THE RIEMANN-ROCH THEOREM

LAUREN TSAI

Abstract. The Riemann-Roch Theorem connects topological and algebraic data and thus has many important applications in algebraic geometry and complex analysis. In this paper, we introduce the theorem for invertible sheaves on the regular projective curve. We first explain some of the background necessary to understand the Riemann-Roch theorem for line bundles on a regular projective curve and then prove the theorem.

Contents

1. Introduction

The Riemann-Roch Theorem has many applications in algebraic geometry and complex analysis. For example, it has as a corollary Clifford's Theorem. Clifford's theorem gives a relationship between the dimension and degree of a divisor D on a curve. In particular, it states the following for a divisor D [1]:

$$\dim |D| \leq \frac{1}{2} \deg D.$$

As such, Clifford's Theorem, and therefore transitively the Riemann-Roch Theorem, can be used to classify curves in $\mathbb{P}^3$.

In this paper, we introduce the Riemann-Roch Theorem for invertible sheaves on a regular projective curve. This theorem is stated as follows: Suppose C is a regular projective curve and let $D := \sum_{p \in C} a_p[p]$ be a Weil divisor on C. (Note that the notation $[p]$ means we are considering p as an irreducible closed subset of C, which is valid because we associate curves in the affine plane with prime ideals in k.) Define the degree of D by

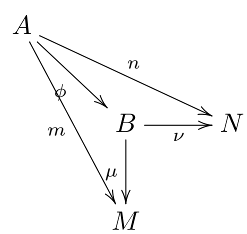
$$\deg D := \sum_{p \in C} a_p \deg p.$$

Date: September 25, 2023.

Then $\chi(C, \mathscr{O}_C(D)) = \deg D + \chi(C, \mathscr{O}_C)$, where all of our notation from before holds. In other words, for any divisor D on C, the Euler characteristic of a projective curve C with the sheaf $\mathscr{O}_C(D)$ is equal to the sum of the degree of D and the Euler characteristic of the projective curve with the structure sheaf on the projective curve.

To introduce this theorem, we first introduce the space in which Vakil's presentation of the Riemann-Roch Theorem for line bundles on a projective curve is set. We begin by introducing the language of sheaves (section 2) and schemes (section 3). We then use this to define the projective space as a scheme (Definition 5.3) and identify the invertible sheaves and regular curves on the projective space (section 5). Finally, we introduce the technology used to formulate and prove the Riemann-Roch Theorem, namely Čech cohomology and the Euler characteristic (section 6), and then prove this theorem.

We make use of commutative diagrams to explain different phenomena within this paper. To make this notion precise, we say that a diagram commutes if the composition of different arrows from one space in the diagram to another are equal. For example, the following diagram commutes if $m = \mu \circ \phi$ and $n = v \circ \phi$:



We also assume the reader has some familiarity with group theory, ring theory, point-set topology, and real analysis.

With this said, we can begin the bulk of the paper by introducing the notion of a sheaf.

2. Sheaves

To learn information about the structure of a topological space, we construct objects called sheaves. This first requires us to define a presheaf and then from the presheaf, construct the sheaf.

Definition 2.1 (Presheaf)**.** Let X be a topological space. We define a **presheaf** $\mathscr{F}$ on X to be a collection of sets satisfying the following properties:

(1) For each open set $U \subset X$, there exists some set in the collection $\mathscr{F}$ that is associated to the open set U. For each open $U \subset X$, we call this set $\mathscr{F}(U)$, $\Gamma(U, \mathscr{F})$, or $H^0(U, \mathscr{F})$, depending on context. Note that if the U is omitted, then we take the open set to be X by convention.

(2) For any open sets $V, U \subset X$ with $U \hookrightarrow V$, there exists a restriction map $\mathrm{res}_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$ such that the following conditions are satisfied:
   (a) The map $\mathrm{res}_{U,U}$ is the identity, and
   (b) If for some open sets $U, V, W \subset X$, we have $U \hookrightarrow V \hookrightarrow W$, then the restriction maps commute, i.e. $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$.

Definition 2.2 (Sheaf)**.** Let X be a topological space. We say that a collection of sets $\mathscr{F}$ is a **sheaf** on X if $\mathscr{F}$ is a presheaf on X that also satisfies two more properties:

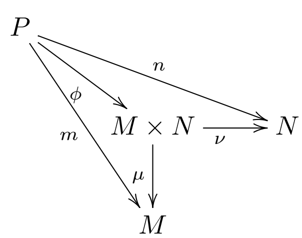
(1) The Identity Axiom: Suppose $\mathscr{F}(U)$ is a set associated to the open set $U \subset X$ in the presheaf $\mathscr{F}$. Suppose $\{U_i\}$ is an open cover of U and $f_1, f_2 \in \mathscr{F}(U)$. If $\text{res}_{U,U_i}(f_1) = \text{res}_{U,U_i}(f_2)$ for all i, then we have $f_1 = f_2$.
(2) The Gluability Axiom: Suppose $\{U_i\}$ is an open cover of the open set U with indexing set I, and suppose that there is a collection of sections $\{f_i \in \mathscr{F}(U_i) \colon i \in I\}$ such that for all i, j, we have $\text{res}_{U_i,U_i\cap U_j}(f_i) = \text{res}_{U_j,U_i\cap U_j}(f_j)$. Then there exists some $f \in \mathscr{F}(U)$ such that $\text{res}_{U,U_i}(f) = f_i$ for all $i \in I$.

These are the technical definitions of sheaves and presheaves. However, it is sometimes difficult to manipulate sheaves and presheaves as they are presented in Definition 2.1 and Definition 2.2. As a result, we often choose to represent these objects in terms of their stalks and germs.

Definition 2.3 (Stalk and germs of a presheaf)**.** Let $\mathscr{F}$ be a presheaf on a topological space X. Let $p \in X$. We define the **stalk** of $\mathscr{F}$ at a point $p \in X$ to be the collection of pairs $\mathscr{F}_p \colon = \{(f, U) \colon p \in U, f \in \mathscr{F}(U)\}/\sim$ where $\sim$ is defined as follows: for any $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$, $(f, U) \sim (g, V)$ if for some open set $W \subset (U \cap V)$ with $p \in W$, we have $\text{res}_{U,W}(f) = \text{res}_{V,W}(g)$. The image of a section f in the stalk of $\mathscr{F}$ at a point p is called the **germ** of f at p.

We can define a more concrete the relationship between presheaves and sheaves using this definition. However, we first need to give some preliminary definitions.

Definition 2.4 (Product)**.** Suppose M and N are sets. A product of M and N consists of a set $M \times N$ and maps $\mu \colon M \times N \to N$ and $\nu \colon M \times N \to N$ such that for all sets P and maps $m \colon P \to M$ and $n \colon P \to N$, there exists a unique map $\phi \colon P \to M \times N$ such that the following diagram commutes:



For a fixed indexing set I, the product of an arbitrary collection of sets $\{A_i \colon i \in I\}$ with maps $f_i \colon A_i \to A_j$ and $i, j \in I$ is defined similarly. It consists of a set $\prod_{i\in I} A_i$ and maps $\mu_i \colon \prod_{i\in I} A_i \to A_i$ such that for all sets P and maps $m_i \colon P \to A_i$ that commute with the f_i for all i, there exists a unique map $\phi \colon P \to \prod_{i\in I} A_i$ such that $m_i = \mu_i \circ \phi$.

Definition 2.5 (Compatible germs)**.** Let X be a topological space with an open subset $U \subset X$ and let $\mathscr{F}$ be a presheaf. Let $(s_p)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$. We say that this is a **compatible set of germs** if for all $p \in U$, there is some open set $U_p \subset U$ containing p and some $t_p \in \mathscr{F}(U_p)$ such that for all $q \in U_p$, the germ of t_p at q is s_q.

We can now concretely define the relationship between presheaves and sheaves.

Theorem 2.6. *Let F be a presheaf on a topological space X. For each open set $U \subset X$, define $\mathscr{F}(U)$ to be the set of compatible germs over U. Then the collection $\mathscr{F}$ forms a sheaf.*

Proof. Let F be a presheaf on a topological space X and for all open sets $U \subset X$, define $\mathscr{F}(U)$ as in the theorem statement. For each $V \subset U$ and each $f = (s_p)_{p \in U} \in \mathscr{F}(U)$, define $\text{res}_{U,V}(f) := ((s_p)_{p \in V})$. Fix some open sets $U, V, W \subset X$ satisfying the property $U \subset V \subset W \subset X$ Then for all $f = (s_p)_{p \in U} \in \mathscr{F}(U)$, $\text{res}_{U,U}(f) = (s_p)_{p \in U} = f$, so $\text{res}_{U,U}$ is the identity map. Moreover, for all $w = (w_q)_{q \in W}$, $\text{res}_{V,U} \circ \text{res}_{W,V}(w) = \text{res}_{V,U}((w_q)_{q \in V}) = (w_q)_{q \in U} = \text{res}_{W,U}(w)$. So, the restriction maps commute for arbitrary, i.e. all, open sets $U, V, W \subset X$ satisfying $U \subset V \subset W$. With this, we have, for all open $U \subset X$, defined a set $\mathscr{F}(U)$, as well as restriction maps satisfying the properties specified in part 2 of Definition 2.1. Therefore, $\mathscr{F}$, as we have defined it, forms a presheaf with our given restriction maps.

What remains is to show that $\mathscr{F}$ satisfies the identity and gluability axioms. We begin with the identity axiom. Suppose $\{U_i\}$ is an open cover of an open set $U \subset X$ and $f_1, f_2 \in \mathscr{F}(U)$. Suppose further that $\text{res}_{U,U_i}(f_1) = \text{res}_{U,U_i}(f_2)$ for all i. Then $(f_1, U) \sim (f_2, U)$ where $\sim$ is given by the equivalence relation in Definition 2.5. So, $f_1 = f_2$ in $\mathscr{F}(U)$. We chose $f_1, f_2 \in \mathscr{F}(U)$ arbitrarily, so this applies for all f_1, f_2, meaning $\mathscr{F}$ satisfies the identity axiom. Now we will show $\mathscr{F}$ satisfies the gluability axiom. Suppose once again $U \subset X$ is an open set and $\{U_i\}$ is an open cover of U. Moreover, suppose there is a collection of sections $\{f_i \in \mathscr{F}(U_i)\}$ such that for all i, j, we have $\text{res}_{U_i, U_i \cap U_j}(f_i) = \text{res}_{U_j, U_i \cap U_j}(f_j)$. Then for all i, j, f_i is associated the same set of germs as f_j on $U_i \cap U_j$. So, there exists an element $(s_p)p \in U \in \prod_{p \in U} F_p = \mathscr{F}(U)$ such that for all $p \in U_i$, s_p is the germ of f_i at p. By definition, $\text{res}_{U,U_i}((s_p)_{p \in U}) = (s_p)_{p \in U_i} = f_i$ for all i. Thus, $\mathscr{F}$ satisfies the gluability axiom.

With this, we have shown $\mathscr{F}$ is a presheaf satisfying the identity and gluability axioms. As such, $\mathscr{F}$ is a sheaf. □

This concludes our introduction of sheaves and presheaves. We can now turn to the larger setting in which they are relevant: schemes.

3. Schemes

To define a scheme, we must first define what is known as the affine scheme, consisting of a topological space (Spec $A, \mathscr{T}$) known as the spectrum of a ring A, and a sheaf of rings on that space known as the structure sheaf.

Definition 3.1 (Spec A as a Set)**.** Let A be a ring. We define the set Spec A to be the set of prime ideals on the ring A. We follow Vakil's convention of adding hard brackets around prime ideals when they are considered as elements of Spec A. For instance, $[\mathfrak{p}] \in$ Spec A, but $p \subset A$. We say that points $a \in A$ are **functions on** Spec A, and their values at the point $[\mathfrak{p}] \in$ Spec A are given by $a(\text{mod } p)$.

We will now define a topology on Spec A, namely the Zariski topology.

Definition 3.2 (The Zariski Topology on Spec A)**.** Let A be a ring and suppose $S \subset A$. Then we define the **Vanishing Set** of S as follows:

$$V(S) = \{[\mathfrak{p}] \in \text{Spec } A \colon S \subset \mathfrak{p} \subset A\}.$$

In the **Zariski Topology**, a set B is closed if and only if there exists some $S \subset A$ such that $B = V(S)$. On the other hand, a set is open in this topology if and only if that set is the complement of the vanishing set of some set $S \subset A$.

Theorem 3.3 (Vakil [2], 3.4.C)**.** *The Zariski Topology is a topology on* Spec A*.*

Having defined the topology on Spec A, it will now prove useful to have a base on the topology with which to work.

Definition 3.4 (The Distinguished Open Sets)**.** Suppose $f \in A$. Then we define the **distinguished open set** associated with f as follows:

$$\begin{aligned} D(f) &= \{[\mathfrak{p}] \in \text{Spec } A \colon f \notin p\} \\ &= \{[\mathfrak{p}] \in \text{Spec } A \colon f([p]) \neq 0\} \end{aligned}$$

Theorem 3.5 (Vakil [2], section 3.5)**.** *The distinguished open sets form a base for the Zariski topology on* Spec A*.*

We can now consider Spec A as a topological space equipped with the Zariski topology and a base on that topology, namely the collection of distinguished open sets as defined in Definition 3.4. This allows us to begin the process of defining the structure sheaf on Spec A.

Definition 3.6. (Multiplicative Set) Let A be a ring and let $S \subset A$. We say S is a **multiplicative subset of** A if S contains $1_A \in A$ and is closed under multiplication.

Definition 3.7 (The Localization of a Ring at a Multiplicative Subset)**.** Let A be a ring and let $S \subset A$ be a multiplicative subset of A. Let $S^{-1}A \colon = \{as^{-1} \colon a \in A, s \in S\}$ and if there exists some $s \in S$ such that $s(s_2a_1 - s_1a_2) = 0$, then say $a_1s_1^{-1} = a_2s_2^{-1}$. Define addition and multiplication in the usual way, i.e. $a_1s_1^{-1} + a_2s_2^{-1} = (a_1s_2 + a_2s_1)(s_1s_2)^{-1}$ and $a_1s_1^{-1} \cdot a_2s_2^{-1} = a_1a_2(s_1s_2)^{-1}$. This set, equipped with these operations and equivalence relations, is called the **localization of** A **at** S.

Example 3.8.
(1) Let A be a ring and let $f \in A$. Then the set $S = \{1, f, f^2, ...\}$ is a multiplicative subset of A. We say $A_f := S^{-1}A$.
(2) Let A be a ring and let $\mathfrak{p}$ be a prime ideal of A. Then $A \backslash \mathfrak{p}$ is a multiplicative subset of A and we write $A_{\mathfrak{p}} := S^{-1}A$.

Remark 3.9. In the previous example, it is worthwhile to note that the notations described are very similar but denote opposite localizations. If f is an element of A, every multiple of f is invertible in A_f. However, if $\mathfrak{p}$ is a prime ideal, then everything not in $\mathfrak{p}$ is invertible in $A_{\mathfrak{p}}$.

Theorem 3.10. *Let $f \in A$ where A is a ring and define the set $S := \{g \in A \colon D(f) \subset D(g)\}$. (In other words, we let S be the set of elements g of A such that $f \in \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ if $g \in \mathfrak{p}$ for that prime ideal.) This set is multiplicative.*

Proof. Let A, f, S be defined as in the theorem statement. There does not exist any prime ideal $\mathfrak{p}$ such that $1 \in \mathfrak{p}$ by definition. So, if $f \notin \mathfrak{p}$, then $1 \notin \mathfrak{p}$. This implies $1 \in S$, for all f. Moreover, suppose $g, h \in S$ and let p be a prime ideal. If $gh \in \mathfrak{p}$, then either $g \in \mathfrak{p}$ or $h \in \mathfrak{p}$, since $\mathfrak{p}$ is prime. Since $g, h \in S$, this implies $f \in \mathfrak{p}$. We chose $\mathfrak{p}$ to be an arbitrary prime ideal, so this applies for all prime ideals $\mathfrak{p} \subset A$ and $D(f) \subset D(gh)$. Moreover, we chose $g, h \in S$ arbitrarily, so this implies $gh \in S$ for all $g, h \in S$. With this, we have shown $1 \in S$ and S is closed under multiplication in A. So, S is a multiplicative subset of A. □

Using these preliminaries, we can finally define the structure sheaf of Spec A on the base of the topology and then use that to define the structure sheaf on all open sets in the topology.

Definition 3.11 ($\mathscr{O}_{\text{Spec } A}$ on the distinguished open sets)**.** Let $D(f)$ be a distinguished open set as defined above and let the set S be defined as above, i.e. $S\colon=\{g \in A\colon D(f) \subset D(g)\}$. Define $\mathscr{O}_{\text{Spec } A}(D(f))$ to be the localization of A at S (this is allowed by Theorem 4.7).

For any $f, f' \in A$ with $D(f') \subset D(f)$, define the restriction map $res_{D(f),D(f')}$ as follows:

$$\text{res}_{D(f),D(f')}(x) = x.$$

In other words, the restriction of an element $x \in \mathscr{O}_X(D(f))$ to $\mathscr{O}_X(D(f'))$ is simply the image of that element in $\mathscr{O}_X(D(f'))$. (We know that this image exists in $\mathscr{O}_X(D(f'))$ since $\mathscr{O}_X(D(f'))$ is simply a further localization of the first ring, given $D(f') \subset D(f)$.)

Definition 3.12 (Sheaf on a base for a topology)**.** We define a **sheaf on a base** $\{B_i\}$ in the same way we define a sheaf on the collection of open sets in a topology. First we define a presheaf on a base. A **presheaf on a base** $\mathscr{F}$ satisfies the following properties: (1) For each i, there exists some set $\mathscr{F}(B_i)$ in the collection $\mathscr{F}$ that is associated to the open set $B_i \in \{B_i\}$, and (2) For all i, j with $B_i \subset B_j$, we have a restriction map res_{B_j,B_i}. For all i, we must have $\text{res}_{B_i,B_i} = \text{id}$, and for all i, j, k with $B_i \subset B_j \subset B_k$, we must have $\text{res}_{B_k,B_i} = \text{res}_{B_j,B_i} \circ \text{res}_{B_k,B_j}$. In a similar way to the presheaf on open sets, we say a presheaf on a base is a **sheaf on a base** if it satisfies the **base identity axiom** and the **base gluability axiom**. These axioms are stated as follows: (1) If $B = \cup B_i$ and $f, g \in F$ satisfy $\text{res}_{B,B_i} f = \text{res}_{B,B_i} g$ for all i, then $f = g$ (the base identity axiom), and (2) if we have $B = \cup B_i$ where I is the indexing set of the union, and for all $i, j \in I$, $\text{res}_{B_i,B_i \cap B_j} f_i = \text{res}_{B_j,B_i \cap B_j} f_j$, then there is some $f \in \mathscr{F}(B)$ such that $\text{res}_{B,B_i} f = f_i$ for all i (the base gluability axiom).

Theorem 3.13. *If A is a ring, then the collection of sets given in Definition 3.11 forms a sheaf on a base (the distinguished open sets) for* Spec A*, when paired with the restriction maps given above.*

Proof. We first prove that the given collection of sets and restriction maps forms a presheaf on a base. For each distinguished open set $D(f)$, we defined the set $\mathscr{O}(D(f))$. Moreover, for all $f \in$ Spec A and $x \in D(f)$, $\text{res}_{D(f),D(f)}(x) = x$, meaning that for all $f \in$ Spec A, $\text{res}_{D(f),D(f)}$ is the identity map. Furthermore, for all $f, g, h \in$ Spec A with $D(f) \subset D(g) \subset D(h)$ and $x \in D(h)$, we have $\text{res}_{D(g),D(f)} \circ \text{res}_{D(h),D(g)}(x) = \text{res}_{D(g),D(f)}(x) = x = \text{res}_{D(h),D(f)}$. Based on our choice of f, g, and h, this implies restriction maps commute. So, $\mathscr{O}_{\text{Spec } A}$ defined as in Definition 3.11 forms a presheaf on a base.

Now we prove that $\mathscr{O}_{\text{Spec } A}$ satisfies the base identity and base gluability axioms. We begin with the base identity axiom. First suppose $B = \cup_i D(f_i) = \cup_i B_i$ for some collection $\{f_i\} \subset$ Spec A (where each $B_i = D(f_i)$) and that $g, h \in \mathscr{O}_{\text{Spec } A}(B)$ satisfy $\text{res}_{B,B_i}(g) = \text{res}_{B,B_i}(h)$ for all i. Then $g = h$, by our definition of the restriction function. Since we chose $g, h \in \mathscr{F}(B)$ arbitrarily, this implies $\mathscr{O}_{\text{Spec } A}$ satisfies the base identity axiom. For the base gluability axiom, suppose B is defined as before and $\{g_i\}$ is a collection of objects such that for all i, $g_i \in \mathscr{O}_{\text{Spec } A}(B_i)$ and

$\text{res}_{B_i,B_i\cap B_j}(g_i) = \text{res}_{B_j,B_i\cap B_j}(g_j)$. Then for all i, j, $g_i = g_j$. For all i, define $g = g_i$ on $\mathscr{O}_{\text{Spec } A}(B_i)$. Then since the g_i agree on overlaps, g is a localization of B_i at $S_i := \{g \in A\colon B_i \subset D(g)\}$ for all i. Therefore, g is an element of the localization of B at the set $S := \{g \in A\colon B \subset D(g)\}$, meaning $g \in \mathscr{O}_{\text{Spec } A}(B)$. Then by our definition of g, for all i, $\text{res}_{B,B_i}(g) = g = g_i$. So, we have found a $g \in \mathscr{O}(B)$ such that the restriction $\text{res}_{B,B_i}(g) = g_i$ for all i. Based on our choice of collection $\{g_i\}$, this result can be applied for all collections $\{g_i\}$ satisfying our specified conditions, meaning $\mathscr{O}_{\text{Spec } A}$ satisfies the base gluability axiom.

With this, we have shown $\mathscr{O}_{\text{Spec } A}$ is a presheaf on a base satisfying the base identity and base gluability axioms in Spec A. Therefore, $\mathscr{O}_{\text{Spec } A}$ is a sheaf on the base of distinguished open sets in Spec A. ☐

We can now extend this sheaf on a base to a sheaf on the whole space Spec A.

Theorem 3.14. *Let $(X, \mathscr{T})$ be a topological space with base $\mathscr{B}$. Suppose $\mathscr{F}$ is a sheaf defined on this base. Then there is a sheaf $\mathscr{G}$, unique up to isomorphism, such that $\mathscr{G}$ extends $\mathscr{F}$ to a sheaf on X. More precisely, there exists a sheaf $\mathscr{G}$ on X such that $\mathscr{G}(B) \cong \mathscr{F}(B)$ for all basis elements $B \in \mathscr{B}$ and the restriction maps associated with $\mathscr{G}$ agree with the restriction maps associated with $\mathscr{F}$.*

Proof. For all $U \subset X$ open, let $\mathscr{G}(U)$ be defined as follows:

$$\mathscr{G}(U) := \{(f_p \in \mathscr{F}_p)_{p\in U}\colon \forall p \in U, \exists B \text{ such that } p \in B \subset U, s \in \mathscr{F}(B), \text{ and } s_q = f_q \forall q \in B\}.$$

Since $\mathscr{F}$ is a presheaf and $\mathscr{G}$ inherits restriction maps from $\mathscr{F}$, $\mathscr{G}$ forms a presheaf. Then for each $U \subset X$ open, the set of compatible germs of $\mathscr{G}$ over U is simply $\mathscr{G}(U)$ by our definition of $\mathscr{G}$. Thus $\mathscr{G}$ is a collection of sets of compatible germs of a presheaf over open sets, meaning $\mathscr{G}$ is a sheaf by Theorem 2.6.

Now we prove $\mathscr{G}(B) \cong \mathscr{F}(B)$ for all basis elements $B \in \mathscr{B}$. Suppose $B \in \mathscr{B}$. Then define a map $\Phi\colon \mathscr{F}(B) \to \mathscr{G}(B)$ such that for all $f \in \mathscr{F}(B)$, $\Phi(f) = (f_p)_{p\in B}$ where, for each $p \in B$, f_p is the germ of f at p. Now fix $f \in \mathscr{F}(B)$ and consider $\Phi(f)$. For all $p \in B$, the following are true:

(1) $p \in B'$
(2) $f_p = \Phi(f)$

So, for all $p \in B$, there exists a basis element B', namely B itself, such that $p \in B' \subset B$ and there exists some $s \in \mathscr{F}(B')$, namely f, such that $s_q = f_q$ for all $q \in B$. With this we have shown Φ is defined for all $f \in \mathscr{F}(B)$ and that for all $f \in \mathscr{F}(B)$, $\Phi(f) \in \mathscr{G}(B)$. So, Φ is well defined. We now prove Φ is injective. Suppose $a, b \in \mathscr{F}(B)$ and $\Phi(a) = \Phi(b)$. Then for all $p \in B$, $a_p = b_p$. Thus, for all $p \in B$, there exists some neighborhood $U \subset B$ containing p such that $\text{res}_{B,U}a = \text{res}_{B,U}b$ in $\mathscr{F}(U)$. Since this applies for all $p \in B$, this implies $a = b$. So, Φ is injective, since we chose $a, b \in \mathscr{F}(B)$ arbitrarily. We will now prove surjectivity. Suppose $(s_q)_{q\in p}$ is a set of compatible germs in $\mathscr{G}(B)$. Then for all $p \in B$, there exists some basis element $B_p \subset B$ with $p \in B_p$ and $f_p \in \mathscr{F}(B_p)$ such that $f_{p,q} = s_q$ for all $q \in B_p$ by our definition of $\mathscr{G}(B)$ (Note, here we define $f_{p,q}$ as the germ of f_p at q.) Then $\cup_p U_p = B$. Consider the collection $\{f_p\colon p \in U\}$. For all $p \in B$ and $q \in B_p$, the germ $f_{p,q} = s_q$. So, for any $m, n \in B$, $\text{res}_{B_m,B_m\cap B_n} f_m = \text{res}_{B_n,B_m\cap B_n} f_n$. Since $\mathscr{F}$ is a sheaf and satisfies the gluability axiom, this implies there exists some $f \in \mathscr{F}(B)$ such that for all $p \in B$, $\text{res}_{B,B_p} f = f_p$. Choose this $f \in \mathscr{F}(B)$. Then for all $p \in B$, the germ of f at p is s_p, since the restriction of f to B_p is f_p, whose germ at p is s_p. So,we have found some $f \in \mathscr{F}(B)$ such that $\Phi(f) = (s_q)_{q\in p}$. Since we chose $(s_q)_{q\in p}$ arbitrarily from $\mathscr{G}(B)$, this applies for all such sets of germs and Φ is surjective. With this, we have found a bijection $\Phi\colon \mathscr{F}(B) \to \mathscr{G}(B)$, meaning $\mathscr{G}(B) \cong \mathscr{F}(B)$. Since the restriction maps on $\mathscr{G}$ are defined in a similar way to the restriction maps on $\mathscr{F}$, the restriction maps agree. Thus, the theorem holds. □

Definition 3.15. The sheaf $\mathscr{O}_{\text{Spec } A}$ given on the distinguished open sets in Definition 3.11 can be uniquely extended to a sheaf on Spec A, since the distinguished open sets form a base for the Zariski topology on Spec A. We call this extended sheaf the **structure sheaf on** Spec A.

Definition 3.16 (The Structure Sheaf of a Ringed Space)**.** Let X be a topological space and suppose the sections of a sheaf $\mathscr{O}_X$ over each open set U on X are rings. Then we say $(X, \mathscr{O}_X)$, or the topological space with the sheaf $\mathscr{O}_X$, is a **ringed space**. The notation $\mathscr{O}_X$ will often be used to denote sheaves of rings. The sheaf of rings on a space X is called the **structure sheaf** of the ringed space.

Theorem 3.17. *The pair $(Spec(A), \mathscr{O}_{Spec(A)})$ is a ringed space.*

Proof. By our definition of the extension of a sheaf from a base on the topological space, for all open sets U in Spec A, we have that the sections of $\mathscr{O}_{\text{Spec } A}(U)$ are compatible germs over the collection of distinguished open sets whose union is U. So, $\mathscr{O}_{\text{Spec } A}(U)$ is the union of a collection of localizations of the same ring at different multiplicative subsets, call them $\{S_i\}$. As such, $\mathscr{O}_{\text{Spec } A}(U)$ is defined as $\cup_i S_i^{-1}A = (\cup_i S_i)^{-1}A$, which is a ring, since it is the localization of a ring at some multiplicative subset of that ring. □

Remark 3.18. From the previous theorem, $(Spec(A), \mathscr{O}_{Spec(A)})$ is a special case of a ringed space. Going forward, the structure sheaf on any topological space X will be denoted by $\mathscr{O}_X$.

Now we are able to define isomorphisms of ringed spaces.

Definition 3.19 (Isomorphism of Ringed Spaces)**.** An **isomorphism of ringed spaces** $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of the following data:

(1) A homeomorphism $\pi\colon X \to Y$
(2) An isomorphism of sheaves $\mathscr{O}_X$ and $\mathscr{O}_Y$ induced by π, ie. an isomorphism $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ of sheaves on X.

Given isomorphisms of ringed spaces, we can define the affine scheme and then the scheme in general.

Definition 3.20 (Affine Scheme)**.** We say a ringed space $(X, \mathscr{O}_X)$ is an **affine scheme** if $(X, \mathscr{O}_X) \cong (Spec(A), \mathscr{O}_{Spec(A)})$ for some ring A.

Definition 3.21. We say that a ringed space $(X, \mathscr{O}_X)$ is a **scheme** if for all $x \in X$, there exists some open neighborhood U such that $(U, \mathscr{O}_{X|_U})$ is an affine scheme.

With this, we have defined a scheme on a topological space. We will now introduces ways to construct new schemes, given schemes and other data on them.

4. Gluing Schemes

In this section, we introduce the idea of gluing schemes together to create new schemes, which will later be important in our definition of the projective space. We begin with a lemma.

Lemma 4.1. *Let A be a ring and let $f \in A$. There is an isomorphism of ringed spaces*

$$(D(f), \mathscr{O}_{\text{Spec } A|_{D(f)}}) \to (\text{Spec } A_f, \mathscr{O}_{\text{Spec } A_f}).$$

Proof. First, we show that there exists an isomorphism of ringed spaces

$$(D(f), \mathscr{O}_{\text{Spec } A|_{D(f)}}) \to (\text{Spec } A_f, \mathscr{O}_{\text{Spec } A_f}).$$

If $p \in D(f)$, then $f \notin p$. So, $[p] \in \text{Spec } A_f$. Moreover, if $[p] \in \text{Spec } A_f$, then p is a prime ideal in A and does not contain f; any prime ideal containing f must be all of A_f, since f is invertible in A_f and the prime ideals of A_f are a subset of the prime ideals of A. So, we can define the map $\pi\colon D(f) \to \text{Spec } A_f$ such that for all $p \in D(f)$, $\pi(p)$ is the image of p in Spec A_f and π is a homeomorphism. So, we have a homeomorphism $\pi\colon D(f) \to \text{Spec } A_f$. In the process of defining this homeomorphism, we essentially identified the prime ideals in $D(f)$ as equivalent to those of Spec A_f. So, their respective structure sheaves are the same and there is an isomorphism between them. This proves there exists an isomorphism of ringed spaces $(D(f), \mathscr{O}_{\text{Spec } A|_{D(f)}}) \to (\text{Spec } A_f, \mathscr{O}_{\text{Spec } A_f})$. □

Using this lemma, we are able to prove the following result, which will allow us to define open subschemes.

Lemma 4.2. *Suppose X is a topological space and $U \subset X$ is an open subset of X. Suppose further $(X, \mathscr{O}_X)$ is a scheme. Then $(U, \mathscr{O}_{X|_U})$ is also a scheme.*

Proof. Suppose X, U, and $\mathscr{O}_X$ are defined as in the lemma statement. Suppose $x \in U$. Then since $(X, \mathscr{O}_X)$ is a scheme, there exists some open neighborhood V with $x \in V$ such that $(V, \mathscr{O}_{X|_V}) \cong (\text{Spec } A, \mathscr{O}_{\text{Spec } A})$. Fix this V and fix this A and call the homeomorphism between them π. Then there exists some distinguished open set $D(f) \subset \text{Spec } A$ such that $D(f) \subset \pi(U \cap V)$ and $\pi(x) \in D(f)$, since the distinguished open sets form a base for the Zariski topology. This yields the following:

$$\begin{aligned}(\pi^{-1}(D(f)), \mathscr{O}_{U|_{\pi^{-1}(D(f))}}) &\cong (D(f), \mathscr{O}_{D(f)})\\ &\cong (\text{Spec } A_f, \mathscr{O}_{\text{Spec } A_f}) \text{ by Lemma 4.1}\end{aligned}$$

Since $D(f) \subset \pi(U \cap V)$, $\pi^{-1}(D(f)) \subset U \cap V$. Moreover, since π is a homeomorphism and $D(f)$ is an open set, $\pi^{-1}(D(f))$ is open in U. So, for arbitrary, i.e. for all, $x \in U$, there exists an neighborhood $\pi^{-1}(D(f))$ containing x such that $(\pi^{-1}(D(f)), \mathscr{O}_{U|_{\pi^{-1}(D(f))}}) \cong (\text{Spec } B, \mathscr{O}_{\text{Spec } B})$ for some ring $B = \text{Spec } A_f$. As such, $(U, \mathscr{O}_{X|_U})$ is a scheme. □

Definition 4.3 (Open Subschemes)**.** Let $U \subset X$ be an open subset of a topological space X with $\mathscr{O}_X$ to denote the sheaf of rings on X. By the previous lemma, $(U, \mathscr{O}_{X|_U})$ is a scheme. We say that $(U, \mathscr{O}_{X|_U})$ is an **open subscheme** of $(X, \mathscr{O}_X)$. We say an open subscheme $(U, \mathscr{O}_{X|_U})$ is an **affine open subscheme** if it is also an affine scheme.

With these definitions, we can "glue together" different schemes satisfying specific conditions.

Definition 4.4 (The Cocycle Condition)**.** Suppose we are given a collection of schemes $\{X_{ij}\}$ and a collection of maps $\{f_{ij}\colon X_{ij} \to X_{ji}\}$, ie. a collection of schemes and a collection of maps indexed such that the map f_{ij} maps X_{ij} to X_{ji}. Suppose also this set of maps is equipped with restriction maps. We say that this collection of maps satisfies the **cocycle condition** if for all $i, j, k \in I$ (where I is the indexing set), we have $f_{ik} |_{X_{ji}\cap X_{jk}} = f_{jk} |_{X_{ji}\cap X_{jk}} \circ f_{ij} |_{X_{ij}\cap X_{ik}}$.

Theorem 4.5. *Suppose (1) $\{X_i\}$ is a collection of schemes, (2) $\{X_{ij} \colon i, j \in I\}$ is a collection of subschemes of X_i, with $X_{ij} \subset X_i$ and $X_{ii} = X_i$, and (3) we have a collection of isomorphisms $f_{ij} \colon X_{ij} \to X_{ji}$ such that the cocycle condition is satified and f_{ii} is the identity. Then "glue" all of these schemes together using the given isomorphisms, i.e. take $X = \sqcup_i X_i/\sim$ where $x \sim y$ if there exists some f_{ij} such that $f_{ij}(x) = y$. The resulting space X forms a scheme.*

Proof. Define X as in the theorem statement. First we make explicit the sheaf $\mathscr{O}_X$. Suppose $U \subset X$. By our definition of X, $U = \sqcup_{j\in J} U_j$ for some indexing set J and some open sets $U_j \subset X_j$. So, we define $\mathscr{O}_X(U)$ as follows:

$$\mathscr{O}_X(U) := \sqcup_{j\in J} \mathscr{O}_{X_j}(U_j)/\sim$$

where $\sim$ is defined as in the theorem statement but with the isomorphisms between sheaves $\mathscr{O}_{ij}$ given by isomorphisms between schemes X_{ij} and X_{ji}.

Now we prove X is a scheme. Let $x \in X$. Then there exists some i such that $x \in X_i$. Then X_i is a scheme, so there exists some open set $U_i \subset X_i$ such that $(U_i, \mathscr{O}_{X_i|_{U_i}})$ is an affine scheme. Take U to be the image of U_i in X. Then $(U, \mathscr{O}_{X|_U}) = (U, \mathscr{O}_{X_i}(U_i)/\sim)$, so $(U, \mathscr{O}_{X|_U})$ is also an affine scheme. So, we have found an open set $U \subset X$ containing x such that $(U, \mathscr{O}_{X|_U})$ is an affine scheme. Since we chose $x \in X$ arbitrarily, this applies for all $x \in X$, meaning X is a scheme. $\square$

We are now ready to define the projective n-space and locally free sheaves on projective space.

5. Locally Free Sheaves on Projective Space

The Riemann-Roch Theorem is set on a regular projective curve, which we can think of as a subscheme of the projective space. As such, we need to construct the projective space.

Throughout the rest of this section, let A be a ring and let $B := A[x_0, ..., x_n]$ for some variables $x_0, ..., x_n$. Ultimately, our goal is to construct the projective n-space, and we do this by gluing a collection of affine schemes together.

Theorem 5.1. *Define n as above. For each $i \in \{0, ..., n\}$, define*

$$X_i := \text{Spec } (A[x_{0/i}, x_{1/i}, ..., x_{n/i}])/(x_{i/i} - 1))$$

where variable $x_{j/i}$ is simply the jth variable in the set X_i. Then for all i, the pair $(X_i, \mathscr{O}_{X_i})$ is well-defined and encodes an affine scheme.

Proof. By definition, each X_i is the prime spectrum of a ring. So, we can apply the discussion of section 3 to each X_i, which gives us a topological space X_i endowed with the Zariski topology and a structure sheaf on that topological space, $\mathscr{O}_{X_i}$. $\square$

Lemma 5.2. *Suppose we are given the following data:*

1. *The collection of schemes given in Theorem 5.1*
2. *For all i, the collection of distinguished open sets $D(x_{j/i}) \subset X_i$, and*
3. *The maps $f_{ij}\colon X_i[1/x_{j/i}] \to X_j[1/x_{i/j}]$ defined by assigning, for all $k \in \{1, ..., n\}$, $f_{ij}(x_{k/i}) = x_{k/j}/x_{i/j}$.*

Then the collection of maps specified by item three satisfies the cocycle condition (given in Definition 4.4). Moreover, for all i, j, the given map f_{ij} is an isomorphism and f_{ii} is the identity map.

Proof. We will begin by proving that the maps satisfy the cocycle condition. Fix i, j, and k. Then for any m and the appropriate restrictions,

$$\begin{aligned}(f_{jk} \circ f_{ij})(x_{m/i}) &= f_{jk}(x_{m/j}/x_{i/j}) \\ &= (x_{m/k}/x_{j/k})/(x_{i/k}/x_{j/k}) \\ &= (x_{m/k})/(x_{i/k}) \\ &= f_{ik}(x_{m/i})\end{aligned}$$

So, compositions agree for arbitrary i, j, k. As such, the collection of f_{ij} satisfies the cocycle condition.

Now, for all i, k, $f_{ii}(x_{k/i}) = x_{k/i}/x_{i/i} = x_{k/i}$, since $x_{i/i} = 1$ in X_i. So, for all i, f_{ii} is the identity map. Then, from the cocycle condition, we see $f_{ii} = f_{ji} \circ f_{ij}$. As such, f_{ji} is the inverse of f_{ij} and both maps are bijections. Additionally, both functions describe a change of variables, meaning the preimage of a distinguished open set under f_{ij} in X_j is a distinguished open set in X_i. In other words, the functions are both continuous (and thus their inverses are continuous, since they are the inverses of each other). Therefore, the f_{ij} are homeomorphisms. Moreover, we get an isomorphism of sheaves induced by f_{ij}. Thus, we have proved the lemma. □

Definition 5.3. By Theorem 4.5, we can use the above data to construct a new scheme, written Proj $(A[x_0, ..., x_n])$, by gluing. We say that this scheme is the **projective n-space** over a ring A, often written $\mathbb{P}^n_A$. The variables $x_0, ..., x_n$ are called the **projective coordinates** on $\mathbb{P}^n_A$.

This defines the projective space. However, this is not all that the Riemann-Roch theorem works with. We must now define locally free sheaves on the projective space, and eventually invertible sheaves.

Definition 5.4 ($\mathscr{O}_X$-module)**.** Suppose $\mathscr{O}_X$ is a sheaf of rings on a topological space X. We say a sheaf $\mathscr{F}$ on X is an **$\mathscr{O}_X$-module** if $\mathscr{F}$ satisfies the following properties:

1. $\mathscr{F}$ is a sheaf of abelian groups
2. For all open sets $U \subset X$, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module
3. If $U \subset V \subset X$, then the following diagram commutes for all maps ϕ that take $\mathscr{O}_X(W) \times \mathscr{F}(W)$ to $\mathscr{F}(W)$:

$$\begin{array}{ccc} \mathscr{O}_X(V) \times \mathscr{F}(V) & \xrightarrow{\phi} & \mathscr{F}(V) \\ \downarrow {\scriptstyle res_{V,U} \times res_{V,U}} & & \downarrow {\scriptstyle res_{V,U}} \\ \mathscr{O}_X(U) \times \mathscr{F}(U) & \xrightarrow[\phi]{} & \mathscr{F}(U) \end{array}$$

Definition 5.5 (Locally free sheaf of rank n on a scheme X)**.** Suppose X is a scheme and fix $n \in \mathbb{N} \cup \{0\}$. An $\mathscr{O}_X$−module $\mathscr{F}$ is a **locally free sheaf of rank** n **on a scheme** X if $\mathscr{F}$ is locally isomorphic to the direct sum of n copies of the the structure sheaf on U_i. More precisely, $\mathscr{F}$ is a locally free sheaf of rank n if there is some open cover $\{U_i\}$ of X such that for all U_i, $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$, where $\mathscr{O}_{U_i}^{\oplus n}$ is the direct sum of n copies of $\mathscr{O}_{U_i}$.

Theorem 5.6 (Vakil[2], 14.2.5)**.** *Suppose $\mathscr{F}$ is a locally free sheaf of rank n and $\{U_i\}$ is an open cover of $\mathscr{F}$ that satisfies the properties given in Definition 5.5, i.e., $\{U_i\}$ is an open cover of X satisfying the property that for all U_i, $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$. Then there exists a unique set of linear functions $T_{ij} \in \mathrm{GL}_n(\mathscr{O}(U_i \cap U_j))$ that satisfies the cocycle condition.*

Theorem 5.7. *Suppose X is a scheme and $\{U_i\}$ is an open cover of X satisfying the property given in Definition 5.5 for some locally free sheaf $\mathscr{F}$. This data determines the sections of $\mathscr{F}$ over any open set U, up to isomorphism. In other words, the sections of a locally free sheaf $\mathscr{F}$ can be determined only from the open cover $\{U_i\}$ for which the property in Definition 5.5 is satisfied.*

Proof. Define X, $\{U_i\}$, and T_{ij} as in the theorem statement. For all U, define $\mathscr{O}_X^{\oplus n}(U \cap U_i)$ to be $\mathscr{F}|_{U_i}$ Then for all i, j, $T_{ij} \in GL_n(\mathscr{O}(U_i \cap U_j))$, so T_{ij} is a bijection mapping one vector space to another. Thus, following the process of $Theorem$ 4.5, we can then glue the schemes on the U_i defined with these sheaves using the transition functions T_{ij}. Define $\mathscr{F}$ to be the sheaf resulting from this gluing process.

Now let $\mathscr{G}$ be a locally free sheaf of rank n. By the definition of a locally free sheaf of rank n, we have that for all i, $\mathscr{G}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$. So, for all $i, \mathscr{F}|_{U_i} \cong \mathscr{G}|_{U_i}$. Moreover, for $\mathscr{G}$ to be well-defined, we require that sections over U_i agree with sections over U_j on $U_i \cap U_j$; otherwise, there exists some section in U defined two ways for a specific $u \in U$. So, for all $U \subset X$ open with $U_i \subset U$ and $U_j \subset U$ for some i, j, we have $\mathscr{G}(U \cap U_i)$ includes only those sections s_i over U_i such that $T_{ij}(s_i) = s_j$ for some section $s_j \in \mathscr{G}(U_j)$ and vice versa. Thus, $\mathscr{F} = \mathscr{G}$, by our definition of $\mathscr{F}$. □

Example 5.8. Suppose we are given the following data for some ring k:

- The collection of affine open sets used to define the projective space $\mathbb{P}_k^m$, i.e. $\{U_i := \text{Spec }(k[x_{0/i}, x_{1/i}, ..., x_{m/i}]/(x_{i/i} - 1)\})$, and
- Transition functions T_{ij} from U_i to U_j given by defining multiplication by $x_{i/j}^n$ as multiplication by $x_{j/i}^{-n}$.

The transition functions given satisfy the cocycle condition. As such, we can construct a locally free sheaf, written $\mathscr{O}(n)$, with these transition functions on the projective space (Theorem 5.7).

Definition 5.9 (Invertible sheaves)**.** An **invertible sheaf** is a locally free sheaf of rank one.

As it turns out, the invertible sheaves on the projective space over a field are limited.

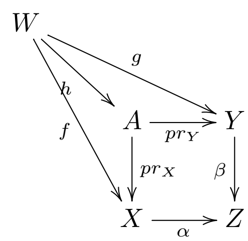
Theorem 5.10 (Vakil [2], pg. 407 and 15.1.D)**.** *Let k be a field. All invertible sheaves on $\mathbb{P}_k^m$ take the form $\mathscr{O}(n)$ and all sheaves of this form are invertible on $\mathbb{P}_k^m$.*

Now, having defined the larger scheme containing the projective curve, we are able to define the projective curve itself. We begin with a few definitions.

Definition 5.11 (Affine Morphism)**.** Suppose X, S are schemes and $f\colon X \to S$ is a map of schemes. If the preimage $f^{-1}(U)$ of every affine open set $U \subset S$ is an affine open subset of X, then we say f is an **affine morphism.**

Definition 5.12 (Closed embedding)**.** Suppose a map $\pi\colon X \to Y$ is an affine morphism and for each affine open subset $U \subset Y$ with $U \cong \text{Spec } B$ for some ring B and $\pi^{-1}(\text{Spec } B) \cong \text{Spec } A$ for some ring A, we have a surjective map $B \to A$. Then this map π is called a **closed embedding.**

Definition 5.13 (Property 1)**.** Let A be a set and let $pr_X\colon A \to X$ and $pr_Y\colon A \to Y$ be maps from A to X and A to Y, respectively. Suppose we have a map $h\colon W \to A$. We say this map satisfies **Property 1** if for any scheme W and for any maps $f\colon W \to X$ and $g\colon W \to Y$ such that $\alpha \circ f = \beta \circ g$, the following diagram commutes:



Note that this is not standard notation.

Definition 5.14 (Fibered Product in the Context of Schemes)**.** Suppose X, Y, Z are schemes and $\alpha\colon X \to Z$ and $\beta\colon Y \to Z$. Then the **fibered product of** X **and** Y **over** Z consists of the following data:

- A set $X \times_Z Y$
- Maps $pr_X\colon X \times_Z Y \to X$ and $pr_Y\colon X \times_Z Y \to Y$ such that there exists some unique map $h\colon W \to X \times_Z Y$ satisfying property 1 (Definition 5.13).

Definition 5.15 (Diagonal morphism)**.** Let X, Z be schemes, let $\pi\colon X \to Z$ be a map from X to Z, and define the fibered product $X \times_Z X$ as in Definition 5.14, but with X in place of Y and $\alpha = \beta = \pi$. Then there exists a map $\delta_\pi\colon X \to X \times_Y X$ defined by $\delta_\pi(x) = (x, x)$. This map is called a **diagonal morphism.**

Definition 5.16 (Separated map)**.** Suppose X and Y are schemes and $\pi\colon X \to Y$ is map of schemes. Then we say π is **separated** if its associated diagonal morphism $\delta_\pi\colon X \to X \times_Y X$ is a closed embedding.

Finally, the regular projective curve is defined as follows.

Definition 5.17 (Regular Projective Curve)**.** A **regular projective curve** is a scheme X such that there is a closed embedding $\pi\colon X \to \mathbb{P}^1_k$, i.e. $X \hookrightarrow \mathbb{P}^1_k$, and X has no singular points.

Remark 5.18. Since a regular projective curve is a scheme with a closed embedding into $\mathbb{P}^1_k$ for some field k, the preimages of affine open subsets of $\mathbb{P}^1_k$ are affine and open. So, X has the structure of a subscheme of $\mathbb{P}^1_k$.

We now prove some results about the regular projective curve.

Lemma 5.19. *Let k be a field and suppose C is a regular projective curve that embeds in a projective space $\mathbb{P}^1_k$. Then there exists some map $f: C \to \text{Spec } k$ such that f is separated.*

Proof. Let k and C be defined as in the lemma statement. By Proposition 11.3.8 in *The Rising Sea*[2], there exists a map $\phi\colon \mathbb{P}^1_k \to \text{Spec } k$. By the definition of a regular projective curve, there exists a closed embedding $\pi\colon C \hookrightarrow \mathbb{P}^1_k$. Since π is a closed embedding, π is separated (Corollary 4.6a, Hartshorne [1]). By Corollary 4.6b in Hartshorne's *Algebraic Geometry*[1], this implies $\phi|_{\text{im}\pi} \circ \pi$ is separated. So,there exists a separated map $\Psi\colon C \to \text{Spec } k$ defined by $\Psi = \phi|_{\text{im}\pi} \circ \pi$, and the lemma holds. □

Lemma 5.20. *Any regular projective curve C can be covered by a finite number of affine open sets.*

Proof. Let C be a regular projective curve and let k be the field for which C embeds in $\mathbb{P}^1_k$. We defined the projective line $\mathbb{P}^1_k$ as the gluing together of two affine open sets. Take these affine open sets. Their images in $\mathbb{P}^1_k$ are still affine because the maps we used to glue them together are isomorphisms. As such, $\mathbb{P}^1_k$ can be covered by a finite number of affine open sets. Since C embeds in $\mathbb{P}^1_k$, this implies C can also be covered by a finite number of affine open sets. □

This concludes our discussion of the space in which the Riemann-Roch Theorem takes place. We now move on to the technology used to prove the theorem. Namely, we introduce the notions of the cohomology of complexes and exact sequences and the Euler characteristic.

6. Cohomology and the Euler Characteristic

To define the cohomology of a complex or an exact sequence, we must first define complexes and exact sequences.

Definition 6.1 (Complexes and Exact Sequences)**.** Suppose A, B, and C are sets and

$$... \to A \to B \to C \to ...$$

is a sequence of maps defined by a map $f\colon A \to B$ and a map $g\colon B \to C$. We say this sequence is a **complex at** B if $g \circ f = 0$, and we say the sequence is **exact at** B if $\ker g = \text{im} f$.

We say a sequence is a **complex** if it is a complex at all sets in the sequence, and we say a sequence is **exact** if it is exact at all sets in the sequence.

This allows us to define the homology and cohomology of these types of sequences.

Definition 6.2 (Homology and cohomology of a complex)**.** Suppose we are given a complex defined as above, i.e. a sequence

$$... \to A \to B \to C \to ...$$

where A, B, and C are sets with maps f and g. The **homology of this sequence at** B is defined to be $\ker g/\text{im} f$. If the sets are reindexed in increasing order, i.e. if we are given the same sequence but relabel the sets $A = A^{i-1}$, $B = A^i$, and $C = A_{i+1}$, then we say the i**th cohomology** object H^i of the sequence is defined as $\ker g/\text{im} f$.

We now need just one more definition and one more theorem before we can define our desired object, the Čech cohomology.

Definition 6.3. Suppose M is an A-module. For all distinguished open sets $D(f)$, define $\widetilde{M}(D(f)) := S^{-1}M$ where S is the multiplicative set of functions that do not vanish outside of $V(f)$. This defines a sheaf.

Definition 6.4 (Quasicoherent Sheaf)**.** Suppose X is a sheaf with structure sheaf $\mathscr{O}_X$. Let $\mathscr{F}$ be an $\mathscr{O}_X$ module. For every affine open set $U \subset X$, $U \cong \text{Spec } A_U$ for some ring A_U. We say $\mathscr{F}$ is a **quasicoherent sheaf** if for every affine open set $U \subset X$, we have $\mathscr{F}|_U \cong \widetilde{M}$ where $U \cong \text{Spec } A_U$ for some ring A_U and $\widetilde{M}$ is defined as in Definition 6.3.

Now that we have defined the quasicoherent sheaf, we will construct the Čech complex.

Let X be a regular projective curve, let $\mathscr{F}$ be a quasicoherent sheaf on X, and suppose $\mathscr{U} := \{U_i\}_{i=1}^n$ is a finite open cover of X consisting of affine open sets. For all $I \subset \{1, ..., n\}$, define the set $U_I := \cap_{i\in I} U_i$. Now, define the map $\delta_{IJ} \colon \mathscr{F}(U_I) \to \mathscr{F}(U_J)$ as follows:

$$\delta_{IJ}(f) := \begin{cases} 0 \text{ if } J \neq I \cup \{j\} \text{ for some } j \in \{1, ..., n\} \\ \sum_{k=1}^{|I|} (-1)^{k-1} \text{res}_{U_I, U_J}(f) \text{ if } J = I \cup \{j\} \text{ for some } j \in \{1, ..., n\} \end{cases}$$

where k is defined such that j is the kth element of J. Now consider the sequence

$$0 \to \prod_{|I|=1, I \subset \{1,...,n\}} \mathscr{F}(U_I) \to ... \to \prod_{|I|=i, I \subset \{1,...,n\}} \mathscr{F}(U_I) \to \prod_{|I|=i+1, I \subset \{1,...,n\}} \mathscr{F}(U_I) \to ...$$

where the map $\Delta_{IJ} \colon \prod_{|I|=i, I\subset\{1,...,n\}} \mathscr{F}(U_I) \to \prod_{|J|=i+1, J\subset\{1,...,n\}} \mathscr{F}(U_J)$ is defined such that $\Delta_{IJ}((f_I)_I) = (\delta_{IJ}(f_I))_I$ where J is an indexing set whose length is $|I| + 1$.

Theorem 6.5. *The sequence given above is well-defined and forms a complex.*

Proof. If $J = I \cup \{j\}$ for some $j \in \{1, ..., n\}$, then $U_J = U_I \cap U_j$ by definition. So, $U_J \subset U_I$ in this case, meaning the restriction map res_{U_I, U_J} is well defined. This implies δ_{IJ} is well-defined. As such, each map Δ_{IJ} is well-defined, meaning the whole sequence is well-defined.

Then for any sets I_{i-1}, I_i, and I_{i+1} with lengths $i - 1$, i and $i + 1$ respectively, $\delta_{I_i I_{i+1}} \circ \delta_{I_{i-1} I_i} = 0$, since

- $\text{res}_{U_{I_i I_{i+1}}}(0) = 0$, and
- $\sum_{k=1}^{i} (-1)^{k-1} \text{res}_{U_I, U_J}(\sum_{k=1}^{i-1} (-1)^{k-1} \text{res}_{U_I, U_J}(f)) = 0$ for all $f \in \mathscr{F}(U_{I_{i-1}})$ with $\sum_{k=1}^{i-1} (-1)^{k-1} \text{res}_{U_I, U_J}(f)) \neq 0$

So, the sequence forms a complex. □

Given that the complex above is in fact a complex, we can now give it a name: the Čech complex for regular projective curves.

Definition 6.6 (The Čech complex for regular projective curves)**.** The complex defined in Theorem 6.5 is called the **Čech complex** on a regular projective curve. Note that as we have presented it, the complex depends on our chosen curve X, our sheaf $\mathscr{F}$, and the cover $\mathscr{U}$. As such, we denote the ith cohomology group of this complex by $H^i_{\mathscr{U}}(X, \mathscr{F})$

Given the Čech complex, we need two more definitions and a theorem before we can define the Čech cohomology group.

Definition 6.7 (Quasicompact)**.** A topological space X is called **quasicompact** if for any open cover $\{U_i\}$ of X, there exists some finite indexing set S such that $\cup_{i\in S} U_i = X$.

Definition 6.8 (Separated Space)**.** We say an "A-scheme" (a scheme generated over a ring A) is **separated** if there exists a separated map from the A-scheme to Spec A.

Theorem 6.9. *Suppose X is a regular projective curve and $\mathscr{F}$ is a quasicoherent sheaf on X. Additionally, suppose $\mathscr{U} \subset \mathscr{V}$. Then there exists an isomorphism $H^i_{\mathscr{V}}(X, \mathscr{F}) \to H^i_{\mathscr{U}}(X, \mathscr{F})$, and the collection*

$$\{H^i_{\mathscr{U}}(X, \mathscr{F}) \colon \mathscr{U} \text{ is an open cover of } X\}$$

forms a group and $H^i_{\mathscr{U}}(\mathscr{F}, X)$ does not depend on our choice of $\mathscr{U}$. We say that $H^i(\mathscr{F}, X)$ represents $H^i_{\mathscr{U}}(\mathscr{F}, X)$ for some open cover $\mathscr{U}$.

Proof. Let X and $\mathscr{F}$ be defined as in the theorem statement. By Lemma 5.19, the projective curve is **separated**. Moreover, by Exercise 5.1.D in *The Rising Sea* [2] and Lemma 5.20, X is quasicompact. So, by Theorem 19.2.2 in *The Rising Sea*[2], our theorem holds. □

This brings us to the definition of the Čech cohomology group on a regular projective curve and the definition of the dimension of this cohomology.

Definition 6.10 (The Čech cohomology group on a regular projective curve)**.** The group described in Theorem 6.9 is called the **Čech cohomology group** on the regular projective curve. We define the ith Čech cohomology group to be $H^i_{\mathscr{U}}(\mathscr{F}, X)$.

Definition 6.11 (Dimension of Cohomology on a Regular Projective Curve)**.** Let X be a regular projective curve with a quasicoherent sheaf $\mathscr{F}$. Then for all open subsets $U \subset X$, $\mathscr{O}_X(U)$ vector space over k. Moreover, since $\mathscr{F}$ is a quasicoherent sheaf, for all affine open sets $U \subset X$, $\mathscr{F}(U) \cong M'$ for some k-module M where M' is defined as in Definition 6.3 and every M is a k-module. As such, for each i, $H^i(X, \mathscr{F})$ can be considered as a vector space over the field k and has a finite dimension over the field k. Define $h^i(X, \mathscr{F}) := \dim_k(H^i(X, \mathscr{F}))$.

We are now ready to define the Euler characteristic, which is essential in the statement and proof of the Riemann-Roch Theorem for line bundles on a regular projective curve.

Definition 6.12 (Euler Characteristic)**.** Let k be a field and suppose $\mathscr{F}$ is a quasicoherent sheaf on a regular projective curve X. Then we define the **Euler characteristic for a regular projective curve** as follows:

$$\chi(X, \mathscr{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathscr{F}).$$

With this definition, we are ready to introduce the main theorem of this paper, the Riemann-Roch Theorem.

7. The Riemann-Roch Theorem

We now introduce the Riemann-Roch Theorem for line bundles on a regular projective curve. But first, we begin with a few more definitions and lemmas directly relevant to the Riemann-Roch theorem.

Definition 7.1 (Irreducible Set)**.** An **irreducible** topological space X is a topological space X that is nonempty and cannot be represented by $X = A \cup B$ where A and B are proper closed subsets of X.

Definition 7.2 (Dimension of a Set)**.** Let X be a topological space. We say the **dimension** of X is the supremum of the lengths of chains of closed irreducible sets in X.

Definition 7.3 (Codimension of a subset)**.** Let X be a topological space. The **codimension** of a subspace $Y \subset X$ is defined to be $\dim X - \dim Y$.

Definition 7.4 (Weil Divisor)**.** Let X be a topological space. A **Weil divisor** on X is defined as an object that takes the form

$$\sum_{Y \in E} n_Y [Y]$$

where E is the set of codimension 1 irreducible subsets of X, $n_Y \in \mathbb{Z}$ for each $Y \in E$, and $n_Y = 0$ for all but finitely many Y.

Example 7.5 (Weil Divisors on a Regular Projective Curve)**.** Suppose C is a regular projective curve. Then a Weil divisor on C takes the form

$$\sum_{Y \in E} n_Y [Y]$$

where E is the set of codimension 1 irreducible subsets of C, $n_Y \in \mathbb{Z}$ for each $Y \in E$, and $n_Y = 0$ for all but finitely many Y. Then, since C is a regular projective curve, C has dimension 1 and each point of C is closed, so the codimension 1 irreducible subsets of C are just the points of C. As such, Weil divisors on C take the form $\sum_{p \in C} a_p [p]$.

Definition 7.6 (Degree of a closed point p)**.** Let X be a topological space. Let $\kappa(p) := \mathscr{O}_{X,p}/\mathfrak{m}_p$. Then **degree of a closed point** $p \in X$ is defined to be the degree of the field extension $\kappa(p)/k$.

Definition 7.7. Let X be a normal topological space and suppose

$$D := \sum_{Y \subset X \text{ irreducible}} a_Y [Y]$$

for some integers a_Y is a Weil Divisor. Then we define the sections of a sheaf $\mathscr{O}_X(D)$ over an open set $U \subset X$ as follows:

$$\Gamma(U, \mathscr{O}_X(D)) := \{t \in K(X)^\times : \mathrm{div}|_U t + D|_U \geq 0\} \cup \{0\}$$

where $\mathrm{div}|_U : = \sum_{Y \subset U} \mathrm{val}_Y(s)[Y]$, where s is defined as a rational section of an invertible sheaf that does not vanish everywhere for any irreducible component of U, the sum indexes over all irreducible subsets Y of U and $D|_U$ is defined as $\sum_{Y \subset X \text{ irreducible}} a_{Y|_U}[Y|_U]$, i.e. $D|_U$ is the divisor D for which the irreducible subsets in the sum are restricted to U.

Remark 7.8. For a regular projective curve C, C is one dimensional, meaning the closed points of C are irreducible subsets of C. Thus, $-p$ is a Weil divisor on C. Define $\mathscr{O}_C(-p)$ as in Definition 7.7, with $X = C$ and $D = -p$.

Lemma 7.9 (Vakil [2], 9.1.2.1)**.** *Suppose C is a regular projective curve with an element $p \in C$. Then the sequence*

$$0 \to \mathscr{O}_C(-p) \to \mathscr{O}_C \to \mathscr{O}|_p \to 0$$

where $\mathscr{O}|_p$ is the structure sheaf of the scheme $\{p\}$ is exact.

Lemma 7.10. *(Vakil [2], 5.10 and 14.2.E) Suppose C is a regular projective curve and let $D := \sum_{p \in C} a_p[p]$ be a Weil divisor on C. Then the sequence*

$$0 \to \mathscr{O}_C(-p) \otimes \mathscr{O}_C(D) \to \mathscr{O}_C \otimes \mathscr{O}_C(D) \to \mathscr{O}|_p \otimes \mathscr{O}_C(D) \to 0$$

is exact.

Lemma 7.11 (Vakil[2], 19.4.A)**.** *Suppose $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$ are sheaves on a topological space X, and suppose the sequence*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

is exact. Then $\chi(X, \mathscr{G}) = \chi(X, \mathscr{F}) + \chi(X, \mathscr{H})$.

This brings us, finally, to the Riemann-Roch Theorem for line bundles on a regular projective curve.

Theorem 7.12 (The Riemann-Roch Theorem for Line Bundles on a Regular Projective Curve)**.** *Suppose C is a regular projective curve and let $D := \sum_{p \in C} a_p[p]$ be a Weil divisor on C. (Note that the notation $[p]$ means we are considering p as an irreducible closed subset of C, which is valid because we associate curves in the affine plane with prime ideals in k.) Define the degree of D by*

$$\deg D := \sum_{p \in C} a_p \deg p.$$

Then $\chi(C, \mathscr{O}_C(D)) = \deg D + \chi(C, \mathscr{O}_C)$, where all of our notation from before holds.

Proof. For this proof, we follow the outline of Vakil (19.4.B, [2]). Suppose C is a regular projective curve. We prove this theorem by induction on the sum of the absolute value of the coefficients of the divisor, i.e. on $\sum |a_p|$ if $D = \sum_{p \in C} a_p[p]$

First, we prove the base case, when $\sum_{p \in C} |a_p| = 0$. In this case,

$$\begin{aligned} \chi(C, \mathscr{O}_C(D)) &= \chi(C, \mathscr{O}_C(0)) \\ &= \chi(C, \mathscr{O}_C) \text{ since } \mathscr{O}_C(0) = \mathscr{O}_C \\ &= \chi(C, \mathscr{O}_C) + \deg D \text{ since } \deg D = 0 \end{aligned}$$

So, the base case holds true.

We now prove the inductive step. Suppose the statement holds when $\sum_{p \in C} |a_p| = n$ for some $n \in N \cup \{0\}$ and suppose that we are given a divisor D with $\sum_{p \in C} |a_p| = n + 1$. Then for all $p \in C$, the sequence

$$0 \to \mathscr{O}_C(-p) \to \mathscr{O}_C \to \mathscr{O}|_p \to 0$$

is exact by Lemma 7.9. So, the sequence

$$0 \to \mathscr{O}_C(-p) \otimes \mathscr{O}_C(D) \to \mathscr{O}_C \otimes \mathscr{O}_C(D) \to \mathscr{O}|_p \otimes \mathscr{O}_C(D) \to 0$$

is also exact by Lemma 7.10. By Lemma 7.11, this implies

$$\chi(C, \mathscr{O}_C \otimes \mathscr{O}_C(D)) = \chi(C, \mathscr{O}_C(-p) \otimes \mathscr{O}_C(D)) + \chi(C, \mathscr{O}|_p \otimes \mathscr{O}_C(D)).$$

Thus,

$$\begin{aligned} \chi(C, \mathscr{O}_C(D)) &= \chi(C, \mathscr{O}_C(D-p)) + \sum_{i=0}^{\dim X} (-1)^i h^i(X, \mathscr{O}|_p) \\ &= \chi(C, \mathscr{O}_C(D-p)) + \deg p \end{aligned}$$

By the inductive hypothesis, $\chi(C, \mathscr{O}_C(D-p)) = \deg(D-p) + \chi(C, \mathscr{O}_C)$. So,

$$\begin{aligned} \chi(C, \mathscr{O}_C(D-p)) + \deg p &= \deg(D-p) + \chi(C, \mathscr{O}_C) + \deg p \\ &= \deg D - \deg p + \chi(C, \mathscr{O}_C) + \deg p \\ &= \chi(C, \mathscr{O}_C) + \deg D. \end{aligned}$$

This implies $\chi(C, \mathscr{O}_C(D)) = \chi(C, \mathscr{O}_C(D-p)) + \deg p = \chi(C, \mathscr{O}_C) + \deg D$ when $\sum_{p \in C} |a_p| = n+1$ for a divisor $D := \sum_{p \in C} a_p[p]$.

With this, we have shown that $\chi(C, \mathscr{O}_C(D)) = \deg D + \chi(C, \mathscr{O}_C)$ for a divisor $D = \sum_{p \in C} a_p[p]$ when $\sum_{p \in C} |a_p| = 0$ and that if $\sum_{p \in C} |a_p| = n$, then $\sum_{p \in C} |a_p| = n+1$. So, the theorem holds by induction. □

This proves our main object of the paper, the Riemann-Roch theorem. Given this result, we can now prove Clifford's theorem and use it to classify line bundles on $\mathbb{P}^3$.

Acknowledgments

I would like to thank my mentor, Livia Xu, for all her help in completing this paper. Her guidance allowed me to understand nuances in the material that I would not have been able to see otherwise. I would additionally like to thank Peter May and the UChicago Math REU for pairing me with Livia and providing a space in which I could write this paper. Finally, I would like to thank my family for providing me with the opportunities that have allowed me to learn enough math to write this paper.

References

[1] Robin Hartshorne. Algebraic Geometry. Springer. 1977.
[2] Ravi Vakil. The Rising Sea: Foundations of Algebraic Geometry. Aug 29, 2022. https://math.stanford.edu/ vakil/216blog/FOAGaug2922public.pdf
[3] Otto Forster. Lectures on Riemann Surfaces. Springer. 1981.
[4] William Fulton. Algebraic Curves: An Introduction to Algebraic Geometry. Jan 28, 2008.