{w}|, \quad (z,w) \in \mathbb{H}^2;$$
$$G^{\mathbb{C}}(z,w) = -\log|z-w|, \quad (z,w) \in \mathbb{C}^2.$$

Therefore, it follows from calculations with respect to Green's functions that the restriction of h^O (resp. h^E) to $\mathbb{H}$ has the same law as the zero-boundary GFF (resp. free-boundary GFF, defined modulo constants) on $\mathbb{H}$. In other words, a zero-boundary GFF (resp. free-boundary GFF) on $\mathbb{H}$ is the odd/even part of a whole-plane GFF.

Using the similar technique, we can also show that the "even part" of a Dirichlet GFF on $B(0,1)$ is a GFF on $\mathbb{H} \cap B(0,1)$ with zero boundary conditions on $\mathbb{H} \cap \partial B(0,1)$ and free boundary conditions on $(-1,1)$.

The following proposition is about another decomposition of free-boundary GFF on $\mathbb{H}$, and is an application of the above facts.

Proposition 3.26. *Suppose that h is a free-boundary GFF on $\mathbb{H}$ (viewed modulo constants). Let $\mathbb{D}^+ := \mathbb{H} \cap B(0,1)$. Then h can be decomposed into an independent sum of h^{ZF} and h_{harm}, where h^{ZF} is a GFF on $\mathbb{D}^+$ with zero boundary conditions on $\mathbb{H} \cap \partial\mathbb{D}^+$ and free boundary conditions on $(-1,1)$, and h_{harm} is a harmonic function (viewed modulo constants) on $\mathbb{D}^+$ with free boundary conditions on $(-1,1)$.*

Proof. Proposition 3.26 follows directly from taking the even part of the Markov property with regard to the whole-plane GFF (counterpart of Proposition 3.12). Note that the even part of a harmonic function has free bonndary conditions on $(-1,1)$. □

The following proposition is an application of Proposition 3.26, and will be useful in Section 3.3 and Section 4. Roughly speaking, it says that for a free-boundary GFF h on $\mathbb{H}$, its behavior close to the origin is approximately independent with that far away from the origin. Moreover, if the additive constant of h is fixed so that $(h,\rho_0) = 0$ for some $\rho_0 \in \mathcal{M}(\mathbb{H}) \setminus \bar{M}(\mathbb{H})$, then different choices of ρ_0 will produce the same behavior at local neighbourhoods of the origin as long as ρ_0 is bounded away from the origin, which is not surprising given Proposition 3.26.

Proposition 3.27. *Let h be a free-boundary GFF on $\mathbb{H}$. Fix $\rho_0 \in \mathcal{M}(\mathbb{H}) \setminus \bar{M}(\mathbb{H})$ that is compactly supported outside of $\mathbb{D}^+$ and declare $(h, \rho_0) = 0$. Let h^{ZF} be a GFF on $\mathbb{D}^+$ such that it is independent with h and has zero boundary conditions on $\mathbb{H} \cap \partial\mathbb{D}^+$ and free boundary conditions on $(-1, 1)$. Then the total variation distance between the law of*

$$(h|_{\mathbb{H}\setminus\mathbb{D}^+}, h|_{\delta\mathbb{D}^+}) \quad \text{and} \quad (h|_{\mathbb{H}\setminus\mathbb{D}^+}, h^{ZF}|_{\delta\mathbb{D}^+})$$

tends to zero as $\delta \to 0$.

Proof. Given Proposition 3.26, the proof of Proposition 3.27 actually boils down to controlling the Radon-Nikodym derivative of $(h^{ZF} + h_{harm})|_{\delta\mathbb{D}^+}$ with respect to $h^{ZF}|_{\delta\mathbb{D}^+}$ (with h_{harm} being the same as in Proposition 3.26); see [2, Corollary 5.37] for detail. Note that the condition on ρ_0 ensures that h_{harm} is independent with h^{ZF}; see Remark 3.23. □

A corollary of Proposition 3.27 is the zero-one law of free-boundary GFF, which has the same flavour as the Blumenthal zero-one law for Brownian motion.

Proposition 3.28. *Suppose that h is a free boundary GFF on $\mathbb{H}$ with arbitrary fixed additive constant and $x \in \mathbb{R}$. For each $\delta > 0$, let $\mathcal{F}_{x,\delta}$ be the σ-algebra generated by the restriction of h to $\mathbb{H} \cap B(x, \delta)$. Then $\cap_{\delta>0}\mathcal{F}_{x,\delta}$ is trivial.*

Proposition 3.28 is also directly proved in [7, Lemma 7.2].

3.2. **Liouville Quantum Gravity.** Suppose h is a zero-boudary GFF or free-boundary or whole-plane GFF (with arbitrary fixed additive constant) on a open regular set $D \subseteq \mathbb{C}$. Roughly speaking, a γ-LQG surface is the random two-dimensional Riemannian manifold parameterized by a domain $D \subseteq \mathbb{C}$ whose Riemannian metric tensor is $e^{\gamma h}(dx^2 + dy^2)$, where $\gamma \in [0, 2)$ is a fixed parameter. To be more specific, we will consider the LQG area measure on D, which should be intuitively defined by $\mu_h(dz) := e^{\gamma h(z)}dz$. However, h is very rough, so we have to somehow mollify h and let the oscillations cancel with each other. In light of Proposition 3.11 and its counterparts, we consider for $\varepsilon > 0$ the following approximation procedure

$$\mu_\varepsilon(dz) := e^{\gamma h^{(\varepsilon)}(z)}\varepsilon^{\gamma^2/2}dz, \tag{3.29}$$

where $h^{(\varepsilon)}(z)$ is a jointly continuous version of the circle averages. Note that here we choose the scaling constant $\gamma^2/2$ because $\mathbb{E}[e^{\gamma h^{(\varepsilon)}(z)}\varepsilon^{\gamma^2/2}]$ is a constant for every $z \in D$. The following is both a theorem and a definition of LQG area measure.

Definition 3.30 (LQG area measure)**.** Let D, h, μ_ε be defined as above. It is almost surely the case that as $\varepsilon \to 0$, the measures μ_ε converges weakly in D to a limiting measure, which we call **LQG area measure** and denote by $\mu = \mu_h = e^{\gamma h(z)}dz$. This remains true if we replace h with a non-centered GFF on D, that is, if we replace h with $h + f$ where f is a deterministic, non-zero continuous function on D. Note that if h is a free-boundary GFF or whole-plane GFF and we do not fix the additive constant of h, then μ_h is only defined modulo multiple constants.

[8] shows that the limit measure exists almost surely if ε is restricted to powers of two, and [43] extends the result to unrestricted case. The proof of existence of limit measure is somewhat technical, and requires a deep understanding of the continuity properties of the map $z \to h^{(\varepsilon)}(z)$.

Heuristically, under $\mu_h(dz)$ (which is intuitively $e^{\gamma h(z)}dz$), points where h is big correspond to relatively big portions of the surface, while points where h is small correspond to relatively small portions of the surface. The first kind of points are "typical" points under the measure μ_h, while the second kind of points are points that geodesics tend to to through.

In this paper, we will focus more on the first kind of points, which are called *thick points* of the Gaussian free field h. Suppose $U \subseteq D$ is a bounded domain and z is a random point sampled according to $\mu_h(U)$ (normalized to be a probability measure). (Note that $\mathbb{E}(\mu_h(U))$ is finite, making $\mu_h(U)$ almost surely finite.) It turns out in Proposition 3.31 that conditioned on z, h has an extra log-singularity of order γ at z, or in other words, z is a γ-*thick point* (see Remark 3.33). It also implies that almost surely, μ_h is supported on γ-thick points.

Proposition 3.31. *Let D, h, μ_h be as in Definition 3.30. When h is a free-boundary GFF or whole-plane GFF, we arbitrarily fix the constant for h. Let $U \subseteq D$ be a bounded domain, and z be a point sampled according to $\mu_h(U)$ (normalized to be a probability measure). Then with probability one we have*

$$\lim_{\varepsilon\to 0} \frac{h^{(\varepsilon)}(z)}{\log(1/\varepsilon)} = \gamma, \tag{3.32}$$

where $h^{(\varepsilon)}(z)$ denotes the circle average of h on $D \cap B(z, \varepsilon)$.

Proof. We only explain the intuition here. For a detailed proof, see [15, 8]. Recall that the covariance structure of h is given by a Green's function (see (3.1) and (3.13)). Therefore, by (heuristically) applying the Girsanov's theorem, tilting h by $e^{\gamma h(z)}$ should shift the mean of h by $\gamma G_0^D(\cdot, z)$ (or $G^D(\cdot, z)$ in the free-boundary case), which is equal to $-\gamma \log|\cdot - z|$ plus some bounded smooth terms. □

Remark 3.33. We actually know more about the "thick points", which are atypical from the point of view of Euclidean geometry but more typical from the view of the associated quantum geometry. Indeed, call a point z α-*thick* if the Gaussian free field h has an extra log-singularity of order α at z. Then the Hausdorff dimension of the set of α-thick points is $2 - \alpha^2/2$. See [15] for a precise statement and proof.

Now let h be a free-boundary GFF on a regular domain $D \neq \mathbb{C}$ where ∂D has a linear segment L, with the additive constant of h fixed in an arbitrary way. We still fix $\gamma \in [0, 2)$. In this case, we want to associate with h an LQG length measure ν_h, which is supported on ∂D and should be used to measure the quantum length of ∂D. This LQG boundary measure can be intuitively defined by $\nu_h(dz) = e^{\gamma h(z)/2}dz$. Note that here the multiplicative factor in the exponential is $\gamma/2$ instead of γ. The reason for this choice is rather deep and is related with the KPZ formula; see [8] for detail. One rough explanation is that we want to describe length, which is a "one-dimensional" object, instead of area, a "two-dimensional" object.

Again, to rigorously define ν_h, some mollification is needed. In light of Proposition 3.21, we consider for $\varepsilon > 0$ the following approximation procedure

$$\nu_\varepsilon(dz) := e^{\gamma h^{(\varepsilon)}(z)/2} \varepsilon^{\gamma^2/4} dz,$$

where $h^{(\varepsilon)}(z)$ is a jointly continuous version of the semicircle averages. We choose the scaling constant $\gamma^2/2$ because $\mathbb{E}[e^{\gamma h^{(\varepsilon)}(z)/2}\varepsilon^{\gamma^2/4}]$ is a constant for every $z \in D$. The following result, which is proved in [8], is both a theorem and a definition of LQG boundary measure.

Definition 3.34 (LQG boundary measure)**.** Let D, L, h, ν_ε be defined as above. It is almost surely the case that as $\varepsilon \to 0$ (along the sequence of powers of two), the measures ν_ε converges weakly in D to a limiting measure, which we call **LQG boundary measure** and denote by $\nu = \nu_h = e^{\gamma h(z)/2}dz$. This remains true if we replace h with a non-centered GFF on D, that is, if we replace h with $h + f$ where f is a deterministic, non-zero continous function on D.

For any finite segment $E \subseteq L$, $\mathbb{E}(\nu_h(E))$ is finite, making $\nu_h(L)$ almost surely finite. We can then consider the counterpart of Proposition 3.31. Note that Proposition 3.35 implies that ν_h is supported on "γ-thick points" on the boundary. This intuition is helpful for understanding our main theorems, Theorem 4.17 and Theorem 4.29.

Proposition 3.35. *Let D, L, h, ν_h be as in Definition 3.34 (with the additive constant for h fixed in an arbitrary way). Let $E \subseteq L$ be a finite segment, and z be a point sampled according to $\nu_h(E)$ (normalized to be a probability measure). Then with probability one we have*

$$\lim_{\varepsilon\to 0} \frac{h^{(\varepsilon)}(z)}{\log(1/\varepsilon)} = \gamma, \tag{3.36}$$

where $h^{(\varepsilon)}(z)$ denotes the semicircle average of h on $D \cap \partial B(z, \varepsilon)$.

Proof. The proof is the same as in Proposition 3.31, except that the exponent of ν_h is $\gamma/2$ (instead of γ), and the free-boundary Green's function has a log-singularity of order 2 at (z, z). □

Remark 3.37. The measures μ_h and ν_h are special cases of a more general family of random measures associated with log-correlated Gaussian fields in dimensions $d \geq 2$, called **Gaussian multiplicative chaos**.

Now in view of the conformal invariance of GFF (Proposition 3.10 and its counterparts), we can try to parameterize the same LQG surface with different choices of domain. From now on, we set for $\gamma \in (0, 2)$

$$Q := \frac{\gamma}{2} + \frac{2}{\gamma}.$$

Let $\phi : D \mapsto \tilde{D}$ be a conformal map. Our regularization then implies a rule for changing coordinates from D to $\tilde{D}$. Indeed, set

$$\tilde{z} := \phi(z) \quad \text{and} \quad \tilde{\varepsilon} := |\phi'(z)|\varepsilon.$$

Then under ϕ, a small circle of radius ε centered at z is mapped approximately into a small circle of radius $\tilde{\varepsilon}$ centered at $\tilde{z}$, and we have

$$e^{\gamma h^{(\varepsilon)}(z)}\varepsilon^{\gamma^2/2}dz \approx e^{\gamma h^{(\tilde{\varepsilon})}(z)} \frac{\tilde{\varepsilon}^{\gamma^2/2}}{|\phi'(z)|^{\gamma^2/2}} \frac{d\tilde{z}}{|\phi'(z)|^2}.$$

This strongly suggests the following proposition,, which was first proved in [8].

Proposition 3.38 (Coordinate change)**.** *Let D, h and μ_h be as in Definition 3.30. Suppose ϕ is a conformal map from D to $\tilde{D}$. We set*

$$\phi(h) := h \circ \phi^{-1} + Q \log |(\phi^{-1})'|. \tag{3.39}$$

Then $\mu_{\phi(h)}$ is almost surely the image under ϕ under the measure μ_h, that is, almost surely, $\mu_{\phi(h)}(\phi(A)) = \mu_h(A)$ holds for all $A \subseteq D$. A similar result also holds for D, h and ν_h in Definition 3.34 if ∂D and $\partial(\phi(D))$ are both Jordan domains with piecewise linear boundaries and ϕ extends to a homeomorphism from $\overline{D}$ to $\overline{\tilde{D}}$.

Proposition 3.38 actually suggests a way to define the LQG boundary measure ν_h on those D (whose boundaries may not be piecewise linear) such that D is a simply connected Jordan domain on the Riemann sphere. In this case, one can use the Riemann mapping theorem to construct a conformal map from the upper half plane $\mathbb{H}$ to D which extends continuously to the boundary, and then define the LQG boundary measure on D to be the pushforward of the LQG boundary measure on $\mathbb{H}$. Note that by Proposition 3.38 this definition is consistent with Definition 3.34.

Remark 3.40. Indeed, [43] showed that, if the regularized field $h^{(\varepsilon)}(z)$ is defined via bump function averages instead of circle averages, then the we can still define μ_h in the same way as Definition 3.30, and the coordinate change formula in Proposition 3.38 almost surely hold for all the conformal maps *simultaneously*.

Given Proposition 3.38, it is then rather natural to define an LQG surface as an equivalence class.

Definition 3.41 (Quantum surface)**.** A **quantum surface** is an equivalence class of pairs (D, h) under the equivalence relation

$$(D, h) \sim (\phi(D), \phi(h)), \tag{3.42}$$

where ϕ is a conformal map and $\phi(h)$ is as in (3.39). An **embedding** of a quantum surface is a choice of representative (D, h) from the equivalence class. A transformation of the kind described in (3.42) is called a **coordinate change**.

For $k \geq 1$, a **quantum surface with k marked points** is an equivalence class of elements of the form $(D, h, x_1, \ldots, x_k)$ with $x_i \in \overline{D}$ for $i = 1, 2, \ldots, k$, under the equivalence relation

$$(D, h, x_1, \ldots, x_k) \sim (\phi(D), \phi(h), \phi(x_1), \ldots, \phi(x_k)). \tag{3.43}$$

A **curve-decorated quantum surface** is an equivalence class of triples (D, h, η) under the equivalence relation

$$(D, h, \eta) \sim (\phi(D), \phi(h), \phi(\eta)), \tag{3.44}$$

where we see a curve as a continuous map from a subset of $\mathbb{R}$ to D. For $k \geq 1$, We similarly define a **curve-decorated quantum surface with k marked points** as a combination of (3.43) and (3.44).

3.3. **Quantum wedge.** Fix $\gamma \in (0, 2)$. Suppose that h is a *deterministic* distribution and one somehow defines μ_h (resp. ν_h) as $e^{\gamma h(z)} dz$ (resp. $e^{\gamma h(z)/2} dz$). Then replacing h for $h + C$ changes μ_h (resp. ν_h) by a factor of $e^{\gamma C}$ (resp. $e^{\gamma C/2}$). In other words, the surface described by $h + C$ is a "zoomed in" version of the surface described by h. Heuristically, when $C \to \infty$, one only need to care about the local property of the original quantum surface. For this reason, it will be nice to have a quantum surface (where h is now random and μ_h/ν_h are now LQG area/boundary measure associated with h) that is invariant (modulo embeddings of quantum surfaces) in law under this zooming. Such a property may be thought of as a type of scale invariance of quantum surfaces.

To be more specific, we want a random quantum surface $(\mathbb{H}, \tilde{h})$ such that:

(1) For every $C > 0$, the law of $(\mathbb{H}, \tilde{h})$ and $(\mathbb{H}, \tilde{h} + C)$ is the same (modulo embeddings).
(2) The surface $(\mathbb{H}, \tilde{h})$ is associated with both an LQG area measure and an LQG boundary measure, which are both *not* viewed modulo constants.

Note that (2) rules out $\tilde{h}$ as a zero-boundary GFF or a free-boundary GFF with arbitrary fixed additive constant. First, although a zero-boundary GFF is scale-invariant (modulo embeddings) by conformal invariance (Proposition 3.10) and the coordinate change formula (Proposition 3.38), it is not associated with an LQG boundary measure. Second, although scale-invariance trivially holds when a free-boundary GFF is viewed modulo constants, it does not hold when the additive constant is fixed.

[40] first introduced quantum wedge, which does have this invariance property. Roughly speaking, a quantum wedge is the limiting surface that one obtains by "zooming in" to a (non-centered) free-boundary GFF with a certain type of log-singularity (with the additive constant fixed in an arbitrary way) close to a point on the boundary; see Proposition 3.53. In particular, in view of Proposition 3.35, zooming in to a (non-centered) free boundary GFF near a "quantum typical" point sampled from the LQG boundary measure should yield a quantum wedge; see Remark 3.54.

In the rest of this subsection, we fix $\gamma \in (0, 2)$ and

$$Q := \frac{\gamma}{2} + \frac{2}{\gamma}.$$

For $\theta > 0$, define

$$W^\theta := \{z \in \mathbb{C} : \arg(z) \in (0, \theta)\},$$

where we view W^θ as a Riemann surface when $\theta \geq 2\pi$. Let h^θ be a free-boundary GFF on W^θ with arbitrary fixed constant. We refer to the quantum surface (W^θ, h^θ) as an **unscaled quantum wedge**. Although (W^θ, h^θ) itself is not scale-invariant, analyzing it helps us to better understand the construction of scale-invariant quantum wedges.

To start, set

$$\alpha := Q\left(1 - \frac{\theta}{\pi}\right).$$

Applying the coordinate-change formula on the conformal map $z \mapsto (\pi/\theta)\log(z)$, we can parameterize (W^θ, h^θ) by $(S, h + (\alpha - Q)\Re(z))$, where S is the infinite strip $\mathbb{R} \times (0, \pi)$ and h is a free-boundary GFF on S (with its additive constant fixed). Working on the strip S turns out to be convenient since we can decompose h into a Brownian motion plus a "lateral noise". To be precise, we can apply the coordinate-change formula on the radial-lateral decomposition (Proposition 3.22) and decompose h into

$$h = h_{rad} + h_{circ}, \tag{3.45}$$

where h_{rad} (resp. h_{circ}) is constant (resp. has average zero) on each vertical line. Note that both h_{rad} and h_{circ} have additive constant fixed, but they might not be independent; see Remark 3.23. In addition, the circle-average property (Proposition 3.21, adapted to coordinate-change) says that $(h_{rad}(t))_{t\in\mathbb{R}}$ behaves like $\sqrt{2}$ times a a two-sided standard Brownian motion (plus some random constant brought by the extra additive constant). So the radial part of $h + (a - Q)\Re(z)$ is actually a drifted Brownian motion.

In the above setting, zooming in $(S, h + (a - Q)\Re(z))$, or in other words, adding constants to $h + (a - Q)\Re(z)$, can be seen as keeping the lateral part h_{circ} fixed and vertically translating the figure of the drifted Brownian motion.

Now, for every $c \in \mathbb{R}$, the horizontal translation $z \mapsto z + c$ is a conformal bijection from S to itself. Therefore, since quantum surfaces are defined modulo embeddings (see Definition 3.41), two samples of drifted Brownian motion are also in the same equivalence class if their figures can be horizontally translated into each other. The problem is that, even if a drifted Brownian motion is defined modulo horizontal translations, its law still changes after vertical translations. To settle this, we replace the drifted Brownian motion with a closely related process.

Definition 3.46. Fix $\alpha \in (-\infty, Q)$. Let $\tilde{h}_{circ}$ be the same as h_{circ} defined above (note that h_{circ} does not depend on the way of fixing the additive constant for h). We futher define

$$\tilde{h}_{rad} := \begin{cases} B_{2s} + (\alpha - Q)s & \text{if}\, \Re(z) = s \,\text{and}\, s \geq 0, \\ W_{2s} + (\alpha - Q)s & \text{if}\, \Re(z) = s \,\text{and}\, s < 0, \end{cases} \tag{3.47}$$

where B_t is a standard Brownian motion and W_t is another independent Brownian motion conditioned so that $W_{2t} - (\alpha - Q)t > 0$ for all $t > 0$. Then an α-**quantum wedge** is the quantum surface $(S, \tilde{h}, -\infty, \infty)$ such that $\tilde{h} = \tilde{h}_{rad} + \tilde{h}_{circ}$.

Remark 3.48. Note that in (3.47) we are actually conditioning on an zero probability event. This can actually be made rigorous. Suppose B_t is a standard Brownian motion and $a > 0$. Then $(B_t + at)_{t\geq 0}$ conditioned to stay positive on $\mathbb{R}^+$ can be defined by the weak limit as $\varepsilon \to 0$ of the conditional law of $(B_t + at)_{t\geq 0}$ conditioned to stay above $-\varepsilon$. An equivalent definition begins with a standard Brownian motion $(W_t)_{t\geq 0}$, letting τ be the last time that $W_t + at$ hits zero (this time is a.s. finite due to the positive drift), and then taking $(B_t)_{t\geq 0} := (W_{t+\tau})_{t\geq 0}$.

Remark 3.49. There is actually another definition of α-quantum wedge which uses the Bessel process. Under this view, one can also define α-quantum wedge for $\alpha \in [Q, Q + \gamma/2)$. The quantum wedge in the $\alpha < Q$ case is called thick quantum wedge and in the $\alpha \geq Q$ case called thin quantum wedge; see Section 5.1 for details.

From Definition 3.46 we see that the law of α-quantum wedge is scale-invariant.

Proposition 3.50. *For any $C \in \mathbb{R}$, $(S, \tilde{h}, 0, \infty)$ and $(S, \tilde{h} + C, 0, \infty)$ have the same law as quantum surfaces.*

Proof. Given Definition 3.46, the proof actually boils down to analyzing the radial part $\tilde{h}_{rad}$. We only mention the key observation here and omit the technical details.

Observation: Suppose $a < 0$ and $(B_t + at)_{0\leq t\leq \tau}$ is a drifted Brownian motion started from $c > 0$ and stopped at the first time it hits zero. Then the reversal process $(B_{\tau-t} + a(\tau - t))_{0\leq t\leq \tau}$ evolves as a Brownian motion with drift $-a$, started from zero, conditioned on staying positive for $t > 0$, and stopped at the last time it hits c. □

For convenience of future reference, we now apply the coordinate-change formula to the conformal map $z \mapsto -e^{-z}$ and shift the setting from $(S, -\infty, \infty)$ to $(\mathbb{H}, 0, \infty)$, where $\mathbb{H}$ is the upper half plane. Note that in Definition 3.46, (3.47) actually specifies the embedding of quantum wedge. We call this particular embedding the **circle average embedding**. In the upper half plane case, if we use $\mathsf{h}^{(r)}(z)$ to denote the semicircle average of a distribution h on $\partial B(z, r) \cap \mathbb{H}$, then the circle average embedding of h requires

$$\sup\{r > 0 : \mathsf{h}^{(r)}(0) + Q \log r\} = 0.$$

Fix $\alpha \in (-\infty, Q)$. Let h be a free-boundary GFF on $\mathbb{H}$, with the additive constant fixed so that its average on $\mathbb{H} \cap \partial B(0,1)$ is zero. The circle average embedding has the benefit that when restricted to $\mathbb{H} \cap B(0,1)$, an α-quantum wedge (under the unit circle embedding) and $h - \alpha \log|\cdot|$ has the same law. Combining this fact with the scale-invariance of quantum wedge, we see that an α-quantum wedge $\tilde{h}$ has a well-defined LQG area measure $\mu_{\tilde{h}}$ and LQG boundary measure $\nu_{\tilde{h}}$ as in Definition 3.30 and Definition 3.34, which are both not defiend modulo constants and both satisfy the coordinate-change formula as in Proposition 3.38.

Proposition 3.50 says that $\tilde{h}$ is scale-invariant. We actually have a stronger result, that is, if we zoom in $(\mathbb{H}, h - \alpha \log|\cdot|)$ close to the origin, then the limit surface will again be an α-quantum wedge. Moreover, the way of fixing the additive constant for h is in fact unimportant, which is not surprising given Proposition 3.27. In addition, if we replace h (with arbitrary fixed additive constant) with $h + \varphi$ where φ is a deterministic smooth function on a neighbourhood of the origin, then the limit result still holds. Roughly speaking, this is because the deterministic function is approximately constant around the origin. In one word, the additive constant of h and the extra function φ are minor parts, and the only thing that matters after zooming in is the order of log-singularity at the origin.

Before stating this result, we shall specify the topology of convergence on the set of quantum surfaces first. In the rest of this paper, we will always consider the following topology.

Definition 3.51. We say that a sequence of doubly marked quantum surfaces $(\mathbb{H}, \mathsf{h}_n, 0, \infty)$ converges to the doubly marked surface $(\mathbb{H}, \mathsf{h}, 0, \infty)$ if the distribution which describes the circle average embedding of $(\mathbb{H}, \mathsf{h}_n, 0, \infty)$ converges to the distribution which describes the circle average embedding of $(\mathbb{H}, \mathsf{h}, 0, \infty)$.

Remark 3.52. Note that there are different ways to induce a topology on the set of quantum surfaces, which does not make a big difference; see [40].

We now give the precise statement with regard to an α-quantum as a natural limit surface under zooming in (which implies Proposition 3.50). In the following proposition, when ρ_0 is the uniform measure on $\mathbb{H} \cap B(0,1)$ and $\varphi \equiv 0$, the proof is almost the same as in Proposition 3.50, while the complete proof for general cases is subtle and will not be provided here.

Proposition 3.53. *Let $\alpha \in (-\infty, Q)$ and h be a free boundary GFF on $\mathbb{H}$. Fix $\rho_0 \in \mathcal{M}(D) \setminus \bar{M}(D)$ where ρ_0 is compactly support away from the origin and declare $(h, \rho_0) = 0$. Let φ be a deterministic smooth function on a neighbourhood of the origin. Let $\bar{h} := h - \alpha \log|\cdot| + \varphi$. Then as $C \to \infty$, the quantum surfaces $(\mathbb{H}, \bar{h} + C, 0, \infty)$ converge to the α-quantum wedge $(\mathbb{H}, \tilde{h}, 0, \infty)$ in the sense of Definition 3.51.*

Remark 3.54. Let h be as in Proposition 3.53 and ν_h be the LQG boundary measure associated with h. In light of Proposition 3.35 and Proposition 3.53, if z is a quantum typical point of ν_h, then zooming in near z should yield a γ-quantum wedge. This viewpoint will be useful in understanding Theorem 4.17 and Theorem 4.29.

3.4. **Schramm-Loewner Evolution.**

3.4.1. *Ordinary* SLE_κ. SLE_κ is a one parameter family of random fractal curves introduced by Schramm [36] as a family of potential scaling limits for interfaces in critical statistical physics models. SLE_κ satisfies two properties that make them appropriate for such limits: conformal invariance and domain Markov property. In fact, these two properties also characterise SLE_κ as a one parameter family.

Let us begin with some definitions with regard to the Loewner's theorem.

Definition 3.55 ((Forward) Loewner evolutions in $\mathbb{H}$)**.** Suppose $(\xi_t)_{t\geq 0}$ is a continuous real value function. For each $z \in \mathbb{H}$ define $(g_t(z))_{0\leq t\leq \zeta(z)}$ to be the maximal solution to the **(forward) Loewner equation**

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z, \tag{3.56}$$

which exists by classical ODE theory. For $t \geq 0$ define

$$K_t := \{z \in \mathbb{H} : \zeta(z) \leq t\}.$$

We call $(K_t)_{t\geq 0}$ the **(forward) Loewner chain** with **driving function** ξ_t. We call $(g_t)_{t\geq 0}$ the **(forward) Loewner flow**. For every $z \in \mathbb{H}$ and $0 \leq t \leq \zeta(z)$ define

$$\tilde{g}_t(z) := g_t(z) - \xi_t.$$

We call $(\tilde{g}_t)_{0\leq t\leq \zeta t}$ the **centered (forward) Loewner flow**.

Define for each $t \geq 0$

$$H_t := \mathbb{H} \setminus K_t.$$

Then H_t is the image of $\mathbb{H}$ under the conformal map g_t^{-1}. For each $t \geq 0$, K_t is bounded and H_t is a simply connected domain. We call sets like $(K_t)_{t\geq 0}$ **compact** $\mathbb{H}$**-hulls**. Here in Definition 3.55 the time parameter t is parameterized by **half-plane capacity**, meaning that for each $t \geq 0$,

$$\lim_{t\to\infty} z(g_t(z) - z) = 2t.$$

A continuous path $\eta(t), t \geq 0$ in $\mathbb{H}$ is said to generate an increasing family of compact $\mathbb{H}$-hulls $(K_t)_{t\geq 0}$ if $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \eta([0, t])$ for all $t \geq 0$. The following is both the definition of chordal SLE_κ on $(\mathbb{H}, 0, \infty)$. and a very hard theorem, which was proved in [35] for case $\kappa \neq 8$ and in [21] for case $\kappa = 8$.

Definition 3.57 (Chordal SLE_κ on $(\mathbb{H}, 0, \infty)$)**.** For $\kappa > 0$, SLE_κ on $(\mathbb{H}, 0, \infty)$ is the random Loewner chain driven by $\xi_t = \sqrt{\kappa}B_t$, where B_t is a standard Brownian motion. An SLE_κ-path on $(\mathbb{H}, 0, \infty)$ is the random curve $(\eta(t))_{t\geq 0}$ given by

$$\eta(t) := g_t^{-1}(\xi_t),$$

which is continuous and generates SLE_κ almost surely.

In some cases it will be more convenient to refer to the centered SLE_κ, which is obtained by translating the random hull K_t in Definition 3.55 by ξ_t for each $t \geq 0$, or in other words, substituting the maps $(g_t)_{t\geq 0}$ with $(\tilde{g}_t)_{t\geq 0}$.

Remark 3.58. In many cases, people use the term SLE_κ to denote the SLE_κ-path. In this paper, the main theorems only care about SLE_κ for $\kappa \in (0,4)$, in which case SLE_κ and SLE_κ-path actually means the same thing, that is, a simple continuous curve in $\mathbb{H}$. Indeed, there are three phases for an SLE_κ-path η.

(1) For $\kappa \in (0.4]$, η almost surely does not touch itself or the real line. In addition, $\mathbb{H} \setminus \eta$ is almost surely a Hölder domain.
(2) For $\kappa \in (4,8)$, η almost surely hits (but not crosses) itself and the real line but does not hit a specific point, thus creating "holes" in $\mathbb{H}$
(3) For $\kappa > 8$, η is almost surely space-filling.

Note that here we just explain the rough idea and omit the precise statement; see [3] for detail. We also remark here that the result in (1) will be useful in the proof of Theorem 4.29.

Two basic properties of SLE_κ on $(\mathbb{H}, 0, \infty)$ is scale invariance and domain Markov property, which is guaranteed by the scale invariance and domain Markov property of Brownian motion. In fact, these two properties also determine that the driving function must be a Brownian motion of some diffusivity $\kappa \in [0,\infty)$.

Proposition 3.59 (Scale invariance and domain Markov property)**.** *Fix $\kappa > 0$. Suppose a curve $\eta(t)_{t\geq 0}$ is an SLE_κ-path on $(\mathbb{H}, 0, \infty)$. Then the following holds.*

*(1) **(Scale invariance)**. $(\eta(at))_{t\geq 0}$ has the same distribution as $a^{1/2}(\eta(t))_{t\geq 0}$.*
*(2) **(Domain Markov property)**. For each stopping time τ for the driving function $\sqrt{\kappa}B_t$, the conditional law of the image of $(\eta(t+\tau))_{t\geq 0}$ under $\tilde{g}_\tau$ given $(\eta(at))_{t\in[0,\tau]}$ has the same law as $(\eta(t))_{t\geq 0}$.*

Scale invariance actually allows us to use conformal mapping to define SLE_κ (both as a curve and the hulls) on (D, a, b) for any simply connected domain D and two marked points a, b on the boundary. In this case, we gets **conformal invariance** and the domain Markov property still holds. We omit the precise definitions here.

Remark 3.60. When $a \in \partial D$ and $b \in D$, we may similarly define radial SLE_κ using radial Loewner chains.

The standard Loewner evolutions in Definition 3.55 should really be refered to as *forward* Loewner evolutions, because they also have counterparts: *reverse Loewner evolutions*. As we can see in Definition 3.55, the forward Loewner evolutions can be encoded via an increasing family of compact $\mathbb{H}$-hulls, where in each infinitesimal increment of time, an infinitesimal new piece of hull is added "at the top". In the reverse Loewner evolutions, however, we get the new hull by adding the infinitesimal new piece "at the root", and the whole previous hull is "pushed into" the interior of $\mathbb{H}$ by a conformal map. In particular, now the family of hulls may not be increasing.

Definition 3.61 (Reverse Loewner evolutions in $\mathbb{H}$)**.** Suppose $(\xi_t)_{t\geq 0}$ is a continuous real value function. For each $z \in \mathbb{H}$ define $(f_t(z))_{t\geq 0}$ to be the solution to the **reverse Loewner equation**

$$\frac{\partial f_t(z)}{\partial t} = \frac{-2}{f_t(z) - \xi_t}, \quad f_0(z) = z - \xi_t, \tag{3.62}$$

which exists by classical ODE theory. In this case, for all $t \geq 0$, f_t is a conformal map from $\mathbb{H}$ to some domain H_t. For $t \geq 0$ define

$$K_t := \mathbb{H} \setminus H_t.$$

We call $(K_t)_{t\geq 0}$ the **reverse Loewner chain** with **driving function** ξ_t. We call $(f_t)_{t\geq 0}$ the **reverse Loewner flow**. For every $z \in \mathbb{H}$ and $t \geq 0$ define

$$\tilde{f}_t(z) := f_t(z) - \xi_t.$$

We call $(\tilde{f}_t)_{t\geq 0}$ the **centered reverse Loewner flow**.

Compare (3.56) with (3.62), we can find that sign of the dt term has changed from positive to negative, explaining why we use the term "reverse".

We now give the definition of reverse SLE_κ. Note that here we cannot construct a reverse SLE_κ-path as in the forward case since the curve that generates the hull always grows "from the root".

Definition 3.63 (Reverse SLE_κ)**.** For $\kappa > 0$, reverse SLE_κ is the random reverse Loewner chain driven by $\xi_t = \sqrt{\kappa} B_t$, where B_t is a standard Brownian motion. We also define the centered reverse SLE_κ, which is obtained by translating the random hull K_t in Definition 3.61 by ξ_t for each $t \geq 0$, or in other words, substituting the maps $(f_t)_{t\geq 0}$ with $(\tilde{f}_t)_{t\geq 0}$.

In parallel with Proposition 3.59, we can still obtain scale invariance of (centered) reverse SLE_κ, but the domain Markov property no longer holds because of the "growing from the root" nature.

One might guess that the reverse Loewner flow is actually a "reversed" forward Loewner flow. This heuristics is true in some sense, as is shown in the following (deterministic) proposition.

Proposition 3.64 (Symmetry for centered foward/reverse Loewner flow)**.** *Suppose that $(\tilde{g}_t)_{t\geq 0}$ is the centered forward Loewner flow with driving function $(\xi_t)_{t\geq 0}$. Fix $T > 0$ and set for $0 \leq t \leq T$*

$$(\hat{\xi}(t))_{t\geq 0} := \xi_T - \xi_{T-t}.$$

Let $(\tilde{f}_t)_{0\leq t\leq T}$ be the centered reverse Loewner flow with driving function $(\xi_t)_{0\leq t\leq T}$. Then for every $0 \leq t \leq T$,

$$\tilde{f}_t = \tilde{g}_{T-t} \circ \tilde{g}_T^{-1}.$$

Proposition 3.64 can be proved by using (3.56) and the conditions to verify (3.62). We can then use Proposition 3.64 and the reversibility of Brownian motion to get the symmetry between forward SLE_κ and centered reverse SLE_κ. Note that they are both "centered at zero".

Proposition 3.65 (Symmetry for forward/reverse SLE_κ)**.** *For any fixed $T > 0$, the curve generated by a centered reverse SLE_κ run up to time T and the curve generated by a forward SLE_κ run up to time T are equal in law.*

Note that in Proposition 3.64 and Proposition 3.65 the result only holds for a fixed $T > 0$ but not for a dynamic T. We emphasize again that this is because the growing natures of the foward/reverse Loewner chain are very different.

3.4.2. *$SLE_\kappa(\underline{\rho})$*. $\mathrm{SLE}_\kappa(\underline{\rho})$ was first introduced in [19, 38] as a variant of ordinary SLE_κ process, where $\underline{\rho} = (\rho_{(1)}, \dots, \rho_{(n)})$ is a vector in $\mathbb{R}^n$. An $\mathrm{SLE}_\kappa(\underline{\rho})$ keeps track of some additional force points (or marked points) on the boundary or in the interior of the domain. These points are pulled toward or pushed away from the origin, and the value of $\underline{\rho}$ encodes the direction and intensity of these additional forces. Similar to SLE_κ, $\mathrm{SLE}_\kappa(\underline{\rho})$ also arises as scaling limits of interfaces in critical statistical models, but these models may involve more complicated boundary behaviors.

In the following, we will give the definitions of (forward) $\mathrm{SLE}_\kappa(\underline{\rho})$ and reverse $\mathrm{SLE}_\kappa(\underline{\rho})$, and then analyze a specific case that we will encounter in the proofs of our main theorems. We will not provide rigorous statements or hard and technical proofs about the uniqueness and existence of the solutions of the SDE's (3.67) and (3.71), and the readers can just focus on the form of these SDE's. For more details about $\mathrm{SLE}_\kappa(\underline{\rho})$, see [19, 35].

Definition 3.66 ((Forward) $\mathrm{SLE}_\kappa(\underline{\rho})$ on $(\mathbb{H}, 0, \infty)$)**.** Fix $\rho_{(1)}, \dots, \rho_{(n)} \in \mathbb{R}$, $x_1, \dots, x_n \in \overline{\mathbb{H}} \setminus \{0\}$ and $\kappa > 0$. Let $\underline{\rho} = (\rho_{(1)}, \dots, \rho_{(n)})$. (Forward) $\mathrm{SLE}_\kappa(\underline{\rho})$ (or similarly $\mathrm{SLE}_\kappa(\rho_{(1)}, \dots, \rho_{(n)})$) with **force points** $x_1, \dots, x_n$ is the random Loewner chain (recall Definition 3.55 and (3.56)) with driving function $(\xi_t)_{t\ge 0}$ satisfying the following SDE's:

$$
\begin{aligned}
d\xi_t &= \sum_{i=1}^{n} \Re\Big(\frac{-\rho_{(i)}}{g_t(x_i) - \xi_t}\Big)dt + \sqrt{\kappa}dB_t, \\
dg_t(x_i) &= \frac{2}{g_t(x_i) - \xi_t}dt, g_0(x_i) = x_i; \quad i = 1, \dots, n,
\end{aligned}
\tag{3.67}
$$

where B_t is a standard Brownian motion. Similar to Definition 3.57, we also define the centered (forward) $\mathrm{SLE}_\kappa(\underline{\rho})$, which is obtained by substituting the maps $(g_t)_{t\ge0}$ with $(\tilde{g}_t)_{t\ge0}$. In particular, when $\underline{\rho}$ is the zero vector, we just get the ordinary case.

Remark 3.68. Definition 3.66 can also be extended to the cases when there are force points located infinitesimally to the left/right of zero (denoted 0^- and 0^+). This is done by taking a limit in Definition 3.66. We will not give the precise definition here.

Remark 3.69. Similar to Remark 3.58, in the specific setting of the proofs of our main theorem, $\mathrm{SLE}_\kappa(\underline{\rho})$ is actually a simple curve in $\bar{\mathbb{H}}$ (which may hit the real line for some $\underline{\rho}$).

In parallel with Proposition 3.59, from (3.67) we can see that the $\mathrm{SLE}_\kappa(\underline{\rho})$ still satisfies a form of scale invariance and domain Markov property as long as we keep track of the force points and adapt the definitions. We can then use conformal maps to define $\mathrm{SLE}_\kappa(\underline{\rho})$ on general (D, a, b) as in the ordinary case. We omit the precise statements here.

We then introduce the reverse $\mathrm{SLE}_\kappa(\underline{\rho})$.

Definition 3.70 (Reverse $\mathrm{SLE}_\kappa(\underline{\rho})$)**.** Fix $\rho_{(1)}, \dots, \rho_{(n)} \in \mathbb{R}$, $x_1, \dots, x_n \in \overline{\mathbb{H}} \setminus \{0\}$ and $\kappa > 0$. Let $\underline{\rho} = (\rho_{(1)}, \dots, \rho_{(n)})$. Reverse $\mathrm{SLE}_\kappa(\underline{\rho})$ (or similarly $\mathrm{SLE}_\kappa(\rho_{(1)}, \dots, \rho_{(n)})$)

with **force points** $x_1, \ldots, x_n$ is the random reverse Loewner chain (recall Definition 3.61 and (3.62)) with driving function $(\xi_t)_{t\geq 0}$ satisfying the following SDE's:

$$\begin{aligned} d\xi_t &= \sum_{i=1}^{n} \Re(\frac{-\rho_{(i)}}{f_t(x_i) - \xi_t})dt + \sqrt{\kappa} dB_t, \\ df_t(x_i) &= \frac{-2}{f_t(x_i) - \xi_t} dt, f_0(x_i) = x_i - \xi_t; \quad i = 1, \ldots, n, \end{aligned} \tag{3.71}$$

where B_t is a standard Brownian motion. Similar to Definition 3.63, we also define the centered reverse $\text{SLE}_\kappa(\underline{\rho})$, which is obtained by substituting the maps $(f_t)_{t\geq 0}$ with $(\tilde{f}_t)_{t\geq 0}$. In particular, when $\underline{\rho}$ is the zero vector, we just get the ordinary case.

Remark 3.72. In Definition 3.70 there might exist some random time τ such that $\tilde{f}_\tau(x_i) = 0$ for some $i \in \{1, \ldots, n\}$. We will only consider (3.71) until this time. How, we can still try to define the solution to (3.71) after τ; see [7] for details.

Though the force points $x_1, \ldots, x_n$ still move under the forward or reverse Loewner flow, they will feel some additional attraction or repulsion. Fix $i \in \{1, \ldots, n\}$. In the forward case, the Loewner drift (the dt term in the second line of (3.67)) is pushing $g_t(x_i)$ away from the driving function ξ_t, or in other words, pushing $\tilde{g}_t(x_i)$ away from the origin. In the reverse case, the Loewner drift (the dt term in the second line of (3.71)) is pulling $\tilde{f}_t(x_i)$ to the origin. On the other hand, from the first line in (3.67) and (3.71) we see that $\rho_{(i)}$ always indicates the intensity of an additional repulsion that pushes the i-th force point away from ξ_t.

In the main theorems of this paper, we will focus on the special case when there is only one force point x_1 on the real line with weight $\rho_{(1)}$. In this case we will simply write forward/reverse $\text{SLE}_\kappa(\rho_{(1)})$. For these two specific processes, the existence and uniqueness results for (3.67) and (3.71) can be found in [7] and [27]. Moreover, the forward/reverse $\text{SLE}_\kappa(\rho_{(1)})$ both have some relations with the Bessel process. To be concrete, a Bessel process of dimension δ is the solution of the SDE

$$dZ_t = dB_t + \frac{\delta - 1}{2} \frac{dt}{Z_t}, \quad Z_0 > 0. \tag{3.73}$$

Then from (3.67) we can see that $\tilde{g}_t(x_1)$ is a Bessel process of dimension

$$\delta = 1 + \frac{2(\rho_{(1)} + 2)}{\kappa}. \tag{3.74}$$

Similarly, from (3.71) we can see that $\tilde{f}_t(x_1)$ is a Bessel process of dimension

$$\delta = 1 + \frac{2(\rho_{(1)} - 2)}{\kappa}. \tag{3.75}$$

Remark 3.76. If $\delta \geq 2$, then a Bessel process of dimension δ started from zero will almost surely be strictly positive for all positive times. If $\delta < 2$, then from any starting point the process will return to zero in finite time almost surely. This is suggested by the fact that, for any integer $n \geq 2$, $Z_t := \|B_t\|$ is a Bessel process with dimension n, where B_t is a n-dimensional Brownian motion and $\|\cdot\|$ is the Euclidean norm.

In Proposition 3.65 we use reversibility of a standard Brownian motion to get the symmetry between forward SLE_κ and reverse SLE_κ. Here we will use a form of reversibility of Bessel process, which is essentially the reversibility of a drifted Brownian motion, to obtain a similar result. Indeed, if Z_t is a Bessel process of dimension δ, then from (3.73) and Itô's formula we see that the process $\log Z_t$, when parameterized by quadratic variation, is a Brownian motion with drift $(\delta-2)/2$. The time reversal of this drifted Brownian motion is a Brownian motion with drift $(2-\delta)/2$. Combining this fact with (3.74) and (3.75), one may guess that there exists a symmetry between forward $\mathrm{SLE}_\kappa(\rho'_{(1)})$ and reverse $\mathrm{SLE}_\kappa(\rho_{(1)})$ with

$$\rho_{(1)} + \rho'_{(1)} = \kappa.$$

In fact, we have the following proposition.

Proposition 3.77 (Symmetry for forward/reverse $\mathrm{SLE}_\kappa(\rho_{(1)})$)**.** *Fix $\kappa > 0$. Suppose that $(\tilde{f}_t)_{t\geq 0}$ is the centered reverse flow for a reverse $SLE_\kappa(\rho_{(1)})$, with a single force point of weight $\rho_{(1)} < \kappa/2 + 2$ located at $x_1 > 0$. Let τ be the first time that $\tilde{f}_\tau(x) = 0$. Then $\mathbb{H} \setminus \tilde{f}_\tau(\mathbb{H})$ has the same law as an initial segment of a forward $SLE_\kappa(\kappa - \rho_{(1)})$, with a single force point of weight $\kappa - \rho_{(1)}$ located at 0^+, run until the last time σ that $\tilde{g}_\sigma(0^+) = x_1$.*

Proof. Here the condition $\rho_{(1)} < \kappa/2+2$ guarantees that the stopping time τ is a.s. finite; see Remark 3.76. The main idea has already been explained. We just point out two key observations here. Note that Observation 1 has a similar flavor as the key observation in Proposition 3.50, but they are not the same.

Observation 1: The time reversal of a Brownian motion with drift $a > 0$, started from zero and stopped at its last hitting time of $x > 0$, has the same law of a Brownian motion with drift $-a$, started form x and stopped at its last hitting time of zero.

Observation 2: The time reversal of a Bessel process of dimension $\delta \in (0, 2)$, started from $y > 0$ and stopped at its first hitting time of zero, has the same law as a Bessel process of dimension $4 - \delta$, started at zero and stopped at its last hitting time of y. □

We end this subsection with Proposition 3.78, which heuristically says that a reverse $\mathrm{SLE}_\kappa(\kappa)$ and a reverse $\mathrm{SLE}_\kappa(\kappa, -\kappa)$ are roughly the same if they share a force point and the second force point of the reverse $\mathrm{SLE}_\kappa(\kappa, -\kappa)$ is far enough. Note that the choice of interval $[1, 2]$ and the number 10 is rather arbitrary, which only requires that the number has much longer distance to the origin than the endpoints of the interval.

Proposition 3.78. *Fix $\kappa > 0$. Suppose that $(\tilde{f}_t)_{t\geq 0}$ (resp. $(\tilde{f}'_t)_{t\geq 0}$) is the centered reverse flow for a reverse $SLE_\kappa(\kappa)$ (resp. $SLE_\kappa(\kappa, -\kappa)$) with a single force point at $x_1 \in [1, 2]$ (resp. with two force points at x_1 and 10). Let τ_{x_1} (resp. τ'_{x_1}) be the first time that $\tilde{f}_{\tau_{x_1}} = 0$ (resp. $\tilde{f}'_{\tau_{x_1}} = 0$). Then for every the total variation distance between*

$$\left(\tilde{f}_t(\mathbb{H}) \cap B(0, \delta)\right)_{0\leq t\leq \tau_{x_1}} \quad \textit{and} \quad \left(\tilde{f}'_t(\mathbb{H}) \cap B(0, \delta)\right)_{0\leq t\leq \tau'_{x_1}}$$

goes to zero as $\delta \to 0$, where we use $B(0, \delta)$ to denote the disk of radius δ around the origin.

The proof of Proposition 3.78 is omitted here; see [2] for a detailed proof.

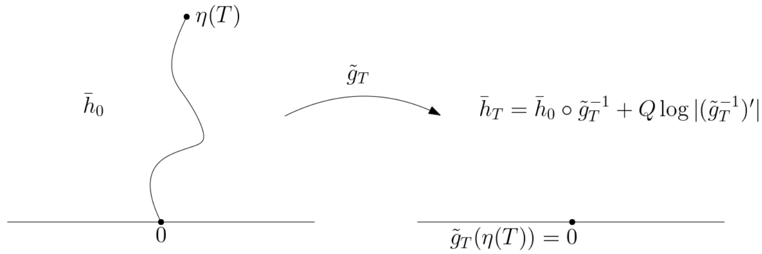

FIGURE 2. An illustration of Theorem 4.1, which says that $\bar{h}_T$ has the same law as $\bar{h}_0$ when viewed as random distributions modulo constants.

4. Conformal welding theorem

We divide this section into four parts, in correspondence with the four aspects mentioned in Section 1. Section 4.1 describes one of the two couplings between SLE and GFF stated in [40], which can be interpreted as the stationarity property of an SLE-decorated LQG surface. Section 4.2 focuses on two ways of zipping up one LQG surface along an SLE curve, the stationary capacity zipper and the stationary quantum zipper. Section 4.3 further introduces a natural length measure of SLE, that is, the quantum length of SLE on an LQG surface. Section 4.4 deals with the conformal welding of two independent LQG surfaces into an LQG surface decorated with an independent SLE-curve, or in other words, slicing an SLE-decorated LQG into two LQG surfaces.

4.1. **Stationarity property of SLE-decorated LQG.**

In this subsection, our main goal is to prove the following theorem; see Figure 2 for an illustration. Though first appearing in [40], the idea of this theorem goes back to two previous papers [37, 6], and the technique of the proof is much similar to that of [37].

Theorem 4.1 (SLE-GFF coupling). *Fix $\kappa > 0$. Let $(\tilde{g}_t)_{t\geq 0}$ be the centered forward Loewner flow associated with a chordal SLE$_\kappa$ on $(\mathbb{H}, 0, \infty)$. Let h be a free-boundary GFF on $\mathbb{H}$ (view modulo constants) independent with $(\tilde{g}_t)_{t\geq 0}$. Write*

$$\varphi(z) := \frac{2}{\sqrt{\kappa}} \log |z| \quad \text{and} \quad Q := \frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}}. \tag{4.2}$$

Then for any fixed time $T > 0$, the following two random distributions (viewed modulo constants) on $\mathbb{H}$ agree in law:

$$\bar{h}_0 := h + \varphi, \tag{4.3}$$

$$\bar{h}_T := \bar{h}_0 \circ \tilde{g}_T^{-1} + Q \log |(\tilde{g}_T^{-1})'|. \tag{4.4}$$

Under the setting of Theorem 4.1, let η_0 be the SLE curve associated with $(\tilde{g}_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ be the Loewner chain associated with η_0 (which is equivalent to $(\eta_0[0,t])_{t\geq 0}$ when $\kappa \leq 4$). Then $\tilde{g}_T$ is a conformal mapping from $\mathbb{H} \setminus K_T$ to $\mathbb{H}$, sending $\eta_0(t)$ to 0 and ∞ to ∞. In addition, (4.4) is essentially the coordinate change equation (3.39). Therefore, if $\gamma \in \{\sqrt{\kappa}, 2/\sqrt{\kappa}\}$, then Theorem 4.1 actually says that $(\mathbb{H}, \bar{h}_0, 0, \infty)$ (surface without any exploration) and $(\mathbb{H} \setminus K_T, \bar{h}_0|_{\mathbb{H}\setminus K_T}, \eta_0(T), \infty)$ (surface explored along η_0 until time T) have the same law as quantum surfaces with two marked points (recall Definition 3.41). Hence the theorem can be seen as a stationarity property of SLE-decorated LQG. However, it is worth pointing out that the statement of Theorem 4.1 is unrelated with LQG or quantum surfaces, and unlike other main theorems, the result of Theorem 4.1 remains true even if $\kappa \geq 4$.

Remark 4.5. From the derivation above we can see that the "right" relationship between κ (as the parameter of an SLE curve) and γ (as the parameter of an LQG surface) is $\kappa \in \{\gamma^2, 16/\gamma^2\}$, since under this condition we have

$$\frac{\sqrt{\kappa}}{2} + \frac{2}{\sqrt{\kappa}} = \frac{\gamma}{2} + \frac{2}{\gamma}, \tag{4.6}$$

and (4.4) can thus be translated into the coordiate change in (3.39).

To prove Theorem 4.1, we use the uniqueness of GFF as a Gaussian process (recall Definition 3.15). The key point of the proof is to construct two (local) martingales by means of Itô's calculus. This technique is widely used to study SLE [1, 35, 22, 23], and similar constructions also appeared in [37]. To construct the first martingale, one need to select a proper constant Q, while the construction of the second martingale is kind of a miracle and relies heavily on the boundary condition (of Green's function).

Proof of Theorem 4.1. Let the driving function of $(\tilde{g}_t)_{t\geq 0}$ be $\sqrt{\kappa}B_t$, where B_t is a standard Brownian motion independent with h. Let $(\hat{B}_t)_{0\leq t\leq T} := (B_T - B_{T-t})_{0\leq t\leq T}$ be the time reversal of B_t. Let $(\tilde{f}_t)_{0\leq t\leq T}$ be the centered reverse Loewner flow (run until time T) with driving function $(\hat{B}_t)_{0\leq t\leq T}$. Then $(\tilde{f}_t)_{0\leq t\leq T}$ satisfy the SDE

$$d\tilde{f}_t = -\frac{2}{\tilde{f}_t} - \sqrt{\kappa} d\hat{B}_t, \quad \text{for } 0 \leq t \leq T. \tag{4.7}$$

By Proposition 3.64 we know that $\tilde{f}_T = \tilde{g}_T^{-1}$. In the rest of this proof, we always consider the filtrations generated by the Brownian motion $\hat{B}_t$ (not B_t!), that is, $\mathcal{F}_t := \sigma(\hat{B}_s, 0 \leq s \leq t)$. We then have the following two key observations that can be derived via Itô's formula. We omit the calculations here.

Observation 1: The first martingale.

For $t \leq T$ set

$$M_t^* := \frac{2}{\sqrt{\kappa}} \log \tilde{f}_t + Q \log \tilde{f}_t'. \tag{4.8}$$

Then for any fixed $t \geq 0$, by Itô's formula we have

$$dM_t^* = \frac{1}{\tilde{f}_t^2}\left(2Q - \frac{4}{\sqrt{\kappa}} - \sqrt{\kappa}\right) dt - \frac{2}{\tilde{f}_t} d\hat{B}_t. \tag{4.9}$$

In particular, by our choice of Q, $M_t^*(z)$ is a continuous local martingale for every fixed $z \in \mathbb{H}$. Moreover, if we let $M_t := \Re(M_t^*)$, then by (4.9) the quadratic variation between two continuous local martingales $M_t(z)$ and $M_t(w)$ is given by

$$d\langle M(z), M(w)\rangle_t = 4\Re\left(\frac{1}{\tilde{f}_t(z)}\right)\Re\left(\frac{1}{\tilde{f}_t(w)}\right) dt. \tag{4.10}$$

Observation 2: The second martingale.

We then set

$$G^{\mathbb{H}}(z,w) := -\log|z-w| - \log|z-\bar{w}| \quad \text{and} \quad G_t(z,w) := G^{\mathbb{H}}(f_t(z), f_t(w)), \tag{4.11}$$

where $G^{\mathbb{H}}(\cdot,\cdot)$ is one instance of Green's function with Neumann boundary conditions on $\mathbb{H}$. Then for every $z, w \in \mathbb{H}$ we have

$$dG_t(z,w) = -4\Re\left(\frac{1}{\tilde{f}_t(z)}\right)\Re\left(\frac{1}{\tilde{f}_t(w)}\right)dt. \tag{4.12}$$

In particular, $M_t(z)M_t(w) + G_t(z,w)$ is a continuous local martingale.

Conclusion of proof.

We now take an arbitrary smooth compactly supported function ρ on $\mathbb{H}$ with zero average. By uniqueness of GFF, it suffices to prove that $(\bar{h}_T, \rho)$ is Gaussian with mean (φ, ρ) and variance $\Gamma(\rho,\rho)$ (recall $\Gamma(\rho,\rho) = \iint_{\mathbb{H}^2} G^{\mathbb{H}}(z,w)\rho(z)\rho(w)\, d^2z\, d^2w$ in (3.14)). We have

$$(\bar{h}_T, \rho) = (M_T, \rho) + (h \circ \tilde{f}_T, \rho). \tag{4.13}$$

We claim that $(M_t, \rho)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted continuous square-integrable martingale with quadratic variation

$$\langle M, \rho\rangle_t = \iint_{\mathbb{H}^2} (G^{\mathbb{H}}(z,w) - G_t(z,w))\rho(z)\rho(w)\, d^2z\, d^2w. \tag{4.14}$$

If this is true, then by (3.16) and independence between h and $(\hat{B}_t)_{t\geq 0}$, conditioned on $\mathcal{F}_T$, $(h \circ \tilde{f}_T)$ is a Gaussian random variable with mean zero and covariance

$$\iint_{\mathbb{H}^2} G_t(z,w)\rho(z)\rho(w)\, d^2z\, d^2w.$$

Combined with (4.13) and (4.14), it yields the desired conclusion.

Given (4.10) and (4.12), (4.14) is essentially a Fubini calculation, but it requires some justification. We sketch the justification here. Indeed, from the centered reverse Loewner equation (4.7), one can show that $\Im(\tilde{f}_t)(z)$ is increasing for any $z \in \mathbb{H}$. Therefore, $\Re(f_t^{-1})$, and thus $d\langle M(z)\rangle_t$ and dG_t, are uniformly bounded in the support of ρ and for all times t. As a result, for any z in the support of ρ and any time t, $M_t(z)$ represents the value of a Brownian motion stopped at a random time that is strictly less than a constant times t, and thus has a law that decays exponentially, uniformly in z. One can then combine this and the fact that G_t has uniformly bounded increments to justify the Fubini calculation. □

Remark 4.15. Theorem 4.1 is one of the two couplings between SLE and GFF stated in [40]. The other coupling, which is closely related to imaginary geometry but is independent with other main theorem in this paper, will not be given here. But we emphasize that the flavours of these two results and techniques of their proofs are essentially the same, and it won't take us much more effort to prove these two couplings together than proving only one of them. In particular, in Theorem 4.1 we consider the free-boundary GFF and take the real part of the martingale M_t^*, while in the other coupling we consider the zero-boundary GFF and take the imaginary part of another complex martingale.

4.2. **Zipping up one LQG surface along the SLE curve.** In the rest of Section 4, we fix $\kappa \in (0,4)$ and $\gamma \in (0,2)$ such that $\kappa = \gamma^2$. This subsection is devoted to proving the following two theorems; see Figure 3 for an illustration.

Theorem 4.16 (Capacity Zipper)**.** *There exists a two-sided stationary process* $((\bar{h}_t, \eta_t))_{t\in\mathbb{R}}$ *such that:*

(1) ***(Marginal law).*** *$\bar{h}_0$ is the same as in (4.3) (a free-boundary GFF (viewed modulo constants) plus the function $2\gamma^{-1}\log|z|$), and η_0 is a chordal SLE$_\kappa$ curve on $(\mathbb{H}, 0, \infty)$ independent with $\bar{h}_0$;*

(2) ***(Zipping).*** *Let $(\tilde{g}_t)_{t\geq 0}$ be the centered forward Loewner flow associated with η_0 (recall Definition 3.55). Then for any $t > 0$, $(\bar{h}_{-t}, \eta_{-t}) = (\tilde{g}_t(\bar{h}_0), \tilde{g}_t(\eta_0))$ (recall (3.39) for notation). Note that by stationarity, this defines the law of process for all time.*

To state Theorem 4.17, the quantum zipper theorem, we need to recall some definitions and properties in Section 3. Recall that for an α-quantum wedge $\tilde{h}$ on $\mathbb{H}$ with parameter $\alpha < Q$, one can associate it with an LQG boundary measure $\nu_{\tilde{h}}$ as in Definition 3.34 that satisfies the coordinate-change formula as in Proposition 3.38. Moreover, thanks to coordinate change and Riemann mapping theorem, we can actually define $\nu_{\tilde{h}}$ for those D (whose boundary may not be piecewise linear) such that D is a simply connected Jordan domain on the Riemann sphere. In Theorem 4.17, we will take D as $\mathbb{H} \setminus K_t$ for any $t \geq 0$ where $(K_t)_{t\geq 0}$ is the Loewner chain generated by an SLE$_\kappa$ for some fixed $\kappa < 4$. Here we will use the fact that SLE$_\kappa$ is a Hölder continuous curve; see Remark 3.58. Note that $\nu_{\tilde{h}}$ is defined on the martin boundary of $H \setminus K_t$, so a priori we do not know that under $\nu_{\tilde{h}}$, the left hand side of the curve has the same LQG boundary length as the right hand side of the curve.

Theorem 4.17 (Quantum zipper)**.** *There exists a two-sided process $(\mathbb{H}, \tilde{h}_t, 0, \infty, \zeta_t)_{t\in\mathbb{R}}$ that is stationary as a process of curve-decorated quantum surface with two marked points (recall Definition 3.41) such that:*

(1) ***(Marginal law).*** *$(\mathbb{H}, \tilde{h}_0, 0, \infty)$ is a $(\gamma - 2/\gamma)$-quantum wedge in the circle average embedding (recall Definition 3.46), and η_0 is a chordal SLE$_\kappa$ curve on $(\mathbb{H}, 0, \infty)$ independent with $\tilde{h}_0$;*

(2) ***(Zipping).*** *Let $\nu_{\tilde{h}_0}$ be the LQG boundary measure associated with $\tilde{h}_0$. For any $t > 0$, define $\sigma(t)$ to be the first time that the LQG boundary measure length (under $\nu_{\tilde{h}_0}$) of the right hand side of $\zeta_0([0, \sigma(t)])$ reaches t, that is,*

$$\sigma(t) := \inf\{s \geq 0 : \nu_{\tilde{h}_0}(\textit{RHS of } \zeta_0([0, s]) \geq t\}.$$

Let $(\tilde{g}_t)_{t\geq 0}$ be the centered forward Loewner flow associated with ζ_0. Then for any $t > 0$, $(\tilde{h}_{-t}, \zeta_{-t}) = (\tilde{g}_{\sigma(t)}(\tilde{h}_0), \tilde{g}_{\sigma(t)}(\zeta_0))$. Note that by stationarity, this defines the law of process for all time.

Similar to Section 4.1, Theorem 4.16 can also be interpreted into a result about a two-sided stationary process of curve-decorated quantum surface (viewed modulo additive constants). In this case, Theorem 4.16 and Theorem 4.17 have the same flavour: moving forward in time corresponds to "zipping up", welding the positive and negative real line together, while moving backward in time corresponds to "zipping down", cutting the upper half plane open along the SLE curve; see also Figure 3. However, unlike Theorem 4.16, which is stated without introducing LQG surfaces, Theorem 4.17 essentially uses quantum wedge as well as the equivalence class of quantum surfaces modulo embeddings.

We also make some remarks on the parameters in Theorem 4.17. There are three parameters that matter: the parameter κ for the SLE curve, the parameter γ for the LQG surface, and the parameter α for the quantum wedge. Note that α can

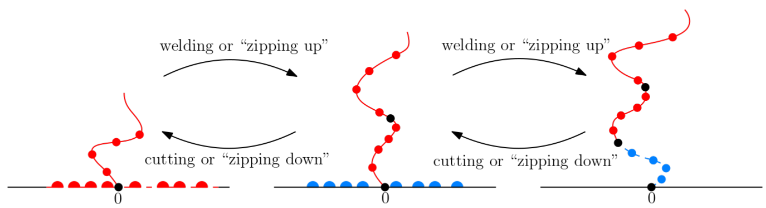

FIGURE 3. An illustration of the two zippers in Theorem 4.16 and Theorem 4.17. Here the negative and positive real line are two "strands" of the zippers. Under welding or "zipping up", semicircular dots on $\mathbb{R}$ will be "zipped together" into circular dots at the origin, while under cutting or "zipping down", circular dots will be pulled apart into semicircular dots at the origin.

also be interpreted as the order of the log-singularity at the origin. It turns out that we must have

$$\alpha = -\frac{2}{\sqrt{\kappa}} + \gamma.$$

Roughly speaking, the term $-2/\sqrt{\kappa}$ comes from Theorem 4.16, while the term γ comes from the order of log-singularity of a quantum typical point. To be more specific, intuitively, $\nu_{\bar{h}_0}$ should be supported on $\{\zeta_0(\sigma(t))\}_{t\geq 0}$. Therefore, by Proposition 3.35, $\bar{h}_0$ should have log-singularity of order γ on $\{\zeta_0(\sigma(t))\}_{t\geq 0}$. So for every $t > 0$, $\tilde{h}_{-t} = \tilde{g}_{\sigma(t)}(\tilde{h}_0)$ should have another log-singularity of order γ at the origin. The only place that $\kappa = \gamma^2$ is needed here is to guarantee that (4.6) holds, which ensures that (4.4) is equivalent to coordinate changes of γ-LQG surfaces.

Proof of Theorem 4.16. Let $(\bar{h}_0, \eta_0)$ be as in Section 4.1. Suppose $(\tilde{g}_t)_{t\geq 0}$ is the centered reverse Loewner flow associated with η_0. For any $t > 0$, let $(\bar{h}_{-t}, \eta_{-t}) := (\tilde{g}_t(\bar{h}_0), \tilde{g}_t(\eta_0))$. Let T be a positive number. By Proposition 3.59, Theorem 4.1 and the independence between $\bar{h}_0$ and η_0, the laws of the one-sided processes $((\bar{h}_{t-T}, \eta_{t-T}))_{t\in(-\infty,T]}$ are consistent as T increases to ∞, and the result follows from Kolmogorov's extension theorem. □

For future reference, we state as a proposition a property of the stationary process $(\bar{h}_t, \eta_t)_{t\in\mathbb{R}}$, which is a corollary of Theorem 4.16 and Proposition 3.64.

Proposition 4.18. *Let $((\bar{h}_t, \eta_t))_{t\geq 0}$ be the nonnegative part of the stationary two-sided process in Theorem 4.16. Then there exists a family of conformal maps $(\tilde{f}_t)_{t\geq 0} : \mathbb{H} \mapsto \mathbb{H} \setminus \eta_t([0,t])$ such that $\bar{h}_t|_{\tilde{f}_t(\mathbb{H})} = \tilde{f}_t(\bar{h}_0)$ and $\eta_t([t,\infty]) = \tilde{f}_t(\eta_0)$. In addition, the marginal law of $(\tilde{f}_t)_{t\geq 0}$ is that of a centered reverse Loewner flow associated with a reverse SLE$_\kappa$, parameterized by capacity. Note that for any $t > 0$, since $\bar{h}_t$ is independent with η_t, $(\tilde{f}_s)_{0\leq s\leq t}$ are independent with $\bar{h}_t$.*

We shall emphasize that $(\tilde{f}_t)_{t\geq 0}$ are *not* independent with $\bar{h}_0$. Indeed, they are determined by $\bar{h}_0$; see Remark 4.33.

Theorem 4.17 is a much harder result. To prove Theorem 4.17, it suffices to prove the following lemma, which can be viewed as the counterpart of Theorem 4.1.

Lemma 4.19. *Let $(\mathbb{H}, \tilde{h}, 0, \infty)$ be a $(\gamma - 2/\gamma)$-quantum wedge in the circle average embedding, and $\nu_{\tilde{h}}$ be the LQG boundary measure associated with $\tilde{h}$. Let ζ be a* *chordal SLE$_\kappa$ curve on $(\mathbb{H}, 0, \infty)$ independent with $\tilde{h}$, and $(\tilde{g}_t)_{t\geq 0}$ be the centered forward Loewner flow assoiated with ζ. For any $T > 0$, denote by $\sigma(T)$ the smallest time such that the $\nu_{\tilde{h}}$ boundary length of the right hand side of $\zeta([0, \sigma(T)])$ reaches T. Then for any $T > 0$, $(\tilde{g}_{\sigma(T)}(\tilde{h}), \tilde{g}_{\sigma(T)}(\zeta))$ has the same law as $(\tilde{h}, \zeta)$ (modulo embeddings).*

Proof of Theorem 4.17 given Lemma 4.19. The proof follows in exactly the same way as in the proof of Theorem 4.17 given Theorem 4.1. □

Using the notations in Proposition 4.18, we now briefly explain the proof of Lemma 4.19, which consists of three steps; see also Figure 4. The first step is "reweighting": we consider a "quantum typical point" X sampled from the LQG boundary measure $\nu_{\bar{h}_0}$, and then compute the conditional law of $((\bar{h}_t, \eta_t))_{t\geq 0}$ given X. Since considering the conditioned law of h is equivalent to tilting h, we can use Girsanov's theorem on the Gaussian field h as well as the driving Brownian motion of $(\tilde{f}_t)_{t\geq 0}$. Let τ_X be the first time that $\tilde{f}_{\tau_X}(X) = 0$. Combining Girsanov's theorem with the same techniques as in the proof of Theorem 4.16, it turns out that for $t \in [0, \tau_X]$, the conditional law of $\bar{h}_t$ has an extra γ-singularity at the point $\tilde{f}_t(X)$ (which is reasonable given Proposition 3.35), and $(\eta_t[0,t])_{0\leq t\leq \tau_X}$ has the law of a reverse SLE$_\kappa(\kappa, -\kappa)$ process (recall Definition 3.70) where X is one of the force points. In particular, $\bar{h}_{\tau_X}$ has a $(\gamma - 2/\gamma)$-singularity at the origin and an initial segment of η_{τ_X} behaves like a forward SLE$_\kappa$ curve due to the symmetry for forward/reverse SLE (recall Proposition 3.65 and Proposition 3.77). However, for $t \in [0, \tau_X]$, $\bar{h}_t$ and η_t are *not* independent after this reweighting.

The second step is "zooming": one zooms in $(\bar{h}_{\tau_X}, \eta_{\tau_X})$ at zero, or in other words, adds a constant C to the field $\bar{h}_{\tau_X}$ and let C tend to ∞. The key point here is that through zooming in one can ignore the secondary terms, thus creating a $(\gamma - 2/\gamma)$-quantum wedge decorated with an *independent* SLE$_\kappa$ curve (both measured under the reweighted measure). Then in the third step we construct a coupling to prove stationarity. The key observation is that, the point X_C that has quantum distance 1 to X under the zoomed-in measure $\nu_{\bar{h}_{\tau_X}+C}$ has almost the same law as X when C is very large.

Proof of Lemma 4.19. Let $(\bar{h}_t, \eta_t)_{t\geq 0}$ and $(\tilde{f}_t)_{t\geq 0}$ be as in Proposition 4.18, where we fix the constant for $\bar{h}_0$ (which further fixes the constants for $(\bar{h}_t)_{t\geq 0}$ due to their construction) so that its unit upper semicircle average around 10 is zero. Note that the way of fixing the constant is rather arbitrary as long as the measure is supported a good distance away from the origin (recall Proposition 3.27 and Proposition 3.53). We now elaborate the three steps mentioned above. We denote the law of $(\bar{h}_t, \eta_t)_{t\geq 0}$ as $\mathbb{P}$ and set $\mathbf{P} := \mathbb{P} \times \mathrm{Leb}[1,2]$ where Leb$[1,2]$ is the Lebesgue measure on $[1,2]$. For every $x \in \mathbb{R}$, let τ_x be the first time that $\tilde{f}_t(x) = 0$. For $x \in \mathbb{R}$ and $r > 0$, write $\rho_{x,r}$ for the uniform measure on the upper semicircle of radius r around x and $\bar{\rho}_{x,r} := \rho_{x,r} - \rho_{10,1}$.

Step 1: Reweighting.

We consider the law $\mathbf{Q}$ on the process $(\bar{h}_t, \eta_t)_{t\geq 0}$ plus a point $Z \in [1,2]$ such that

- the marginal law of $(\bar{h}_t, \eta_t)_{t\geq 0}$ is given by $\nu_{\bar{h}_0}([1,2])d\,\mathbb{P}$ (normalized to be a probability measure);
- conditioned on $(\bar{h}_t, \eta_t)_{t\geq 0}$, X is chosen uniformly from $\nu_{\bar{h}_0}$ on $[1,2]$.

Our goal is to show that the following holds:

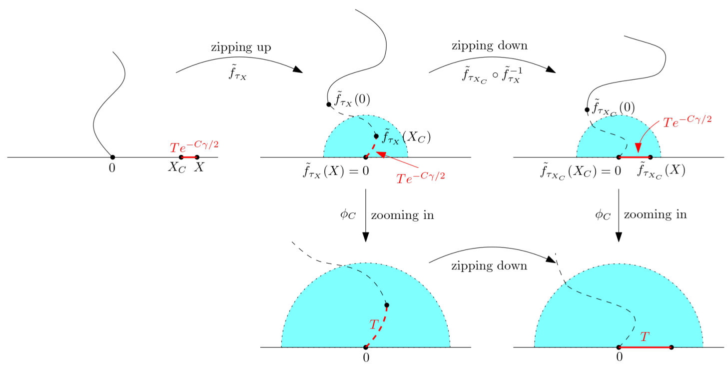

FIGURE 4. An illustration of the proof of Lemma 4.19. Pictures in the same row are in the same zipping process, while pictures in the same column can be transferred into each other via zooming. In addition, X_C has almost the same law as X as $C \to \infty$, and the quantum boundary distances of $[X_C, X]$, which are marked with red, remain the same in each row due to coordinate-change formula. Lemma 4.19 follows from the fact that the diagram consisting of the four pictures on the right is commutative. **Top left:** From Proposition 3.35 we know that $\bar{h}_0$ should have a log-singularity of order γ at quantum typical points X_C and X, and $(\eta_t[0,t])_{t\geq 0}$ becomes a reverse $\mathrm{SLE}_\kappa(\kappa, -\kappa)$. **Top middle and top right:** When X or X_C is absorbed to the origin, the order of log-singularity at the origin becomes $\gamma - 2/\gamma$, and the initial segment of the reverse $\mathrm{SLE}_\kappa(\kappa, -\kappa)$ behaves like a forward SLE_κ. **Bottom middle and bottom right:** A $(\gamma - 2/\gamma)$-quantum wedge decorated with an independent forward SLE_κ, obtained after zooming in the top middle or top right picture.

(a) Conditioned on X, for every $t \in [0, \tau_X]$, the law of $\eta_t[0,t]$ is that of a reverse $\mathrm{SLE}_\kappa(\kappa, -\kappa)$ with force points $X, 10$, run up to time t;

(b) Conditioned on X, for every $t \in [0, \tau_X]$ we can write

$$\bar{h}_t \overset{d}{=} h + \frac{2}{\gamma}\log|\cdot| + \frac{\gamma}{2}G^{\mathbb{H}}(\cdot, \tilde{f}_t(X)) - \frac{\gamma}{2}\int_{\mathbb{H}} G^{\mathbb{H}}(\cdot, \tilde{f}_t(z))\rho_{10,1}(z)d^2z,$$

where h has the law of a free-boundary GFF that is independent of $(\tilde{f}_s)_{0\leq s\leq t}$, and the way of fixing the additive constant for h is determined by that of $\bar{h}_t$.

To prove (a) and (b), we first prove similar results where $\bar{h}_0$ is substituted by its circle averages $\bar{h}_0^{(\varepsilon)}$, and then take $\varepsilon \to 0$. For every $\varepsilon > 0$, define

$$\bar{h}_0^{(\varepsilon)}(dx) := (\bar{h}_0, \bar{\rho}_{x,\varepsilon}) \quad \text{and} \quad \nu_{\bar{h}_0^{(\varepsilon)}}(x) := e^{\gamma \bar{h}_0^{(\varepsilon)}(x)}\varepsilon^{\gamma^2/4}dx.$$

For every $\varepsilon > 0$,define the probability measure $\mathbf{Q}_\varepsilon$ in the same way as $\mathbf{Q}$, except that we change $\bar{h}_0$ and $\nu_{\bar{h}_0}$ into $\bar{h}_0^{(\varepsilon)}$ and $\nu_{\bar{h}_0^{(\varepsilon)}}$. Then the Radon-Nikodym measure of $\mathbf{Q}_\varepsilon$ with respect to $\mathbf{P}$ is $e^{\gamma \bar{h}_0^{(\varepsilon)}(X)/2}$ times some normalizing constant.

Using the same notations as in the proof of Theorem 4.16, we define $(\mathcal{F}_t)_{t\geq 0} := (\sigma\{\hat{B}_s, 0 \leq s \leq t\})_{t\geq 0}$, where $(\hat{B}_t)_{t\geq 0}$ is the driving function of $(\tilde{f}_t)_{t\geq 0}$. Note that from Proposition 4.18, $\mathcal{F}_t$ is independent with $\bar{h}_t$ but is not independent with $\bar{h}_0$. For $t \le \tau_{X-\varepsilon}$, we define M_t^*, $G^{\mathbb{H}}$, G_t as in (4.8), (4.11), and also define $M_t := \Re(M_t^*)$. Then similar to (4.13), we have the decomposition

$$\bar{h}_0^{(\varepsilon)}(X) = (\bar{h}_0, \bar{\rho}_{X,\varepsilon}) = (M_t, \bar{\rho}_{X,\varepsilon}) + (\bar{h}_t - \frac{2}{\gamma}\log|\cdot|, \bar{\rho}_{X,\varepsilon}). \tag{4.20}$$

Since $\mathbf{P}$ is a product measure, under $\mathbf{P}(\cdot \mid X)$, the first term of the right hand side of (4.20) is still an $(\mathcal{F}_t)_{t\ge 0}$-adapted continuous square-integrable martingale with quadratic variation $\langle M, \bar{\rho}_{X,\varepsilon}\rangle_t$, and the second term is still a Gaussian random variable independent with $\mathcal{F}_t$ with mean zero and variance

$$-\langle M, \bar{\rho}_{X,\varepsilon}\rangle_t + \text{some constant } \left(\text{which is } \iint_{\mathbb{H}^2} G^{\mathbb{H}}(z,w)\rho(z)\rho(w)\,d^2z\,d^2w\right).$$

Combine this with (4.20) and we get that for $t \le \tau_{X-\varepsilon}$, the Radon-Nikodym derivative of $\mathbf{Q}_\varepsilon(\cdot \mid X)|_{\mathcal{F}_t}$ with respect to $\mathbb{P}|_{\mathcal{F}_t}$ is exactly the exponential martingale

$$\exp\left(\frac{\gamma}{2}(M_t, \bar{\rho}_{X,\varepsilon}) - \frac{\gamma^2}{8}\langle M, \bar{\rho}_{X,\varepsilon}\rangle_t\right).$$

Moreover, by Schwartz reflection and the mean value theorem we have $(M_t, \bar{\rho}_{X,\varepsilon}) = M_X - M_{10}$ for $t \le \tau_{X-\varepsilon}$. Then by Girsanov's theorem we get that under $\mathbf{Q}_\varepsilon(\cdot \mid X)$, $\hat{B}_t - \frac{\gamma}{2}\langle \hat{B}, M_X - M_{10}\rangle_t$ is a standard Brownian motion for $t \le \tau_{X-\varepsilon}$. It then follows from (4.9) that under $\mathbf{Q}_\varepsilon(\cdot \mid X)$, $d\hat{B}_t$ has a drift term of $\gamma\Re(\tilde{f}_t(10)^{-1} - \tilde{f}_t(X)^{-1})dt$. Combining this with with Definition 3.70 and (3.71), we find that (a) holds under $\mathbf{Q}_\varepsilon$ until time $\tau_{X-\varepsilon}$. By taking $\varepsilon \to 0$ we find that (a) holds for $\mathbf{Q}$.

Note that by Proposition 4.18 we have

$$\bar{h}_0^{(\varepsilon)}(X) = (\bar{h}_0, \bar{\rho}_{X,\varepsilon}) = (\bar{h}_t|_{\tilde{f}_t(\mathbb{H})}, \bar{\rho}_{x,\varepsilon} \circ \tilde{f}_t^{-1}). \tag{4.21}$$

By Proposition 4.18 we know that $\bar{h}_t$ is independent with $(\tilde{f}_s)_{0\le s\le t}$, and is distributed like a free boundary GFF plus a deterministic function $\gamma\log|\cdot|/2$. Applying Cameron-Martin theorem on the Gaussian process $\bar{h}_t$ and using change of variable formula, for $0 \le t \le \tau_X$, under measure $\mathbf{Q}_\varepsilon(\cdot \mid X, (\tilde{f}_s)_{0\le s\le t})$,

$$\bar{h}_t \overset{d}{=} h + \frac{2}{\gamma}\log|\cdot| + \frac{\gamma}{2}\int_{\mathbb{H}} G^{\mathbb{H}}(\cdot, \tilde{f}_t(z))\bar{\rho}_{X,\varepsilon}(z)d^2z. \tag{4.22}$$

Since Green's function is harmonic, for any $0 \le t \le \tau_{X-\varepsilon}$ and any $w \in \tilde{f}_t(\mathbb{H} \setminus B(w,\varepsilon))$ (here we use $B(w,\varepsilon)$ to denote the circle with radius ε around w),

$$\int_{\mathbb{H}} G^{\mathbb{H}}(w, \tilde{f}_t(z))\rho_{X,\varepsilon}(z)d^2z = G^{\mathbb{H}}(w, \tilde{f}_t(X)).$$

Combining this with (4.22), we find that (b) holds under $\mathbf{Q}_\varepsilon$ until time $\tau_{X-\varepsilon}$. By taking $\varepsilon \to 0$ we find that (b) holds for $\mathbf{Q}$.

Step 2: Zooming.

Taking $t = \tau_X$ in (b) we get that under $\mathbf{Q}(\cdot \mid X)$,

$$\bar{h}_{\tau_X} \overset{d}{=} h - \left(\gamma - \frac{2}{\gamma}\right)\log|\cdot| + \text{ some smooth terms}, \tag{4.23}$$

where h is a free-boundary GFF on $\mathbb{H}$ (with additive constant fixed) that is independent of $(\tilde{f}_t)_{0\le t\le \tau_X}$, and the extra log-singularity of order γ comes from the Green's function. To use Proposition 3.53, we need to analyze the additive constant of $\bar{h}_{\tau_X}$ and also the last term on the right of (4.23). We claim that $\tilde{f}_{\tau_X}(B(10,1))$ is supported away from the origin. This is because by the reverse Loewner equation (3.71), under $\mathbf{Q}(\cdot \mid X)$, for $w \in B(10,1)$ and $X \in (1,2)$,

$$\frac{\partial \left(\Re(\tilde{f}_t(w) - \tilde{f}_t(z)\right)}{\partial t} = \frac{2}{\tilde{f}_t(z)} - \Re\left(\frac{2}{\tilde{f}_t(w)}\right) = \frac{2}{\tilde{f}_t(z)} - \frac{2\Re\left(\tilde{f}_t(w)\right)}{\left|\tilde{f}_t(w)\right|^2},$$

which is positive as long as $\Re(\tilde{f}_t(w)) > \tilde{f}_t(X) > 0$. Since this inequality is true at time zero, we get that $\Re(\tilde{f}_t(w)) - \tilde{f}_t(X)$ is increasing for all $t \le \tau_X$, and the claim thus follows.

Now for every $C > 0$, let $\phi_C(\bar{h}_{\tau_X} + C)$ be the circle average embedding of $\bar{h}_{\tau_X} + C$. By Proposition 3.53, as $C \to \infty$, $\phi_C(\bar{h}_{\tau_X} + C)$ converges (in the sense of Definition 3.51) to a $(\gamma - 2/\gamma)$-quantum wedge $\tilde{h}$.

It also follows by Lemma 4.19 that under $\mathbf{Q}(\cdot \mid X)$, $B(0,\delta) \cap \eta_{\tau_X}$ can be coupled with the intersection of $B(0,\delta)$ and a reverse $\mathrm{SLE}_\kappa(\kappa)$ with a single force point at X and run up to the first time X is absorbed to the origin, so that they are equal with arbitrary high probability as $\delta \to 0$. By Proposition 3.77, the latter can further be coupled with the intersection of $B(0,\delta)$ and a chordal forward SLE_κ so that they are equal with arbitrary high probability as $\delta \to 0$. Therefore as $C \to \infty$, under $\mathbf{Q}(\cdot \mid X)$, $\phi_C(\eta_{\tau_X})$ converges in law to an SLE_κ curve ζ on $(\mathbb{H}, 0, \infty)$. The independence of $\tilde{h}$ and ζ follows from the conditional independence between h and η_{τ_X} since the former is independent of $\mathcal{F}_{\tau_X}$ while the latter is measurable with respect to $\mathcal{F}_{\tau_X}$.

Step 3: Coupling.

Note that for any X, under $\mathbf{Q}(\cdot \mid X)$, the pair $(\tilde{h}, \zeta)$ constructed in Step 2 has the law of a $(\gamma - 2/\gamma)$ quantum wedge decorated by an independent SLE_κ curve. Therefore, $(\tilde{h}, \zeta)$ still has this law without any conditioning under the reweighted measure $\mathbf{Q}$. Let $\sigma(T)$ be as in Lemma 4.19. Now given a sample $((\tilde{h}_t, \eta_t)_{0 \le t \le \tau_X}, X)$ from $\mathbf{Q}$ and $C > 0$, let $X_C \in [0, X]$ be such that $\nu_{\bar{h}_0}([X_C, X]) = Te^{-C\gamma/2}$. If this is not possible, then we set $X_C = 0$. Note that X_C exists with arbitrary high probability as $C \to \infty$. In addition, if X_C exists, then $\nu_{\bar{h}_0 + C}([X_C, X]) = T$.

Since the total variation distance between X_C and X goes to zero as $C \to \infty$ due to the fact that $\nu_{\bar{h}_0}([1,2])$ is finite a.s., the total variation distance (modulo embeddings and measured under $\mathbf{Q}$) between the law of $(\bar{h}_{\tau_{X_C}} + C, \eta_{\tau_{X_C}})$ and the law of $(\tilde{h}, \zeta)$ tends to zero as $C \to \infty$. By Proposition 3.38, if X_C exists, zipping down $(\bar{h}_{\tau_X} + C, \eta_{\tau_X})$ along η_{τ_X} for LQG boundary length T (in a similar sense as in the statement of Lemma 4.19) yields $(\bar{h}_{\tau_{X_C}} + C, \eta_{\tau_{X_C}})$. The conclusion then follows from taking C to infinity. □

Remark 4.24. Fix $\gamma \in [0.2)$. Using the same technique as the last proof, one can show that a γ-quantum wedge on $\mathbb{H}$ can be obtained by zooming in at a quantum typical point of a free-boundary GFF (where the LQG boundary length has parameter γ), and that its law is stationary with respect to shifting the origin by a given amount of quantum length. Note that the latter result is a generalization of the translation invariance of $\mathbb{H}$, which is simply the $\gamma = 0$ case.

4.3. **A natural random length measure of SLE.** In this subsection, we will prove the following theorem; see Figure 5 for an illustration.

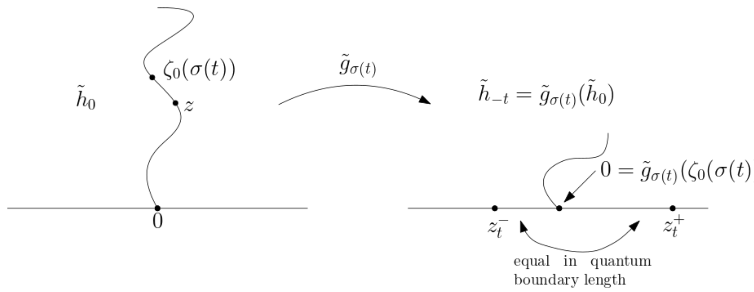

FIGURE 5. An illustration of Theorem 4.25, which says that the line segments $[z_t^-, 0]$ and $[0, z_t^+]$ have the same quanmtum boundary length under $\nu_{\tilde{h}_{-t}}$. Note that by Corollary 4.26, the result still holds if $\tilde{h}_0$ is replaced with some general non-centered GFF.

Theorem 4.25 (Quantum length of SLE)**.** *Fix $\gamma \in (0,2)$ and $\kappa = \gamma^2$. Let $(\mathbb{H}, \tilde{h}_t, 0, \infty, \zeta_t)_{t\in\mathbb{R}}$, $\nu_{\tilde{h}_0}$, $(\tilde{g}_t)_{t\geq 0}$, and $(\sigma(t))_{t\geq 0}$ be as in Theorem 4.17. For every $t \in \mathbb{R}$, let $\nu_{\tilde{h}_t}$ be the LQG boundary measure (on $\mathbb{R}$) associated with $\tilde{h}_t$. For every $t > 0$ and $z \in \zeta_0([0, \sigma(t)])$, let $z_t^- < z_t^+$ (they are both in $\mathbb{R}$) be the two images of z under $\tilde{g}_{\sigma(t)}$. Then for every $t > 0$, almost surely,*

$$\nu_{\tilde{h}_{-t}}([z_t^-, 0]) = \nu_{\tilde{h}_{-t}}([0, z_t^+])$$

holds for all $z \in \zeta_0([0, \sigma(t)])$.

In one word, Theorem 4.25 says that, under the LQG boundary measure $\nu_{\tilde{h}_0}$, the right hand side of ζ_0 has the same quantum boundary length as the left hand side of ζ_0. Note that then we can unambiguously regard the LQG boundary length measure as a natural (random) length measure of SLE.

Before proving Theorem 4.25, let us state a corollary of it.

Corollary 4.26. *The result of Theorem 4.25 still holds if we replace $\tilde{h}_0$ with $h + \varphi$ and $(\tilde{h}_t)_{t\geq 0}$ as $(\tilde{g}_t(h+\varphi))_{t\geq 0}$, where h is a free boundary GFF on $\mathbb{H}$ (with arbitrary fixed additive constant) and φ is a smooth function on $\mathbb{H}$.*

Proof. First, since multiplying the quantum boundary length by a random constant does not affect the result, we see that the additive constant of h can be arbitrary. Now if $\varphi = -(\gamma - 2/\gamma)\log|\cdot|$ and the additive constant for h is fixed so that its unit semicircle average is zero, then the restriction of h to $\mathbb{H} \cap B(0,1)$ has the same law as that of $\tilde{h}_0$ (under the circle average embedding). Therefore the statement is true, at least when restricted to $\mathbb{H} \cap B(0,1)$. Scale-invariance of h (viewed modulo constants) implies that the result also holds when the field is restricted to any large disk. Finally, for any compact subset D of $\mathbb{H}$, adding another smooth function φ to h affects the law of the restriction of $h + \varphi$ to D in an absolutely continuous way. So for general h and φ, the left and right quantum boundary length of ζ_0 on D a.s. agree. The conclusion then follows from expanding D to $\mathbb{H}$. $\square$

Given Theorem 4.17, we can use ergodic theorem to prove that the ratio of the left quantum boundary length of $\zeta_0[(0, \sigma(t)]$ to t (the right quantum boundary length of $\zeta_0[(0,\sigma(t)])$ is a (random) constant independent of t. We then use an zero-one law argument to prove that this constant is deterministic, which then must be one by symmetry.

Proof of Theorem 4.25. Denote by $L(t)$ the left quantum boundary length of $\zeta_0([0,\sigma(t)])$, that is, $\nu_{\tilde{h}_{-t}}([\zeta_0(\sigma(t))_t^-, 0])$. It suffices to show that $L(t) \equiv t$.

By stationarity of the quantum zipper (see Theorem 4.17), we know that $(L(t))_{t\geq 0}$ has stationary increments. Then by ergodic theorem we have

$$\frac{L(n)}{n} \to Y \text{ a.s. as } n \to \infty \text{ in } \mathbb{Z}, \tag{4.27}$$

where Y is a random variable that is allowed to be infinity. (Note that in the ergodic theorem, we can use a truncation argument and the monotone convergence theorem to weaken the requirement that $L(1)$ is L^1-integrable to the condition that $L(1)$ is almost surely positive, which is the case here.) Then by the scale-invariance of quantum wedge and SLE (see Proposition 3.50 and Proposition 3.59), for every $C>0$ and $t>0$,

$$\frac{L(Ct)}{Ct} \overset{d}{=} \frac{L(t)}{t}.$$

Combine this with (4.27) we see that almost surely,

$$\frac{L(t)}{t} = Y, \quad \forall t \in \mathbb{Q}. \tag{4.28}$$

Let $\mathcal{G}_\delta$ denote the σ-algebra generated by the restriction of $\tilde{h}_0$ and ζ_0 to $\mathbb{H}\cap B(0,\delta)$. Then by (4.28), Y is $\cap_{\delta>0}\mathcal{G}_\delta$-measurable.

Let h be a free boundary GFF with the additive constant fixed so that its average on the unit semicircle is zero. Then by Definition 3.46, when restricted to $\mathbb{H}\cap B(0,1)$, $\tilde{h}_0$ (under the circle average embedding) has the same law as $h-(\gamma - 2/\gamma)\log|\cdot|$. So by Proposition 3.28 and the Blumenthal zero-one law we see that Y is deterministic, and then must be one by left-right symmetry. □

4.4. **Conformal welding of two LQG surfaces.** We will try to prove the following theorem in this subsection; see Figure 6 for an illustration.

Theorem 4.29 (Conformal welding)**.** *Fix $\gamma\in(0,2)$ and $\kappa=\gamma^2$. Let $(\mathbb{H},\tilde{h},0,\infty)$ be a $(\gamma-2/\gamma)$-quantum wedge in the circle average embedding, and ζ be an independent SLE_κ on $(\mathbb{H},0,\infty)$. Let D^L and D^R be the left and right components of $\mathbb{H}\setminus\zeta$. Let $\tilde{h}^{D^L}$ and $\tilde{h}^{D^R}$ be the restrictions of $\tilde{h}$ to D^L and D^R. Then the following holds.*

(1) $(D^L, h^{D^L}, 0, \infty)$ and $(D^R, h^{D^R}, 0,\infty)$ are two independent γ-quantum wedges.
(2) $(\tilde{h},\zeta)$ is almost surely determined by the two quantum surfaces $(D^L, h^{D^L}, 0,\infty)$ and $(D^R, h^{D^R}, 0,\infty)$.

Remark 4.30. In fact, since ζ is independent of $\tilde{h}$ and is scale-invariant in law, the choice of embedding for the quantum wedge $\tilde{h}$ does not matter as long as it is independent with ζ. The same thing also hold for Theorem 4.17 and Theorem 4.25.

Let us first consider the proof of (1); see also Figure 7. We appeal to the proof of Theorem 4.17 and consider the one-sided stationary process $((\bar{h}_t,\eta_t))_{t\geq 0}$ in Proposition 4.18 as well as a quantum typical point X, under the measure $\mathbf{Q}$. Let X^L be the point on the negative real line such that $\nu_{\bar{h}_0}[(X^L,0)] = \nu_{\bar{h}_0}[(0,X)]$.

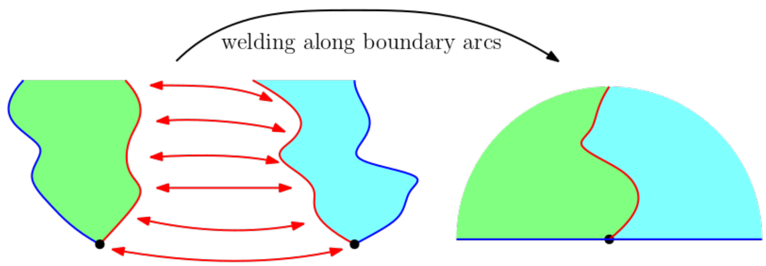

FIGURE 6. An illustration of Theorem 4.29. **Left:** two independent γ-quantum wedges (under some arbitrary embedding), which will be glued together along boundary arcs in a quantum-boundary-length-preserving way. **Right:** A $(\gamma - 2/\gamma)$ quantum wedge (under the circle average embedding) decorated with an independent SLE_κ on $(\mathbb{H}, 0, \infty)$, obtained after welding.

By Theorem 4.25 we know that X and X^L will be welded into one point, and the right or the left quantum surface is obtained from zooming in near X or X^L respectively. Note that both X and X_L are quantum typical points, so by Proposition 3.35, $\bar{h}_0$ should have a log-singularity of order γ at X (here the log-singularity at the origin should not be counted). Then heuristically, after zooming in under $\mathbf{Q}(\cdot \mid X)$ (see Step 2 of the proof in Lemma 4.19), as the whole surface becomes a $(\gamma - 2/\gamma)$-quantum wedge, the left and right quantum surface also become a γ-quantum wedges (modulo embeddings). In our proof, we will use Proposition 3.53 to prove that the right quantum surface is a γ-quantum wedge, and then use left-right symmetry of SLE-docorated quantum wedge to quickly show that the left quantum surface is also a γ-quantum wedge.

Now to prove (1) we only need to show that the restrictions of $\bar{h}_0$ to tiny neighbourhoods of X and X_L are approximately independent under $\mathbf{Q}(\cdot \mid X)$, which should not be surprising in light of a similar result in Proposition 3.27. However, the proof is somewhat technical.

Proof of Theorem 4.29(1). Consider the one-sided stationary process $((\bar{h}_t, \eta_t))_{t\geq 0}$, the quantum typical point X, the measure $\mathbf{Q}$ and the conformal maps $(\tilde{f}_t)_{t\geq 0}$ as in the proof of Lemma 4.19. Let D_L and D_R be the left and right components of $\mathbb{H}\backslash\eta_0$, and X^L be the point on the negative real line such that $\nu_{\bar{h}_0}[(X^L, 0)] = \nu_{\bar{h}_0}[(0, X)]$. Then we know that under $\mathbf{Q}(\cdot \mid X)$, $(D^R, h^{D^R}, 0, \infty)$ (resp. $(D^L, h^{D^L}, 0, \infty)$) is the limit surface (in the sense of Definition 3.51) of $(D_R, \bar{h}_0|_{D_R} + C, X, \infty)$ (resp. $(D_L, \bar{h}_0|_{D_L} + C, X^L, \infty)$) as $C \to \infty$.

Since $\tilde{f}_0$ is the identity map and $G^{\mathbb{H}}$ has a log-singularity of order 2 in the diagonal, by Step 1 (a) we know that under $\mathbf{Q}(\cdot \mid X)$,

$$\bar{h}_0 \overset{d}{=} h - \gamma \log|\cdot - X| + \text{ some smooth terms,} \tag{4.31}$$

where h is a free-boundary GFF on $\mathbb{H}$ with the additive constant fixed so that the its average on $\mathbb{H} \cap \partial B(10, 1)$ is zero, and the smooth terms are bounded in a neighbourhood of X. Let $0 < \delta < 1$. Note that since $X \geq 1$, conditioned on $\nu_{\bar{h}_0}([X-\delta, X])$ and the restriction of $\bar{h}_0$ to $\mathbb{H}\backslash B(X, \delta)$, the behavior of the restriction

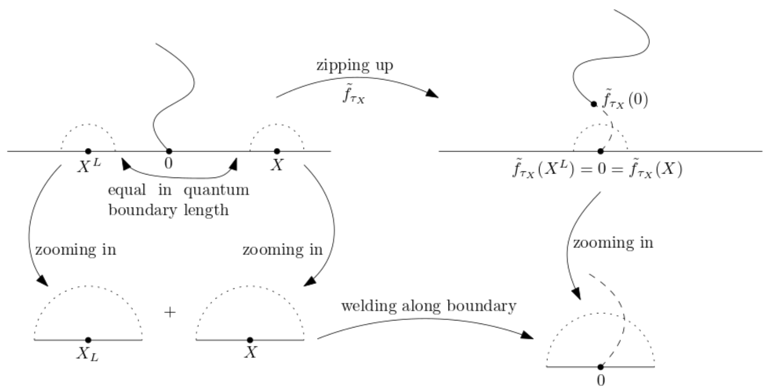

FIGURE 7. An illustration of proof of Theorem 4.29(1). Pictures in the same row are in the same zipping process, while pictures in the same column can be transferred into each other via zooming. In addition, in the top left picture, from Proposition 3.35 we know that $\bar{h}_0$ should have a log-singularity of order γ at quantum typical points X_L and X. Therefore, in the bottom left picture the two quantum surfaces are γ-quantum wedges. Theorem 4.29(1) follows from the fact that the above figure is commutative and an independence argument.

of $\bar{h}_0$ to $B(X^L, \delta)$ is known. We then consider the influence of this conditioning on the behavior of $\bar{h}_0$ near X. We split this into two steps.

Step 1: First round of conditioning.

We first condition on the restriction to of $\bar{h}_0$ to $\mathbb{H} \setminus B(X, \delta)$. In this case, by Markov property (see Proposition 3.26) and (4.31) we can write

$$\bar{h}_0 = h^{ZF} - \gamma \log|\cdot - X| + \text{ some smooth terms,} \tag{4.32}$$

where h^{ZF} is a GFF on $\mathbb{H}$ with zero boundary conditions on $\mathbb{H} \cap \partial B(X, \delta)$ and free boundary conditions on $(X - \delta, X + \delta)$. Note that here the harmonic extension part is added to the smooth terms, and this part is also bounded due to Schwartz reflection. From now on, we only need to consider the influence of conditioning on $\nu_{\bar{h}_0}([X - \delta, X])$.

Step 2: Second round of conditioning.

After the first round of conditioning, condition on $\nu_{\bar{h}_0}([X - \delta, X])$ is equivalent to condition on $\nu_{h^{ZF}}([X - \delta, X])$. So from now on we only work with h^{ZF}.

Consider a smooth function φ that is supported on $U \subseteq B(X, \delta)$ with $\overline{U} \cap \mathbb{R} \subseteq (X - \delta, X)$. We also assume that φ is equal to one on some interval of $\overline{U} \cap \mathbb{R}$ and $\varphi(z) \in [0, 1]$ for all $z \in B(X, \delta)$. By the definition of GFF with mixed boundary conditions (see Definition 3.24) we can write $h^{ZF} = X_1 \varphi + h_\varphi$, where X_1 is a standard Gaussian variable and h_φ is the projection of h^{ZF} onto the complement of the span of φ, and X_1 and h_φ are independent. Then the restriction of $\nu_{h^{ZF}}$ to $\partial U \cap \mathbb{R}$ is given by

$$\nu_{h^{ZF}} = e^{X_1 \varphi} \nu_{h_\varphi},$$

where ν_{h_φ} is the LQG boundary measure associated with h_φ.

In particular, this implies that once we further condition on h_φ, $\nu_{h^{ZF}}(\partial U \cap \mathbb{R})$ is almost surely given by an increasing smooth function of X_1, where the smoothness can be verified by differentiating with respect to X_1 and noting that no matter how many times we differentiate we obtain a compactly supported test function integrated against ν_{h_φ}. This then implies that conditioned on h_φ, the quantity $\nu_{h^{ZF}}$ has a law which is absolutely continuous with respect to the Lebesgue measure and has a smooth density function. Denote this densify function as ψ.

Pick $\delta' < \delta$ such that $B(X, \delta') \cap U = \emptyset$. Let

$$\ell := \nu_{h^{ZF}}([X - \delta, X]).$$

Given the restriction of h^{ZF} to $\mathbb{H} \setminus B(X, \delta')$, rerandomizing $h^{ZF}|_{\mathbb{H} \cap B(X,\delta')}$ under the conditioning of the value of ℓ is equivalent to rerandomizing $h^{ZF}|_{\mathbb{H} \cap B(X,\delta')}$ without conditioning on ℓ, except that we need to reweight the law by (a quantity proportional to)

$$\psi(\ell - \nu_{h^{ZF}}([X - \delta, X] \setminus \partial U)),$$

where the value above is viewed as a function of $h^{ZF}|_{\mathbb{H} \cap B(X,\delta')}$. As $\delta' \to 0$, the amount by which resampling $h^{ZF}|_{\mathbb{H} \cap B(X,\delta')}$ changes $\nu_{h^{ZF}}([X - \delta, X])$ is a quantity that converges to zero in probability. We can then conclude by smoothness of ψ that the second round of conditioning affects the law of $h^{ZF}|_{\mathbb{H} \cap B(X,\delta')}$ by an amount that tends to zero (in total variation sense) as $\delta' \to 0$.

Conclusion of proof.

By (4.31) and Proposition 3.53 we know that as $C \to \infty$, $(D_R, \bar{h}_0|_{D_R} + C, X, \infty)$ converges to a γ-quantum wedge (in the sense of Definition 3.51). Then by the left-right symmetry of a $(\gamma - 2/\gamma)$-quantum wedge we know that $(D_L, \bar{h}_0|_{D_L} + C, X^L, \infty)$ also converges to a γ-quantum wedge as $C \to \infty$. It then follows from Proposition 3.53 and the two rounds of conditioning above that, even conditioned on $(D_L, \bar{h}_0|_{D_L}, X^L, \infty)$, $(D_R, \bar{h}_0|_{D_R} + C, X, \infty)$ still converges to a γ-quantum wedge (Note that here we need to adapt Proposition 3.53 by changing h into h^{ZF}, which is also true due to Proposition 3.27). The independence of the left and right limiting γ-quantum wedge, and thus the conclusion, then follows. □

We now move to the proof of Theorem 4.1(2), which follows from the conformal removability of SLE_κ for $\kappa \in (0, 4)$. Here we say that a set $E \subseteq \mathbb{C}$ is *comformally removable* if E satisfies that, for every homeomorphism $\phi : \mathbb{C} \mapsto \mathbb{C}$, if ϕ is conformal on $\mathbb{C} \setminus E$, then it is actually conformal on $\mathbb{C}$.

Proof of Theorem 4.29(2). Let ϕ^L (resp. ϕ^R) be the unique conformal map from D^L (resp. D^R) to $\mathbb{H}$ such that the image of $\Big(\mathbb{H}, \phi^L(h^{D^L}), \phi^L(0), \phi^L(\infty)\Big)$ (resp. $\Big(\mathbb{H}, \phi^R(h^{D^R}), \phi^R(0), \phi^R(\infty)\Big)$) is a γ-quantum wedge in the circle average embedding. Suppose there is another candidate $((\mathbb{H}, \tilde{h}_c, 0, \infty), \zeta_c)$ that also gives rise to the same two quantum surfaces (viewed modulo embeddings) $(D^L, h^{D^L}, 0, \infty)$ and $(D^R, h^{D^R}, 0, \infty)$ when slicing $\mathbb{H}$ along ζ_c. We similarly define ϕ_c^L and ϕ_c^R.

Now by Theorem 4.25, $(\phi_c^L)^{-1} \circ \phi^L$ and $(\phi_c^R)^{-1} \circ \phi^R$ together extend to a homeomorphism ϕ from $\overline{\mathbb{H}}$ itself, and ϕ can be further extended to a homeomorphism from $\mathbb{C}$ to itself via Schwarz reflection. In addition, ϕ is also conformal on $\mathbb{C} \setminus (\zeta \cup \zeta')$, where ζ' is the reflection of ζ with respect to the real line. It is shown in [35] that almost surely, $\mathbb{C} \setminus (\zeta \cup \zeta')$ is a Hölder domain and the simple curve $\zeta \cup \zeta'$ is the boundary of this domain (see also Remark 3.58) (using $\kappa < 4$). By [16, Corollary

2] we know that $\zeta \cup \zeta'$ is a.s. removable, and therefore ϕ is actually a.s. conformal. Since ϕ maps the origin to itself and maintains the real line, it must be a scaling. So if we require that $(\mathbb{H}, \tilde{h}_c, 0, \infty)$ is in the circle average embedding, then ϕ must be the identity map, and the conclusion follows. □

Remark 4.33. Using exactly the same technique as in the proof of Theorem 4.29(2), one can also show that under the setting of Theorem 4.16, $\bar{h}_0$ actually determines $(\tilde{f}_t)_{t\geq 0}$.

5. General comformal welding therory of LQG surfaces

In this section, we will briefly review some results in the groundbreaking paper [7]. These results serve as generalizations of the theorems in [40]. Note that another import part of [7] is the mating-of-trees theory which substantially rely on these generalizations for its proof. To be self-contained, we do not include the mating-of-trees theory here; see [9] for a detailed survey.

This section will be divided into two parts. Section 5.1 deals with more scale-invariant quantum surfaces, and is a generalization of Section 3.3. Section 5.2 consists of generalized SLE/GFF couplings (generalizations of Theorem 4.1) and generalized conformal weldings between LQG surfaces (generalizations of Theorem 4.29).

5.1. **Quantum cones and thin quantum wedges.** Recall that in Section 3.3 we introduce the α-quantum wedge for $\alpha \in (-\infty, Q)$ as a zoomed-in surface of (W^θ, h^θ), where W^θ is an infinite wedge with opening angle θ and h^θ is a free-boundary GFF (with arbitrary additive constant). Roughly speaking, the **quantum cone** (with parameter α) is obtained in a same way, except that we replace W^θ by a cone C^θ obtained from identifying the left and right side of W^θ according to Lebesgue measure, and replace h^θ by the whole-plane GFF on C^θ. Indeed, for $\alpha \in (-\infty, Q)$, an α-quantum cone can be defined by the analog of Definition 3.46, where the strip S is replaced by a cylinder, the $\tilde{h}_{circ}$ part is replaced by the circular part in the radial-circular decomposition of whole-plane GFF (which is an analog of Proposition 3.22), and $\tilde{h}_{rad}$ remains the same.

We then show that, for $\alpha \in (-\infty, Q)$, the radial part $\tilde{h}_{rad}$ can be encoded via a Bessel process that starts at zero and has dimension (recall (3.73))

$$\delta = 2 + \frac{2(Q-\alpha)}{\gamma}. \tag{5.1}$$

Indeed, let Z_t be a Bessel process with dimension δ, started at $\varepsilon > 0$. Then since $\delta > 2$, almost surely, Z_t does not hit zero and will fly to infinity as $t \to \infty$. In addition, by Itô's formula, reparameterizing $-2\gamma^{-1}\log(Z_t)$ so that it has quadratic variation $2dt$ and satisfies $\inf\{t : Z_t = 1\} = 0$ yields $B_{2t} + (\alpha - Q)t$, where B_t is a standard Brownian motion that starts from $\log(\varepsilon)$ and hits 0 for the first time at time zero. We then get the desired result by sending ε to 0.

The above paragraph actually suggests a way to introduce α-quantum wedge for $\alpha \in [Q, Q+\gamma/2)$, which corresponds to $\delta \in (1, 2]$. In general, we will call a quantum wedge with parameter $\alpha \in (-\infty, Q)$ (resp. $\alpha \in (Q, Q+\gamma/2)$) as a **thick quantum wedge** (resp. **thin quantum wedge**). Note that although we can encode a quantum cone via a Bessel process in a similar way, it is unnecessary (at least in [7]) to define α-quantum cone for $\alpha \in [Q, Q+\gamma/2)$.

To be more precise, when $\delta = 2$, we actually have the same story as the paragraph before last. When $\delta \in (1,2)$, Z_t has infinitely many excursions from zero. For each excursion $(Z_t)_{a \le t \le b}$, $(2\gamma^{-1}\log(Z_t))_{a \le t \le b}$ has a unique reparameterization that has quadratic variation $2dt$ and hits its (unique) maximum at time zero. We can then define a finite area quantum surface by taking its radial part as this reparameterized process, and define an α-quantum wedge (with α given by (5.1)) by concatenating the chain of finite area quantum surfaces in the same order as their associating excursions. Note that by Itô excursion decomposition we see that the excursions form a Poisson point process on $\mathcal{E} \times (0,\infty)$, where $\mathcal{E}$ is the space of excursions from zero. In other words, a thin quantum wedge is a concatenation of infinite Poissonian "bubbles" of finite area quantum surfaces. We also remark here that the law of γ-LQG area of these bubbles coincides with the law of the lengths of the excursions of Z from zero.

It is worth mentioning that since a Bessel process of dimension $\delta < 2$ conditioned to be non-negative yields a Bessel process of dimension $4-\delta$, the behavior of the "bubble" near one of its marked points (that is, the point corresponding to the starting point of the excursion) locally looks like a thick quantum wedge. In addition, when $\alpha = 2Q - \gamma$ or equivalently $\delta = 3 - 4/\gamma^2$, the "bubbles" are actually called **quantum disks**.

5.2. **Extensions of Theorem 4.1 and Theorem 4.29.** For conciseness, we might not give the precise statement of the theorems; see [7] for details.

We begin with the generalized SLE/GFF coupling, where we only state the result for chordal SLE. Fix $\kappa > 0$ and $Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}$. Suppose that we have a reverse $\mathrm{SLE}_\kappa(\underline{\rho})$ process (recall Section 3.4.2) with force points located at $x_1, \ldots, x_k \in \overline{\mathbb{H}}$ of weights $\rho_{(1)}, \ldots, \rho^{(k)} \in \mathbb{R}$, associated with its centered reverse Loewner flow $(\tilde{f}_t)_{t \ge 0}$. We also set

$$\bar{h}_0 := h + \frac{2}{\sqrt{\kappa}} \log|\cdot| + \frac{1}{2\sqrt{\kappa}} \sum_{i=1}^{k} \rho^{(i)} G^{\mathbb{H}}(\tilde{f}_t(x_i, \tilde{f}_t(\cdot)),$$

where h is a free-boundary GFF on $\mathbb{H}$ (viewed modulo constants) and $G^{\mathbb{H}}$ is as in (4.11). Theorem 5.1 of [7] says that, for any time $t > 0$, we have

$$\bar{h}_0 \overset{d}{=} \bar{h}_0 \circ \tilde{f}_t + Q \log|\tilde{f}_t'|,$$

where both sides are viewe as distributions modulo constants. Just like the reweighting step in the proof of Theorem 4.17 (see (a)), this theorem can be seen as the generalization of Theorem 4.1 in that the results are the same, except that we need to measure them in a tilted measure. In addition, the proof is also almost the same and we need to cook up two martingales. Besides, although the theorem itself is not related with LQG surfaces, we can understand it as the Domain Markov property of certain γ-LQG surfaces where $\gamma \in \{\sqrt{\kappa}, 2/\sqrt{\kappa}\}$.

We now move on to the generalized conformal welding theorems. From now on, we fix $\gamma \in (0,2)$, $\kappa = \gamma^2$ and $Q = \gamma/2 + 2/\gamma$. We define the weight of an α-quantum wedge (with $\alpha \in (-\infty, Q + \gamma/2)$) and an α-quantum cone (with $\alpha \in (-\infty, Q)$) as

$$W^{wedge} := \gamma\left(\frac{\gamma}{2} + Q - \alpha\right) \quad \text{and} \quad W^{cone} := 2\gamma(Q - \alpha).$$

Here W^{wedge} and W^{cone} are always positive numbers. Note that the thick (resp. thin) quantum wedge corresponds to $W^{wedge} \ge \gamma^2/2$ (resp. $W^{wedge} \in (0, \gamma^2/2)$).

Let positive numbers W, W_1, W_2 satisfy $W = W_1 + W_2$. In one word, Theorem 1.2 of [7] says that, slicing a quantum wedge with weight W (paramerized by $\mathbb{H}$ under arbitrary embedding) by an independent chordal $\mathrm{SLE}_\kappa(W_1-2, W_2-2)$ on $(\mathbb{H}, 0, \infty)$ with force points located at 0^- and 0^+ yields two independent quantum wedges with weights W_1 and W_2. When $W < \gamma^2$, the SLE curve is replaced by a concatenation of independent $\mathrm{SLE}_\kappa(W_1-2, W_2-2)$ curves on the chain of quantum disks. In other words, gluing two quantum wedges with weights W_1 and W_2 along the boundary in a (γ-LQG) boundary-length-preserving way yields a quantum wedge with weight W decorated by the interface $\mathrm{SLE}_\kappa(W_1-2, W_2-2)$. Moreover, the curve-decorated wedge and the two independent wedges with weights W_1 and W_2 determine each other. In particular, when $W_1 = W_2 = 2$, we get exactly Theorem 4.29. It is also worth noting that the $\mathrm{SLE}_\kappa(W_1-2, W_2-2)$ on $(\mathbb{H}, 0, \infty)$ almost surely hits $(-\infty, 0)$ (resp. $(0, \infty)$) if and only if $W_1 < \gamma^2$ (resp. $W_2 < \gamma^2$), which coincides with the threshold of a thin/thick quantum wedge. When the curve does hit one side of the real line, it then creates a thin quantum wedge on that side.

Theorem 1.5 of [7] also have a similar flavour, which says that for any $W > 0$, slicing a quantum cone with weight W (parameterized by $\mathbb{C}$ under arbitrary embedding) by an independent **whole-plane $\mathbf{SLE}_\kappa(\mathbf{W}-\mathbf{2})$** on $(\mathbb{C}, 0, \infty)$ yields a quantum wedge of weight W. In other words, gluing the boundary of a quantum wedge with weight W to itself in a (γ-LQG) boundary-length-preserving way yields a quantum cone with the same weight W. Here the whole-plane $\mathrm{SLE}_\kappa(W)$ on $(\mathbb{C}, 0, \infty)$ is a SLE-like random continuous curve between 0 and ∞, driven by a while-plane Loewner evolution and being sacle-invariant in law. In addtion, it almost surely has self-intersections if and only if $W < \gamma^2$, also coinciding with the threhold of quantum wedge. See [30, 7] for detail.

We end up with a final remark. The proof of the last two theorems in [7] is much analogous to the proof of Theorem 4.29. In particular, it takes the SLE/GFF coupling as an input, and introduces the "quantum natural time" as the LQG analog of the natural parameterization for $\mathrm{SLE}_\kappa(\underline{\rho})$, which is the counterpart of γ-LQG-boundary-length parameterization as in the quantum zipper theorem (Theorem 4.17). Similar to Theorem 4.17, [7] proves that the laws of certain types of quantum wedges are invariant under zipping/unzipping according to quantum natural time. In addition, this proof also relies on understanding the local behavior of the LQG at a "quantum typical point" (except that the LQG measure has changed), playing around with the order of log-singularities.

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Appendix A. More relationships with the discrete models

This appendix is devoted to explaining some relationships between SLE, LQG and the discrete models, and can be seen as a complement of Section 2.

Random planar maps may converge to LQG surfaces in different types of topology; see [42] for an overview on the different perspectives. Random planar maps can be seen as compact metric spaces equipped with graph distances, and may converge to LQG surfaces equipped with their LQG metrics, in the sense of Gromov-Hausdorff topology. So far, this type of convergence has been established for uniform random planar maps, and the scaling limit is called Brownian map (see [24, 26]), which is equivalent to certain LQG surfaces with $\gamma = \sqrt{8/3}$ (see [31, 32, 33]). Some random planar maps (including those decorated by uniform spanning trees, critical percolations, bipolar orientations, Schnyder woods, etc.) can also be encoded by random walks via mating-of-trees bijections, and are shown to converge to certain LQG surfaces (seen as peanospheres) encoded by Brownian motions via continuum analog of mating-of-trees bijections, in the sense of peanosphere topology; see [41, 4, 11, 25, 17]. Random planar maps reweighted by partition functions of statistical physics models can as well be embedded into the sphere $\mathbb{C} \cup \{\infty\}$, and one can ask whether the map equipped with normalized counting measure on vertices converge to certain LQG surfaces equipped with LQG area measures, in the sense of weak topology; see [14, 13]. Note that no matter what kind of topology we are considering, the limit object, as a scaling limit, should be scale-invariant, leading to the natural construction of quantum wedge discussed in Section 3.3 (and also other LQG surfaces enjoying conformal invariance).

Although the scaling limits are all LQG surfaces, they feature different structures because of the difference in topology of convergence: Brownian maps are endowed with metric space structures; LQG surfaces seen as peanospheres are endowed with tree structures; LQG surfaces seen as measure spaces are endowed with conformal structures. In addition, the first two convergence results were directly motivated by discrete models, and their proofs rely heavily on discrete bijections between random planar maps and simpler objects like trees or random walks. The third perspective, on the other hand, was motivated more from the continuum side, closely related to the KPZ formula introduced in the influential paper [18].

Though we believe these different definitions are consistent, or in other words, they are different aspects of the same universal object, proving this turns out to be very difficult and complicated, and there are still numerous conjectures with regard to these equivalence results.

In this article, we choose to focus on the third kind of convergence because of its closer and more direct connection with conformal structures and SLE. In this case, there are still various ways of embedding planar maps into the sphere, including Tutte embedding, Cardy embedding, and circle packing, etc. We save the precise definitions here, but shall refer to the notion of universality. Indeed, random planar maps weighted by the partition function of a same statistical physics model are believed to lie in the same universality class, meaning that the normalized counting measure shall converge to an LQG area measure with the same parameter γ, no matter what type of planar maps we consider and what type of embedding we choose. Examples of universality class include: LQG surface with $\gamma = \sqrt{2}$ corresponding to random planar maps decorated by a uniform spanning tree; $\gamma = \sqrt{8/3}$ corresponding to (and proved in certain cases, see [14, 10]) random planar map decorated by a critical Bernoulli site percolation configuration, etc. Note that in the latter case, if we fix the number of vertices of the planar map, then the conditional law of the underlying map is uniform.

Oded Schramm [36] introduced the Schramm-Loewner evolution, which is uniquely characterized by conformal invariance and domain Markov property, as a potential candidate of scaling limits of uniform spanning tree (UST) and loop-erased random walk (LERW) on two-dimensional square grids of a given domain when the grids become finer. The scaling limit result has the flavour much similar to the convergence to two-dimensional simple random walk to two-dimensional Brownian motion, which is also conformally invariant and Markovian. We note here that an SLE curve may touch itself but has no self-crossing, but the Brownian motion has many "trasversal" intersections.

Meanwhile, physicists were also able to make a number of predictions with respect to two-dimensional statistical physics model on lattice. They believe that the model at criticality has a conformally invariant continuum scaling limit as mesh of the lattice becomes finer, and SLE is then naturally conjectured to be the scaling limit of certain interfaces of discrete models. Just as the fact that different random walks with the same covariance structures converge to one same kind of Brownian motion, SLE are also conjectured to enjoy universality, meaning that an identical model on different deterministic discrete lattice has one type of SLE as their common scaling limit. Examples of universality class include: SLE_6 as limit of critical two-dimensional critical Bernoulli percolation (proved in the case of triangular lattice based on the famous Cardy's formula, see [44]); $\mathrm{SLE}_8/\mathrm{SLE}_2$ as limit of UST/LERW (proved independent of lattice, see [21]); $\mathrm{SLE}_{8/3}$ as limit of self-avoiding walk; see [20].

It is then quite natural to consider the joint scaling limit of both the random planar maps and the statistical physics model whose partition function is used for reweighting the map, and to conjecture that this limit should be an conformally-invariant LQG surface (with its LQG area measure) decorated by an SLE curve. In addition, since we believe that universality holds on deterministic discrete lattice models, the scaling limit result should hold in a quenched sense, that is, the conditional law of the statistical physics model given the random planar map should converge. If this is true, then in the continuum phase, the SLE curve should be independent with the LQG surface. Through a spectacular program, this joint convergence is demonstrated in one specific case: under the Cardy embedding, the uniform random triangulations decorated by critical Bernoulli percolation converges to a $\sqrt{8/3}$-LQG disk decorated by an independent SLE_6; see [14].