AN INTRODUCTION TO FOURIER SERIES AND TRANSFORMS

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Abstract. This paper offers a brief introduction to the theory, calculation, and application of Fourier series and transforms. First, we define the trigonometric and exponential representations of the Fourier series, coupled with some examples of its use. We then define the Fourier transform, followed by an illustrative example of its function and distinctness from the Fourier Series. Thirdly, we establish the definition and properties of the Dirac Delta Function, which we proceed to use in taking the Fourier transform of constant functions. Lastly, we briefly discuss the application of Fourier analysis to signal processing through the advent of the magnitude Fourier transform. This paper is designed for readers willing to accept Fourier theory at face value, merely motivated to know its formulae and methods of use. This paper seeks to satisfy such motivation by providing a straightforward and relatively concise walkthrough of the core formulae and real-world application of Fourier analysis.

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1. Introduction

Named after its founder, the great French mathematician Joseph Fourier (1768-1830), Fourier analysis allows for the decomposition of periodic and aperiodic functions alike into a series of sinusoidal and exponential functions. This is done by taking advantage of the orthogonality between the sine and cosine functions, which act as a basis in the space of functions. The pillars of Fourier analysis are Fourier Series and Fourier Transforms. The first deals with periodic functions, and the second deals with aperiodic functions. Fourier series and transforms have powerful real-world applications in signal processing, seismology, econometrics, and physics, to name a few. Fourier analysis is embedded in the technology we find so essential to our modern lifestyle, for example, in the storage and transmission of digital images.

2. Fourier Series

Theorem 2.1 (Trigonometric Fourier Representation)**.** *Let a function* $f : \mathbb{R} \to \mathbb{R}$ *be periodic for* $0 \le x \le L$*,* L *being the length of the period. This means that for all* $x \in \mathbb{R}$*,* $f(x) = f(x + L)$*. Fourier's Theorem states that* $f(x)$ *can be written as the series*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right) \right] \tag{2.2}$$

where

$$a_0 = \frac{1}{L} \int_0^L f(x) dx, \tag{2.3}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n x}{L}\right) dx, \tag{2.4}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi n x}{L}\right) dx. \tag{2.5}$$

a_0, a_n, and b_n are called **Fourier coefficients**.

Remark 2.6. The bounds of integration for (2.3), (2.4), and (2.5) are from 0 to L. However, Fourier coefficients can be determined just the same as long as you integrate over a period length. For example, you can integrate from $-\frac{L}{2}$ to $\frac{L}{2}$, $-\frac{L}{4}$ to $\frac{3L}{4}$, $-7L$ to $6L$, and in general, for $p \in R$, from p to $L + p$. One may want to use alternate bounds of integration in order to avoid discontinuities or if it produces a cleaner way of evaluating an antiderivative.

The definition just outlined can be daunting. Therefore, working through a couple of examples will be useful for understanding how it works and that it works.

Example 2.7. (Sawtooth Function) Consider a sawtooth function where $f(x) = Ax$ for some $A \in \mathbb{R}$ and for $-\frac{L}{2} < x < \frac{L}{2}$ with a period L. Such a function looks like

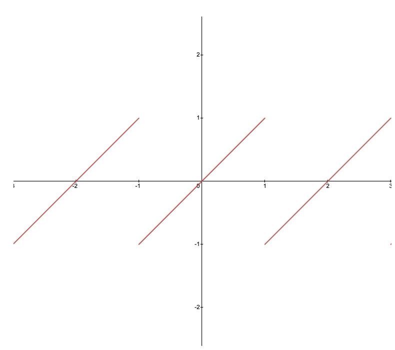


Figure 1. Odd sawtooth function $f(x) = x$ with a period of 2

This function is odd because we have defined it such that it is symmetric across the origin. Therefore, by integrating from $\frac{-L}{2}$ to $\frac{L}{2}$, it is clear that a_0 will integrate to 0 since we only want to integrate over the original function. Furthermore, all of our a_n terms will also integrate to 0 since an odd function multiplied by an even function remains an odd function. However, an odd function multiplied by an odd function is an even function. Therefore, we only need to consider b_n since it is the only term that does not integrate to 0. Plugging in our function Ax, we get

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} Ax \sin\left(\frac{2\pi n x}{L}\right) dx.$$

By integrating by parts for a fixed constant $r \in \mathbb{R}$, we find that

$$\int x\sin(rx)dx = x\left(\frac{-1}{r}\cos(rx)\right) - \int \frac{-1}{r}\cos(rx)dx.$$

$$= \frac{-x}{r}\cos(rx) + \frac{1}{r^2}\sin(rx)$$

Letting $r = \frac{2\pi n}{L}$, we have

$$b_n = \frac{2A}{L}\left(-\frac{Lx}{2\pi n}\cos\left(\frac{2\pi n x}{L}\right) + \frac{L^2}{4\pi^2 n^2}\sin\left(\frac{2\pi n x}{L}\right) dx\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{2A}{L}\left(-\frac{L^2}{4\pi n}\cos(\pi n x) + \frac{L^2}{4\pi^2 n^2}\sin(\pi n x) - \frac{L^2}{4\pi n}\cos(-\pi n x)\frac{L^2}{4\pi^2 n^2}\sin(\pi n x)\right).$$

Since $\sin(\pi n) = 0$ for any integer n, we're left with

$$b_n = -\frac{AL}{2\pi n}\cos(\pi n) - \frac{AL}{2\pi n}\cos(-\pi n).$$

Since $\cos(x) = \cos(-x)$ because it is an even function,

$$b_n = -\frac{AL}{\pi n}\cos(\pi n)$$

$$\implies b_n = (-1)^{n+1}\frac{AL}{\pi n}.$$

Therefore, our Fourier Series for $f(x)$ will look like,

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{AL}{\pi n}\sin\left(\frac{2\pi n x}{L}\right)$$

$$\implies f(x) = \frac{AL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{2\pi n x}{L}\right). \tag{2.8}$$

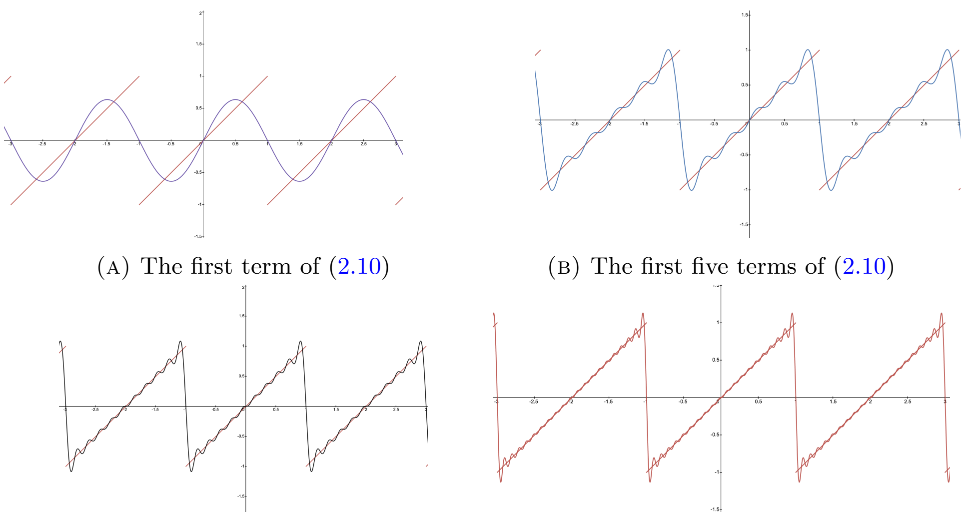
Writing out the first few terms of this series, we get

$$f(x) = \frac{AL}{\pi}\left[\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) - \cdots\right]. \tag{2.9}$$

Plugging in $L = 2$ and $A = 1$ for this series, we get

$$f(x) = \frac{2}{\pi}\left[\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) - \cdots\right], \tag{2.10}$$

which graphically looks like the following:



(A) The first term of (2.10)

(B) The first five terms of (2.10)

(C) The first ten terms of (2.10)

(D) The first twenty terms of (2.10)

FIGURE 2. Increasing terms of our series (2.10) approaches $f(x) = x$ when $L = 2$ and $A = 1$

As the number of terms increases, our series looks more and more like the function it is intended to describe. This is exactly the purpose of Fourier analysis.

We can restate Fourier's Theorem by reinterpreting the sine and cosine terms as components of complex exponential functions. This results in a series of exponential functions consisting of a real part and an imaginary part.

Theorem 2.11 (Exponential Fourier Representation)**.** *Let a function $f : \mathbb{R} \to \mathbb{R}$ be periodic for $0 \le x \le L$, L being the length of the period. This means that for all $x \in \mathbb{R}$, $f(x) = f(x + L)$. Fourier's Theorem states that $f(x)$ can be written as the following series:*

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{2i\pi n x/L} \tag{2.12}$$

where the C_n coefficients are given by

$$C_n = \frac{1}{L}\int_0^L f(x) e^{-2i\pi n x/L} dx. \tag{2.13}$$

Proof. We will start by examining Euler's identity for sine and cosine, which states that for $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \tag{2.14}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2.15}$$

Plugging this in to (2.2) by letting $z = \frac{2\pi n x}{L}$, we have

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{iz} + e^{-iz}}{2} + b_n \frac{e^{iz} - e^{-iz}}{2i} \\ &= a_0 + \sum_{n=1}^{\infty} \frac{a_n e^{iz} + a_n e^{-iz} - i b_n e^{iz} + i b_n e^{-iz}}{2} \\ &= a_0 + \sum_{n=1}^{\infty} e^{iz} \left(\frac{a_n - i b_n}{2} \right) + e^{-iz} \left(\frac{a_n + i b_n}{2} \right). \end{aligned}$$

Now, we can take both terms in the parentheses and assign them to the same variable by letting

$$C_{-n} = \frac{a_n + i b_n}{2}, \quad C_n = \frac{a_n - i b_n}{2}. \tag{2.16}$$

Thus, we can rewrite our series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} C_n e^{iz} + C_{-n} e^{-iz}.$$

Plugging back in our definition for z, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} C_n e^{2\pi i n x/L} + C_{-n} e^{-2\pi i n x/L}.$$

We can conglomerate our exponential terms and coefficients by accounting for only C_n while allowing the sum to operate along all integers. Doing so multiplies $2\pi - inx/L$ by -1, and thus both exponential terms go to $e^{2\pi i n x/L}$. Lastly, we can let our $a_0 = C_0$ to account for $n = 0$. Therefore, we have

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n x/L}.$$

Now, we can calculate C_n based on how we defined it. In (2.16), $C_n = \dfrac{a_n - i b_n}{2}$. By plugging in (2.4) and (2.5), we have that

$$C_n = \frac{1}{2} \left[\frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi n x}{L} \right) dx - \frac{2i}{L} \int_0^L f(x) \sin\left(\frac{2\pi n x}{L} \right) dx \right].$$

Recalling (2.14) and (2.15), we then have

$$C_n = \frac{1}{L}\int_0^L f(x)\left(\frac{e^{2\pi inx/L}+e^{-2\pi inx/L}}{2}\right)dx$$
$$-\frac{i}{L}\int_0^L f(x)\left(\frac{e^{2\pi inx/L}-e^{-2\pi inx/L}}{2i}\right)dx$$

Combining terms and canceling out the i's in the second term, we get

$$C_n = \frac{1}{L}\int_0^L f(x)\left(\frac{e^{2\pi inx/L}+e^{-2\pi inx/L}-e^{2\pi inx/L}+e^{-2\pi inx/L}}{2}\right)dx$$
$$\implies C_n = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}dx$$

□

Remark 2.17. We can now switch between exponential and trigonometric representations of the Fourier Series relatively easily. This process of switching is even simpler when one knows a_n and b_n for the trigonometric series and C_n for the exponential series. The following formulae allow us to relate all three:

$$C_n = \frac{a_n + ib_n}{2}, \tag{2.18}$$

$$a_n = C_n + C_{-n}, \tag{2.19}$$

$$b_n = i(C_n - C_{-n}). \tag{2.20}$$

Let's work through an example, though, this time using the exponential representation of a Fourier series.

Example 2.21. We will repeat the sawtooth function example. Plugging in Ax to (2.13),

$$C_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} Axe^{-2\pi nx/L}. \tag{2.22}$$

Integrating by parts, we know that for a constant r,

$$\int xe^{-rx}dx = -\frac{xe^{-rx}}{r} - \frac{e^{-rx}}{r^2}.$$

This is valid except for the case in which $r = 0$. Therefore, by letting $r = i2\pi n/L$, we need to consider the special case in which $n = 0$. If $n = 0$, (2.22) will be $\int_{-\frac{L}{2}}^{\frac{L}{2}} Ax$ which just integrates to 0 since its an odd function. Therefore, considering all n except for 0, we have

$$C_n = \frac{-A}{L}\left(\frac{Lxe^{-2\pi inx/L}}{i2\pi n} + \frac{L^2e^{-i2\pi nx/L}}{4\pi^2n^2}\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}\right).$$

The second term evaluates to 0 because it's limits are equal, and we are left with

$$\begin{aligned} C_n &= -\frac{A}{L}\left(\frac{L^2 e^{-\pi i n}}{i4\pi n} + \frac{L^2 e^{\pi i n}}{i4\pi n}\right) \\ &= -\frac{A}{L} \cdot \frac{L^2}{i4\pi n}(e^{-\pi i n} + e^{\pi i n}). \end{aligned}$$

The exponential terms sum to $2(-1)^n$. So, factoring i out, we get

$$C_n = (-1)^n \frac{iAL}{2\pi n} \quad \text{for } n \neq 0.$$

Therefore, our series can be expressed as

$$f(x) = \sum_{n\neq 0} (-1)^n \frac{iAL}{2\pi n} e^{-2\pi n x/L}.$$

We can double-check that this answer agrees with our previous result by writing this exponential in terms of sines and cosines:

$$f(x) = \sum_{n\neq 0} (-1)^n \frac{iAL}{2\pi n}\left(\cos\left(\frac{2\pi n}{L}\right) + i \sin\left(\frac{2\pi n}{L}\right)\right).$$

Since $(-1)^n/n$ is an odd function of n and the cosine terms are even functions of n, those terms sum to 0. However, since the sine terms are odd in sine, we only consider the positive terms and then double the result. Lastly, $i^2(-1)^n = (-1)^{n+1}$, leaving us with

$$f(x) = \frac{AL}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{2\pi n x}{L}\right), \tag{2.23}$$

which agrees with (2.8).

3. Fourier Transforms

The Fourier series we have been working with applies to periodic functions. However, an appropriate next question is asking if we can do the same for a function that is not periodic: does there exist a Fourier series for an aperiodic function? At first, this question may sound like it necessitates an entirely different approach, but in reality, we can extend Fourier series over periodic functions to aperiodic functions by initially examining a function over a period length, $-\frac{L}{2} \leq x \leq \frac{L}{2}$, but then extending $L \to \infty$. This move intuitively makes sense because we can think of an aperiodic function as a function that repeats only at infinity, which essentially means it doesn't repeat at all.

Theorem 3.1 (Exponential Fourier Transform Representation)**.** *Let a function $f\colon \mathbb{R} \to \mathbb{R}$ be aperiodic. We can express this function as*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \tag{3.1}$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{3.2}$$

Proof. We know by Theorem 2.11 that a periodic function $f(x)$ with period L can be expressed as

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{2\pi i n x/L}.$$

Let $k_n = 2\pi n/L$. Taking the derivative of this, we have that $dk_n = 2\pi \; dn/L$. However, recall that we are still operating over a discrete sum of integers, and thus $\frac{d}{dn} = 1$ because that is the amount we are changing from one n term to another. Therefore, we have that $dk_n = 2\pi/L$. If we take the limit of $L \to \infty$, we have that $\lim_{L\to\infty} dk_n = \lim_{L\to\infty} 2\pi/L = 0$ Now, since dn is 1, it is equivalently true to write our function as

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{ik_n x} dn$$

$$\implies f(x) = \sum_{-\infty}^{\infty} C_n e^{ik_n x} \left(\frac{L}{2\pi} dk_n \right).$$

Here is the critical step. We want to take the limit of L as it approaches infinity. By doing this, we get the following:

$$\lim_{L\to\infty} dk_n = \lim_{L\to\infty} 2\pi/L \to 0,$$

$$\lim_{L\to\infty} f(x) = \lim_{L\to\infty} \left[\sum_{-\infty}^{\infty} C_n e^{ik_n x} \left(\frac{L}{2\pi} dk_n \right) \right].$$

As L approaches infinity, dk_n gets really small, eventually transforming our discrete sum into a continuous sum. Thus we can take the integral:

$$f(x) = \int_{-\infty}^{\infty} C_n \left(\frac{L}{2\pi} \right) e^{ik_n x} dk_n.$$

Lastly, because we are no longer operating on a discrete sum, we can drop the index n and let $k_n = k$ and for notation's sake, we can let $\hat{f}(k) = C_n \frac{L}{2\pi}$. Therefore, plugging back in for k, we end up with

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) \left(\frac{L}{2\pi} \right) e^{ikx} dk_n$$

where

$$\hat{f}(k) = \frac{L}{2\pi} C_n = \frac{L}{2\pi} \cdot \frac{1}{L} \int_0^L f(x) e^{-2\pi i n x/L} dx$$

$$\implies \hat{f}(k) = \frac{1}{2\pi} \int_0^L f(x) e^{-ikx} dk.$$

□

Let's move through some examples to further understand how this works and that it works.

Example 3.3. First, consider a periodic, odd, step function. For $n \in \mathbb{N}$, let

$$f(x) = \begin{cases} A & \text{if } \frac{(n-1)L}{2} < x < \frac{nL}{2} \\ -A & \text{if } -\frac{nL}{2} < x < \frac{(1-n)L}{2} \end{cases}$$

The graph for this looks like

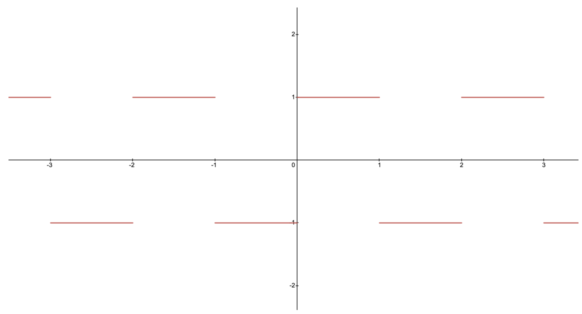


FIGURE 3. Odd periodic function where $A = 1$ and $L = 2$

This function is periodic, and so by taking the trigonometric representation of the Fourier Series for this function, we need to only consider the b_n terms since this is an odd function, and therefore the a_0 and a_n equal 0. We thus have

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2\pi nx}{L}\right) = \frac{2}{L}\cdot 2\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi nx}{L}\right)$$

$$= -\frac{4A}{L}\cdot\frac{L}{2\pi n}\left(\cos\left(\frac{2\pi nx}{L}\right)\Big|_0^{\frac{L}{2}}\right) = \frac{2A}{\pi n}(1-\cos(\pi n)).$$

Because for all even n, $\cos(\pi n) = 1$, we need to only consider summing odd n terms. Thus we have

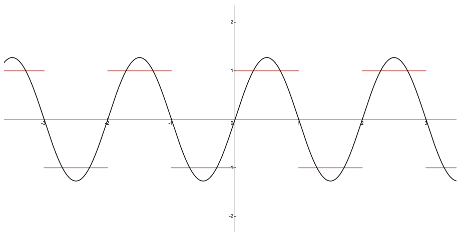
$$f(x) = \frac{2A}{\pi}\sum_{\text{odd } n}^{\infty}\frac{1}{n}\sin\left(\frac{2\pi nx}{L}\right) \tag{3.4}$$

$$= \frac{2A}{\pi}\left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{6\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{10\pi x}{L}\right) + \cdots\right]. \tag{3.5}$$

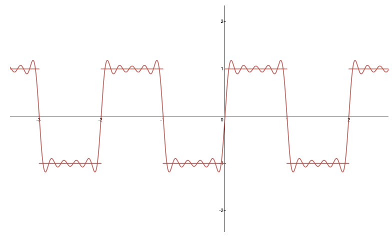
In exponential terms, this series looks like

$$f(x) = \frac{-iA}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{\pi nx}{L}\right)\right)e^{2i\pi nx/L}. \tag{3.6}$$

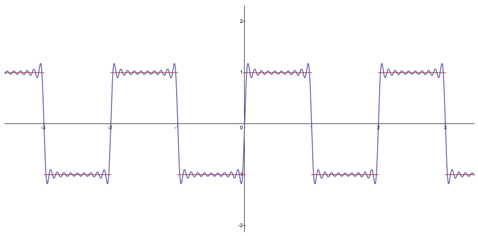
Plugging in $1 = A$ and $L = 2$ for (3.6), our calculation will yield expected results:



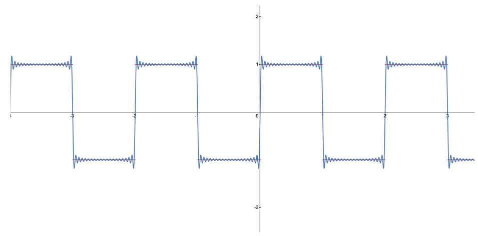
(A) The first term of (3.6)



(B) The first five terms of (3.6)



(C) The first ten terms of (3.6)



(D) The first twenty terms of (3.6)

Now, equipped with Fourier transforms, let us now consider a step function with just one step and then zero everywhere else, i.e., consider a constant A and a period L in which

$$f(x) = \begin{cases} A & \text{if } 0 < x < \frac{L}{2} \\ -A & \text{if } -\frac{L}{2} < x < 0 \\ 0 & \text{if } x < -\frac{L}{2} \text{ or } 1 < \frac{L}{2} \end{cases} \tag{3.7}$$

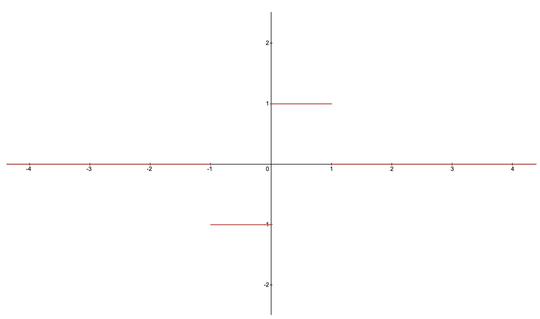


FIGURE 5. Odd aperiodic function where $A = 1$

Here,

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dk$$

$$\Longrightarrow \hat{f}(k) = \frac{1}{2\pi} \int_{-\frac{L}{2}}^{0} -Ae^{-ikx} dk + \frac{1}{2\pi} \int_{0}^{\frac{L}{2}} Ae^{-ikx} dk$$

$$\implies \hat{f}(k) = \frac{-A}{2\pi}\left(\frac{e^{-ikx}}{-ik}\bigg|_{-\frac{L}{2}}^{0}\right) + \frac{A}{2\pi}\left(\frac{e^{-ikx}}{-ik}\bigg|_{0}^{\frac{L}{2}}\right)$$

$$\implies \hat{f}(k) = \frac{-A}{2\pi ik}\left(-(1-e^{-ikL/2}) + (e^{ikL/2}-1)\right)$$

$$\implies \hat{f}(k) = \frac{-Ai}{\pi k}\left(1-\cos\left(\frac{kl}{2}\right)\right).$$

So for $f(x)$ we get,

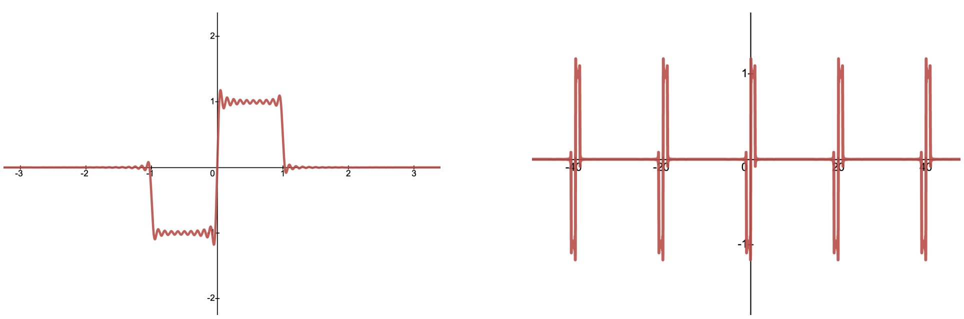
$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{-ikx}dk$$

$$\implies f(x) = \int_{-\infty}^{\infty} \frac{-Ai}{\pi k}\left(1-\cos\left(\frac{kL}{2}\right)\right)e^{-ikx}dk$$

$$\implies f(x) = \frac{-iA}{\pi}\int_{-\infty}^{\infty} \frac{1}{k}\left(1-\cos\left(\frac{kL}{2}\right)\right)e^{-ikx}dk. \tag{3.8}$$

When we compare (3.8) with (3.6), they are highly similar. The key difference is that in (3.6), we are taking a discrete sum over the integers n whereas in (3.8) we are integrating over a continuous variable k.

A lurking concern may be how we can know that (3.8) accurately describes (3.8). To do this, we can approximate the integral by taking a discrete sum over a very small dk. The smaller dk is, the better the approximation we will have.



(A) Zoomed in view

(B) Zoomed out view

FIGURE 6. Approximation of (3.8) using $dk = .1$, $A = 1$, $L = 2$, using the first 200 terms

Our approximation is highly revealing. We can say that for a small dk, our approximation will look like Figure 5, though only when zoomed in to the origin. When zoomed out enough, however, we can see that the behavior at the origin repeats regularly. The smaller we take our dk to be, the better our approximation for (3.7) will be as the step will repeat at further and further distances from the origin. This is equivalent to letting the period approach infinity, which is exactly what we did originally.

Theorem 3.9. *Fourier transforms can be expressed in terms of trigonometric functions as the following:*

$$f(x) = \int_{-\infty}^{\infty} C_e(k)\cos(kx)dk + i\int_{-\infty}^{\infty} C_o(k)\sin(kx)dk \tag{3.10}$$

where $\hat{f}(k)$ is split into its even component,

$$C_e(k) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} f_e(x)\cos(kx)dx, \tag{3.11}$$

and its odd component:

$$C_o(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_o(x)\sin(kx)dx. \tag{3.12}$$

The even and odd components of the original function can be expressed respectively as

$$f_e(x) = \frac{f(x)+f(-x)}{2} \quad \textit{and} \quad f_o(x) = \frac{f(x)-f(-x)}{2}$$

such that $f_e(x) + f_o(x) = f(x)$.

4. The Dirac Delta Distribution

We have thus determined methods for determining trigonometric and exponential Fourier representations of periodic and aperiodic functions alike. However, we have yet to consider functions that are merely non-zero constants. A non-zero constant function f is intriguing because $f(x)$ is the same everywhere, and thus the period would be so small as to be 0 since the function is always repeating. In order to tackle such functions, we must first consider the Dirac delta function.

Definition 4.1 (Dirac's Delta "Function")**.** The Dirac delta function, $\delta(x)$, is zero everywhere except at the origin and has area 1.

Remark 4.2. This "function" is a function in name alone as it is actually a distribution. Distributions, rather than functions themselves, are generalizations of the concept of a function, describing where functions lie in the differentiable function space. It is analogous to the statistical distribution of probabilities in the probability space. This definition may appear outlandish as it seems impossible for a function that is non-zero except for at a single point to have a non-zero area. However, this function does make sense when the limits of certain functions are taken. We will see what this means later.

Let's begin by considering some properties of the delta function. First, consider letting $f\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then,

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0). \tag{4.3}$$

This is because the delta function is zero everywhere at the origin. Thus, the function integrates to 0 on all points that are not at the origin, and thus we need only consider $f(0)$. Furthermore, $f(0)$ is a constant and thus can be factored out. Lastly, we have defined the area under $\delta(x)$ to be exactly one, and thus we have

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \int_{-\infty}^{\infty} \delta(x) f(0) dx = f(0) \int_{-\infty}^{\infty} \delta(x) dx = f(0).$$

We can further generalize this property by considering the following:

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0).$$

This fact follows from similar logic as laid out in (4.3). However, we result in $f(x_0)$ because $\delta(x - x_0)$ is only non-zero at x_0, and thus it is the only point that does not integrate to 0. With these properties in mind, we can thus conclude with the following.

Definition 4.4. The Dirac delta function is the function that satisfies the following for an arbitrary function f:

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0). \tag{4.5}$$

Let us now examine an example of what the delta function actually looks like. There is only one true delta distribution (zero everywhere except at the origin with an area of 1), though it can be written down in various ways. We will use Fourier analysis to write the delta distribution down explicitly. We will first present proof for one and then merely list examples for others.

Example 4.6. Let us take the Fourier transform of the exponential function $f(x) = Ae^{-b|x|}$ where both A and b are constants in $\mathbb{R}$ (the shape of this function can be found in Figure 7).

Plugging into our formula for Theorem 3.2, we get

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Ae^{-b|x|} e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} Ae^{bx} e^{-ikx} dx + \frac{1}{2\pi} \int_{0}^{\infty} Ae^{-bx} e^{-ikx} dx$$

$$= \frac{A}{2\pi} \left(\int_{-\infty}^{0} e^{(b-ik)x} dx + \frac{1}{2\pi} \int_{0}^{\infty} e^{(-b-ik)x} dx \right)$$

$$= \frac{A}{2\pi} \left(\frac{e^{(b-ik)x}}{b-ik} \Big|_{-\infty}^{0} + \frac{e^{(-b-ik)x}}{(-b-ik)} \Big|_{0}^{-\infty} \right)$$

$$= \frac{A}{2\pi} \left(\frac{1}{b-ik} + 1 {-b-ik} \right)$$

$$= \frac{A}{2\pi} \left(\frac{b+ik+b-ik}{b^2+k^2} \right)$$

$$\hat{f}(k) = \frac{Ab}{\pi(b^2+k^2)}. \tag{4.7}$$

Equation (4.7) has a special name, the Lorentzian function, formalized as

$$f(k) = \frac{c_1}{c_2 + c_3 k} \tag{4.8}$$

where c_1, c_2 and c_3 are constants in $\mathbb{R}$.

With this calculation of (4.7), we can now compute the Fourier transform for $f(x) = 1$. To do so, we will use a "tapering" method, in which we take a function that is equal to 1 or very nearly equal to 1 for very large x, though eventually tapers off to 0. By "tapering" infinitely, we will therefore arrive at a function in which is identical to $f(x) = 1$. We will first taper with the function given above. Let $f(x) = Ae^{-bix}$. If we fix $A = 1$ and let $b \to 0$, our function would approach $f(x) = 1$ because $e^{0 \cdot ix} = 1$. We can see this graphically by letting $b = 0.01$:

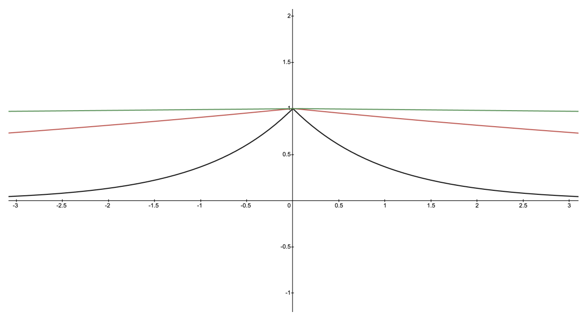


Figure 7. Exponential decay function with $A = 1$ for all lines, $b = 1$ for the black line, $b = 0.1$ for the red line, and $b = 0.01$ for the green line

As b gets smaller and smaller, i.e., as our exponential decay function approaches $f(x) = 1$, our $\hat{f}(k)$ will look more and more like a delta function because a taller and taller spike will appear. We can see this graphically:

Letting b approach 0, the Lorentizan Fourier transform starts to look like a delta function. We can see this by taking the integral of

$$\int_{-\infty}^{\infty} \frac{b\ \ dk}{\pi(b^2 + k^2)}.$$

Using the fact that $\displaystyle\int_{-\infty}^{\infty} \frac{b\,dk}{(b^2 + k^2)} = \tan^{-1}\left(\frac{k}{b}\right)$, we have

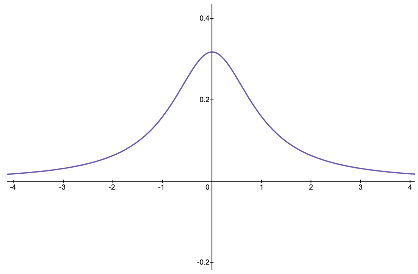
$$\int_{-\infty}^{\infty} \frac{bdk}{\pi(b^2 + k^2)} = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{k}{b}\right)\Bigg|_{-\infty}^{\infty}\right) = \frac{1}{\pi}\left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 1.$$

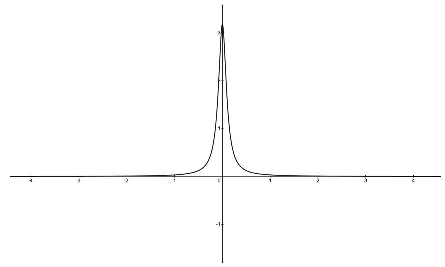
Since we arrived at this result regardless of the choice of b, and because this function becomes sharply peaked at 0 when $b \to 0$, we have found an equivalent representation of the delta function, which we can formalize as

$$\delta(k) = \lim_{b \to 0} \frac{b}{\pi(b^2 + k^2)}. \tag{4.9}$$

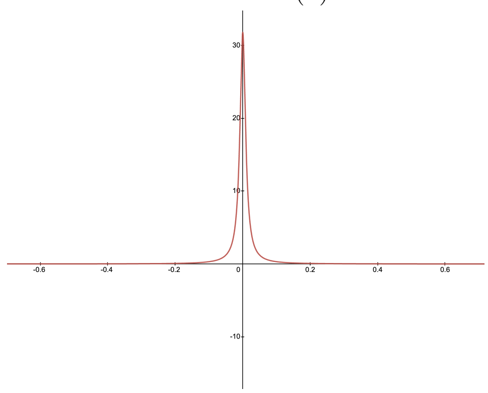
Other representations of the delta function can be found by taking the Fourier transform of other "tapered" functions. Some prominent examples are the square wave function and the Gaussian function. The Gaussian function gains its significance by representing the probability density function of a normal distribution,



(A) Fourier transform when $b = 1$



(B) Fourier transform when $b = 0.1$



(C) Fourier transform when $b = 0.01$

FIGURE 8. Fourier transform for $f(x) = Ae^{-b|x|}$ for $A = 1$ for different values of b

taking on a "bell" shape. It is special because it accurately describes the probability distribution of a wide range of phenomena, such as rolling a pair of dice or height. The Gaussian function can be generalized as $f\colon \mathbb{R} \to \mathbb{R}$ where

$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}} \tag{4.10}$$

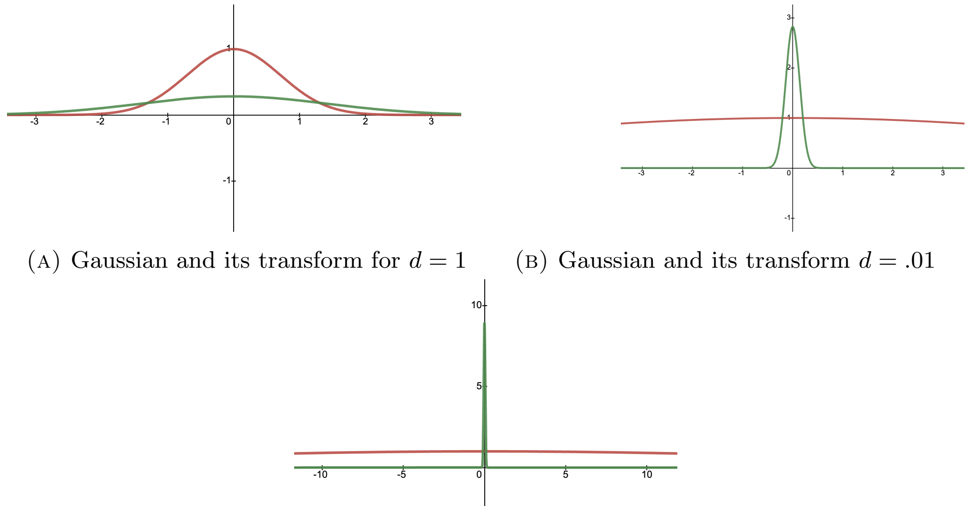
where a describes the height of the peak, b describes the x-coordinate of the center of the peak, and c describes the standard deviation.

Example 4.11. Let's consider the Gaussian function, which has a peak height of 1 and is centered at the origin. Such a function can be expressed as $f(x) = e^{-dx^2}$ where d is a constant in $\mathbb{R}$. We have simplified the exponent such that now, $\sqrt{\frac{1}{2d}} = c$, where c is the standard deviation. The Fourier transform of this Gaussian function is

$$\hat{f}(k) = \frac{e^{-k^2/4d}}{2\sqrt{\pi d}}. \tag{4.12}$$

Its corresponding delta function representation would be

$$\delta(k) = \lim_{d \to 0} \frac{e^{-k^2/4d}}{2\sqrt{\pi d}}. \tag{4.13}$$



(A) Gaussian and its transform for $d = 1$

(B) Gaussian and its transform $d = .01$

(C) Gaussian and its transform for $d = .001$

FIGURE 9. The Gaussian function, $f(x) = e^{-dx^2}$ shown in red and its transform shown in green for different values of d

Remark 4.14. The result from Example 4.11 is particularly notable as the Fourier transform of a Gaussian function is also a Gaussian function. It is one of the few examples of functions that are their own transforms.

Example 4.15. The Fourier transform of a square wave, i.e., for constants $a, A \in \mathbb{R}$,

$$f(x) = \begin{cases} A & \text{if } -a < x < a \\ 0 & \text{if } x < -a \text{ or } x < a \end{cases} \tag{4.16}$$

If we let $A = 1$, our Fourier transform will be

$$\hat{f}(k) = \frac{\sin(ka)}{\pi k}. \tag{4.17}$$

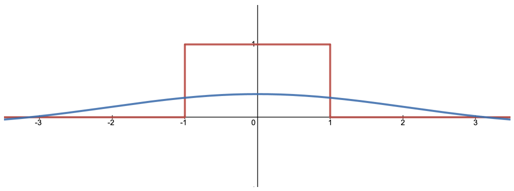
This Fourier transform attains another representation of the delta function by letting $a \to \infty$ (which correspondingly allows the function to approach $f(x) = 1$):

$$\delta(k) = \lim_{a\to\infty} \frac{\sin(ka)}{\pi k}. \tag{4.18}$$

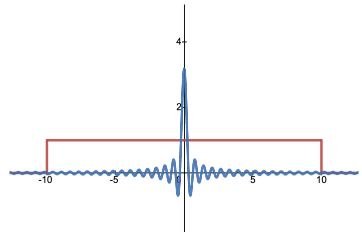
In general, we have seen that the Fourier transform for $f(x) = 1$ is a delta function. Therefore, plugging in 1 to (3.2), we are left with

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dx. \tag{4.19}$$

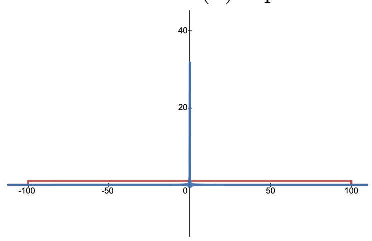
For curious readers, the calculations for (4.12),(4.13), (4.17), and (4.18) are derived in Section 3.5.3 of Morin's *Fourier Analysis*[1].



(A) Square wave and transform for $a = 1$



(B) Square wave and transform for $a = 10$



(C) Square wave and transform for $a = 100$

FIGURE 10. The square wave function shown in red and its transform shown in blue for different values of a

5. Application to Signal Processing

Fourier transforms and series are hugely applicable in signal processing. A signal is a function that expresses the behavior and condition of a physical measure within a system. Signals are measured by the intensity of the behavior over time, which can be highly informative, though even more information about the system can be learned. By taking the Fourier transform, we can also determine the underlying fundamental frequencies a system exhibits. As we have demonstrated, the Fourier transform does this by decomposing a signal into sine and cosine components, which can be reinterpreted as frequency components. In other words, the original signal displays how a system changes over time, and its Fourier transform displays the same change over frequency.

Let's further break down how this works in terms of sound. Sound is produced by a vibrating object which causes surrounding air molecules to undergo compression or rarefaction by bouncing off of each other at various rates. The air molecules remain relatively stable in space though sound energy is transferred through space. This ultimately translates to oscillations in air pressure, and therefore, sound is measured as air pressure over time. The shape it takes on is called its waveform. The waveform of a sound appears, on a macroscopic scale, aperiodic, though it is, in fact, periodic on a microscopic scale. This phenomenon can be seen in Figure 11.

Specifically in regards to sound, "signal tells us when certain notes are played in time, but hides the information about frequencies. In contrast, the Fourier transform of music displays which notes (frequencies) are played but hides the information about when the notes are played."[2] This speaks to the utility of the Fourier transform as it extracts underlying phenomena mostly imperceptible to humans.

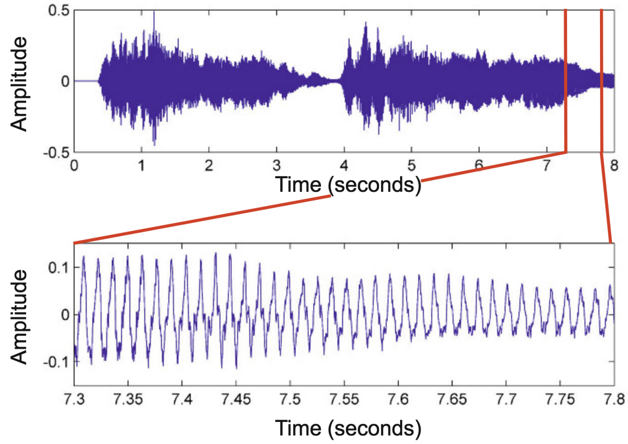


FIGURE 11. The waveform of a recording of the first five measures of Beethoven's Fifth with seconds 7.3 to 7.8 magnified

Graphing the frequencies of a signal is most often done digitally through what is called a Discrete Fourier Transform (DFT). As opposed to analog systems, which take continuous readings of systems, digital technology can only store and process a finite, albeit large, number of data. Therefore, a signal can only be digitized by converting it into what is called a discrete-time signal. A discrete-time signal is a signal expressed over a set of finite points, where points in time equidistant from each other are each associated with the amplitude of the sound at that point in time. The sampling period, T, is the time between samplings, and the sampling rate is the inverse of T, measured in Hertz. The following graphs display the signal as amplitude over time.

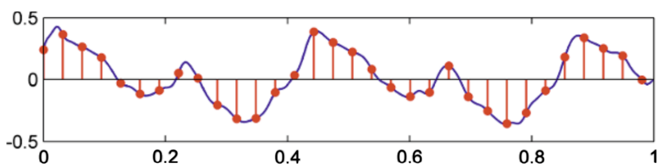


FIGURE 12. The digitizing of an arbitrary signal with a sampling rate of 32Hz

Once a signal has been digitized, a Fourier transform can be taken rather readily. Firstly, we also want to digitize both a sine wave and a cosine wave.

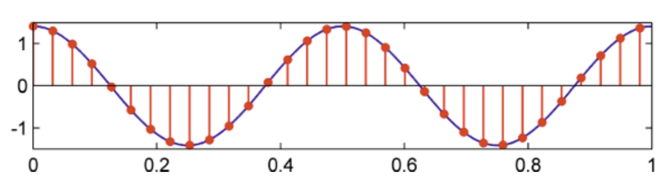


FIGURE 13. The digitizing of a cosine wave with frequency 2Hz

We then want to multiply the signals together.

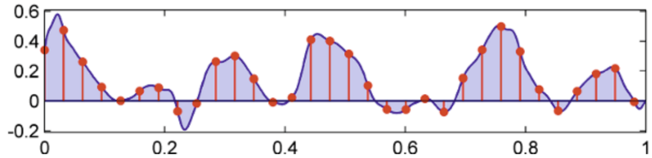


FIGURE 14. The digitizing of a cosine wave multiplied by our arbitrary signal

Our last step deviates from the Fourier transform as we must now take the Riemann sum of our new signal, as opposed to integrating. The greater our sampling rate is, the more closely our DFT will align with the true Fourier transform.

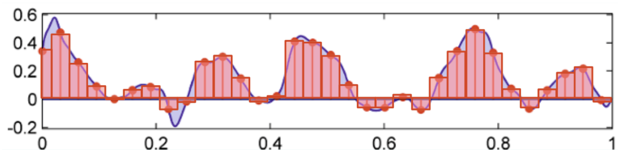


FIGURE 15. The digitizing of a cosine wave multiplied by our arbitrary signal

Once this sum is calculated, we then want to repeat the same process with a sine wave of the same frequency. Using both sums, we can calculate the magnitude of a given frequency by summing the squares of both sums and then taking their square root. This entire process is called a **magnitude Fourier Transform**, whose results are graphed in Figure 16(B). The algorithm for our DFT is a function $g : \mathbb{Z} \to \mathbb{R}$, with a given sampling of T,

$$g(k) = \sum_{n=-\infty}^{\infty} Tf(nT)e^{(-iknT)}. \tag{5.1}$$

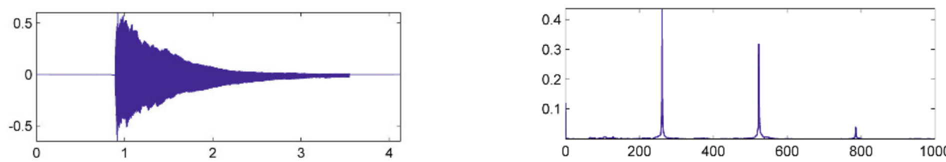
To calculate the magnitude of a frequency using our method of sinusoidal functions, we recall Euler's Formula to get

$$g(k) = \sum_{n=-\infty}^{\infty} Tf(nT)\cos(-knT) + i\sum_{n=-\infty}^{\infty} Tf(nT)\sin(-knT). \tag{5.2}$$

This will yield a complex number. We can calculate the magnitude of any complex number of the form $z = a + bi$ as $|z| = \sqrt{a^2 + b^2}$. We are therefore motivated to define our magnitude function as $|g(k)|: \mathbb{Z} \to \mathbb{R}$ where

$$|g(k)| = \sqrt{\left(\sum_{n=-\infty}^{\infty} Tf(nT)\cos(-knT)\right)^2 + \left(\sum_{n=-\infty}^{\infty} Tf(nT)\sin(-knT)\right)^2}. \tag{5.3}$$

By repeating this process for a range of frequencies, we are able to graph against magnitude, which displays the underlying frequency components of a given waveform.

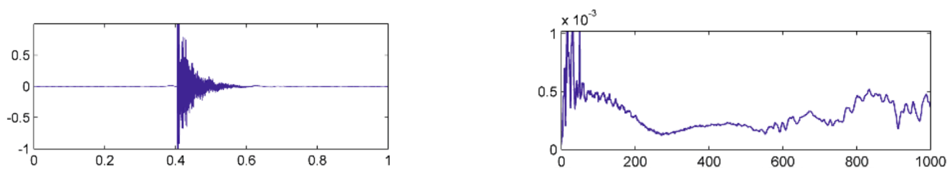


(A) Amplitude over time(seconds) (B) Magnitude over frequency(Hz)

FIGURE 16. Waveform and magnitude Fourier transform of a tone C4("the middle C") played on the piano

The above example of Figure 16 indicates that the tone C4 possesses a fundamental frequency of 262Hz, which is supported by the spikes around 262Hz, 524Hz, and 786Hz, all of which are multiples of 262Hz.

The Dirac delta function explains the physical properties of impulses on the spectrum of sound. Impulse sounds make sharp sounds for a short amount of time, such as a drum beat or a clap of the hand. Therefore, by taking its magnitude Fourier transform, we would expect to see a constant magnitude over a range of frequencies. This is indeed the case, as can be seen in Figure 17, where we take the magnitude Fourier transform of an impulse sound.



(A) Amplitude over time(seconds) (B) Magnitude over frequency(Hz)

FIGURE 17. Waveform and magnitude Fourier transform for a clapping sound

We can now see that the energy of an impulse sound is distributed relatively evenly across the frequency spectrum.

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