INTRODUCTION TO THE LEBESGUE INTEGRAL

JACOB STUMP

Abstract. We provide an introduction to the Lebesgue integral. We begin by discussing measures, and then we define the Lebesgue integral and prove several of its properties. We also work with L^p spaces, vector spaces of integrable functions. Finally, we put everything together to prove the Riesz Representation Theorem, an important result that describes the dual space of L^p.

Contents

1. Introduction

1.1. Motivation.

We will first review the construction of the Riemann integral.

Definition 1.1. Let $[a, b]$ be an interval. A **partition** P of $[a, b]$ is a finite set of points $x_0, x_1, \cdots, x_n$ such that $a = x_0 < x_1 < \cdots < x_n = b$. For each $1 \leq i \leq n$, we write $\Delta x_i = x_i - x_{i-1}$. Given a bounded function f and a partition of $[a, b]$, we denote:

$$M_i = \sup\{f(x) \mid x_{i-1} \leq x \leq x_i\};$$

$$m_i = \inf\{f(x) \mid x_{i-1} \leq x \leq x_i\};$$

Date: February 9, 2024.

$$U(P,f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i;$$
$$L(P,f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i.$$

$U(P,f)$ and $L(P,f)$ are called the **upper** and **lower sums**, respectively. Finally, we define the **upper** and **lower Riemann integrals of** f **over** $[\mathbf{a}, \mathbf{b}]$, respectively, as

$$\overline{\int_a^b} f(x)\, dx = \inf\{U(P,f) \mid P \text{ is a partition of } [a,b]\}$$
$$\underline{\int_a^b} f(x)\, dx = \sup\{L(P,f) \mid P \text{ is a partition of } [a,b]\}.$$

Note that the supremum and infimum are taken over all partitions of $[a,b]$.

We say that f is **Riemann integrable** if the upper and lower integrals are equal in value. In that case, we call this value the **Riemann integral** of f over $[a,b]$ and denote it by $\int_a^b f(x)\, dx$.

For elementary uses, the Riemann integral is invaluable. However, it can fail in numerous scenarios, failing to be defined. For example, consider the function defined on the interval $[0,1]$, given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

The Riemann integral can only be defined if the upper and lower integrals are equal. However, no matter what partition is chosen, each part will contain at least one rational number and one irrational number, due to the density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$. Thus, the upper integral will be equal to 1, while the lower integral will be 0. Hence, the Riemann integral cannot be defined for this function.

This is a serious shortcoming of the Riemann integral. We would like to create a new definition of the integral, which is applicable in more scenarios, and which agrees with the Riemann integral when it is defined. This is the purpose of the Lebesgue integral.

1.2. **Measures.** We must first introduce several definitions in order to be able to define the Lebesgue integral. First is the σ-algebra.

Definition 1.2. A **σ-algebra** χ is a collection of sets in $\mathbb{R}$ satisfying the following properties:
(i) $\mathbb{R}$ belongs to χ;
(ii) If A is in χ, then its complement $\mathbb{R} \setminus A$ is in χ;
(iii) χ is closed under countable unions of sets.

We now review De Mogan's Laws, which describe the relationship between the negation of unions and intersections:

Proposition 1.3. (**De Morgan's Laws**) *Let A and B be propositions. Then*

$$\textit{not } (A \textit{ or } B) \iff \textit{not } (A) \textit{ and not } (B);$$
$$\textit{not } (A \textit{ and } B) \iff \textit{not } (A) \textit{ or not } (B).$$

Proof. These results are easy to prove using truth tables. □

Example 1.4. In the context of this paper, we often take the propositions referenced above to be of the form: x is in E. For example,

$$x \notin \bigcup_{n=1}^{\infty}(E_j) \iff x \notin E_1 \text{ and } x \notin E_2 \text{ and } \cdots \iff x \in \bigcap_{n=1}^{\infty}(E_j)^c.$$

Similarly,

$$x \notin \bigcap_{n=1}^{\infty}(E_j) \iff x \notin E_1 \text{ or } x \notin E_2 \text{ or } \cdots \iff x \in \bigcup_{n=1}^{\infty}(E_j)^c.$$

Remark 1.5. By De Morgan's Laws, properties (ii) and (iii) in Definition 1.2 imply that χ is also closed under countable intersections of sets.

Definition 1.6. A sequence of sets $\{E_j\}_{j=1}^{\infty}$ is **pairwise disjoint** if, for all $i \neq j$, $E_i \cap E_j = \varnothing$.

Next, we provide a definition of a measure.

Definition 1.7. Let χ be a σ-algebra. A **measure** is a function $\lambda\colon \chi \to \widehat{\mathbb{R}}^+$ such that

(i) $\lambda(\varnothing) = 0$;

(ii) λ is countably additive, that is, if $\{E_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets in χ, then

$$\lambda\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \lambda(E_j).$$

We now produce a similar definition of a signed measure:

Definition 1.8. Let χ be a σ-algebra. A **signed measure** is a function $\lambda\colon \chi \to \mathbb{R}$ satisfying properties (i) and (ii) above.

Remark 1.9. The only differences between a signed measure and a measure are that a signed measure may take on positive and negative values, but a signed measure may not take on values in the extended reals.

Definition 1.10. A **measure space** is a triple (X, χ, λ), where X is a subset of $\mathbb{R}$, χ is a σ-algebra, and λ is a measure.

Definition 1.11. Let (X, χ, λ) be a measure space. λ is called **finite** if $\lambda(X) < \infty$. It is called σ-**finite** if X can be written as the union of a countable collection of measurable sets, each with finite measure. A measurable set E is said to be of **finite measure** if $\lambda(E) < \infty$ and is said to be σ-**finite** if E is the union of a countable collection of measurable sets, each with finite measure.
Remark. Here we have used the concept of the measurability of a set without defining it. We will do so in Definition 1.19.

Remark 1.12. Notice that finiteness is a strictly stronger condition than σ-finiteness: for given a finite set E, we can express E as the countable union $\bigcup_{j=1}^{\infty} E_j$, where $E_1 = E$ and $E_j = \varnothing$ for $j \geq 2$.

Remark 1.13. We note that given a σ-finite set E with a countable cover E_j, we can always choose a countable pairwise disjoint cover of E by keeping E_1 and replacing each E_k with $E \setminus \bigcup_{j=1}^{k-1} E_j$.

We now develop a few properties of measures.

Lemma 1.14. *Let (X, χ, λ) be a measure space. If $E, F \in \chi$ and $E \subseteq F$, then $\lambda(E) \leq \lambda(F)$. Moreover, if $\lambda(F) < \infty$, then $\lambda(F \setminus E) = \lambda(F) - \lambda(E)$.*

Proof. Using the identity: $E = (E \setminus F) \cup F$, we see by property (ii) of measures that since $(E \setminus F)$ and F are disjoint, $\lambda(E) = \lambda(E \setminus F) + \lambda(F) \leq \lambda(F)$. To prove the second result, we use a similar method: $\lambda(F) = \lambda(F \setminus E) + \lambda(E)$. Since $\lambda(F) < \infty$, then $\lambda(F \setminus E) < \infty$, so we may rearrange the equation, yielding the desired equality. $\square$

Proposition 1.15. *Let (X, χ, λ) be a measure space.*

(**a**) *If $\{E_j\}_{j=1}^{\infty}$ is a sequence of increasing sets, i.e. $E_1 \subseteq E_2 \subseteq \cdots$, in χ, then*

$$\lambda\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \to \infty} \lambda(E_j).$$

(**b**) *If $\{F_j\}_{j=1}^{\infty}$ is a sequence of decreasing sets in χ, i.e. $F_1 \supseteq F_2 \supseteq \cdots$, and if $\lambda(F_1) < \infty$, then*

$$\lambda\left(\bigcap_{j=1}^{\infty} F_j\right) = \lim_{j \to \infty} \lambda(F_j).$$

Proof. The proof is left as an exercise to the reader. Refer to [3], Lemma 2.4. $\square$

The above proposition will prove useful when we prove convergence theorems for sequences of functions under the Lebesgue integral.

Definition 1.16. For a measurable set E, we say that a property holds **λ-almost everywhere** on E, or it holds for **almost all** $x \in E$, if there is a subset E_0 of E for which $\lambda(E_0) = 0$ and the property holds for all $x \notin E_0$.

One of the most useful types of measures is called the Lebesgue measure, which seeks to provide a notion of the length of sets in $\mathbb{R}$. Desirable properties of such a function would include:

(i) $\mu(E) \geq 0$ for all E in $\mathbb{R}$;
(ii) $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all A, B in $\mathbb{R}$;
(iii) Countable additivity;
(iv) Unaffected by translation, that is, $\mu(E) = \mu(a + E)$, where $a \in \mathbb{R}$, and $\{a + E\}$ is defined as $\{a + e \mid e \in E\}$.

Remark 1.17. We will not describe the construction the Lebesgue measure in this paper, see Chapter 2 of [6] for more detailed explanation.

Notation 1.18. We typically use the symbol μ when referring to the Lebesgue measure while an unspecified measure is often denoted by λ or ν.

Unfortunately, there exist sets in $\mathbb{R}$ for which the Lebesgue measure is not defined. For an example of such a set, see Theorem 1.8 in [3]. Nevertheless, we may still use the Lebesgue measure on sets for which it is defined.

Definition 1.19. If the Lebesgue measure is defined for a set E, then we say that E is **Lebesgue-measurable**, or simply **measurable**.

One can show that the collection of all measurable sets forms a σ-algebra in $\mathbb{R}$. For most purposes, it suffices to consider only a subset of the Lebesgue measurable sets, called the Borel sets.

Definition 1.20. The **Borel σ-algebra** is the smallest σ-algebra which contains every open set in $\mathbb{R}$. This collection is referred to as the Borel sets.

From the definition of a σ-algebra, we see that the Borel sets include open sets, closed sets, and countable unions and intersections of open and closed sets.

Definition 1.21. Let λ be a signed measure on the σ-algebra χ. A set $P \in \chi$ is said to be **positive** with respect to λ if $\lambda(E \cap P) \geq 0$ for all $E \in \chi$. A set $N \in \chi$ is said to be **negative** with respect to λ if $\lambda(E \cap N) \leq 0$ for all $E \in \chi$. A set M is said to be a **null set** for λ if $\lambda(E \cap M) = 0$ for all $E \in \chi$.

Lemma 1.22. *Let λ be a signed measure on the σ-algebra χ on the set X. Then every measurable subset of a positive set is itself positive, and the union of a countable collection of positive sets is also positive.*

Proof. The first statement is true by definition of a positive set. For the second statement, let $E \subseteq \bigcup_{j=1}^{\infty} A_j$, where each A_j is positive. For $k \in \mathbb{N}$, define

$$E_k := \left(E \cap A_k\right) \setminus \left(\bigcup_{j=1}^{k-1} A_j\right).$$

Then, each E_k is a subset of A_k and is therefore positive. Also, $E = \bigcup_{k=1}^{\infty} E_k$. Thus, by the countable additivity of λ, we see that

$$\lambda(E) = \lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} E_k \geq 0.$$

Thus, E is positive. $\square$

Lemma 1.23. (**Hahn's Lemma**) *Let λ be a signed measure on the σ-algebra χ on the set X, and E a measurable set for which $0 < \lambda(E) < \infty$. Then there is a measurable subset F of E that is positive.*

Proof. If E is a positive set, then we are done, so assume otherwise. Then, there exists a subset of E which has negative measure. Let m_1 be the smallest natural number such that E contains a subset with measure less than $-1/m_1$. Choose a subset of E with measure less than $-1/m_1$ and call it E_1. Clearly $\lambda(E \setminus E_1) > 0$, since $\lambda(E \setminus E_1) = \lambda(E) - \lambda(E_1) > \lambda(E) > 0$. If $E \setminus E_1$ is positive, then we are done, so assume otherwise. Then $E \setminus E_1$ has a subset which has negative measure. Now, for let m_2 be the smallest natural number such that $E \setminus E_1$ contains a subset with measure less than $-1/m_2$. Choose such a subset, and call it E_2. We repeat this process, where m_k is the smallest natural number such that there exists a measurable subset of $E \setminus \bigcup_{j=1}^{k-1} E_j$ with measure less than $-1/m_k$, and E_k is a subset of $E \setminus \bigcup_{j=1}^{k-1} E_j$ for which $\lambda(E_k) < -1/m_k$. If this process ends, i.e. there exists n such that E_n does not contain any sets with negative measure, then E_n is a positive set, and we are done.

Otherwise, define $F := E \setminus \bigcup_{k=1}^{\infty} E_k$. We first show that $m_k \to \infty$ as $k \to \infty$. If not, then $1/m_k \nrightarrow 0$ as $k \to \infty$. Hence, $\sum_{k=1}^{\infty} -1/m_k = -\infty$. But, we know that

$|\lambda(E)| < \infty$ by definition of a signed measure, and $\bigcup_{k=1}^{\infty} E_k \subset E$, so

$$-\infty < \lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \lambda(E_k) \leq \sum_{k=1}^{\infty} -1/m_k, \tag{1.24}$$

a contradiction. Therefore, $m_k \to \infty$ as $k \to \infty$. We claim that F is a positive set. If not, then it contains a negative set G. However, for each k, we have $G \subseteq F \subseteq E \setminus \bigcup_{j=1}^{k-1} E_j$. Thus, by definition of k, we have $\lambda(G) \geq -1/(m_k - 1)$ for all $k \in \mathbb{N}$. Hence, $\lambda(G) \geq 0$, and F is a positive set

The last step is to show that $\lambda(F) > 0$. This follows from the countable additivity of λ since

$$0 < \lambda(E) = \lambda(E \setminus F) + \lambda(F) = \lambda\left(\bigcup_{k=1}^{\infty} E_k\right) + \lambda(F) < \lambda(F),$$

since $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) < 0$ by Equation 1.24. $\square$

Theorem 1.25. (**Hahn Decomposition Theorem**) *Let λ be a signed measure on the σ-algebra χ on the set X. Then, there is a positive set P and a negative set N for λ such that $X = P \cup N$ and $P \cap N = \varnothing$.*

Proof. We first construct P. Consider the set $\mathcal{P}$ of all positive subsets of X. Then $\mathcal{P}$ is non-empty since $\mu(\varnothing) = 0 \geq 0$. Let

$$\alpha \coloneqq \sup\{\lambda(A) \mid A \in \mathcal{P}\}.$$

Let $\{A_j\}$ be a sequence in $\mathcal{P}$ such that $\lim_{j\to\infty} \lambda(A_j) = \alpha$ and let $P = \bigcup_{j=1}^{\infty} A_j$. By Lemma 1.22, P is positive, and so $P \in \mathcal{P}$ and $\lambda(P) \leq \alpha$. To prove the reverse inequality, note that for each j, we have $P \setminus A_j \subseteq P$, so $\lambda(P \setminus A_j) \geq 0$ by Lemma 1.22. Thus, for each j, $\lambda(P) = \lambda(P \setminus A_j) + \lambda(A_j) \geq \lambda(A_j)$. Since $\lim_{j\to\infty} \lambda(A_j) = \alpha$, we see that $\lambda(P) \geq \alpha$. Hence $\lambda(P) = \alpha$, and $\alpha < \infty$ since λ does not take on the value of ∞.

Now, we define $N \coloneqq X \setminus P$. Clearly $X = P \cup N$ and $P \cap N = \varnothing$, so we simply need to show that N is a negative set. Assume for the sake of contradiction that there is a subset E of N for which $\lambda(E) > 0$. Then, by Hahn's Lemma, there exists a subset F of E that is positive and has positive measure, i.e. $F \in \mathcal{P}$. Now consider the set $P \cup F$. Since $F \subseteq N = X \setminus P$, P and F are disjoint. In addition, since both P and F are positive, then $P \cup F$ is positive. This implies that

$$\lambda(P \cup F) = \lambda(P) + \lambda(F) > \lambda(P) = \alpha.$$

But this is a contradiction because α was defined to be the supremum of $\lambda(A)$ for positive sets A. Thus, N is a negative set. $\square$

Now that we have defined the concept of a measure for sets in $\mathbb{R}$, we may define the criterion for integrability. We first start with a lemma.

Lemma 1.26. *Let f have a measurable domain E. Then, the following statements are equivalent:*

(i) For each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable.
(ii) For each real number c, the set $\{x \in E \mid f(x) \geq c\}$ is measurable.
(iii) For each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.
(iv) For each real number c, the set $\{x \in E \mid f(x) \leq c\}$ is measurable.

Proof. Let χ be our σ-algebra of measurable sets. Notice that the set described in (i) is the complement of the set described in (iv). Then, by the properties of a σ-algebra, these statements are equivalent. Likewise, (ii) and (iii) are equivalent by being complements of each other. Now, we will show that (i) and (ii) are equivalent.

Let A be defined as the set $\{x \in E \mid f(x) > c\}$, and let B be defined as the set $\{x \in E \mid f(x) \geq c\}$ for some real number c. Then, suppose that for each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable. Then this must certainly hold for the set $A_n \coloneqq \{x \in E \mid f(x) > c - \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Notice that

$$B = \bigcup_{j=1}^{\infty} A_n.$$

Since χ respects countable unions of measurable sets, we see that B is in χ.

Conversely, suppose that B is in χ for each real number c. Then $B_n \coloneqq \{x \in E \mid f(x) \geq c + \frac{1}{n}\}$ is in χ. Now, we see that

$$A = \bigcap_{j=1}^{\infty} B_n.$$

Since χ respects countable intersections of measurable sets, we see that A is in χ.

Thus, all four statements are equivalent. $\square$

Definition 1.27. Let f be a function defined on a measurable set E. Then, f is **Lebesgue measurable**, or simply measurable, if it satisfies one of the properties in Lemma 1.26.

We will later see that all measurable functions are Lebesgue integrable. For now, let us return to the function used as motivation in the introduction and generalize the result.

Definition 1.28. Let E be a subset of $\mathbb{R}$. Define the **characteristic function**,

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \in \mathbb{R} \setminus E \end{cases}$$

We are interested in the special case when E is measurable. Note that the function referenced in the introduction was a special case where $E = \mathbb{Q}$. Suppose we wish to show that statement (ii) from Lemma 1.26 is true. Let $A = \{x \in E \mid f(x) \geq c\}$. If $c \leq 0$, then $A = \mathbb{R}$, if $0 < c \leq 1$, then $A = E$, and if $c > 1$, then $A = \varnothing$. Thus, A is measurable for all real numbers c. Thus, the function χ_E is measurable for all measurable sets E. We will later be able to compute the integral of this function.

Before defining the Lebesgue Integral, we will explore a few more properties of measurable functions.

Proposition 1.29. *Let $\{f_j\}$ be a sequence of measurable functions. Define:*

$$f(x) = \inf_{j \in \mathbb{N}} \{f_j(x)\}$$

$$g(x) = \sup_{j \in \mathbb{N}} \{f_j(x)\}$$

$$f^*(x) = \liminf_{j \to \infty} \{f_j(x)\}$$

$$g^*(x) = \limsup_{j\to\infty}\{f_j(x)\}$$

Then f, g, f^*, and g^* are all measurable.

Proof. By the properties of inf and sup, we have the following equalities:

$$\{x \in \mathbb{R} \mid f(x) < c\} = \bigcup_{j=1}^{\infty}\{x \in \mathbb{R} \mid f_j(x) < c\};$$

$$\{x \in \mathbb{R} \mid g(x) \leq c\} = \bigcap_{j=1}^{\infty}\{x \in \mathbb{R} \mid f_j(x) \leq c\}.$$

Note that each one of the sets on the right side of the equations above is measurable because f_j is measurable for all j. Therefore, it follows that f and g are measurable since χ respects countable intersections and unions. For f^* and g^*, we will rely on an alternative characterization of the lim sup and lim inf:

$$f^*(x) = \liminf_{j\to\infty}\{f_j(x)\} = \sup_{k\geq 1}\left\{\inf_{j\geq k}\{f_j(x)\}\right\};$$

$$g^*(x) = \limsup_{j\to\infty}\{f_j(x)\} = \inf_{k\geq 1}\left\{\sup_{j\geq k}\{f_j(x)\}\right\}.$$

Since we have just shown that the supremum and infimum of a sequence of measurable functions are themselves measurable, it follows that f^* and g^* are measurable. □

Corollary 1.30. *If $\{f_j\}_{j=1}^{\infty}$ is a sequence of measurable functions on $\mathbb{R}$ which converges to a function f on $\mathbb{R}$, then f is measurable.*

Proof. Note that $\{f_j\}$ converges to f if and only if

$$\liminf_{j\to\infty}\{f_j(x)\} = \limsup_{j\to\infty}\{f_j(x)\}.$$

In that case, f is equal to the lim inf and the lim sup. Thus, $f = f^*$, so f is measurable. □

2. The Lebesgue integral

2.1. Simple functions and the Lebesgue integral.

Definition 2.1. A real-valued function φ defined on a measurable set E is called **simple** if it is measurable and takes on only finitely many values.

A simple function defined on E can be represented as a sum of characteristic functions as following:

$$\varphi(x) = \sum_{j=1}^{n} a_j \cdot \chi_{E_j}, \text{ where the } a_j \text{ are distinct and } E_j = \{x \in E \mid \varphi(x) = a_j\}.$$

This characterization of the simple function is called the **canonical representation**. From the expression above, it follows that the E_j are all pairwise disjoint. Since a simple function takes on finitely many values, it follows that linear combinations and products of simple functions are also simple functions.

Before we define the Lebesgue integral for an arbitrary function, we will first restrict our focus to the simple functions.

Definition 2.2. Let φ be a simple function defined on a measurable set E and in its canonical representation. Then, the **integral** of φ with respect to the measure μ is given by

$$\int_E \varphi(x)\, d\mu = \sum_{j=1}^{n} a_j \cdot \mu(E_j),$$

where $\sum_{j=1}^{n} a_j \cdot \chi_{E_j}$ is the canonical representation of φ.

We will now show a few basic properties of the Lebesgue integral for simple functions. To save ourselves a tedious computation, we begin with a lemma.

Lemma 2.3. *Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subsets of a set E with finite measure. For $1 \leq i \leq n$, let a_i be a real number.*

$$\text{If } \varphi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E, \text{ then } \int_E \varphi\, d\mu = \sum_{i=1}^{n} a_i \cdot \mu(E_i).$$

Remark 2.4. In the lemma above, we are not assuming that the simple function φ is in its canonical representation. We allow for repetitions among the numbers a_i.

Proof. Suppose that φ takes on distinct values $\lambda_1, \cdots, \lambda_m$. For $1 \leq j \leq m$, let $A_j \coloneqq \{x \in E \mid \varphi(x) = \lambda_j\}$. Then, we can express our function in canonical representation as follows:

$$\varphi(x) = \sum_{j=1}^{m} \lambda_j \cdot \chi_{A_j}, \text{ so that } \int_E \varphi\, d\mu = \sum_{j=1}^{m} \lambda_j \cdot \mu(A_j).$$

Now, for $1 \leq j \leq m$, define the set $I_j \coloneqq \{i \in \{1, \cdots, n\} \mid a_i = \lambda_j\}$. Then, $\cup_{j=1}^{m} I_j = \{1, \cdots, n\}$, and the sets I_j are pairwise disjoint. We also note that based on our definitions, $A_j = \cup_{i \in I_j} E_i$ for each j. Finally we compute that

$$\sum_{i=1}^{n} a_i \cdot \mu(E_i) = \sum_{j=1}^{m} \left(\sum_{i \in I_j} a_i \cdot \mu(E_i) \right)$$

$$= \sum_{j=1}^{m} \lambda_j \cdot \left(\sum_{i \in I_j} \mu(E_i) \right) = \sum_{j=1}^{m} \lambda_j \cdot \mu(A_j) = \int_E \varphi\, d\mu$$

□

We can now use this result to prove the linearity of the integral.

Proposition 2.5. *Let φ and ψ be simple functions defined on a set of finite measure E. Then, for any real numbers α and β, we have*

$$\int_E (\alpha\varphi + \beta\psi)\, d\mu = \alpha \int_E \varphi\, d\mu + \beta \int_E \psi\, d\mu.$$

In addition, if $\varphi \leq \psi$ on E, then $\int_E \varphi\, d\mu \leq \int_E \psi\, d\mu$.

Finally, the function λ, defined for $E \in \chi$, given by

$$\lambda(E) = \int \varphi \cdot \chi_E\, d\mu, \tag{2.6}$$

is a measure on χ.

Proof. Since φ and ψ each take on only finitely many values, we can choose a finite disjoint collection sets $\{E_i\}_{i=1}^n$ so that $\cup_{i=1}^n E_i = E$ and so that φ and ψ each take on only one value on each set E_i. Namely, the sum $\alpha\varphi + \beta\psi$ is constant on each E_i. Suppose that φ and ψ take on the values a_i and b_i, respectively, on the set E_i. Then, we can use the preceding lemma to achieve the following result:

$$\int_E (\alpha\varphi + \beta\psi)\, d\mu = \sum_{i=1}^n (\alpha \cdot a_i + \beta \cdot b_i) \cdot \mu(E_i) = \sum_{i=1}^n \alpha \cdot a_i \cdot \mu(E_i) + \sum_{i=1}^n \beta \cdot b_i \cdot \mu(E_i)$$

$$= \alpha \sum_{i=1}^n \cdot a_i \cdot \mu(E_i) + \beta \sum_{i=1}^n \cdot b_i \cdot \mu(E_i) = \alpha \int_E \varphi\, d\mu + \beta \int_E \psi\, d\mu.$$

To prove the second part, we use the linearity property from the first part:

$$\int_E \psi\, d\mu - \int_E \varphi\, d\mu = \int_E (\psi - \varphi)\, d\mu \geq 0.$$

This is true since $\varphi \leq \psi$ on E. Rearranging, we get the desired inequality.

The third result is left as an exercise to the reader. See Lemma 2.3 in [3] for a proof. □

Lemma 2.7. (**Simple Approximation Lemma**) *Let f be a bounded, measurable function on E. Then, for all $\varepsilon > 0$, there exist simple functions φ_ε and ψ_ε such that*

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \text{ and } 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. Let (c, d) be an open interval which contains $f(E)$, the image of f over E. We now construct a partition $(y_0, y_1, \cdots, y_n)$ of $[c, d]$ such that for all $1 \leq k \leq n$, $y_k - y_{k-1} < \varepsilon$ and

$$c = y_0 < y_1 < \cdots < y_n = d.$$

Next, for all $1 \leq k \leq n$ define $I_k \coloneqq [y_{k-1}, y_k)$ and $E_k \coloneqq f^{-1}(I_k)$. Then, the I_k are pairwise disjoint, and thus the E_k are pairwise disjoint. In addition, we see that $E = \bigcup_{k=1}^n E_k$ since the I_k cover $F(E)$. Now, we define our two simple functions:

$$\varphi_\varepsilon(x) = \sum_{k=1}^n y_{k-1} \cdot \chi_{E_k} \text{ and } \psi_\varepsilon(x) = \sum_{k=1}^n y_k \cdot \chi_{E_k}.$$

Fix some x in E. Then, there exists a unique $1 \leq k \leq n$ such that $\varphi_\varepsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\varepsilon(x)$. Moreover, since $y_k - y_{k-1} < \varepsilon$, we see that $\psi_\varepsilon(x) - \varphi_\varepsilon(x) < \varepsilon$ since our choice of x was arbitrary. □

We are now ready to define the Lebesgue integral for arbitrary nonnegative functions.

Definition 2.8. Let f be a bounded, nonnegative function defined on a measurable set E. Then, we define the **upper and lower Lebesgue integrals**, respectively, as

$$\inf \left\{ \int_E \psi\, d\mu \;\middle|\; \psi \text{ simple and } f \leq \psi \text{ on E} \right\}$$

and

$$\sup \left\{ \int_E \varphi\, d\mu \;\middle|\; \varphi \text{ simple and } \varphi \leq f \text{ on E} \right\}.$$

We say that f is **Lebesgue integrable** if these two values are equal. In that case, we call this value the **Lebesgue integral** of f over E and denote it by $\int_E f(x)\,d\mu$.

Earlier, we alluded to the following result: the Lebesgue and Riemann integrals agree in value whenever the Riemann integral is defined. We will now prove this.

Proposition 2.9. *Let f be a bounded function defined on the closed and bounded interval [a,b]. If f is Riemann integrable over [a,b], then it is Lebesgue integrable over [a,b], and the two integrals are equal.*

Proof. To prove this, we will use the alternative characterization of the Riemann integral, defined in terms of step functions. The upper and lower Riemann integrals, respectively, are given by

$$\inf\left\{(R)\int_I \psi\,d\mu \;\middle|\; \psi \text{ a step function, } f \le \psi\right\}$$

and

$$\sup\left\{(R)\int_I \varphi\,d\mu \;\middle|\; \varphi \text{ a step function, } \varphi \le f\right\},$$

where $I = [a, b]$. Likewise, the Riemann integral is only defined when these two values are equal. However, we note that this criterion implies the Lebesgue criterion, since the Riemann and Lebesgue integrals of a step function are equivalent. ☐

Certainly the class of Lebesgue integrable functions expands far beyond that of Riemann integrable functions. In the next theorem, we prove that a much larger class of functions can be evaluated using the Lebesgue integral.

Theorem 2.10. *If f is a bounded, measurable function defined on a set of finite measure E, then f is integrable over E.*

Proof. Fix $\varepsilon > 0$. By the Simple Approximation Lemma (2.7), there exist simple functions φ_ε and ψ_ε such that $\varphi_\varepsilon \le f \le \psi_\varepsilon$ and $\psi_\varepsilon - \varphi_\varepsilon < \frac{\varepsilon}{\mu(E)}$ for all $x \in E$. Now, we compute:

$$0 \le \inf\left\{\int_E \psi\,d\mu \;\middle|\; \psi \text{ simple}, f \le \psi\right\} - \sup\left\{\int_E \varphi\,d\mu \;\middle|\; \varphi \text{ simple}, \varphi \le f\right\}$$

$$\le \int_E \psi_\varepsilon\,d\mu - \int_E \varphi_\varepsilon\,d\mu = \int_E (\psi_\varepsilon - \varphi_\varepsilon)\,d\mu < \int_E \frac{\varepsilon}{\mu(E)} = \frac{\varepsilon}{\mu(E)} * \mu(E) = \varepsilon.$$

Since ε was arbitrary, we see that this holds for all $\varepsilon > 0$. Thus, the lower and upper Lebesgue integrals are equal, and f is integrable, as desired. ☐

Remark 2.11. The converse of the theorem above is also true, that is, a bounded function f is integrable if and only if it is measurable. We will not prove the converse here. See [6], Chapter 5, Theorem 7 for the proof.

Now that we have some fundamental knowledge of the Lebesgue integral, we will develop some useful properties in the next section.

2.2. **Properties of the Lebesgue integral.** We will omit a few levels of rigor here for the purpose of keeping this paper to a manageable length. Normally, one develops properties for the integral of nonnegative bounded functions on a set of finite measure. Then, one tackles functions defined on sets of infinite measure. This can only be defined if the function vanishes everywhere except for a set of finite measure. The properties for these more general functions are completely analogous to those of the nonnegative bounded functions. For the purpose of this paper, we consider only the properties of nonnegative bounded functions on a set of finite measure and direct the reader to Section 4.3 of [6] for a more detailed treatment of the subject.

Theorem 2.12. *Let f and g be bounded, nonnegative, measurable functions defined on a set of finite measure E. Then, for any real numbers α and β, we have*

$$\int_E (\alpha f + \beta g)\, d\mu = \alpha \int_E f\, d\mu + \beta \int_E g\, d\mu.$$

In addition, if $f \leq g$ on E, then $\int_E f\, d\mu \leq \int_E g\, d\mu$.

Proof. See Chapter 4, Theorem 5 of [6]. □

Corollary 2.13. *Let f be a bounded, nonnegative, measurable function defined on a set of finite measure E. Suppose that A and B are disjoint subsets of E. Then*

$$\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu$$

Proof. Since f is a measurable function and A and B are measurable, we see that the functions $f \cdot \chi_A$ and $f \cdot \chi_B$ are also measurable functions. Next, it is not difficult to verify that for any subset F of E,

$$\int_F f\, d\mu = \int_E f \cdot \chi_F\, d\mu.$$

This is true because on the right side of the equation, the integrand will be 0 everywhere on $E \setminus F$. Using this fact and the linearity of the integral, we see that

$$\begin{aligned}\int_{A \cup B} f\, d\mu &= \int_E f \cdot \chi_{A \cup B}\, d\mu = \int_E f \cdot (\chi_A + \chi_B)\, d\mu \\ &= \int_E f \cdot \chi_A\, d\mu + \int_E f \cdot \chi_B\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu,\end{aligned}$$

as desired. □

Using Corollary 2.13, we can now return to the question that motivated the discussion of the Lebesgue integral:

Proposition 2.14. *The (Lebesgue) integral of the function $f(x) = \chi_{\mathbb{Q}}$ is 0.*

Proof. We first note that $\chi_{\mathbb{Q}}$ is a simple function, and $\mathbb{Q}$ is measurable and has measure 0. By Corollary 2.13, we can split the integral into two parts:

$$\begin{aligned}\int_{[0,1]} \chi_{\mathbb{Q}}\, d\mu &= \int_{[0,1] \cap \mathbb{Q}} \chi_{\mathbb{Q}}\, d\mu + \int_{[0,1] \setminus \mathbb{Q}} \chi_{\mathbb{Q}}\, d\mu = \int_{[0,1] \cap \mathbb{Q}} 1\, d\mu + \int_{[0,1] \setminus \mathbb{Q}} 0\, d\mu \\ &= [1 \cdot \mu([0,1] \cap \mathbb{Q})] + [0 \cdot \mu([0,1] \setminus \mathbb{Q})] = 0 + 0 = 0\end{aligned}$$

□

At the end of this section, we will prove a much more general result: any nonnegative measurable function has integral 0 if and only if the function equals 0 almost everywhere.

As mentioned at the beginning of this section, we will now take for granted the above properties for nonnegative functions on sets of infinite measure.

We now prove two important results about the relationship between the integral of convergent sequences of functions and the integral of their limit function.

Theorem 2.15. (**Monotone Convergence Theorem**) *Let $\{f_n\}$ be an increasing sequence of nonnegative functions on E. If $\{f_n\} \to f$ pointwise on E, then*

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Proof. By Corollary 1.30, f is measurable and thus integrable by Theorem 2.10.

Because $\{f_j\}$ is an increasing sequence converging to f, we have, for each j, $f_j \leq f_{j+1} \leq f$. By the monotonicity of the integral (Theorem 2.12), we have

$$\int_E f_j \, d\mu \leq \int_E f_{j+1} \, d\mu \leq \int_E f \, d\mu$$

for each j. This implies that $\lim_{j\to\infty} \int_E f_j \, d\mu \leq \int_E f \, d\mu$.

To prove the reverse inequality, let $0 < \alpha < 1$ and let φ be a simple function satisfying $0 \leq \varphi \leq f$ on E. Define $A_j \coloneqq \{x \in X \,\big|\, f_j(x) \geq \alpha\varphi(x)\}$. Since $\{f_j\}$ is an increasing function, we see that $A_j \subset E$, $A_j \subset A_{j+1}$ for all j, and $X = \bigcup_{j=1}^{\infty} A_j$. Thus, we have, for each j,

$$\int_{A_j} \alpha\varphi \, d\mu \leq \int_{A_j} f_j \, d\mu \leq \int_X f_j \, d\mu. \tag{2.16}$$

Now, by Equation 2.6 and Lemma 1.15, and using the measure $\lambda(E) = \int \varphi \cdot \chi_E \, d\mu$, we see that

$$\begin{aligned}\int_X \varphi \, d\mu = \int \varphi \cdot \chi_E \, d\mu = \lambda(E) = \lambda\left(\bigcup_{j=1}^{\infty} A_j\right) &= \lim_{j\to\infty} \lambda(A_j) \\ = \lim_{j\to\infty} \int \varphi \cdot \chi_{A_j} \, d\mu &= \lim_{j\to\infty} \int_{A_j} \varphi \, d\mu,\end{aligned}$$

where the fourth equality is true because $\{A_j\}$ is an increasing sequence. If we take the limit as $j \to \infty$ in Equation 2.16, we see that

$$\lim_{j\to\infty} \int_{A_j} \alpha\varphi \, d\mu = \int_X \alpha\varphi \, d\mu \leq \lim_{j\to\infty} \int_X f_j \, d\mu. \tag{2.17}$$

Since $0 < \alpha < 1$ was chosen arbitrarily, then Equation 2.17 must hold for $\alpha = 1$ as well. Thus,

$$\int_X \varphi \, d\mu \leq \lim_{j\to\infty} \int_X f_j \, d\mu. \tag{2.18}$$

Lastly, since φ was chosen arbitrarily, it follows that Equation 2.18 holds for all φ satisfying $0 \leq \varphi \leq f$. In particular,

$$\int_X f \, d\mu = \sup_{0\leq\varphi\leq f} \int_X \varphi \, d\mu \leq \lim_{j\to\infty} \int_X f_j \, d\mu.$$

Hence,

$$\lim_{n\to\infty}\int_E f_n \, d\mu = \int_E f \, d\mu.$$

□

We shall now prove another convergence theorem that will be useful later.

Theorem 2.19. (**Fatou's Lemma**) *Let $X, \chi, \mu)$ be a measure space. Suppose we have a sequence $\{f_j\}$ of nonnegative measurable functions. Then,*

$$\int (\liminf_{j\to\infty} f_j) \, d\mu \leq \liminf_{j\to\infty} \int f_j \, d\mu.$$

Proof. Let $g_j(x) = \inf\{f_j(x), f_{j+1}(x), \cdots\}$. Then, $g_j \leq f_k$ provided that $j \leq k$. By the monotonicity of the integral, this implies that

$$\int g_j \, d\mu \leq \int f_k \, d\mu$$

for all $k \geq j$. It follows that

$$\int g_j \, d\mu \leq \liminf_{k\to\infty} \int f_k \, d\mu.$$

Now, we notice that g_j is a monotonically increasing function and that $\lim_{j\to\infty} g_j = \liminf_{k\to\infty} f_k$. Thus, we may use the Monotone Convergence Theorem to show that

$$\int (\liminf_{k\to\infty} f_k) \, d\mu = \lim_{j\to\infty} \int g_j \, d\mu \leq \liminf_{k\to\infty} \int f_k \, d\mu.$$

□

We now present a corollary of Fatou's Lemma:

Corollary 2.20. *Let (X, χ, μ) be a measure space. Suppose that f is a nonnegative, measurable function. Then $f(x) = 0$ almost everywhere if and only if $\int f \, d\mu = 0$.*

Proof. Suppose that $\int f \, d\mu = 0$. Then, let us define $E_j \coloneqq \{x \in X \mid f(x) > \frac{1}{j}\}$. Then, $f \geq \frac{1}{j} \cdot \chi_{E_j}$ for all j. Thus,

$$0 = \int f \, d\mu \geq \int \frac{1}{j} \cdot \chi_{E_j} = \int_{E_j} \frac{1}{j} = \frac{1}{j}\mu(E_j) \geq 0$$

for each j. But the string of inequalities tells us that we have $\mu(E_j) = 0$ for each j, so $E = \{x \in X \mid f(x) > 0\} = \bigcup_{j=1}^{\infty} E_j$ also has measure 0.

For the converse, assume that $f(x) = 0$ almost everywhere. Then, $\mu(E) = 0$, where E is defined as above. Let $f_j = j \cdot \chi_E$ for all $j \in \mathbb{N}$. We have $f \leq \liminf_{j\to\infty} f_j$ and therefore, by Fatou's Lemma,

$$0 \leq \int f \, d\mu \leq \int \liminf_{j\to\infty} f_j \, d\mu \leq \liminf_{j\to\infty} \int f_j \, d\mu = \liminf_{j\to\infty} \left(\int_E f_j \, d\mu + \int_{X\setminus E} f_j \, d\mu \right)$$

$$= \liminf_{j\to\infty} \left(\int_E j \cdot \chi_E \, d\mu + 0 \right) = \liminf_{j\to\infty} (j \cdot \mu(E)) = \liminf_{j\to\infty} 0 = 0$$

□

2.3. **General Lebesgue Integral.** We will now extend the Lebesgue integral to all functions, not necessarily nonnegative. To do this, we will define f^+ and f^-.

Definition 2.21. Given a function $f : E \to \mathbb{R}$, where E is a measurable set, define

$$f^+(x) = \max\{f(x), 0\} = \frac{f(x) + |f(x)|}{2}$$

and

$$f^-(x) = \max\{-f(x), 0\} = \frac{|f(x)| - f(x)}{2}.$$

Notice that f^+ and f^- are both nonnegative functions, $f(x) = f^+(x) - f^-(x)$, and $|f(x)| = f^+(x) + f^-(x)$. Thus, f is measurable if and only if f^+ and f^- are measurable.

Proposition 2.22. *Let f be a measurable function on E. Then, f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.*

Proof. First, let us assume that f^+ and f^- are integrable over E. Since $|f(x)| = f^+(x) + f^-(x)$, we see from the linearity of the Lebesgue integral for nonnegative functions that $\int |f|\, d\mu = \int f^+\, d\mu + \int f^-\, d\mu$.

Conversely, assume that $|f|$ is integrable on E. Then, since $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$, and since f^+ and f^- are measurable, the monotonicity of the integral implies that f^+ and f^- are integrable. □

Definition 2.23. Let f be a measurable function on E. Then we say f is **integrable** if $|f|$ is integrable over E. In this case, we define the **integral** of f as follows:

$$\int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu.$$

Proposition 2.24. *Let f be a measurable function on E. Suppose there is a nonnegative function g that dominates f on E, i.e. $|f| \leq g$ on E. then, f is integrable and*

$$\left| \int_E f\, d\mu \right| \leq \int_E |f|\, d\mu.$$

Proof. By the monotonicity of the integral for nonnegative functions, we see that $|f|$, and thus f is integrable. Then, we use the Triangle Inequality to conclude:

$$\left| \int_E f\, d\mu \right| = \left| \int_E f^+\, d\mu - \int_E f^-\, d\mu \right| \leq \left| \int_E f^+\, d\mu \right| + \left| \int_E f^-\, d\mu \right|$$
$$= \int_E f^+\, d\mu + \int_E f^-\, d\mu = \int_E (f^+ + f^-)\, d\mu = \int_E |f|\, d\mu.$$

□

As in the previous section, we will omit the details of the proofs of the properties of the Lebesgue integral for general functions. Such properties include linearity, monotonicity, and additivity over disjoint domains of integration. See Section 4.4 of [6] for the proofs.

Theorem 2.25. (**Lebesgue Dominated Convergence Theorem**) *Let $\{f_j\}$ be a sequence of measurable functions that converge almost everywhere to a measurable function f. If there exists an integrable function g such that $|f_j| \leq g$ for all $j \in \mathbb{N}$, then f is integrable and*

$$\int f \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu.$$

Proof. For the values of x for which $\{f_j\}$ does not converge to f, we will define f_j and f to be 0. Since the set of such values has measure 0, this will not affect the integral by Corollary 2.20 and the additivity over disjoint domains of integration. Thus, we may now assume that $\{f_j\}$ converges everywhere. Next, by Proposition 2.24, we see that each f_j and thus f are integrable. Our assumption that g dominates f_j implies that $g - |f_j| \geq 0$, so $g + f_j \geq 0$ and $g - f_j \geq 0$ by the properties of the absolute value. Thus, we may apply Fatou's Lemma to the sequence $g + f_j$:

$$\int g \, d\mu + \int f \, d\mu = \int (g+f) \, d\mu = \int \liminf_{j\to\infty} (g+f_j) \, d\mu \leq \liminf_{j\to\infty} \int (g+f_j) \, d\mu$$

$$= \liminf_{j\to\infty} \left(\int g \, d\mu + \int f_j \, d\mu \right) = \int g \, d\mu + \liminf_{j\to\infty} \int f_j \, d\mu$$

Hence,

$$\int f \, d\mu \leq \liminf_{j\to\infty} \int f_j \, d\mu.$$

For the reverse inequality, we will apply Fatou's Lemma to the sequence $g - f_j$:

$$\int g \, d\mu - \int f \, d\mu = \int (g-f) \, d\mu = \int \liminf_{j\to\infty} (g-f_j) \, d\mu$$

$$\leq \liminf_{j\to\infty} \int (g-f_j) \, d\mu = \liminf_{j\to\infty} \left(\int g \, d\mu - \int f_j \, d\mu \right)$$

$$= \int g \, d\mu + \liminf_{j\to\infty} \left(-\int f_j \, d\mu \right) = \int g \, d\mu - \limsup_{j\to\infty} \int f_j \, d\mu.$$

Thus,

$$\int f \, d\mu \geq \limsup_{j\to\infty} \int f_j \, d\mu.$$

Putting the two inequalities together, we see that

$$\limsup_{j\to\infty} \int f_j \, d\mu \leq \int f \, d\mu \leq \liminf_{j\to\infty} \int f_j \, d\mu.$$

However, we know by the definition of lim sup and lim inf that we always have lim inf $\leq$ lim sup for any sequence. Thus, the limit exists, and by the squeeze theorem we conclude that,

$$\int f_j \, d\mu = \lim_{j\to\infty} \int f_j \, d\mu.$$

∎

3. L^p SPACES

3.1. **Normed linear spaces.** We will assume a basic understanding of linear algebra here, namely the concept of a vector space. We will now review the definition of a norm.

Definition 3.1. Let V be a vector space over $\mathbb{R}$. A real-valued function N on V is said to be a **norm** if

(i) $N(v) \geq 0$ for all $v \in V$. Moreover, $N(v) = 0$ if and only if $v = 0$.
(ii) $N(\alpha v) = |\alpha| \cdot N(v)$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
(iii) $N(u + v) \leq N(u) + N(v)$ for all $u, v \in \mathbb{R}$.

A vector space equipped with a norm is called a **normed linear space**.

We will be considering infinite-dimensional spaces of functions, called the Lebesgue spaces, or L^p spaces.

Definition 3.2. Let (X, χ, μ) be a measure space, and $1 \leq p < \infty$. The space of functions f such that $|f|^p$ has finite integral is denoted by $L^p(X, \mu)$, or simply L^p. The norm on this space is given by:

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

We will now verify that this is in fact a normed space. Properties (i) and (ii) follow easily from the properties of integration developed in Sections 2.2 and 2.3. One slight deviation is that $\|f\| = 0$ if and only if $f = 0$ almost everywhere, as proven in Corollary 2.20. Property (iii) is more difficult to prove and requires a few steps.

Definition 3.3. The **conjugate** of a number $p \in (1, \infty)$ is the number $q = \frac{p}{p-1}$, which is the unique number $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We define the conjugate of 1 to be ∞ and the conjugate of ∞ to be 1.

Remark 3.4. Unless otherwise stated, q will always refer to the conjugate of p.

We will now prove that the L^p spaces are normed linear spaces.

Lemma 3.5. (**Young's Inequality**) *For $1 < p < \infty$, q, the conjugate of p, and any two positive numbers a and b, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Consider the function $g(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ for $x > 0$. The derivative of g is negative when $x \in (0, 1)$ and is positive for $x \in (1, \infty)$. In addition, $g(1) = 0$. As a result, $g(x) \geq 0$ for $x \in (0, \infty)$, i.e. $x \leq \frac{1}{p}x^p + \frac{1}{q}$. Now, let us take $x = \frac{a}{b^{q-1}}$, which is positive since $a, b > 0$. Then,

$$\frac{a}{b^{q-1}} \leq \frac{1}{p}\left(\frac{a}{b^{q-1}}\right)^p + \frac{1}{q} = \frac{1}{p}\frac{a^p}{b^{p(q-1)}} + \frac{1}{q} = \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q},$$

since $q = p(q - 1)$. Now, multiplying both sides of the inequality by b^q, we get Young's Inequality. $\square$

Proposition 3.6. (**Hölder's Inequality**) *Let $f \in L^p$ and $g \in L^q$, where $1 < p < \infty$. Then, $f \cdot g \in L^1$, and*

$$\|f \cdot g\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}.$$

Proof. If $f = 0$ or $g = 0$ almost everywhere, then the inequality becomes an equality, with both sides being 0. So assume $\|f\|_{L^p} \neq 0$ and $\|g\|_{L^q} \neq 0$. Now, we apply Young's Inequality, letting $a = \frac{|f(x)|}{\|f\|_{L^p}}$ and $b = \frac{|g(x)|}{\|g\|_{L^q}}$:

$$\frac{|f(x) \cdot g(x)|}{\|f\|_{L^p} \cdot \|g\|_{L^q}} \leq \frac{|f(x)|^p}{p \cdot \|f\|_{L^p}^p} + \frac{|g(x)|^q}{q \cdot \|g\|_{L^q}^q}.$$

Since $f \in L^p$ and $g \in L^q$, we see that both terms on the right side of the inequality are integrable, so by Proposition 2.24, $|f \cdot g|$ is integrable, hence $f \cdot g \in L^1$. In addition, by definition of the L^p norm, $\int |f|^p \, d\mu = \|f\|^p$ and $\int |g|^q \, d\mu = \|g\|^q$. Thus, integrating both sides of the inequality, we see that

$$\frac{\|f \cdot g\|_{L^1}}{\|f\|_{L^p} \cdot \|g\|_{L^q}} = \int \left(\frac{|f(x) \cdot g(x)|}{\|f\|_{L^p} \cdot \|g\|_{L^q}} \right) d\mu$$

$$\leq \int \left(\frac{|f(x)|^p}{p \cdot \|f\|_{L^p}^p} \right) d\mu + \int \left(\frac{|g(x)|^q}{q \cdot \|g\|_{L^q}^q} \right) d\mu = \frac{\|f\|_{L^p}^p}{p \cdot \|f\|_{L^p}^p} + \frac{\|g\|_{L^q}^q}{q \cdot \|g\|_{L^q}^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplying out the denominator between the first and last expressions, we get the desired inequality: $\|f \cdot g\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$. □

Definition 3.7. Let $f : E \to \mathbb{R}$. The function $sgn(f) : E \to \mathbb{R}$ is defined to be 1 if $f(x) > 0$, 0 if $f(x) = 0$, and -1 if $f(x) < 0$.

We define the **conjugate function** f^* by $f^* := \|f\|_{L^p}^{1-p} \cdot sgn(f) \cdot |f|^{p-1}$.

An important property of the sgn function is that $sgn(f) \cdot f = |f|$ almost everywhere, given that f is finite almost everywhere. This is easily verified.

Lemma 3.8. *If $f \neq 0$, then f^* belongs to L^q,*

$$\int f \cdot f^* = \|f\|_{L^p}, \text{ and } \|f^*\|_{L^q} = 1.$$

Proof. Since $sgn(f) \cdot f = |f|$ almost everywhere, we see that $f^* \cdot f = \|f\|_{L^p}^{1-p} \cdot |f|^p$ almost everywhere. Then, we compute:

$$\int f \cdot f^* \, d\mu = \|f\|_{L^p}^{1-p} \int |f|^p \, d\mu = \|f\|_{L^p}^{1-p} \cdot \|f\|_{L^p}^p = \|f\|_{L^p}.$$

To compute $\|f^*\|_{L^q}$, we integrate $|f^*|^q$:

$$\left(\int |f^*|^q \, d\mu \right)^{1/q} = \left(\|f\|_{L^p}^{(1-p)q} \int |f|^{(p-1)q} d\mu \right)^{1/q} = \|f\|_{L^p}^{1-p} \left(\int |f|^p d\mu \right)^{1/q}$$

$$= \|f\|_{L^p}^{1-p} \left[\left(\int |f|^p d\mu \right)^{1/p} \right]^{p/q} = \|f\|_{L^p}^{1-p} \cdot \|f\|_{L^p}^{p/q} = \|f\|_{L^p}^{p/q+1-p} = \|f\|_{L^p}^0 = 1.$$

□

Theorem 3.9. (**Minkowski's Inequality**) *If f and g belong to L^p, $1 \leq p < \infty$, then $f + g$ belongs to L^p and*

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Proof. The case $p = 1$ follows directly from the Triangle Inequality for the Euclidean norm, so we consider the case $p > 1$. Since f, g are measurable, then so is $f + g$. In addition, since $|f + g| \le |f| + |g| \le 2\max\{|f|, |g|\}$, it follows that $|f + g|^p \le 2^p \max\{|f|^p, |g|^p\}$. Then, by Proposition 2.24, $f + g \in L^p$. If $f + g = 0$, Minkowski's Inequality always holds, so assume $f + g \neq 0$. Now consider $(f + g)^*$.

$$\|f + g\|_{L^p} = \int (f + g) \cdot (f + g)^* \, d\mu = \int f \cdot (f + g)^* \, d\mu + \int g \cdot (f + g)^* \, d\mu$$
$$\le \|f\|_{L^p} \cdot \|(f + g)^*\|_{L^q} + \|g\|_{L^p} \cdot \|(f + g)^*\|_{L^q} = \|f\|_{L^p} + \|g\|_{L^p}.$$

In the above string of equations, the second equality is due to the linearity of the integral, the inequality is due to Hölder's Inequality, and the final equality is due to Lemma 3.8. □

When $p = 2$, we get a familiar result:

Corollary 3.10. (**Cauchy-Schwarz Inequality**) *Let f and g be measurable functions for which f^2 and g^2 are integrable, i.e. $f, g \in L^2$. Then, their product fg is integrable, and*

$$\int |fg| \, d\mu \le \sqrt{\int f^2 \, d\mu} \cdot \sqrt{\int g^2 \, d\mu}.$$

We now prove another important property of the L^p spaces.

3.2. L^p spaces are Banach spaces.

Definition 3.11. A vector space is **complete** if every Cauchy sequence converges to a limit in the vector space.

A normed linear space that is complete is called a **Banach space**.

We wish to show that the L^p spaces are Banach spaces. We have already shown that they are normed linear spaces, and we will now show that they are complete.

Lemma 3.12. *Let X be a normed linear space. Then every convergent sequence is Cauchy. Moreover, a Cauchy sequence converges if it has a convergent subsequence.*

Proof. Let $\{f_n\}$ be a sequence in X which converges to f, and fix $\varepsilon > 0$. Then, there exists $N > 0$ such that if $n > N$, then $\|f - f_n\| < \frac{\varepsilon}{2}$. Now, let $m, n > N$. Then,

$$\|f_n - f_m\| \le \|f_n - f\| + \|f - f_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by the Triangle Inequality.

For the second result, let $\{f_n\}$ be a Cauchy sequence which has a subsequence f_{n_k} which converges to f in X. Since f_{n_k} converges to f, there exists $N > 0$ so that $n_k > N$ implies that $\|f - f_{n_k}\| < \frac{\varepsilon}{2}$. Moreover, since f_n is Cauchy, there exists $M > 0$ so that if $n, n_k > M$, then $\|f_n - f_{n_k}\| < \frac{\varepsilon}{2}$. Now, choosing $n, n_k > \max\{N, M\}$, we see that

$$\|f - f_n\| \le \|f - f_{n_k}\| + \|f_{n_k} - f_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{f_n\}$ converges to f in X. □

Theorem 3.13. *Every Cauchy sequence in L^p, $1 \le p < \infty$, converges to a limit in the L^p space.*

Proof. Let $\{f_j\}$ be a Cauchy sequence in L^p, and fix $\varepsilon > 0$. Then there exists $J > 0$ so that if $j, k > J$, then $\|f_j - f_k\|_{L^p} < \varepsilon$. Then, there exists a (Cauchy) subsequence $\{f_{j_k}\}$ such that for all $k \in \mathbb{N}$,

$$\|f_{j_{k+1}} - f_{j_k}\|_{L^p} < 2^{-k}. \tag{3.14}$$

Now, define the function g by

$$g(x) = |f_{j_1}(x)| + \sum_{k=1}^{\infty} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big|.$$

Then g is nonnegative, measurable, and thus integrable. Thus, we may apply Fatou's lemma:

$$\begin{aligned}\int |g|^p \, d\mu &= \int \left\{ \liminf_{n\to\infty} \left\{ |f_{j_1}(x)| + \sum_{k=1}^{n} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big| \right\} \right\}^p \, d\mu \\ &= \int \liminf_{n\to\infty} \left\{ |f_{j_1}(x)| + \sum_{k=1}^{n} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big| \right\}^p \, d\mu \\ &\le \liminf_{n\to\infty} \int \left\{ |f_{j_1}(x)| + \sum_{k=1}^{n} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big| \right\}^p \, d\mu.\end{aligned}$$

Now, we take the p-th root on both sides and use Minkowski's inequality:

$$\begin{aligned}\|g\|_{L^p} &= \left\{ \int |g|^p \, d\mu \right\}^{1/p} \\ &\le \liminf_{n\to\infty} \left(\int \left\{ |f_{j_1}(x)| + \sum_{k=1}^{n} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big| \right\}^p \, d\mu \right)^{1/p} \\ &= \liminf_{n\to\infty} \left\{ \left\| \Big| |f_{j_1}(x)| + \sum_{k=1}^{n} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big| \Big| \right\|_{L^p} \right\} \\ &\le \liminf_{n\to\infty} \left\{ \|f_{j_1}\|_{L^p} + \sum_{k=1}^{n} \|f_{j_{k+1}} - f_{j_k}\|_{L^p} \right\} \le \|f_{j_1}\|_{L^p} + 1,\end{aligned}$$

where the final inequality is true by equation 3.14. Thus, g is in L^p and is therefore bounded almost everywhere. Let $E := \{x \in X \mid g(x) < \infty\}$, so that $\mu(X \setminus E) = 0$. This also implies that $\sum_{k=1}^{\infty} \big|f_{j_{k+1}}(x) - f_{j_k}(x)\big|$ converges almost everywhere.

We now define f to be:

$$f(x) = \begin{cases} f_{j_1}(x) + \sum_{k=1}^{\infty} \{f_{j_{k+1}}(x) - f_{j_k}(x)\} & x \in E, \\ 0 & x \notin E. \end{cases}$$

Observe that the sum is telescoping, so that $f_{j_k}(x) = f_{j_1}(x) + \sum_{\ell=1}^{k-1} \{f_{j_{\ell+1}}(x) - f_{j_\ell}(x)\}$. Thus, $\lim_{k\to\infty} f_{j_k} = f(x)$ for $x \in E$. The triangle inequality also tells us that

$$|f_{j_k}| \le |f_{j_1}| + \sum_{\ell=1}^{k-1} \big|f_{j_{\ell+1}}(x) - f_{j_\ell}(x)\big| \le |f_{j_1}| + \sum_{\ell=1}^{\infty} \big|f_{j_{\ell+1}}(x) - f_{j_\ell}(x)\big| = g.$$

Since f_{j_k} converges pointwise almost everywhere to f, we may use the Lebesgue Dominated Convergence Theorem (2.25) to deduce that $f \in L^p$. We are nearly done. We simply need to show that f_{j_k} converges to f in the L^p norm. We compute:

$$\begin{aligned} |f - f_{j_k}| &= \left| f_{j_1}(x) + \sum_{k=1}^{\infty} \{ f_{j_{k+1}}(x) - f_{j_k}(x) \} - \left(f_{j_1}(x) + \sum_{\ell=1}^{k-1} \{ f_{j_{\ell+1}}(x) - f_{j_\ell}(x) \} \right) \right| \\ &= \left| \sum_{\ell=k}^{\infty} \{ f_{j_{\ell+1}}(x) - f_{j_\ell}(x) \} \right| \leq \sum_{\ell=k}^{\infty} \left| f_{j_{\ell+1}}(x) - f_{j_\ell}(x) \right| \leq g. \end{aligned}$$

Thus, we have $|f - f_{j_k}|^p \leq g^p$ and $\lim_{k\to\infty} |f - f_{j_k}| = 0$. Then by the Lebesgue Dominated Convergence Theorem, since $g \in L^p$,

$$\lim_{j\to\infty} \int |f - f_{j_k}|^p \, d\mu = \int \lim_{j\to\infty} |f - f_{j_k}| \, d\mu = \int 0 \, d\mu = 0.$$

We have now shown that f_{j_k} converges to f in the L^p norm. Thus, by Lemma 3.12, f_j converges to f in the L^p norm. □

3.3. **The case $p = \infty$.** Up until now, we have ignored the case where $p = \infty$. This space is defined differently from $1 \leq p < \infty$. Recall that the supremum of a function is defined as the infimum of all upper bounds of that function.

Definition 3.15. An **essential upper bound** for a real-valued function f is a number U such that $f \leq U$ for almost all $x \in X$.

We define the **essential supremum**, **ess sup** f, to be the infimum of the set of all essential upper bounds. If this set is empty, then ess sup f is defined to be $+\infty$. **Essential lower bounds** and the **essential infimum** are defined similarly.

A function is said to be **essentially bounded** if ess sup f and ess inf f are both finite.

Definition 3.16. Let (X, χ, μ) be a measure space. The space $L^\infty(X, \mu)$, or simply L^∞ is defined to be the collection of all essentially bounded functions. We define a norm on the space L^∞ to be:

$$\|f\|_{L^\infty} = \max\{|\text{ess sup } f|, |\text{ess inf } f|\}$$

Remark 3.17. An equivalent definition of the norm is $\|f\|_{L^\infty} = \text{ess sup } |f|$.

Lemma 3.18. *Let $f \in L^\infty$. Then $|f(x)| \leq \|f\|_{L^\infty}$ almost everywhere.*

Proof. Let $\{M_n\}$ be a decreasing sequence of essential upper bounds for f such that $\lim_{n\to\infty} M_n = \|f\|_{L^\infty}$. Now let $E_n \coloneqq \{x \in X \mid |f(x)| > M_n\}$, so that $\mu(E_n) = 0$ for all n. Then, $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 0$. Thus, if $x \notin \bigcup_{n=1}^{\infty} E_n$, then $|f(x)| < M_n$ for all n, hence $|f(x)| \leq \|f\|_{L^\infty}$ almost everywhere. □

Theorem 3.19. *The space L^∞ is a Banach space.*

Proof. The proof that L^∞ is a normed linear space is routine and left as an exercise to the reader.

To show that L^∞ is complete, let $\{f_n\}$ be a Cauchy sequence in L^∞. That is, for all $\varepsilon > 0$ and $x \in X$, there exists $N > 0$ such that $n, m > N$ implies that $|f_n(x) - f_m(x)| < \varepsilon$. We need to show that $\lim_{n\to\infty} \|f - f_n\|_{L^\infty} = 0$. Let $E_n^c = \{x \in X \mid f_n(x) = \infty\}$ for $n \in \mathbb{N}$. Since each $f_n \in L^\infty$, we know that $\mu(E_n^c) = 0$. Now define $E^c \coloneqq \bigcup_{n=1}^{\infty} E_n^c$. Because a countable union of sets with measure 0 also has measure 0, it follows that $\mu(E^c) = 0$. Then, by Lemma 3.18, for all $x \in E$ and $m, n \in \mathbb{N}$, we have $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{L^\infty} = \text{ess sup } |f_n - f_m|$.

However, by the fact that $\{f_n\}$ is Cauchy, we see that this quantity tends to 0 as m, n grow large.

We claim that in fact, $\{f_n\}$ is uniformly Cauchy on E, that is,

$$\text{for all } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that } n, m > N \\ \text{implies that } |f_n(x) - f_m(x)| < \varepsilon \text{ for all } x \in E. \tag{3.20}$$

Suppose this were false. Then, there exists $\varepsilon > 0$ such that for all $N > 0$, there exist $x \in E$ and $n, m \in \mathbb{N}$ for which $n, m > N$ and $|f_n(x) - f_m(x)| \geq \varepsilon$. However, this is a contradiction by the remark at the end of the previous paragraph. Thus, $\{f_n\}$ is uniformly Cauchy.

We define

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n & x \in E, \\ 0 & x \in E^c. \end{cases}$$

Since 3.20 holds for all n, m large enough, we see from the definition of f that $f_n \to f$ uniformly on E. Finally, we conclude:

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty} = \lim_{n\to\infty} \text{ess sup } |f_n - f| = \\ \lim_{n\to\infty} \inf\{U \mid U \text{ is an upper bound of } |f_n - f|\}.$$

The uniform convergence of $\{f_n\}$ tells us that this last quantity is equal to 0. Thus, $f_n \to f$ in the L^∞ norm.

The final step is to show that $f \in L^\infty$:

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \|f - f_n\|_{L^\infty} + \|f_n\|_{L^\infty} \text{ almost everywhere,}$$

so f is essentially bounded. □

4. Radon-Nikodým Theorem

We have now done most of the background work required to prove some important theorems. The rest of the paper will work toward proving the Riesz Representation Theorem.

Definition 4.1. Let λ, μ be measures on a σ-algebra χ. Then we say that λ is **absolutely continuous** with respect to μ if for all sets $E \in \chi$, $\mu(E) = 0$ implies that $\lambda(E) = 0$. We write $\lambda \ll \mu$.

Lemma 4.2. *Let (X, χ, μ) and (X, χ, ν) be finite measure spaces such that $\nu \ll \mu$. Also assume that there exists $E \in \chi$ such that $\nu(E) \neq 0$. Then, there exists a nonnegative function f on X that is measurable with respect to μ such that*

$$\int_X f \, d\mu > 0 \text{ and } \int_E f \, d\mu \leq \nu(E) \text{ for all } E \in \chi.$$

Proof. For $\alpha > 0$, consider the signed measure $\nu - \alpha\mu$. By the Hahn Decomposition Theorem (1.25), there exists a positive set P_α and a negative set N_α such that $P_\alpha \cup N_\alpha = X$ and $P_\alpha \cap N_\alpha = \varnothing$. Now, we claim that there exists $\alpha_0 > 0$ such that $\mu(P_{\alpha_0}) > 0$. Suppose that this is not true. Then, $\mu(P_\alpha) = 0$ for all $\alpha > 0$. Since P_α is positive, this implies that $\mu(E) = 0$ for all $E \in P_\alpha$. By absolute continuity,

$\nu(E) = 0$ as well. Now, we show that $\nu(E) = 0$ for all $E \in N_\alpha$. Since N_α is negative, this implies that $\nu(E) - \alpha\mu(E) \leq 0$ for all $E \in N_\alpha$. Hence

$$0 \leq \nu(E) \leq \alpha\mu(E) \tag{4.3}$$

for all $\alpha > 0$. If $\mu(E) = 0$, then $\nu(E) = 0$ by absolute continuity. If, on the other hand, $\mu(E) > 0$, then we still have $\nu(E) = 0$ since Equation 4.3 must hold for all $\alpha > 0$. So $\nu(E) = 0$ for all $E \in M_\alpha$, and thus we have shown that $\nu(E) = 0$ for all $E \in \chi$, a contradiction to our assumption that ν does not vanish everywhere on χ.

Now, define $f \coloneqq \alpha_0 \cdot \chi_{P_{\alpha_0}}$, where $\chi_{P_{\alpha_0}}$ is the characteristic function of $P_{a\alpha_0}$. So $\int_X f\, d\mu > 0$. Since $\nu - \alpha\mu \geq 0$ for $E \in P_\alpha$, $\alpha\mu(E) \leq \nu(E)$. Hence,

$$\begin{aligned}\int_E f\, d\mu &= \int_{E \setminus P_{\alpha_0}} \alpha_0 \chi_{P_{\alpha_0}} + \int_{E \cap P_{\alpha_0}} \alpha_0 \chi_{P_{\alpha_0}} \\ &= 0 + \alpha_0\mu(P_{\alpha_0} \cap E) \leq \nu(P_{\alpha_0} \cap E) \leq \nu(E) \text{ for all } E \in \chi.\end{aligned}$$

□

Theorem 4.4. (**Radon-Nikodým Theorem**) *Let (X, χ, μ) and (X, χ, λ) be σ-finite measure spaces such that $\lambda \ll \mu$. Then, there exists a nonnegative measurable function f on X that is measurable with respect to μ such that*

$$\lambda(E) = \int_E f(x)\, d\mu \text{ for all } E \in \chi. \tag{4.5}$$

This function is uniquely determined. That is, if f, g both satisfy these conditions, then $f = g$ μ-almost everywhere.

Proof. We first reduce to the case where λ and μ are finite measures. Let $\mathcal{F}$ be the collection of all nonnegative measurable functions f satisfying

$$\int_E f\, d\mu \leq \lambda(E) \text{ for all } E \in \chi$$

and define

$$M \coloneqq \sup_{f \in \mathcal{F}} \int_X f\, d\mu.$$

Then $\mathcal{F}$ is non-empty since $0 \in \mathcal{F}$. We claim that there is a function $f \in \mathcal{F}$ such that $\int_X f\, d\mu = M$. To this end, suppose that $g, h \in \mathcal{F}$ and let E be a measurable set. Then, let $E_1 = \{x \in X \mid g(x) < h(x)\}$ and $E_2 = \{x \in X \mid g(x) \geq h(x)\}$. Then $E = E_1 \cup E_2$. Thus,

$$\int_E \max\{g, h\}\, d\mu = \int_{E_1} h\, d\mu + \int_{E_2} g\, d\mu \leq \lambda(E_1) + \lambda(E_2) = \lambda(E).$$

Hence, $\max\{g, h\} \in \mathcal{F}$.

Now, let $\{f_n\}$ be a sequence of nonnegative measurable functions in $\mathcal{F}$ such that $\lim_{n\to\infty} \int_X f_n\, d\mu = M$. We can choose the sequence to be monotone increasing by replacing f_k by $\max\{f_1, f_2, \cdots, f_k\}$ for each k. Then, define $f \coloneqq \lim_{n\to\infty} f_n$. By the Monotone Convergence Theorem (2.15),

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu = M.$$

Now, for all $E \in \chi$ we define

$$\nu(E) \coloneqq \lambda(E) - \int_E f\, d\mu.$$

By assumption, λ is finite, so by the definition of f, $0 \leq \int_E f\, d\mu \leq \lambda(E) < \infty$. This implies that $\nu(E) \geq 0$ for all $E \in \chi$. Since $\nu(E)$ is the sum of two measures and is always nonnegative, it is itself a measure. Since it does not take on values in the extended reals, it is a signed measure. Moreover, since λ is absolutely continuous with respect to μ, it follows that if $\mu(E) = 0$, then $\nu(E) = 0$. Hence, ν is absolutely continuous with respect to μ. In fact, we claim that $\nu(E) = 0$ for all $E \in \chi$. If not, then by Lemma 4.2, there exists a nonnegative measurable function $\hat{f}$ such that

$$\int_X \hat{f}\, d\mu > 0 \text{ and } \int_E \hat{f}\, d\mu \leq \nu(E) = \lambda(E) - \int_E f\, d\mu \text{ for all } E \in \chi.$$

Thus, rearranging, we see that $\int_E (\hat{f} + f)\, d\mu \leq \lambda(E)$, so $(\hat{f} + f) \in \mathcal{F}$. But

$$\int_E (\hat{f} + f)\, d\mu = \int_E \hat{f}\, d\mu + \int_E f\, d\mu > \int_E f\, d\mu = M.$$

But this contradicts our choice of M as the supremum of $\int_E f\, d\mu$ for $f \in \mathcal{F}$. Therefore, $\nu(E) = 0$ for all $E \in \chi$, so $\int_E f\, d\mu = \lambda(E)$ for all $E \in \chi$, as desired.

Now, suppose that f, g both satisfy Equation 4.5. Then, $\int_E f\, d\mu = \lambda(E) = \int_E g\, d\mu$. This implies that $\int_E (f - g)\, d\mu = 0$, so by Corollary 2.20, $f - g = 0$ μ-almost everywhere.

Now, we consider the case where λ and μ are σ-finite. Let $\{X_n\} \subseteq \chi$ be an increasing sequence such that $X = \bigcup_{n=1}^{\infty} X_n$, $\lambda(X_n) < \infty$, and $\mu(X_n) < \infty$. Then, for each n, by the finite case of the theorem, we get a nonnegative measurable function f_n such that $f_n(x) = 0$ for $x \in X \setminus X_n$, and if $E \subseteq X_n$ is measurable, then

$$\lambda(E) = \int_E f\, d\mu.$$

We now wish to construct a sequence of increasing functions from f_n. If $n \geq m$, then $X_m \subseteq X_n$, and by the finite case, the uniqueness property tells us that $f_m = f_n$ for almost all $x \in X_m$. So define

$$F_n \coloneqq \sup\{f_1, f_2, \cdots, f_n\},$$

so that F_n is an increasing sequence of nonnegative measurable functions. We also have $F_n = f_n$ almost everywhere. We then define $f \coloneqq \lim_{n\to\infty} F_n$. So

$$\lambda(E \cap X_n) = \int_E f_n\, d\mu = \int_E F_n\, d\mu \text{ for all } E \in \chi.$$

Finally, since $(E \cap X_n)$ is monotone increasing and converges to E, we may apply the Monotone Convergence Theorem (2.15) to conclude that

$$\lambda(E) = \lim_{n\to\infty} \lambda(E \cap X_n) = \lim_{n\to\infty} \int_E F_n\, d\mu = \int_E f\, d\mu.$$

The uniqueness of f can be shown by the same reasoning as in the finite case. This concludes the proof. □

5. Riesz Representation Theorem

We begin with a definition of a linear functional:

Definition 5.1. Let V be a vector space over $\mathbb{R}$. A **linear functional** is a linear map $V \to \mathbb{R}$.

Notation 5.2. Let V, W be vector spaces. We denote the space of linear maps from V to W as $L(V, W)$.

Definition 5.3. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed linear spaces. Let $T \in L(V, W)$. Then, we say that T is **bounded** if

$$\sup_{\|x\|_V \leq 1} \{\|T(x)\|_W\} < \infty.$$

In this case, we define

$$\|T\| = \sup_{\|x\|_V \leq 1} \{\|T(x)\|_W\}.$$

The vector space of bounded linear functionals on V is denoted V^*, and it is called the **dual space** of V.

Proposition 5.4. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed linear spaces. Let $T \in L(V, W)$. Then the following are equivalent:*
(i) T is continuous;
(ii) T is continuous at 0;
(iii) T is bounded.

Proof. $(i) \implies (ii)$. This is obvious.

$(ii) \implies (iii)$. If T is continuous at 0, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in V$, $0 < \|y\|_V < \delta$ implies that $\|T(y)\|_W < \varepsilon$. Choose $\varepsilon = 1$. Then there exists δ_1 such that $y \in V$, $0 < \|y\|_V < \delta_1$ implies that $\|T(y)\|_W < 1$.

We wish to compute $\sup_{\|y\|_V \leq 1}\{\|T(y)\|_W\}$. Note that if $0 < \|y\|_V < \delta_1$, then

$$\|\frac{1}{\delta_1} \cdot y\|_V = \frac{1}{\delta_1} \cdot \|y\|_V < \frac{1}{\delta_1} \cdot \delta_1 = 1,$$

and

$$\|T(\frac{1}{\delta_1} \cdot y)\|_W = \|\frac{1}{\delta_1} \cdot T(y)\|_W = \frac{1}{\delta_1} \cdot \|T(y)\|_W < \frac{1}{\delta_1},$$

by the linearity of T and the homogeneity of the norm with respect to scalar multiplication.

$(iii) \implies (i)$. Finally, suppose that T is bounded, i.e. that $\|T\| < \infty$. If $\|T\| = 0$, then $T = 0$ and is thus continuous, so assume $\|T\| > 0$. Fix some $\varepsilon > 0$ and $x \in V$. Then, let $\delta := \frac{\varepsilon}{\|T\|}$ so that if $y \in V$, and $0 < \|y - x\|_V < \delta$, then

$$\|T(y) - T(x)\|_W = \|T(y - x)\|_W \leq \|T\| \cdot \|y - x\|_W < \|T\| \cdot \frac{\varepsilon}{\|T\|} = \varepsilon.$$

Since ε and x were arbitrary, this statement holds for all $\varepsilon > 0$ and $x \in V$. Thus, T is everywhere continuous.

This concludes the proof. $\square$

Proposition 5.5. *Assume that $T : V \to \mathbb{R}$ is bounded. Let*

$$\|T\|^* = \sup_{v \neq 0} \left\{ \frac{|Tv|}{\|v\|_V} \right\}.$$

Then, $\|T\| = \|T\|^*$.

Proof. Observe that we can reformulate $\|T\|^*$ as follows. Since T is bounded, there exists $M > 0$ such that $|Tv| \leq M \cdot \|v\|_V$ for all $v \in V$. Consider the infimum of all such M satisfying this property, i.e. $\inf\{M \geq 0 \mid M \geq \frac{|Tv|}{\|v\|_V} \text{ for all } v \neq 0\}$. We now see that

$$\inf\left\{M \geq 0 \;\middle|\; M \geq \frac{|Tv|}{\|v\|_V} \text{ for all } v \neq 0\right\} = \sup_{v\neq 0}\left\{\frac{|Tv|}{\|v\|_V}\right\} = \|T\|^*.$$

Now, suppose that $M \geq 0$ satisfies $|Tv| \leq M \cdot \|v\|_V$ for all $v \in V$. Then, for all v satisfying $\|v\|_V \leq 1$, we have $Tv \leq |Tv| \leq M \cdot \|v\|_V \leq M$. Thus, $\|T\| = \sup_{\|v\|_V \leq 1} |Tv| \leq M$ for all M satisfying $M \geq \frac{|Tv|}{\|v\|_V}$ for all $v \neq 0$, so we have $\|T\| \leq \|T\|^*$.

To prove the reverse inequality, fix $v \in V$, $v \neq 0$. Then, by the linearity of T,

$$\frac{Tv}{\|v\|_V} = T\left(\frac{v}{\|v\|_V}\right) \leq \|T\|,$$

since $v/\|v\|_V \leq 1$ for all $v \neq 0$. Thus, $Tv \leq \|T\| \cdot \|v\|_V$ for all $v \neq 0$. Hence, $\|T\|^* \leq \|T\|$. □

Remark 5.6. Since we have proven that $\|T\| = \|T\|^*$, in the future, we will refer to both quantities simply as $\|T\|$.

Corollary 5.7. *Let $T : V \to \mathbb{R}$ be a bounded linear functional. Then, for all $v \in V$, $|Tv| \leq \|T\| \cdot \|v\|_V$.*

Proof. If $v = 0$, equality follows easily. Otherwise, since $\|T\| = \sup_{v\neq 0}\left\{\frac{|Tv|}{\|v\|_V}\right\}$, we have $\|T\| \geq \frac{|Tv|}{\|v\|_V}$ for all $v \in V$. Rearranging, we get the desired inequality. □

The goal of the Riesz Representation Theorem is to be able to describe the dual space of the L^p space, i.e. $(L^p)^*$. We begin with three lemmas.

Lemma 5.8. *Let (X, χ, μ) be a measure space, and $1 \leq p \leq \infty$. Let q be the conjugate of p. Then, for every $g \in L^q$, the map $\phi_g : L^p \to \mathbb{R}$ defined by $\phi_g(f) = \int_X fg\, d\mu$ is a continuous linear functional on L^p. Moreover, $\|\phi_g\| \leq \|g\|_{L^q}$, and if $1 < p \leq \infty$, then $\|\phi_g\| = \|g\|_{L^q}$. If μ is σ-finite, then equality holds for $p = 1$.*

Proof. By Proposition 5.4, in order to show that ϕ_g is continuous, it suffices to show that ϕ_g is bounded. We apply Hölder's inequality to see that

$$\|\phi_g\| = \sup_{\|f\|_{L^p}\leq 1} \int_X |fg|\, d\mu \leq \sup_{\|f\|_{L^p}\leq 1} \|f\|_{L^p} \cdot \|g\|_{L^q} = \|g\|_{L^q}.$$

Since $g \in L^q$, $\|g\|_{L^q}$ exists and is bounded.

If $p > 1$, we have already shown that $\|\phi_g\| \leq \|g\|_{L^q}$. We will now prove the reverse inequality. First, if $g = 0$, then the inequality is trivial, since $\phi_0 = 0$ for all $f \in L^p$. So assume that $g \neq 0$. Let φ_k be a monotone increasing sequence of simple functions in L^1 such that $\varphi_k \to |g|^q$ as $k \to \infty$. This is possible since $|g|^q \in L^1$. We may assume without loss of generality that $\varphi_k \geq 0$ for all n. Now, define $\psi_k \coloneqq \varphi_k^{1/p} \cdot sgn(g)$. Then, $\psi_k \in L^p$, and

$$\|\psi_k\|_{L^p} = \left(\int_X |(\varphi_k)^{\frac{1}{p}} sgn(g)|^p\, d\mu\right)^{1/p} = \left(\int_X |\varphi_k|\, d\mu\right)^{1/p} = \|\varphi_k\|_{L^1}^{1/p}.$$

Next, since $|g| \geq \varphi_k^{1/q}$ for all k, it follows that

$$\psi_k g = \varphi_k^{1/p}|g| \geq \varphi_k^{1/p} \cdot \varphi_k^{1/q} = \varphi_k^{1/p+1/q} = \varphi_k.$$

Putting everything together, along with Corollary 5.7, we see that

$$\|\varphi_k\|_{L^1} = \int_X \varphi_k \, d\mu \le \int_X \psi_k g \, d\mu = \phi_g(\psi_k) \le \|\phi_g\| \cdot \|\psi_k\|_{L^p} = \|\phi_g\| \cdot \|\varphi_k\|_{L^1}^{1/p}.$$

Rearranging, we see that $\|\varphi_k\|_{L^1}^{1-1/p} \le \|\phi_g\|$, or $\|\varphi_k\|_{L^1}^{1/q} \le \|\phi_g\|$ for each k. Since $\phi_k \to |g|^q$ as $k \to \infty$, the last inequality tells us that

$$\|\phi_g\| \ge \lim_{k\to\infty} \|\varphi_k\|_{L^1}^{1/q} = \left(\big\||g|^q\big\|_{L^1}\right)^{1/q} = \left(\int_X |g|^q \, d\mu\right)^{1/q} = \|g\|_{L^q},$$

as desired.

Now, let $p = 1$ and μ be σ-finite. If $g = 0$, then as above, equality is trivial. For $g \neq 0$, we have $g \in L^\infty$, so $|g|$ is essentially bounded. Let E be the set outside of which is a set of measure 0 for which $|g(x)| > \|g\|_{L^\infty}$. Then $\|g\|_{L^\infty}$ is an upper bound for g on E. Thus, for any $f \in L^1$, we have

$$\int_X |fg| \, d\mu = \int_E |fg| \, d\mu \le \int_E \big|f\|g\|_{L^\infty}\big| \, d\mu = \|g\|_{L^\infty} \cdot \int_E |f| \, d\mu = \|g\|_{L^\infty} \cdot \|f\|_{L^1}.$$

(Note: this is Hölder's inequality for $p = 1$.). It follows that

$$\|\phi_g\| = \sup_{\|f\|_{L^1} \le 1} \int_X |f \cdot g| \, d\mu \le \sup_{\|f\|_{L^1} \le 1} \{\|g\|_{L^\infty} \cdot \|f\|_{L^1}\} = \|g\|_{L^\infty}.$$

This proves one direction of the inequality. To prove the reverse inequality, suppose for the sake of contradiction that $\|\phi_g\| < \|g\|_{L^\infty}$. Then, there exists $\varepsilon > 0$ such that the set $S_\varepsilon := \{x \in X \mid |g(x)| > \|\phi_g\| + \varepsilon\}$ has positive measure. Otherwise, $\|\phi_g\|$ would be an essential upper bound for g. We may assume without loss of generality that the set $S_\varepsilon^1 := \{x \in X \mid g(x) > \|\phi_g\| + \varepsilon\}$ has positive measure. Corollary 5.7 tells us that

$$\big|\phi_g(\chi_{(S_\varepsilon^1)})\big| \le \|\phi_g\| \cdot \|\chi_{(S_\varepsilon^1)}\|_{L^1} = \|\phi_g\| \cdot \int_X \chi_{(S_\varepsilon^1)} = \|\phi_g\| \cdot \mu(S_\varepsilon^1).$$

So, we have

$$\|\phi_g\| \cdot \mu(S_\varepsilon^1) \ge \big|\phi_g(\chi_{(S_\varepsilon^1)})\big| = \int_X g \cdot \chi_{(S_\varepsilon^1)} \, d\mu = \int_{S_\varepsilon^1} g \, d\mu > (\|\phi_g\| + \varepsilon) \cdot \mu(S_\varepsilon^1).$$

This shows that $\|\phi_g\| > \|\phi_g\| + \varepsilon$, a contradiction. Thus, we must have $\|\phi_g\| \ge \|g\|_{L^\infty}$. This proves the lemma. □

Lemma 5.9. *Let (X, χ, μ) be a finite measure space, and $1 \le p < \infty$. Let g be an integrable function such that there exists a constant M with $|\int \varphi g \, d\mu| \le M\|\varphi\|_{L^p}$ for all simple functions φ. Then $g \in L^q$.*

Proof. We first consider the case when $p > 1$. Let φ_k and ψ_k be defined as in the previous lemma. From above, we have that $\psi_k g \ge \varphi_k$ and $\|\psi_k\|_{L^p} = \left(\int_X \varphi_k \, d\mu\right)^{1/p}$. Thus,

$$\int_X \varphi_k \, d\mu \le \int_X \psi_k g \, d\mu \le M\|\psi_k\|_{L^p} = M\left(\int_X \varphi_k \, d\mu\right)^{1/p}.$$

Simplifying, we see that

$$\frac{\int_X \varphi_k \, d\mu}{\left(\int_X \varphi_k \, d\mu\right)^{1/p}} \le M \implies \left(\int_X \varphi_k \, d\mu\right)^{1-1/p} = \left(\int_X \varphi_k \, d\mu\right)^{1/q} \le M,$$

or $\int_X \varphi_k \, d\mu \le M^q$. This holds for all k. Applying the Monotone Convergence Theorem (2.15), we see:

$$\|g\|_{L^q}^q = \int_X |g|^q \, d\mu = \lim_{n\to\infty} \int_X \varphi_k \, d\mu \le M^q.$$

So $g \in L^q$.

Now, suppose $p = 1$. We want to show that $g \in L^\infty$, i.e. that g is bounded almost everywhere. Let M be as in the hypotheses, and let $E := \{x \in X \mid |g(x)| > M\}$. Suppose for the sake of contradiction that $\mu(E) > 0$. Then, define $f := \frac{1}{\mu(E)} \cdot \chi_E \cdot sgn(g)$. Then, f is a simple function. We have

$$\|f\|_{L^1} = \int_X |f| \, d\mu = \frac{1}{\mu(E)} \int_X \chi_E \, d\mu = \frac{1}{\mu(E)} \int_E \chi_E \, d\mu = \frac{1}{\mu(E)} \cdot \mu(E) = 1.$$

But

$$\begin{aligned}\int_X fg = \int_X \frac{1}{\mu(E)} \cdot \chi_E \cdot sgn(g) \cdot g = \frac{1}{\mu(E)} \int_X \chi_E \cdot |g| \, d\mu \\ = \frac{1}{\mu(E)} \int_E |g| \, d\mu > \frac{1}{\mu(E)} \int_E M \, d\mu = \frac{1}{\mu(E)} \cdot M \cdot \mu(E) = M.\end{aligned}$$

So $|\int_X fg \, d\mu| > M\|f\|_{L^1} = M$, which is a contradiction of the hypotheses. Thus, $\mu(E) = 0$, so g is bounded almost everywhere. $\square$

Lemma 5.10. *Let $1 \le p < \infty$ and $\{X_n\}$ be a sequence of disjoint sets such that $X = \bigcup_{n=1}^\infty X_n$. Let $\{f_n\} \subseteq L^p$ such that for each $n \ge 1$, $f_n(x) = 0$ if $x \notin X_n$. Define $f := \sum_{n=1}^\infty f_n$. Then, $f \in L^p$ if and only if $\sum_{n=1}^\infty (\|f_n\|_{L^p})^p < \infty$. In this case, $(\|f\|_{L^p})^p = \sum_{n=1}^\infty (\|f_n\|_{L^p})^p$.*

Proof. By definition, $f \in L^p$ if and only if

$$\int_X |f|^p \, d\mu = \int_X \left|\sum_{n=1}^\infty f_n\right|^p \, d\mu < \infty.$$

Since the X_n are disjoint and their union is X, we can write

$$\int_X \left|\sum_{n=1}^\infty f_n\right|^p \, d\mu = \int_{X_1} \left|\sum_{n=1}^\infty f_n\right|^p \, d\mu + \int_{X_2} \left|\sum_{n=1}^\infty f_n\right|^p \, d\mu + \cdots .$$

However, each $f_n(x)$ is zero for all $x \notin X_n$, so this simplifies to

$$\int_{X_1} |f_1|^p \, d\mu + \int_{X_2} |f_2|^p \, d\mu + \cdots = \int_X |f_1|^p \, d\mu + \int_X |f_2|^p \, d\mu + \cdots = \sum_{n=1}^\infty \int_X |f_n|^p \, d\mu.$$

Thus, we see that $\int_X |f|^p \, d\mu < \infty$ if and only if $\sum_{n=1}^\infty \int_X |f_n|^p \, d\mu < \infty$, and we have shown these quantities to be equal. $\square$

Now we are finally ready to prove our major result:

Theorem 5.11. (**Riesz Representation Theorem I**) *Let $\Gamma \in (L^p)^*$, where $1 \le p < \infty$ and μ is σ-finite, and let q be the conjugate of p. Then, there exists a unique $g \in L^q$ such that*

$$\Gamma(f) = \int_X fg \, d\mu = \phi_g(f)$$

for all $f \in L^p$. Moreover, $\|\Gamma\| = \|g\|_{L^q}$.

Proof. We first consider the case when μ is finite. Let $\lambda : \chi \to \mathbb{R}$, $\lambda(E) = \Gamma(\chi_E)$. We first show that λ is a signed measure. It is clear that $\lambda(\varnothing) = \Gamma(\chi_\varnothing) = \Gamma(0) = 0$ since Γ is linear by assumption. Fix $E \in \chi$ and let $\{E_n\}$ be a sequence of disjoint sets in χ such that $E = \bigcup_{n=1}^\infty E_n$. We show countable additivity:

$$\lambda\left(\bigcup_{n=1}^\infty E_n\right) = \lambda(E) = \Gamma(\chi_E) = \Gamma\left(\sum_{n=1}^\infty \chi_{E_n}\right) = \sum_{n=1}^\infty \Gamma(\chi_{E_n}) = \sum_{n=1}^\infty \lambda(E_n).$$

It remains to show that $|\lambda(E)| < \infty$. Let $\alpha_n = sgn(\Gamma(\chi_{E_n})$ and define $f_n \coloneqq \alpha_n \cdot \chi_{E_n}$, and $f \coloneqq \sum_{n=1}^\infty f_n$. Observe that $|\alpha_n| \leq 1$ everywhere. We can now compute:

$$\int_X |f_n|^p \, d\mu = \int_X |\alpha_n \cdot \chi_{E_n}|^p \, d\mu \leq \int_X |\chi_{E_n}|^p \, d\mu = \int_X \chi_{E_n} = \mu(E_n),$$

so $f_n \in L^p$ for each n. Thus,

$$\sum_{n=1}^\infty \|f_n\|_{L^p}^p \leq \sum_{n=1}^\infty \mu(E_n) = \mu(E) < \infty,$$

by the assumption that μ is finite. By Lemma 5.10, we conclude that $f \in L^p$ and thus is in the domain of Γ:

$$\begin{aligned}\Gamma(f) = \Gamma\left(\sum_{n=1}^\infty sgn(\Gamma(\chi_{E_n}))\cdot\chi_{E_n}\right) &= \sum_{n=1}^\infty sgn(\Gamma(\chi_{E_n}))\cdot\Gamma(\chi_{E_n}) \\ &= \sum_{n=1}^\infty |\Gamma(\chi_{E_n})| = \sum_{n=1}^\infty |\lambda(E_n)|,\end{aligned} \tag{5.12}$$

where the second equality is true by the linearity of Γ. We observe that $\|\Gamma\| < \infty$ by assumption, and $\|f\|_{L^p} < \infty$ by definition of being in L^p. Thus, by Corollary 5.7, we have $|\Gamma(f)| \leq \|\Gamma\|\cdot\|f\|_{L^p} < \infty$. Thus, by 5.12, we conclude that $\sum_{n=1}^\infty |\lambda(E_n)| < \infty$. Finally,

$$|\lambda(E)| = \left|\lambda\left(\bigcup_{n=1}^\infty E_n\right)\right| = \left|\sum_{n=1}^\infty \lambda(E_n)\right| \leq \sum_{n=1}^\infty |\lambda(E_n)| < \infty.$$

Since E was arbitrary, these properties hold for all $E \in \chi$. Thus, λ is a signed measure. Next, if $\mu(E) = 0$, then $\chi_E = 0$ almost everywhere, so $0 = \Gamma(\chi_E) = \lambda(E)$. Hence, $\lambda \ll \mu$.

We may now apply the Radon-Nikodým Theorem (4.4) to λ in order to produce a unique function g such that $\lambda(E) = \int_E g \, d\mu$ for all $E \in \chi$. Now if φ is a simple function, then

$$\Gamma(\varphi) = \Gamma\left(\sum_{k=1}^n c_k \cdot \chi_{E_k}\right) = \sum_{k=1}^n ((c_k \cdot \Gamma(\chi_{E_k})) = \sum_{k=1}^n (c_k \cdot \lambda(E_k)) =$$

$$\sum_{k=1}^n \left(c_k \int_{E_k} g \, d\mu\right) = \sum_{k=1}^n \int_X c_k \cdot g \cdot \chi_{E_k} \, d\mu = \int_X \left(\sum_{k=1}^n (c_k \cdot \chi_{E_k})\right) g \, d\mu = \int_X \varphi g \, d\mu.$$

By Corollary 5.7, we have $|\Gamma(\varphi)| \leq \|\Gamma\|\|\varphi\|_{L^p}$, so by Lemma 5.9, $g \in L^q$. Since we have $\Gamma, \phi_g \in (L^p)^*$, with ϕ_g defined as in Lemma 5.8, then $\Gamma - \phi_g \in (L^p)^*$ and $\Gamma - \phi_g = 0$ for simple functions in L^p. Since simple functions are dense in L^p (see the Simple Approximation Lemma, 2.7), then we can say $\Gamma - \phi_g = 0$ on L^p. Thus $\Gamma = \phi_g$, which also implies that $\|\Gamma\| = \|\phi_g\| = \|g\|_{L^q}$ by Lemma 5.8.

We now turn to the case when μ is σ-finite. By definition of σ-finite, we can write $X = \bigcup_{n=1}^{\infty} X_n$, where $\mu(X_n) < \infty$ and $X_n \subseteq X_{n+1}$ for all n. For each n, the finite case gives rise to $g_n \in L^q$, where $g_n = 0$ for all $x \notin X_n$ and $\Gamma(f) = \int_X f g_n \, d\mu$ for all f which vanish outside X_n. Moreover, $\|g_n\|_{L^q} \leq \|\Gamma\|$. By the uniqueness of the g_n, we can assume that $g_{n+1} = g_n$ for all $x \in X_n$. Now, we define $g \coloneqq \lim_{n\to\infty} g_n$ so that $\{|g_n|\}$ is a monotone increasing sequence converging to $|g|$. Thus, we apply the Monotone Convergence Theorem (2.15) to see that

$$\int_X |g|^q \, d\mu = \lim_{n\to\infty} \int_X |g_n|^q \, d\mu = \lim_{n\to\infty} (\|g_n\|_{L^q})^q \leq \|\Gamma\|^q.$$

This implies that $g \in L^q$. Now, let $f \in L^p$, and define $f_n \coloneqq f \cdot \chi_{X_n}$ so that $f_n \to f$ pointwise on X and $f_n \in L^p$ for all n. We also observe that $|fg| \in L^1$ by Hölder's inequality, and $|f_n g| \leq |fg|$. Lastly, notice that

$$\int_X f_n g_n \, d\mu = \int_{X_n} f_n g_n \, d\mu + \int_{X \setminus X_n} f_n g_n \, d\mu = \int_{X_n} f_n g \, d\mu + 0 = \int_X f_n g \, d\mu$$

since $g = g_n$ on X_n and $f_n = 0$ on $X \setminus X_n$.

Putting everything together and applying the Monotone Convergence Theorem (2.15) and Lebesgue Dominated Convergence Theorem (2.25), we see that

$$\int_X fg \, d\mu = \lim_{n\to\infty} \int_X f_n g \, d\mu = \lim_{n\to\infty} \left(\int_X f_n g_n \, d\mu \right) = \lim_{n\to\infty} \Gamma(f_n) = \Gamma(f).$$

We are allowed to pull the limit through Γ in the final step because Γ is a continuous map by Lemma 5.8. □

Theorem 5.13. (**Riesz Representation Theorem II**) *Let $\Gamma \in (L^p)^*$, where $1 < p < \infty$. Then, there exists a unique $g \in L^q$ such that*

$$\Gamma(f) = \int_X fg \, d\mu$$

for all $f \in L^p$. Moreover, $\|\Gamma\| = \|g\|_{L^q}$.

Proof. Suppose that $E \subseteq X$ is σ-finite. Then, from Riesz Representation Theorem I, there exists a unique $g_E \in L^q$ which vanishes outside of E and satisfies $\Gamma(f) = \int_X f g_E \, d\mu$ for all $f \in L^p$ such that $f = 0$ for all $x \notin E$. The goal of the proof is to construct such a set E.

For all σ-finite E, let $\lambda(E) = \int_X |g_E|^q \, d\mu$. We claim that λ is a measure. First, $g_\varnothing = 0$ everywhere outside of $\varnothing$, which is all of X. Thus, $\lambda(\varnothing) = 0$. Now we show countable additivity: let $\{E_n\}$ be a sequence of disjoint sets whose union is E.

$$\begin{aligned}
\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \lambda(E) = \int_X |g_E|^q d\mu = \int_E |g_E|^q d\mu = \int_{E_1} |g_E|^q d\mu + \int_{E_2} |g_E|^q d\mu + \cdots \\
= \int_{E_1} |g_{E_1}|^q \, d\mu + \int_{E_2} |g_{E_2}|^q \, d\mu + \cdots = \int_X |g_{E_1}|^q \, d\mu + \int_X |g_{E_2}|^q \, d\mu + \cdots \\
= \sum_{n=1}^{\infty} \int_X |g_{E_n}|^q \, d\mu = \sum_{n=1}^{\infty} \lambda(E_n).
\end{aligned}$$

The first and fourth equalities are true because $E = \bigcup_{n=1}^{\infty} E_n$. The third equality is true because g_E vanishes everywhere outside E. The fifth equality is true because (i) $g_E = \sum_{n=1}^{\infty} g_{E_n}$, and (ii) for $i = j$, $g_E = g_{E_i}$ on E_j, but for $i \neq j$, $g_{E_i} = 0$ on

E_j. The sixth equality is true because for each n, g_{E_n} vanishes everywhere outside E_n. Observe that if $A \subseteq E$, then $\lambda(A) \leq \lambda(E) \leq \|\Gamma\|^q$.

We now define $M := \sup\{\lambda(E) \mid E\ \sigma\text{-finite}\}$ and choose $\{E_n\}$ to be a sequence of σ-finite sets such that $\lim_{n\to\infty} \lambda(E_n) = M$. We now let $H := \bigcup_{n=1}^{\infty} E_n$, so that H is σ-finite and $\lambda(H) = M$. Suppose a set E is σ-finite with $H \subseteq E$. Then, $g_H = g_E$ almost everywhere on H due to the uniqueness property, and

$$\int_X |g_E|^q \, d\mu = \lambda(E) \leq M = \lambda(H) = \int_X |g_H|^q \, d\mu,$$

since M is the supremum of $\lambda(E)$ for σ-finite E. This implies that $g_E = 0$ almost everywhere on $E \setminus H$. Otherwise, we would have $\lambda(E) > \lambda(H)$, a contradiction.

We now define our function g by $g = g_H$. Then, $g \in L^q$, and if E is σ-finite with $H \subseteq E$, then $g_E = g$ almost everywhere. This is true because $g_E = g$ almost everywhere on H, $g = 0$ for all $x \notin H$, and $g_E = 0$ for all $x \notin E$ and almost all $x \in E \setminus H$.

Fix $f \in L^p$ and define $E := \{x \in X \mid f(x) \neq 0\}$ so that f vanishes outside E. We claim that E is σ-finite. Suppose not. Then, $\mu(E) = \infty$, and there does not exist a sequence of sets $E_1, E_2, \cdots$ such that $\bigcup_{n=1}^{\infty} E_n = E$ and $\mu(E_n) < \infty$ for each n. Thus, we have

$$\infty > \int_X |f|^p \, d\mu = \int_E |f|^p \, d\mu + \int_{X\setminus E} |f|^p \, d\mu = \int_E |f|^p \, d\mu = \sup_{\varphi \leq |f|^p} \int_E \varphi \, d\mu,$$

for simple functions φ. Since $f > 0$ on E, we can assume that $\varphi > 0$ on E. By definition of the integral of a simple function, we can write

$$\int_E \varphi \, d\mu = \sum_{n=1}^{\infty} a_n \cdot \mu(E_n).$$

Since $\varphi > 0$, each $a_n \neq 0$. Since E is not σ finite, there exists an n such that $\mu(E_n) = \infty$. Thus, the whole sum must be ∞. Hence, the integral of every positive simple function is ∞, so the supremum of all positive simple functions $\varphi \leq f$ must also be ∞. Thus, $\int_X |f|^p \, d\mu = \infty$. But we know from the definition of the L^p norm that $\int_X |f|^p \, d\mu < \infty$, so this is a contradiction. Thus, E must be σ-finite.

Now, let $E_1 := E \cup H$. Since E and H are σ-finite, so is E_1. As before, we obtain a function g_{E_1} such that g_{E_1} vanishes outside E_1 and $\Gamma(f) = \int_X f g_{E_1} \, d\mu$. Since $H \subseteq E_1$, we see that, as above, $g_{E_1} = g$ almost everywhere. Thus,

$$\Gamma(f) = \int_X f g_{E_1} \, d\mu = \int_X fg \, d\mu = \phi_g(f).$$

Since f was arbitrary, this equation holds for all $f \in L^p$. Thus, $\Gamma = \phi_g$ and as above, $\|\Gamma\| = \|g\|_{L^q}$. □

Acknowledgments

I would like to thank my mentor, Becky Zhang, for all of her help this summer, including helping me choose a topic, pointing me to the proper resources, giving consistent guidance as I learned about the Lebesgue integral, and editing this paper. I would not have been able to do this project without her. I would also like to thank Professor Peter May for the work he put into organizing the 2023 University of Chicago Math REU and Professors Daniil Rudenko and László Babai for the time they invested in teaching the Apprentice Program. I am indebted to the

Jeff Metcalf Internship Program for providing me with the funding necessary to participate in the math REU. Finally, I would like to thank my parents for all the love and support they have given me over last twenty years. I would not be here without them.

6. Bibliography

References

[1] Brian Forrest, University of Waterloo. PMath 451 Course Notes, Chapter 4, Fall 2014. https://www.math.uwaterloo.ca/~beforres/PMath451/Course_Notes/Chapter4.pdf

[2] Brian Forrest, University of Waterloo. PMath 451 Course Notes, Chapter 6, Fall 2014. https://www.math.uwaterloo.ca/~beforres/PMath451/Course_Notes/Chapter6.pdf

[3] Steven G. Krantz. Elementary Introduction to the Lebesgue Integral. CRC Press. 2018.

[4] David J. Malison. Solutions to Part I of Real Analysis (Fourth edition) by Royden and Fitzpatrick. 2022. https://drive.google.com/file/d/0B-ilAo2YE_lQYXFLU3pzZVRkX1k/view?pli=1&resourcekey=0-v9uleHrsc6tVwgYHQOzjPg

[5] Hamid Meziani, Florida International University. Real Analysis MAA 6616 Lecture 31: The Dual Space of L^p. https://faculty.fiu.edu/~meziani/Lecture31.pdf

[6] H.L. Royden and P.M. Fitzpatrick. Real Analysis, 4e. Pearson Education, Inc. 2010.

[7] Walter Rudin. Principles of Mathematical Analysis, 3e. McGraw-Hill Education. 1976.

[8] Anton R. Schep. Supplemental Notes for Math 704, Spring 2012. University of Southern California. https://people.math.sc.edu/schep/Banach.pdf