THE DISCRETE SPECTRUM OF THE AUTOMORPHIC LAPLACIAN

SIDNEY STANBURY

Abstract. Automorphic forms gives rise to a large crossover between analysis and number theory. Central to their study is the automorphic Laplacian whose spectral decomposition allows for complete description of them. In this paper, we study cusp forms by studying the discrete part of the spectral decomposition of the automorphic Laplacian.

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1. Background on $\mathrm{SL}(2,\mathbb{R}), \mathrm{PSL}(2,\mathbb{R})$ and the Laplacian

We begin by introducing some vocabulary which will be essential to Sections 2 and 3. Section 3 will heavily rely on some fundamental results from Functional Analysis and Spectral Theory. We refer the reader to Appendix A for a synopsis of the material that will be used from these areas of study. One of the focal objects of study in this paper is the complex upper half plane:

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\},$$

which we will use as a model for the hyperbolic plane. First note that $\mathbb{H}$ is a metric space with metric

$$\rho(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| + |z - w|},$$

and it is more practical to re-write this as

$$\cosh \rho(z,w) = 1 + 2u(z,w), \qquad \text{where} \qquad u(z,w) = \frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}. \tag{1.1}$$

Let G be the group $G = \mathrm{SL}(2,\mathbb{R})$. It is known that G acts faithfully and transitively by Möbius transformations on the upper half plane $\mathbb{H}$. Furthermore one can show that the elements of G that fix i is $G_i = \mathrm{SO}(2,\mathbb{R})$, and thus we have that $\mathbb{H} \cong G/\mathrm{SO}(2,\mathbb{R})$. Writing $K = \mathrm{SO}(2,\mathbb{R})$, we get a one-to-one correspondence between points $z \in \mathbb{H}$, and cosets gK, which send i to z. Using this correspondence, we can understand that action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}$, as multiplication on itself. In order to make better light of this correspondence, we use the Iwasawa decomposition (see [10] p. 373 for the statement in full generality).

Proposition 1.1 (Iwasawa Decomposition of $\mathrm{SL}(2,\mathbb{R})$)**.** *The group $G = SL(2,\mathbb{R})$ has the following decomposition*

$$G = NAK,$$

where

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$
$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^+ \right\},$$
$$K = SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}.$$

In other words, for every $g \in G$, there exist unique n, a, k in N, A, K respectively, such that $g = nak$.

We call elements of N, A, K **translations**, **dilations** and **rotations** respectively.

Note 1.2. Note that under the above correspondence, the point $z = x + iy$ corresponds to the coset gK, where $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$.

For any $g \in \mathrm{PSL}(2,\mathbb{R})$, the action of g on $\mathbb{H}$ can be understood by considering the action of any element conjugate to g. An important invariant of these conjugacy classes is the trace. It useful to distinguish the conjugacy classes of $\mathrm{PSL}(2,\mathbb{R})$ by their action on $\mathbb{H}$, and the trace allows us to do so.

Definition 1.3 (Parabolic, Hyperbolic and Elliptic motions)**.** Let $g \in \mathrm{PSL}(2,\mathbb{R})$. We say that

(a) g is **hyperbolic** if $|\mathrm{tr}(g)| > 2$,
(b) g is **parabolic** if $|\mathrm{tr}(g)| = 2$,
(c) g is **elliptic** if $|\mathrm{tr}(g)| < 2$,

and the same is said about conjugacy classes in $\mathrm{PSL}(2,\mathbb{R})$.

Note 1.4. Note that the groups N, A, K defined above are parabolic, hyperbolic and elliptic respectively. Furthermore, any $g \in \mathrm{SL}(2,\mathbb{R})$ has eigenvalues $\{\lambda, \lambda^{-1}\}$, and characteristic polynomial: $x^2 - \mathrm{tr}(g)x + 1$. Put $t = \mathrm{tr}(g)/2$, then $\lambda = t \pm \sqrt{t^2-1}$. By considering the three possibilities: $i)\, |t| > 1, ii)\, |t| = 1, iii)\, |t| < 1$, the Jordan Normal Form Theorem gives us

(a) g is hyperbolic iff g is conjugate to a dilation iff g fixes two points on $\hat{\mathbb{R}}$,

(b) g is parabolic iff g is conjugate to a translation iff g fixes one point in $\hat{\mathbb{R}}$,
(c) g is elliptic iff g is conjugate to a rotation iff g fixes one point in $\mathbb{H}$, and its complex conjugate in $\overline{\mathbb{H}}$.

We finish the background section by introducing the Laplace operator on $\mathbb{H}$ (for further details see [11]). For a Riemannian Manifold (M, g) with or without boundary, the **geometric Laplacian** operator is the linear map $\Delta : C^\infty(M) \to C^\infty(M)$ given by $\Delta = \operatorname{grad} \circ \operatorname{div}$, and it is obtained in any smooth local coordinates (x^i) by

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j} \right),$$

where $g = g_{ij}\, dx^i\, dx^j$, and $g^{ij} = (g^{-1})_{ij}$. The upper half plane is a Riemannian surface with boundary, and its Riemannian metric is given by $g = y^{-2}(\,dx^2 + dy^2)$. From this we obtain the Laplacian on $\mathbb{H}$

$$\Delta = \frac{1}{y^2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{1.2}$$

From now on, by Δ we will mean the operator defined in (1.2). There is another way to describe the Laplacian using **geodesic polar coordinates**. These arise from the Cartan decomposition of the Lie Group $G = \mathrm{SL}(2, \mathbb{R})$. The geodesic polar coordinates give a unique expression for every $z = x + iy \in \mathbb{H}$ as a pair (r, θ), where r is the hyperbolic distance between i and z, and $\theta \in [0, 2\pi)$ is chosen such that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} e^{-r} i = z$. From here we get an expression for x and y in terms of (r, θ), which allows us to transform (1.2) into (r, θ)-coordinates. Finally, using that $\cosh r = 1 + 2u$ (where u is as in (1.1)), we get an expression for Δ in (u, θ)-coordinates

$$\Delta = u(u+1) \frac{\partial^2}{\partial u^2} + (2u+1) \frac{\partial}{\partial u} + \frac{1}{16u(u+1)} \frac{\partial^2}{\partial \theta^2}. \tag{1.3}$$

2. Automorphic Forms and Cusp Forms

There are various treatments of automorphic forms. For example, in [5], the attention is restricted to congruence subgroups. These are subgroups of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{H}$ which for some N contain the subgroup

$$\Gamma(N) = \left\{ \gamma \in \mathrm{SL}(2, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

The objects of study are the meromorphic functions on $\mathbb{H}$ which satisfy an automorphy condition (analogous to Definition 2.10) and some growth conditions. These functions and congruence groups have a lot of interesting properties, and their study provides a lot of intuition for the generalised approach to automorphic forms which we will follow.

2.1. **Fuchsian Groups and Cusps.** Let G be the group $G = PSL(2, \mathbb{R})$. It is known that G acts faithfully and transitively by Möbius transformations on the upper half plane $\mathbb{H}$. In order to define automorphic forms, we will need the notion of a lattice in G. For this we first need the notion of a wandering group action.

Definition 2.1 (Wandering group action)**.** The action of a Hausdorff topological group G on a Hausdorff topological space X is **wandering** if every point $x \in X$, has a neighbourhood U_x such that $g(U_x) \cap U_x \neq \emptyset$, for only finitely many $g \in G$.

With this definition we can define Fuchsian groups.

Definition 2.2 (Fuchsian Group)**.** Let $G = \mathrm{PSL}(2, \mathbb{R})$. A subgroup $\Gamma \leq G$ is **Fuchsian** if Γ has a wandering action on $\mathbb{H}$.

We have an alternative characterisation for Fuchsian groups (see [13] p. 641).

Proposition 2.3. *A subgroup in $SL(2, \mathbb{R})$ is discrete if and only if it has a wandering action on $\mathbb{H}$ when considered as a subgroup of $PSL(2, \mathbb{R})$.*

Note 2.4. By identifying $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, it follows from the Heine-Borel Theorem: $\Gamma \leq \mathrm{SL}(2, \mathbb{R})$ is a discrete subgroup iff for all $M > 0$, the set $\{\gamma \in \Gamma \mid \|\gamma\| \leq M\}$ is finite. Hence, by Proposition 2.3, it follows that Fuchsian groups are countable. This result will often be used without mention when carrying out summations over a Fuchsian group.

It is worth noting the following fact.

Proposition 2.5. *Every discrete subgroup of a Hausdorff group is closed. Hence in particular, every Fuchsian group is closed.*

We can now define a lattice in $\mathrm{PSL}(2, \mathbb{R})$.

Definition 2.6 (Lattice)**.** A Fuchsian group that has finite covolume, $\mathrm{vol}(\Gamma \backslash \mathbb{H})$ is called a **lattice** in $\mathrm{PSL}(2, \mathbb{R})$.

It follows that for a lattice Γ, we can put the quotient topology on $\Gamma \backslash \mathbb{H}$ by using the projection $\pi : \mathbb{H} \to \Gamma \backslash \mathbb{H}$. With this topology $\Gamma \backslash \mathbb{H}$ is a Hausdorff connected space, and with appropriate charts it becomes a Riemann surface. From now on, unless stated otherwise, Γ will be a lattice. Before moving to automorphic functions, we first give more terminology related to Fuchsian groups.

Definition 2.7 (Fundamental Domain)**.** Let Γ be a Fuchsian group. A set $F \subset \mathbb{H}$ is a **fundamental domain** for Γ if it satisfies:

(1) F is a domain,
(2) Distinct points in F are not equivalent under Γ, i.e. if $z, w \in F$, then $w \notin \Gamma z$.
(3) Any orbit of Γ contains at least one point in $\overline{\Gamma}$, where the closure is taken in the $\hat{\mathbb{C}}$-topology.

It is a well known fact that any Fuchsian group has a (not necessarily unique) fundamental domain, and that by unimodularity, the hyperbolic measure of any fundamental domain of a Fuchsian group is equal to the co-volume of that Fuchsian group. For our purposes it is enough to know that any lattice has a fundamental domain, which can be taken as the **Dirichlet polygon**

$$F(w) = \{z \in \mathbb{H} \mid \rho(w, z) < \rho(\gamma w, z), \text{ for all } \gamma \in \Gamma\},$$

for some arbitrary $w \in \mathbb{H}$ with trivial stabilizer group. We can now define cusps of a Fuchsian group.

Definition 2.8 (Cusp)**.** Let Γ be a Fuchsian group. A **cusp** of Γ is an element of the set $F \cap \hat{\mathbb{R}}$, where F is a fundamental domain for Γ.

Example 2.9. Consider the lattice $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. It has a fundamental domain given by

$$F = \left\{ z \in \mathbb{H} \mid |Re(z)| \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

and thus it only has one cusp, namely ∞ (see [5] p. 52).

2.2. Automorphic Forms and Cusp Forms.

We begin by introducing the property of automorphy, which is a notion common in all definitions of automorphic forms. The automorphy property will crucial to many proceeding constructions.

Definition 2.10 (Automorphic Function)**.** Let Γ be a lattice. A function $f : \mathbb{H} \to \mathbb{C}$ is an **automorphic function** with respect to Γ if

$$f(\gamma z) = f(z), \quad \forall \gamma \in \Gamma.$$

The set of automorphic functions with respect to Γ is denoted $\mathcal{A}(\Gamma \backslash \mathbb{H})$.

With this above definition, and the definition of a lattice, we can define automorphic forms.

Definition 2.11 (Automorphic Form)**.** Let Γ be a lattice. An **automorphic form** with respect to Γ, is an automorphic function $f \in \mathcal{A}(\Gamma \backslash \mathbb{H})$ which is an eigenfunction of the Laplace operator

$$(\Delta + \lambda) f = 0, \quad \lambda = s(1-s).$$

We denote by $\mathcal{A}_s(\Gamma \backslash \mathbb{H})$, the space of automorphic forms with respect to Γ with eigenvalue $\lambda = s(1-s)$.

The introduction of the Laplacian in the definition of automorphic forms can seem arbitrary. One reason for this, is that the Laplacian has a large number of interesting spectral properties. However, a more important reason is related to **invariant** operators. These are linear operators T acting on functions $f : \mathbb{H} \to \mathbb{C}$ which satisfy for all $g \in G$

$$T(f(gz)) = (Tf)(gz).$$

Of particular interest are the invariant differential operators, and Δ is such an operator. In fact, it is a very special operator, and a first important property is the following (see [8] p. 387 for a more general result).

Theorem 2.12. *Let M be a Riemannian manifold with geometric Laplacian operator Δ. A diffeomorphism $\Phi : M \to M$ is an isometry iff Φ commutes with Δ.*

Another important property of the geometric Laplacian is that any invariant differential operator on $\mathbb{H}$ is a polynomial in Δ. This follows from a much more general result about symmetric spaces

In order to introduce cusp forms we need more notation concerning cusps of lattices. We first consider the example $\mathrm{PSL}(2,\mathbb{Z})$. Recall that for an element $\gamma =\in \mathrm{PSL}(2,\mathbb{R})$, the point ∞ is mapped to $\frac{a}{c} \in \mathbb{Q}$, when $c \neq 0$, and is mapped to ∞ when $c = 0$. Thus its stabiliser group is the subgroup of translations in $\mathrm{PSL}(2,\mathbb{Z})_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$. Thus, for any lattice Γ, contained in $\mathrm{PSL}(2,\mathbb{Z})$, the stabiliser group Γ_∞ is a subgroup of the translations $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & h_\Gamma \\ 0 & 1 \end{pmatrix} \right\rangle$. The constant h_Γ is sometimes called the period of ∞ in Γ, and we write h when the context is clear. For the general case, we have the following result, (see [1] p. 189)[1].

Proposition 2.13. *Let Γ be a Fuchsian group and let $z \in \hat{\mathbb{C}}$. Then the stability group Γ_z is cyclic.*

Now for a general Fuchsian group Γ, the stabilizers of ∞ are all parabolic. Hence, (after conjugation if necessary) we get $\Gamma_\infty = \left\langle \begin{pmatrix} 1 & h_\Gamma \\ 0 & 1 \end{pmatrix} \right\rangle$, for some h_Γ. We will make use of this convention throughout. Also, for a cusp $\mathfrak{a}$, there exists a $\sigma_\mathfrak{a}$ such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$. Then

$$\Gamma_\mathfrak{a} = \left\langle \sigma_\mathfrak{a} \begin{pmatrix} 1 & h_\Gamma \\ 0 & 1 \end{pmatrix} {\sigma_\mathfrak{a}}^{-1} \right\rangle . \tag{2.1}$$

Note that $\sigma_\mathfrak{a}$ is not unique, as we can multiply it on the right by any translation to get the same relation. We call such $\sigma_\mathfrak{a}$, a **scaling matrix**. Now suppose that $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$, then for any $m \in \mathbb{Z}$ we have

$$f\left(\sigma_\mathfrak{a} \begin{pmatrix} 1 & mh_\Gamma \\ 0 & 1 \end{pmatrix} z\right) = f\left(\sigma_\mathfrak{a} \begin{pmatrix} 1 & mh_\Gamma \\ 0 & 1 \end{pmatrix} \sigma_\mathfrak{a}^{-1}\sigma_\mathfrak{a} z\right) = f(\sigma_\mathfrak{a} z).$$

Thus we have a Fourier expansion

$$f(\sigma_\mathfrak{a} z) = \sum_n f_{\mathfrak{a}n}(y) e\left(\frac{nx}{h}\right),$$

where $e(y) = \exp(2\pi i y)$, and $f_{\mathfrak{a}n}$ is given by

$$f_{\mathfrak{a}n}(y) = \int_0^h f(\sigma_\mathfrak{a} z) e\left(\frac{-nx}{h}\right) dx\,.$$

Here we use the notation; $f_\mathfrak{a} = f_{\mathfrak{a}0}$. One of the main goals of the spectral theory of automorphic forms is to expand automorphic functions into automorphic forms. In order to do this we use tools of spectral theory in a Hilbert space, called $\mathcal{L}(\Gamma\backslash\mathbb{H})$.

Definition 2.14 ($\mathcal{L}(\Gamma\backslash\mathbb{H})$)**.** Let Γ be a lattice. Define the inner product space $\mathcal{L}(\Gamma\backslash\mathbb{H})$ by

$$\mathcal{L}(\Gamma\backslash\mathbb{H}) = \{f \in \mathcal{A}(\Gamma\backslash\mathbb{H}) : \|f\| < \infty\},$$

where we define the inner product by

$$\langle f, g\rangle = \int_F f(z)\overline{g(z)}\, d\mu z\,,$$

where F is a fundamental domain of Γ.

We will sometimes write $\Gamma\backslash\mathbb{H}$ as the domain of integration, and this can be understood as integrating over a fundamental domain.

We note that since Γ is a lattice, it has finite covolume, and thus every bounded element of $\mathcal{A}(\Gamma\backslash\mathbb{H})$ is in $\mathcal{L}(\Gamma\backslash\mathbb{H})$. This naturally leads to the definition of the following dense subspace of $\mathcal{A}(\Gamma\backslash\mathbb{H})$.

Definition 2.15 ($\mathcal{B}(\Gamma\backslash\mathbb{H})$)**.** Let Γ be a lattice. Define $\mathcal{B}(\Gamma\backslash\mathbb{H})$ to be the set of smooth bounded automorphic functions.

[1] The proof in [1] uses the fact that discrete subgroups of N, A, K, are cyclic. This can be shown for N by noting that any such subgroup will have a minimal $(1,2)$-entry, and the idea is similar for A and K

We now define an important subspace of $\mathcal{B}(\Gamma\backslash\mathbb{H})$.

Definition 2.16 (Incomplete Eisenstein Series)**.** Let Γ be a lattice and $\mathfrak{a}$ be a cusp for Γ. If $\psi \in C_0^\infty(\mathbb{R}_{>0})$, a series of the form

$$E_\mathfrak{a}(z \mid \psi) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)),$$

is called an **incomplete Eisenstein series**. The set of all such series is denoted by $\mathcal{E}(\Gamma\backslash\mathbb{H})$.

Note 2.17. (1) Note that for $\gamma, \gamma' \in \Gamma$ by (2.1), we have that

$$\Gamma_\mathfrak{a}\gamma = \Gamma_\mathfrak{a}\gamma' \iff \gamma \in \Gamma_\mathfrak{a}\gamma' \iff \gamma = \sigma_\mathfrak{a}\begin{pmatrix}1 & mh\\ 0 & 1\end{pmatrix}\sigma_\mathfrak{a}^{-1}\gamma', \tag{2.2}$$

for some $m \in \mathbb{Z}$. Hence, using the invariance of $\mathrm{Im}(z)$ under translations, we have that $\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z) = \mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma' z)$, and thus the sum is well defined. Furthermore, it is clear from its definition that $\mathcal{E}(\Gamma\backslash\mathbb{H})$ is a subspace of $\mathcal{A}(\Gamma\backslash\mathbb{H})$.

(2) It is easy to check that for any $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$, we have that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}. \tag{2.3}$$

Also, if $\psi \in C_0^\infty(\mathbb{R}_{>0})$, then there exists $R, \varepsilon > 0$ such that $\mathrm{supp}(\psi) \subset [\varepsilon, R]$. Thus if $\gamma \in \Gamma$

$$\mathrm{Im}(\gamma z) \geqslant \varepsilon \iff \frac{\mathrm{Im}(z)}{|cz+d|^2} \geqslant \varepsilon \implies \frac{\mathrm{Im}(z)}{\varepsilon} \geqslant |c|\mathrm{Im}(z), \tag{2.4}$$

and thus for the bound to hold, we need $c \in [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$. But Γ is discrete and closed, and thus this holds for only finitely many values of c, irrespective of z. Furthermore, for $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, it follows by (2.2) that $(\sigma_\mathfrak{a}^{-1}\tilde{\gamma})_{2,1}$ is the same for all $\tilde{\gamma} \in \Gamma_\mathfrak{a}\gamma$. Also, for $m \in \mathbb{Z}$, we have that

$$\begin{pmatrix}1 & hm\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a+hmc & b+hmd\\ c & d\end{pmatrix},$$

and thus we can identify the set of distinct $\Gamma_\mathfrak{a}\gamma$ which satisfy $(\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\gamma)_{2,1} = c$ with a subset of $[0, hc]$. But this is compact, and thus again by discreteness and closedness of Γ, there are finitely many $\Gamma_\mathfrak{a}\gamma$ which satisfy $(\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\gamma)_{2,1} = c$. Thus, the sum for $E_\mathfrak{a}(z \mid \psi)$ has finitely many terms, each bounded by $\|\psi\|$ which is finite. Hence in particular, we have shown that $\mathcal{E}(\Gamma\backslash\mathbb{H}) \subseteq \mathcal{B}(\Gamma\backslash\mathbb{H})$.

In fact, we have a much stronger result whose proof follows by studying the geometry of the fundamental domain (see [9] Lemma 2.10). The above note can be thought of as a motivation for the following result.

Lemma 2.18. *Let Γ be a lattice, and let $\mathfrak{a}$ be a cusp for Γ. For any $z \in \mathbb{H}$, and $Y > 0$, we have*

$$\#\left\{\gamma \in \Gamma_\mathfrak{a}\backslash\Gamma : Im(\sigma_\mathfrak{a}^{-1}\gamma z) > Y\right\} < 1 + \frac{10}{c_\mathfrak{a} Y}.$$

We wish to study the orthogonal complement of $\mathcal{E}(\Gamma\backslash\mathbb{H})$ in $\mathcal{B}(\Gamma\backslash\mathbb{H})$, as the cusp forms will arise from this subspace. We define the subspace $\mathcal{C}(\Gamma\backslash\mathbb{H})$ and show that this is indeed the orthogonal complement.

Definition 2.19 ($\mathcal{C}(\Gamma\backslash\mathbb{H})$)**.** Let Γ be a lattice. Define $\mathcal{C}(\Gamma\backslash\mathbb{H})$ to be the subspace of $\mathcal{B}(\Gamma\backslash\mathbb{H})$ consisting of all f such that $f_{\mathfrak{a}} \equiv 0$, for any cusp $\mathfrak{a}$.

Proposition 2.20. *Let Γ be a lattice. Then, $\mathcal{C}(\Gamma\backslash\mathbb{H})$ is the orthogonal complement of $\mathcal{E}(\Gamma\backslash\mathbb{H})$ in $\mathcal{B}(\Gamma\backslash\mathbb{H})$. Hence*

$$\mathcal{L}(\Gamma\backslash\mathbb{H}) = \overline{\mathcal{E}(\Gamma\backslash\mathbb{H})} \oplus \overline{\mathcal{C}(\Gamma\backslash\mathbb{H})},$$

where the closures are taken in $\mathcal{L}(\Gamma\backslash\mathbb{H})$.

Proof. Let $f \in \mathcal{B}(\Gamma\backslash\mathbb{H})$ and $E_{\mathfrak{a}}(z \mid \psi) \in \mathcal{E}(\Gamma\backslash\mathbb{H})$. We compute the inner product to get

$$\begin{aligned}\langle f, E_{\mathfrak{a}}(z \mid \psi)\rangle &= \int_{\Gamma\backslash\mathbb{H}} f(z) \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))}\, d\mu z \\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma F} f(\sigma_{\mathfrak{a}} z)\overline{\psi(y)}\, d\mu z\end{aligned}$$

where in the last equality we made the substitution $z \mapsto \gamma^{-1}\sigma_{\mathfrak{a}} z$. Now let $\gamma \in \Gamma, z \in F$. Then using the translations from (2.2), it follows that for some unique $\gamma' \in \Gamma_{\mathfrak{a}}\gamma$, we have that $\sigma_{\mathfrak{a}}^{-1}\gamma' z \in P$, where $P = \{\tau \in \mathbb{H} : 0 < \mathrm{Re}(\tau) < h\}$. Clearly, by the definition of F, this mapping is onto. Also, if $\gamma, \gamma' \in \Gamma$, and $z, z' \in F$, then $\sigma_{\mathfrak{a}}^{-1}\gamma z = \sigma_{\mathfrak{a}}^{-1}\gamma' z'$ if and only if $z = \gamma^{-1}\gamma' z'$, and by the definition of F, this holds iff $z = z'$. Thus this covering is one-to-one and onto and we have

$$\langle f, E_{\mathfrak{a}}(z \mid \psi)\rangle = \int_0^\infty \left(\int_0^h f(\sigma_{\mathfrak{a}} z)\, dx\right) \overline{\psi(y)} y^{-2}\, dy \; = \int_0^\infty f_{\mathfrak{a}}(y)\overline{\psi(y)} y^{-2}\, dy\,.$$

Finally, the result follows from the fact that $f \in \mathcal{E}(\Gamma\backslash\mathbb{H})^{\perp}$ iff $\langle f, E_{\mathfrak{a}}(z \mid \psi)\rangle = 0$, for all $\psi \in C_0^\infty(\mathbb{R}_{>0})$. ☐

Definition 2.21 (Cusp Form)**.** Let Γ be a lattice. The automorphic forms in $\mathcal{C}(\Gamma\backslash\mathbb{H})$ are called **cusp forms** with respect to Γ. We denote by $\mathcal{C}_s(\Gamma\backslash\mathbb{H})$, the space of automorphic forms with respect to Γ with eigenvalue $\lambda = s(1-s)$.

3. Discrete Spectrum

In this section, we study the crucial ideas in order to show that $\mathcal{C}(\Gamma\backslash\mathbb{H})$ is spanned by cusp forms. This is done by using the theory of symmetric operators and Hilbert-Schmidt operators. We will often use fundamental results from Functional Analysis and Spectral Theory, where we will refer the reader to the relevant places in Appendix A.

3.1. **Compactification of Automorphic Kernels.**

We begin by introducing the notion of automorphic kernels.

Definition 3.1 (Point-Pair Invariant)**.** A smooth function $k : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is a **point-pair invariant** if $k(\sigma z, \sigma w) = k(z, w)$, for all $\sigma \in \mathrm{PSL}(2, \mathbb{R})$.

Note 3.2. Note that a point-pair invariant is only a function of $\rho(z,w)$ (the hyperbolic distance between z and w). Hence we can make use of (1.1) to write such a function as $k(z,w) = k(u(z,w))$, for some function $k(u)$. For now we will always assume that $k(u) \in C_0^\infty(\mathbb{R}^+)$.

Definition 3.3 (Invariant integral operator)**.** An integral operator on $L^2(\mathbb{H}, \mathcal{B}_{\mathbb{H}}, \mu)$, with a point-pair invariant kernel $k(z,w)$, is called an **invariant integral operator**.

Definition 3.4 (Automorphic Kernel)**.** Let Γ be a lattice, and let $k(z,w)$ be a point-pair invariant. Its **automorphic kernel** is the function

$$K(z,w) = \sum_{\gamma \in \Gamma} k(z, \gamma w), \tag{3.1}$$

and has associated to it the integral operator $L_K : \mathcal{A}(\Gamma\backslash\mathbb{H}) \to \mathcal{A}(\Gamma\backslash\mathbb{H})$ defined by

$$(L_K f)(z) = \int_{\Gamma\backslash\mathbb{H}} K(z,w) f(w)\, d\mu(w)\,.$$

Note 3.5. The above definition is well defined. Indeed, for fixed $z \in \mathbb{H}$, the number of nonzero terms in (3.1) is at most the number of distinct fundamental domains that intersect the support of $k(z,\cdot)$, which is compact.

We have a first interesting result which further motivates the use of these operators in order to study $\mathcal{C}(\Gamma\backslash\mathbb{H})$.

Proposition 3.6. *Let L_K be an integral operator with automorphic kernel K. Then L_K maps the subspace $\mathcal{B}(\Gamma\backslash\mathbb{H})$ into itself, and moreover, L_K maps the subspace $\mathcal{C}(\Gamma\backslash\mathbb{H})$ into itself.*

Proof. The fact that L_K maps $\mathcal{B}(\Gamma\backslash\mathbb{H})$ to itself follows directly from our condition on $k(u)$, as well as the definition of $\mathcal{B}(\Gamma\backslash\mathbb{H})$. To see why the subspace $\mathcal{C}(\Gamma\backslash\mathbb{H})$ gets to itself, we simply calculate the constant term in the Fourier expansion of $(L_K f)(z)$ at a cusp $\mathfrak{a}$. By using that f is automorphic, we have that

$$\begin{aligned}
(L_K f)_{\mathfrak{a}}(y) &= \int_0^h (L_K f)(\sigma_{\mathfrak{a}} z)\, dx \\
&= \int_0^h \left(\int_{\Gamma\backslash\mathbb{H}} K(\sigma_{\mathfrak{a}} z, w) f(w)\, d\mu(w) \right) dx \\
&= \int_0^h \left(\int_{\mathbb{H}} k(\sigma_{\mathfrak{a}} z, w) f(w)\, d\mu(w) \right) dx \\
&= \int_{\mathbb{H}} k(z,w) \left(\int_0^h f(\sigma_{\mathfrak{a}} w)\, dx \right) d\mu(w)\,,
\end{aligned}$$

where in the last equality, after changing the order of integration, we made the substitution $w \mapsto \sigma_{\mathfrak{a}} w$, and used the point-pair invariance of k. The result follows from noting that the bracketed term in the final expression is $f_{\mathfrak{a}}(y)$. $\square$

We now move towards defining our Hilbert-Schmidt operators. No matter how small the support of $k(u)$ is made, we cannot have the guarantee that $K(z,w)$ is bounded on $F \times F$, for a fundamental domain F, as the number non-zero of terms in the series for $K(z,w)$ grows to infinity, as z and w approach cusps. In order to define a new operator from L_K, we need the notion of a principal part.

Definition 3.7 (Principal Part)**.** Let k be a point-pair invariant for a lattice Γ, and let $\mathfrak{a}$ be a cusp. Define the **principal part** of k at $\mathfrak{a}$ by the series

$$H_{\mathfrak{a}}(z,w) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\mathbb{R}} k(z, t+\sigma_{\mathfrak{a}}^{-1}\gamma w)\, dt\,. \tag{3.2}$$

We define the **principal part** of the automorphic kernel $K(z,w)$ to be

$$H(z,w) = \sum_{[\mathfrak{a}]} H_{\mathfrak{a}}(z,w),$$

where the sum is taken over all the in-equivalent cusps of Γ, which is well defined by the following

Note 3.8. Note that for fixed $z \in \mathbb{H}$, the series $H_{\mathfrak{a}}(z,\cdot)$ is an automorphic function for Γ. Furthermore, if $\mathfrak{a}$ is a cusp of Γ, then $\Gamma_{\tilde{\gamma}\mathfrak{a}} = \tilde{\gamma}\Gamma_{\mathfrak{a}}\tilde{\gamma}^{-1}$, and we can take $\sigma_{\tilde{\gamma}\mathfrak{a}} = \gamma\sigma_{\mathfrak{a}}$. Hence we have

$$\begin{aligned} H_{\tilde{\gamma}\mathfrak{a}}(z,w) &= \sum_{\gamma\in\Gamma_{\tilde{\gamma}\mathfrak{a}}\backslash\Gamma} \int_{\mathbb{R}} k(z, t+\sigma_{\tilde{\gamma}\mathfrak{a}}^{-1}\gamma w)\, dt \; = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\mathbb{R}} k(z, t+\sigma_{\mathfrak{a}}^{-1}\tilde{\gamma}^{-1}\tilde{\gamma}\gamma w)\, dt \\ &= H_{\mathfrak{a}}(z,w), \end{aligned}$$

where here we have used that $\Gamma_{\tilde{\gamma}\mathfrak{a}}\gamma = \Gamma_{\tilde{\gamma}\mathfrak{a}}\gamma'$ if and only if $\Gamma_{\mathfrak{a}}\tilde{\gamma}^{-1}\gamma = \Gamma_{\mathfrak{a}}\tilde{\gamma}^{-1}\gamma'$. Hence $H_{[\mathfrak{a}]}(z,w)$, is well defined, where $[\mathfrak{a}]$ is the equivalence class of $\mathfrak{a}$ under the action of Γ.

Proposition 3.9. *Let Γ be a lattice, let k be a point-pair invariant for Γ, and let $\mathfrak{a}$ be a cusp. For fixed $z \in \mathbb{H}$, the principal part $H_{\mathfrak{a}}(z,\cdot)$ is in $\mathcal{B}(\Gamma\backslash\mathbb{H})$. Moreover, it is orthogonal to the space $\mathcal{C}(\Gamma\backslash\mathbb{H})$, and is thus in the subspace $\overline{\mathcal{E}(\Gamma\backslash\mathbb{H})}$, where here the closure is taken in the space $\mathcal{B}(\Gamma\backslash\mathbb{H})$.*

A proof of this result can be found in [9] p. 58. Instead of reproducing the proof, we discuss the important elements of the proof. Fix $z \in \mathbb{H}$. We first note that smoothness of $H_{\mathfrak{a}}(z,\cdot)$ follows directly from our restriction on k. For boundedness, fix $w \in \mathbb{H}$, and note that since k has compact support in $\mathbb{R}^+$, it follows that for some $R = R(k) > 0$, a restriction on γ and t for the integrand in (3.2) to be non-zero is $|z - t - \sigma_{\mathfrak{a}}^{-1}\gamma w|^2 \leqslant R\cdot \mathrm{Im}(z)\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma w)$. Also, using Lemma 2.18 we get

$$\begin{aligned} |H_{\mathfrak{a}}(z,w)| &\leqslant |\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\mathbb{R}} k(z, t+\sigma_{\mathfrak{a}}^{-1}\gamma w)\, dt\,| \leqslant \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\mathbb{R}|} k(z, t+\sigma_{\mathfrak{a}}^{-1}\gamma w)|\, dt \\ &\leqslant \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} 2(R\cdot\mathrm{Im}(z))^{\frac{1}{2}}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma w)^{\frac{1}{2}}\,\|k\| \leqslant C_{z,k} \sum_{\gamma\in\mathfrak{a}(n)} (\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma w)^{\frac{1}{2}} \\ &\leqslant C_{z,k}\sum_{n}\frac{1}{n^2}\left(1+\frac{1}{(n+1)^4}\right) < +\infty, \end{aligned}$$

where $C_{z,k}$ is some relabelled constant, and

$$\mathfrak{a}(n) = \left\{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma \mid \frac{1}{(n+1)^2} \leq (\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma w))^{\frac{1}{2}} \leqslant \frac{1}{n^2}\right\}.$$

The final part of the proposition follows by computing $\langle H_{\mathfrak{a}}(z,\cdot), f\rangle$, where by suitably changing the order of integration, the term $f_{\mathfrak{a}}$ appears in the integrand. We can now define a new integral operator which will turn out to be Hilbert-Schmidt.

Definition 3.10 (Compact Part)**.** Let Γ be a lattice, and let $K(z,w)$ be an automorphic kernel. Define the **compact part** of $K(z,w)$ to be

$$\hat{K}(z,w) = K(z,w) - H(z,w).$$

The compact part of an automorphic kernel naturally defines an integral operator $L_{\hat{K}}$ on $\mathcal{A}(\Gamma\backslash\mathbb{H})$. As we shall see, this operator is Hilbert-Schmidt on the Hilbert space $\mathcal{L}(\Gamma\backslash\mathbb{H})$. Moreover, from Proposition 3.9, we have the following result.

Proposition 3.11. *Let K be an automorphic kernel with respect to a lattice Γ. Then $L_{\hat{K}} = L_K$ on the subspace $\mathcal{C}(\Gamma\backslash\mathbb{H})$.*

We now arrive to one of the crucial points of this section.

Proposition 3.12. *Let Γ be a lattice with a fundamental domain F. Then $\hat{K}(z,w)$ is bounded on the set $F \times F$. Hence, since Γ has finite co-volume, it follows that $L_{\hat{K}}$ is a Hilbert-Schmidt operator on $\mathcal{L}(\Gamma\backslash\mathbb{H})$.*

The most important takeaway from the result is that $L_{\hat{K}}$ is Hilbert-Schmidt. For a full proof of this result, see [9] p. 67, where a slightly more general result is proved. In the proof of [9], there is the use of the following version of the Euler-Maclaurin formula: for $f \in C_0^\infty(\mathbb{R})$, we have that

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{\mathbb{R}} f(t)\,dt\, + \int_{\mathbb{R}} (t - \lfloor t \rfloor - 1/2) f'(t)\,dt\,,$$

which can be checked by integration by parts.

The main idea is that the non-parabolic motions carry a uniformly bounded weight in the sum for $K(z,w)$, and thus we must only examine the parabolic motions. Furthermore, using Poisson summation and integration by parts, one can use the compact support of all derivatives of $k(u)$ to show that for any N

$$\sum_{m\in\mathbb{Z}} k(z, w+m) = \int_{\mathbb{R}} k(z, w+t)\,dt\, + O((\mathrm{Im}(z)\mathrm{Im}(w))^{-N}).$$

Thus, when defining $\hat{K}(z,w)$, we have essentially taken away the unbounded part in the sum for $K(z,w)$, leaving only the bounded part.

We make a final important remark concerning automorphic kernels.

Note 3.13. Throughout this section, we have used point-pair invariants $k(z,w)$ under the assumption that $k(u)$ be compactly supported. This is quite a strong condition, and it turns out that we can impose weaker growth conditions on $k(u)$. With care, one can show that all preceding results of this section hold true when $k(u) \in C^\infty(\mathbb{R}^+)$ subject to the growth condition

$$|k(u)|, |k'(u)| \ll \frac{1}{(u+1)^2}. \tag{3.3}$$

We assumed compactness in order to give a more intuitive account of the construction of the compact part of an automorphic kernel, but from now on, we assume that our point-pair invariants are subject to the growth condition in (3.3).

3.2. **The Automorphic Laplacian.** In order to get a spectral decomposition of the space $\mathcal{C}(\Gamma\backslash\mathbb{H})$, we need a self-adjoint operator. This will come from the Laplacian. More precisely, we will first define the Laplacian on a dense subspace of $\mathcal{L}(\Gamma\backslash\mathbb{H})$, and then use the theory of unbounded operators to extend it to a self-adjoint operator on $\mathcal{L}(\Gamma\backslash\mathbb{H})$. The Laplacian acts on all smooth automorphic functions, but we do not have much control of its range as of yet. In this pursuit, we first define Δ on a dense subset of $\mathcal{L}(\Gamma\backslash\mathbb{H})$, so that the range of Δ restricted to this set is contained in the dense subspace $\mathcal{B}(\Gamma\backslash\mathbb{H})$.

Definition 3.14 ($\mathcal{D}(\Gamma\backslash\mathbb{H})$)**.** Let Γ be a lattice. Define the subspace $\mathcal{D}(\Gamma\backslash\mathbb{H})$ of $\mathcal{L}(\Gamma\backslash\mathbb{H})$ by

$$\mathcal{D}(\Gamma\backslash\mathbb{H}) = \{f \in \mathcal{B}(\Gamma\backslash\mathbb{H}) \mid \Delta^n f \in \mathcal{B}(\Gamma\backslash\mathbb{H}),\ \text{for all } n\}\,.$$

Note 3.15. It follows its definition that $\mathcal{D}(\Gamma\backslash\mathbb{H})$ contains all compactly supported functions in $\mathcal{B}(\Gamma\backslash\mathbb{H})$ and thus $\mathcal{D}(\Gamma\backslash\mathbb{H})$ is a dense subset of $\mathcal{L}(\Gamma\backslash\mathbb{H})$. Furthermore, all cusp forms are in $\mathcal{D}(\Gamma\backslash\mathbb{H})$.

As it turns out, the space $\mathcal{D}(\Gamma\backslash\mathbb{H})$ is exactly the space we need to get a self-adjoint extension of Δ.

Proposition 3.16. *The restriction of* $-\Delta$ *on* $\mathcal{D}(\Gamma\backslash\mathbb{H})$ *is symmetric and non-negative.*

One can deduce the symmetry of $-\Delta$ from Green's First identity on $\mathbb{R}^2$, which still holds true on a fundamental domain $F \subseteq \mathbb{H}$ due to the fact that functions in $f \in \mathcal{D}(\Gamma\backslash\mathbb{H})$ have that both f and Δf are smooth and bounded, and that Γ has finite co-volume. The expression that we obtain is

$$\int_F \Delta f \overline{g}\, d\mu(z) = -\int_F \nabla f \cdot \overline{\nabla g}\, dx\ dy + \int_{\partial F} \frac{\partial f}{\partial \mathbf{n}} \overline{g}\, d\mathbf{s}\,, \tag{3.4}$$

where $\frac{\partial}{\partial \mathbf{n}} = y\frac{\partial}{\partial}$ and $d\mathbf{s} = \frac{ds}{y}$. The reason for writing the expression in such a way is that $d\mathbf{s}$ is a $\mathrm{PSL}(2,\mathbb{R})$-invariant differential form: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R})$ we have using (2.3)

$$\gamma^*(\,d\mathbf{s}\,) = \gamma^*\left(\frac{1}{y}\,ds\right) = \frac{|cz+d|^2}{y}\left(\frac{1}{|cz+d|^2}\right) ds \,=\, d\mathbf{s}\,.$$

and thus, (3.4) is independent of the chosen fundamental domain. Finally symmetry of Δ follows from the fact that the integral along the boundary will be cancelled out by the opposite sides of ∂F. Non-negativity of $-\Delta$ is immediate from (3.4). Note that for an eigenfunction $f \in \mathcal{D}(\Gamma\backslash\mathbb{H})$ with eigenvalue $\lambda = s(1-s)$, since $-\Delta$ is symmetric it follows that λ must be real and non-negative. If s is real, it follows that $s \in [0,1]$. If $s \notin \mathbb{R}$, then it follows that $1 - s = \overline{s}$, and thus s must lie on the line: $\mathrm{Re}(s) = \frac{1}{2}$.

We have now shown that $-\Delta$ is symmetric and non-negative on $\mathcal{D}(\Gamma\backslash\mathbb{H})$, and we can thus apply Friedrich's Theorem (see Theorem A.17) to obtain a self-adjoint extension of Δ. We call this extension of Δ, the **automorphic Laplacian**, which is densely defined, and which we will still denote by Δ.

Note 3.17. The automorphic Laplacian is a self-adjoint operator on $\mathcal{L}(\Gamma\backslash\mathbb{H})$, and it thus has a spectral decomposition on that space. In the remainder of the section, we consider the subspace $\mathcal{C}(\Gamma\backslash\mathbb{H})$, and we show that the automorphic Laplacian has pure-point spectrum in this case. Throughout the discussion, it will be worth keeping in mind that since Δ (our original Laplacian) is defined on a dense subset of the closed subspace $\mathcal{C}(\Gamma\backslash\mathbb{H})$, any result that holds true for Δ on this dense subset, holds for its self-adjoint extension.

3.3. **Discrete spectrum of Δ.** The spectral resolution of the extension Δ restricted to $\mathcal{C}(\Gamma\backslash\mathbb{H})$ will come from the following sequence of results:

Theorem 3.18 (Hilbert-Schmidt Theorem for integral operators.)**.** *Let $(X, \mathcal{A}, \mu)$ be a seperable σ-finite measure space, and let A_k be a self adjoint Hilbert-Schmidt integral operator. Then A_k has pure point spectrum in $L^2(X, \mathcal{A}, \mu)$. The eigenspaces of A_k are finite dimensional. The range of A_k is spanned by eigenfunctions of A_k, and any maximal system $\{f_n\}$ of eigenfunctions of A_k is an orthonormal basis for $\overline{Im(A_k)}$.*

For the following two results, see [9], p. 26-30. Note that Proposition 3.20 follows from the fact that the results in this section of [9] still hold true subject to the conditions in (3.3).

Proposition 3.19. *Let $k(z, w)$ be a smooth point-pair invariant on $\mathbb{H} \times \mathbb{H}$. We have*

$$\Delta_z k(z, w) = \Delta_w k(z, w).$$

Hence, the invariant integral operators commute with the Laplace operator.

Proof. First center geodesic polar coordinates at w (i.e. send w to i), and consider the expression for $\Delta_z k(z, w)$ using the expression found in (1.3). Repeating the same procedure with z and w switched, and using that $k(u)$ only depends on u gives the result. □

Proposition 3.20. *Any eigenfunction of Δ is also an eigenfunction of all invariant integral operators.*

Theorem 3.21. *Let A, B be a pair of commuting symmetric operators in a Hilbert pace H, and suppose that the with eigenspaces of A are finite dimensional. Then there exists a maximal orthonormal system of eigenvectors of A which are also eigenvectors of B.*

Note 3.22. Theorem 3.21 follows from the Spectral Theorem for Hermitian matrices applied to the eigenspaces of A. Note that the Theorem does not assert the existence of a Hilbert basis. Furthermore, the reason why we only need to consider symmetric operators and not self adjoint operators is precisely because the proof restricts its attention to the eigenspaces of A, and the restrictions of A and B to these spaces are self-adjoint, which is all that we need.

Suppose now that A is compact and defined on the whole space H. Then by Theorem A.25, the image of A is spanned by its eigenfunctions, and there is an orthonormal basis of $\overline{\mathrm{Im}(A)}$ consisting entirely of eigenvectors of A. Hence (if necessary), restricting the domain of A to a domain for which it commutes with B, by Theorem 3.21, it follows that the image of the restriction of A will be spanned by eigenvectors of B, and that its closure will have an orthonormal basis consisting entirely of eigenvectors of A which are also eigenvectors for B. With this in mind, if we can find a compact self-adjoint operator $A : \mathcal{D}(\Gamma\backslash\mathbb{H}) \to \mathcal{D}(\Gamma\backslash\mathbb{H})$ which commutes with the automorphic Laplacian, and satisfies $\mathcal{C}(\Gamma\backslash\mathbb{H}) \subseteq \overline{\mathrm{Im}(A)}$, then we will be able to deduce that $\mathcal{C}(\Gamma\backslash\mathbb{H})$ is spanned by cusp forms. We will do this by making use of all of the theory described above.

Motivated by Proposition 3.19, we will consider a symmetric invariant integral operator L, from which we will obtain a self-adjoint Hilbert-Schmidt operator $\hat{L}$ which is equal to L on $\mathcal{C}(\Gamma\backslash\mathbb{H})$. We will then show that L maps $\mathcal{C}(\Gamma\backslash\mathbb{H})$ into a dense subspace of itself. From there, we get the main result of this section (see [9] Theorem 4.7).

Theorem 3.23 (Spectral Resolution of Δ in $\mathcal{C}(\Gamma\backslash\mathbb{H})$)**.** *Let Γ be a lattice. The space $\mathcal{C}(\Gamma\backslash\mathbb{H})$ is spanned by cusp forms. Hence, the automorphic Laplacian Δ has pure point-spectrum in $\mathcal{C}(\Gamma\backslash\mathbb{H})$. The eigenspaces have finite dimension. For any complete orthonormal system of cusp forms $\{u_j\}$, every $f \in \mathcal{C}(\Gamma\backslash\mathbb{H})$ has the expansion*

$$f(z) = \sum_j \langle f, u_j \rangle u_j(z),$$

converging in the norm topology. Moreover, if $f \in \mathcal{C}(\Gamma\backslash\mathbb{H}) \cap \mathcal{D}(\Gamma\backslash\mathbb{H})$, then the series converges absolutely on compact subsets of $\mathbb{H}$.

To finish this section, we will break down some of the key steps that allow us to find the suitable invariant operator from which we can deduce Theorem 3.23. Recall that the automorphic Laplacian is a self-adjoint operator, and hence its eigenvalues are all real. Thus, for $s \in \mathbb{C}\backslash\mathbb{R}$, it follows from Proposition A.33, that the resolvent, R_s is defined on $\mathcal{L}(\Gamma\backslash\mathbb{H})$. As it turns out, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we can express R_s as an invariant integral operator. In order to do so, we must first describe the kernel of R_s.

Definition 3.24 (The Green's function for Δ)**.** For $\mathrm{Re}(s) > 1$, define the **Green's function** to be the function $G_s(u)$ on $\mathbb{R}^+$ defined by the integral

$$G_s(u) = \frac{1}{4\pi} \int_0^1 (\xi(1-\xi))^{s-1} (\xi + u)^{-s} \, d\xi \,. \tag{3.5}$$

The integral in (3.5) clearly converges absolutely for $\mathrm{Re}(s) > 0$, but for our purposes, only the values of s in the region $\mathrm{Re}(s) > 1$, will be useful to consider. One can show that $G_s(u)$ is an eigenfunction of Δ corresponding to the eigenvalue $\lambda = s(1-s)$. Note that in this context, we are considering Δ acting on smooth automorphic functions in $\mathcal{A}(\Gamma\backslash\mathbb{H})$, and it is not symmetric in this space.

We now give a result which justifies the naming of $G_s(u)$. A proof of this result can be found in [9] Theorem 1.17.

Theorem 3.25. *Let $s \in \mathbb{C}$ with Re$(s) > 1$. If f is smooth and bounded on $\mathbb{H}$, then*

$$-(R_s f)(z) = \int_{\mathbb{H}} G_s(u(z,w)) f(w) \, d\mu(w) \,. \tag{3.6}$$

The main idea in proving the above result is to show that the integral operator on the rhs of 3.6 defined on smooth and bounded functions on $\mathbb{H}$ commutes with $(\Delta + s(1-s))$, and then to use an argument involving Green's Theorem. The way one deduces the commutativity of the two operators, is by making use of the $\mathrm{SL}(2,\mathbb{R})$-invariance of the integral operator on the rhs of 3.6. Since $\mathrm{SL}(2,\mathbb{R})$ is a matrix Lie group, for each $X \in \mathfrak{sl}(2,\mathbb{R})$, we use the map $t \mapsto \exp(tX)$ to define a linear operator $\mathcal{L}_X : C^\infty(G) \to C^\infty(G)$, which satisfies Leibniz rule. We then use the three basis elements X_1, X_2, X_3 for $\mathfrak{sl}(2,\mathbb{R})$ to write Δ as an expression in terms of the operators $\mathcal{L}_{X_1}, \mathcal{L}_{X_2}, \mathcal{L}_{X_1}$. This is when we use $\mathrm{SL}(2,\mathbb{R})$-invariance from 3.6, and from here the commutativity follows. For more details on Lie algebras, see [7], and for a detailed account of differential operators derived from Lie algebras, see [10].

In order to define the operator for the proof of Theorem 3.23, we must study the behaviour of $G_s(u)$. A first important property of the Green's function is the following easy result.

Proposition 3.26. *Let $s \in \mathbb{C}$ with Re$(s) = \sigma > 1$, then for any $x > 0$, we have*

$$\sup_{u \geq x} \left| \frac{G_s(u)}{(1+u)^\sigma} \right| < \infty.$$

Clearly, $G_s(u)$ has a singularity at 0. Motivated by Proposition A.34, one might wonder whether this singularity is cancelled out when considering $G_a(u) - G_s(u)$. This turns out to be the case.

Proposition 3.27. *Let $s \in \mathbb{C}$ with Re$(s) > 1$. Then, $G_s(u)$ satisfies the following bounds*

$$G_s(u) = \frac{1}{4\pi} \log \frac{1}{u} + O(1), \qquad u \to 0, \tag{3.7}$$

$$G_s^{(n)}(u) = \frac{C_n}{u^n} + O(1), \qquad u \to 0, \tag{3.8}$$

where $C_n = \frac{(-1)^n (n-1)!}{4\pi}$.

For a proof of (3.7), see [9] Lemma 1.7. The proof of (3.8) is done by an almost identical argument. We can now construct the operator we wished for in the discussion preceding the statement of Theorem 3.23.

Let $a, s \in \mathbb{C}$ with $\mathrm{Re}(s) > \mathrm{Re}(a) \geq 2$. Consider the invariant integral operator $L = R_a - R_s : \mathcal{L}(\Gamma\backslash\mathbb{H}) \to \mathcal{L}(\Gamma\backslash\mathbb{H})$, which has kernel $k(u) = G_s(u) - G_a(u)$. Note that in Theorem 3.25 , it was only proved that the restriction of the operator $-R_s$ to the subspace of smooth and bounded functions on $\mathbb{H}$, is an integral operator with kernel equal to $G_s(u)$. However, if L_K is the integral operator on $\mathcal{L}(\Gamma\backslash\mathbb{H})$ with automorphic kernel obtained from $k(u) = G_s(u) - G_a(u)$, then by Proposition 3.27, it follows that L_K is bounded. Hence, since the space $\mathcal{B}(\Gamma\backslash\mathbb{H})$ is dense in $\mathcal{L}(\Gamma\backslash\mathbb{H})$, it follows that L is defined on the entire space $\mathcal{L}(\Gamma\backslash\mathbb{H})$, and its restriction to $\mathcal{B}(\Gamma\backslash\mathbb{H})$ is an integral operator, with automorphic kernel obtained from $k(u)$. Furthermore, by the definition of $G_s(u)$, one can see that $k(u(z,w)) = \overline{k(u(w,z))}$, and thus $\hat{L}$ is a self-adjoint Hilbert-Schmidt operator. Note that L is also self-adjoint. We need one final result before giving our concluding argument.

Proposition 3.28. *With the above definitions, the operator $\hat{L}$ maps the subspace space $\mathcal{C}(\Gamma\backslash\mathbb{H})$ densely into itself.*

Proof. We prove this by showing that the image of L restricted to this space contains $\mathcal{D}(\Gamma\backslash\mathbb{H}) \cap \mathcal{C}(\Gamma\backslash\mathbb{H})$, and then appeal to Proposition 3.11, to get the result. Let $f \in \mathcal{D}(\Gamma\backslash\mathbb{H})$, and define

$$g = (a(1-a) - s(1-s))^{-1}(\Delta + s(1-s))(\Delta + a(1-a))f.$$

Then by Proposition A.34, since $\mathcal{C}(\Gamma\backslash\mathbb{H}) \subseteq \mathcal{B}(\Gamma\backslash\mathbb{H})$, it follows that $Lg = f$. Now suppose that $f \in \mathcal{C}(\Gamma\backslash\mathbb{H})$. Then by Proposition 2.20, $g = e + c$, for some unique

$e \in \overline{\mathcal{E}(\Gamma\backslash\mathbb{H})}, c \in \overline{\mathcal{C}(\Gamma\backslash\mathbb{H})}$. But g is in the subspace $\mathcal{D}(\Gamma\backslash\mathbb{H}) \subseteq \mathcal{D}(\Gamma\backslash\mathbb{H})$, and thus $c \in \mathcal{C}(\Gamma\backslash\mathbb{H})$. Now since both Lg and Lc are in $\mathcal{C}(\Gamma\backslash\mathbb{H})$, so is Le, and hence $L^2 e \in \mathcal{C}(\Gamma\backslash\mathbb{H})$. Since L is self-adjoint, we have

$$0 = \langle L^2 e, e\rangle = \langle Le, Le\rangle = \|Le\|^2 ,$$

and thus $Le = 0$. Finally, using the Proposition A.34 and the fact that R_a is injective gives the result. □

Note that $k(u)$ is a point-pair invariant, and by Propositions 3.26 and 3.27, by our choice of a and s, it follows that $k(u)$ respects the sufficient conditions in Note 3.13. We can thus apply Proposition 3.12 to L, to obtain a self-adjoint Hilbert-Schmidt operator $\hat{L}$.

Finally, recall that $\mathcal{C}(\Gamma\backslash\mathbb{H})$ is the orthogonal complement of $\mathcal{E}(\Gamma\backslash\mathbb{H})$ in $\mathcal{B}(\Gamma\backslash\mathbb{H})$, and it is thus a closed subspace of $\mathcal{B}(\Gamma\backslash\mathbb{H})$. Let us now restrict our attention to the Hilbert space $H = \mathcal{C}(\Gamma\backslash\mathbb{H})$. Since L is an invariant integral operator, by Proposition 3.19, it follows that Δ and L commute. By Proposition 3.11, $\hat{L} = L$ on H, and thus Δ and $\hat{L}$ commute on H. Applying Theorem 3.18 to $\hat{L} : H \to H$, using the fact that its image is dense in H, and combining this with Theorem 3.21, gives us Theorem 3.23.

Appendix A. Essential Background on Spectral Theory

A.1. **Spectral Decomposition.** The spectral decomposition of self adjoint operators will be in the background of much of the motivation for the work done on Eisenstein series. We give a brief summary of the important notions leading to its construction. For a more detailed account on this topic, see [12] Chapter 7, and for more details on Spectral Theory see [14]. We first recall some terminology concerning measures. For a topological space X, we will write $\mathcal{B}_X$ for the σ-algebra generated by the open sets in X.

Definition A.1 (Absolutely Continuous)**.** Let μ and ν be measures on a measurable space $(X, \mathcal{A})$. We say that μ is **absolutely continuous** with respect to ν, written $\mu \ll \nu$ if

$$\nu(A) = 0 \implies \mu(A) = 0, \forall A \in \mathcal{A}.$$

Definition A.2 (Singular)**.** Let μ and ν be measures on a measurable space $(X, \mathcal{A})$. We say that μ and ν are **singular**, written $\mu \perp \nu$ if there exist disjoint sets $A, B \in \mathcal{A}$ satisfying:

$$A \cap B = \emptyset, \quad A \cup B = X, \quad \mu(B) = \nu(A) = 0.$$

Definition A.3 (Pure Points)**.** Let X be a topological space, and let μ be a **Baire** measure on $(X, \mathcal{B}_X)$, i.e μ is finite on compact subsets of X. The **pure points** of μ is the set

$$P = \{x \in X : \mu(x) \neq 0\}.$$

A Borel measure μ on $\mathbb{R}$ is called **continuous** if it has no pure points, and is called a **pure point** measure if $\mu(X) = \sum_{x\in X} \mu(\{x\})$, for any Borel set X.

Note A.4. For a Baire measure on $(X, \mathcal{B}_X)$, P is countable, and defining the measures μ_{pp} on $(X, \mathcal{B}_X)$ by

$$\mu_{\text{pp}} = \sum_{x\in X\cap P} \mu(\{x\}), \quad \mu_{\text{cont}} = \mu - \mu_{\text{pp}},$$

we can write $\mu = \mu_{\text{cont}} + \mu_{\text{pp}}$, where μ_{cont} is continuous and μ_{pp} is pure point.

Theorem A.5 (Lebesgue Decomposition Theorem)**.** *For any two σ-finite signed measures μ and ν on a measurable space $(X, \mathcal{A})$, there exist two σ-finite signed measures ν_0 and ν_1 such that ν_0 is absolutely continuous with respect to μ, ν_1 and μ are singular, and $\nu = \nu_0 + \nu_1$. Moreover, ν_0 and ν_1 are uniquely determined by μ and ν.*

The above theorem admits the following refinement in the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, which follows by combining the Lebesgue Decomposition Theorem and Note A.4 (see Theorems 1.13 and 1.14 in [12] for further details).

Theorem A.6. *For any σ-finite measure on the measurable space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, we can write ν as a sum of three mutually singular measures*

$$\nu = \nu_{ac} + \nu_{sc} + \nu_{pp}$$

where $\nu_{ac} \ll \lambda^1$, ν_{sc} is continuous and singular with respect to λ^1, and ν_{pp} is pure point.

In order to discuss the spectral decomposition, we need to introduce spectral measures. The construction of these makes use of the following result, whose proof can be found in [6]. Here, by a **positive** linear functional ψ, we mean that $\psi(f) \geqslant 0$, whenever $f \geqslant 0$.

Theorem A.7 (Riesz-Markov-Kakutami Representation Theorem)**.** *Let X be a compact Hausdorff space. For any positive linear functional ψ on $C(X)$, there is a unique Baire measure μ on $(X, \mathcal{B}_X)$ such that $\forall f \in C(X)$*

$$\psi(f) = \int_X f(x)\, d\mu\, .$$

Now let H be a Hilbert space, and let $T \in \mathcal{B}(H)$ be self-adjoint, where $\mathcal{B}(H)$ is the C^*-algebra of linear bounded operators on H. Recall that for $A \in \mathcal{B}(H)$, if $\sigma(A)$ is the spectrum of A, then $\sigma(A)$ is a non-empty compact subset of $\mathbb{C}$. Moreover, since T is self-adjoint, $\sigma(T) \subset \mathbb{R}$. Furthermore, for any polynomial $P \in \mathbb{C}[t]$, since T is self-adjoint we have

$$\|P(T)\| = \sup_{\lambda \in \sigma(T)} |P(\lambda)|.$$

Now by Stone-Weierstrass, if $f \in C(\sigma(T))$, there exists a sequence of polynomials P_n which converge to f uniformly. We can thus define $f(T)$ as the limit of the sequence $(P_n(T))$ in $\mathcal{B}(H)$. Now fix $h \in H$, and note that the map $\psi_h : C(\sigma(T)) \to \mathbb{R}$ given by $C(\sigma(T)) \ni f \mapsto (h, f(T)h)$, is a positive linear functional. The image of ψ_h being $\mathbb{R}$ is due to the self-adjointess of $f(T)$[2] for any $f \in C(\sigma(T))$:

$$(h, f(T)h) = (f(T)h, h) = \overline{(h, f(T)h)},$$

and the positivity of ψ_h follows similarly after noting that for positive f we can write $f = (\sqrt{f})^2$. We can thus apply the Riesz-Markov-Kakutami Representation Theorem to obtain a measure μ_h on $\sigma(T)$ such that

$$\int_{\sigma(T)} f\, d\mu_h = (h, f(T)h).$$

[2]Theorem VII.1 in [12] gives more details as to why $f(T)$ is self-adjoint

The measure μ_h is called the **spectral measure** associated to h. By the refinement of Lebesgue's Decomposition Theorem, μ_h can be decomposed into three mutually singular parts, $\mu_{\mathrm{ac}}, \mu_{\mathrm{sc}}$ and μ_{pp}, which satisfy the analogous properties as in Theorem A.6. For each $\alpha \in \{$ac,sc,pp$\}$, define the subspace H_α of H by

$$H_\alpha = \{h \in H : \mu_h = \mu_\alpha\}. \tag{A.1}$$

These subspaces are invariant under T since for $h \in H$ and $A \in \mathcal{B}_{\sigma(T)}$, we have

$$\begin{aligned}\mu_{T(h)}(A) &= \int_{\sigma(T)} \mathbb{1}_A \, d\mu_{T(h)} = (T(h), \mathbb{1}_A(T)T(h)) = (h, T\mathbb{1}_A(T)T(h)) \\ &= \int_{\sigma(T)} x^2 \mathbb{1}_A \, d\mu_h(x),\end{aligned}$$

and thus $\mu_{T(h))}(A) = 0$ whenever $\mu_h(A) = 0$. We can now make use of the Spectral Theorem in its multiplication operator form to obtain a decomposition of H in terms of the subspaces defined in $(A.1)$.

Theorem A.8 (Spectral Theorem)**.** *Let H be a seperable Hilbert space, and let T be self adjoint operator in $\mathcal{B}(H)$. Then there exists a sequence of spectral measures $(\mu_{h_n})_{n=1}^N$, (where $N = 1, 2, ...$ or ∞) on $\sigma(A)$ and a unitary operator, $U : H \to \bigoplus_{n=1}^N L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n}))$, such that for every $f = (f_n)_{n=1}^N$ we have*

$$(UTU^{-1}f)_n(\lambda) = \lambda f_n(\lambda).$$

With the above setup, we can make use of Theorem A.6, noting that the three measures obtained are mutually singular to write for each n

$$L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n}) = L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n:\mathrm{ac}}) \oplus L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n:\mathrm{sc}}) \oplus L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n:\mathrm{pp}}).$$

Furthermore, by the Spectral Theorem, UTU^{-1} acts on $f(x)$ as multiplication by x and it follows that for $\alpha \in$\{ac,sc,pp\}, for $\psi \in L^2(\mathbb{R}, \mu_{h_n})$ and for any n

$$\psi \in L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n})_\alpha \iff \psi \in L^2(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mu_{h_n:\alpha})$$

This leads us to the spectral decomposition theorem.

Theorem A.9 (Spectral Decomposition)**.** *Let H be a seperable Hilbert space, and let T be self adjoint operator in $\mathcal{B}(H)$. Then H can be decomposed into invariant subspaces*

$$H = H_{ac} \oplus H_{sc} \oplus H_{pp}.$$

A.2. **Unbounded Operators.** We now give a brief account of the results concerning unbounded operators that are needed in order to construct the automorphic Laplacian. In the following definitions, by an **operator** A on a Hilbert space H, we mean a map $A : D(A) \to H$, where $D(A) \subseteq H$ is closed under addition and scalar multiplication, and A is linear on $D(A)$. A subset of H that satisfies the conditions on $D(A)$ is sometimes called a linear manifold. We will say that an operator A is **non-negative** if $\langle Af, f\rangle \geqslant 0$ for all $f \in D(A)$. Note that the eigenvalues of these operators are always non-negative.

Definition A.10 (Symmetric Operator)**.** Let H be a Hilbert space, and let A be a linear operator, with a dense domain $D(A) \subseteq H$. Then A is **symmetric** if $\forall x, y \in D(A)$

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

We have the following easy result.

Proposition A.11. *Let A be a symmetric operator on a Hilbert space H*

(1) The eigenvalues of A are real.
(2) Eigenvectors of A in $D(A)$ corresponding to distinct eigenvalues are orthogonal.

Definition A.12 (Closed Operator)**.** Let A be an operator on a Hilbert space H. We say that A is **closed** if its graph is closed in $H \oplus H$.

Definition A.13 (Adjoint)**.** Let A be an operator on a Hilbert space H, with a dense domain $D(A) \subseteq H$. Define the set

$$D(A^*) = \{k \in H : h \mapsto \langle Ah, k\rangle \text{is a bounded linear functional on } D(A)\}.$$

If $k \in D(A^*)$, then by the Riesz-Fréchet Theorem (see [2] Theorem 5.2), there exists a unique $f_k \in H$ such that $\langle A\cdot, k\rangle = \langle \cdot, f\rangle$ on $D(A)$. Denote this unique f_k by $f_k = A^*k$. Then the **adjoint** of A, is the map $A^* : D(A^*) \to H$. We say that A is **self-adjoint** if $A = A^*$.

We have the following interesting result, which in particular says that self-adjoint operators are closed (see [3] p. 305).

Proposition A.14. *Let A be a densely defined operator on a Hilbert space H. Then,*

(a) A^* is a closed operator.
(b) A^* is densely defined iff A is closable.
(c) If A is closable, then its closure is $(A^*)^*$.

Note that self-adjoint operators are always symmetric, but the converse is not always true.

Definition A.15 (Extension)**.** If A, B are operators on a Hilbert space H, then A is an **extension** of B, written $B \subseteq A$ if $\mathrm{gr}(B) \subseteq \mathrm{gr(A)}$, where $\mathrm{gr}(A) \subseteq H \oplus H$ is the graph of A.

We have the following equivalent formulations for symmetric operators.

Proposition A.16. *If A is a densely defined operator on a Hilbert space H, TFAE:*

(a) *A is symmetric*
(b) *$\langle Af, f\rangle$ is real for all $f \in D(A)$*
(c) *$A \subseteq A^*$.*

For a deeper discussion on the unbounded operators, see Chapter 10 of [3]. One of the most important results from the theory of bounded operators is that the non-negative symmetric operators have a self-adjoint extension.

Theorem A.17 (Friedrich's Theorem)**.** *Let A be a non-negative densely defined operator on a Hilbert space. Then A has a positive self-adjoint extension.*

For a proof of this result, see [4], (p. 329-334). Note that in their proof, it is assumed that $\langle Af, f\rangle \geqslant \langle f, f\rangle$, for every $f \in D(A)$, and an inner product is defined on $D(A)$ by

$$(f, g) = \langle Af, g\rangle. \tag{A.2}$$

In order to prove Theorem A.17 (where instead we assume that $(Af, f) \geqslant 0$ on $D(A)$), one can mimic the argument in [4], by substituting the inner product on $D(A)$ given by A.2, with the inner product

$$(f, g) = \langle f, g \rangle + \langle Af, g \rangle,$$

defined on $D(A)$.

Note A.18. It is worth noting that no proper extension of a self-adjoint operator is symmetric. This follows from the fact that $A \subseteq B \implies B^* \subseteq A$. Thus, if A is self-adjoint and B is a symmetric extension of A, we have

$$A \subseteq B \subseteq B^* \subseteq A^* = A.$$

A.3. **Compact Operators.** In the spectral resolution of Δ in $\mathcal{C}(\Gamma \backslash \mathbb{H})$, we use the Hilbert-Schmidt integral operators. The most important property is that they are compact operators, and we thus have a good understanding of their spectra. We briefly outline some of the most important results of this theory. See [2] Chapter 7 for an introductory development to compact operators, and see [14] Section 2.8 for a development of Hilbert-Schmidt integral operators. We begin by recalling terminology concerning the spectrum.

Definition A.19 (Point Spectrum, Continuous Spectrum, Residual Spectrum, Resolvent)**.** Let $T \in \mathcal{B}(H)$, we define

(1) The **point spectrum**, denoted by $\sigma_{\mathrm{p}}(T)$, is the set of eigenvalues of T.
(2) The **continuous spectrum**, denoted by $\sigma_{\mathrm{c}}(T)$, is the set complex numbers λ, which are not eigenvalues of T, but for which the range of $T - \lambda$ is a proper dense subset of H.
(3) The **residual spectrum**, denotes by $\sigma_{\mathrm{r}}(T)$, consists of all remaining elements in $\sigma(T)$.

Finally, the **resolvent set** of T is the set $\rho(T) = \mathbb{C} \setminus \sigma(T)$.

We now define compact operators in the context of Banach spaces, although for our purposes we will only study those acting on Hilbert spaces.

Definition A.20 (Compact Operator)**.** Let X and Y be normed spaces. A linear transformation $T : X \to Y$ is compact if for any bounded sequence $(x_n) \in X$, the sequence (Tx_n) in Y has a convergent subsequence. We denote the space of these by $\mathcal{K}(X, Y)$. Equivalently, one can define T to be compact if for every bounded set $A \subset X$, the set $\overline{T(A)} \subset Y$ is compact.

It is immediate that all compact operators are bounded. More interestingly, when Y is Banach, one can prove using a Cantor diagonalization argument that the space $\mathcal{K}(X, Y)$ is closed in $\mathcal{B}(X, Y)$. There are many other interesting results concerning these operators on Banach spaces, and one of particular interest is the following.

Theorem A.21. *Let X be a normed space, H be a Hilbert space, and let $T \in \mathcal{K}(X, Y)$. Then there is a sequence of finite rank operators which converge to T in $\mathcal{B}(X, H)$.*

Note that by our previous remark, the converse of the above Theorem is true, even when H is just a normed space. We now restrict our attention to Hilbert spaces. One can show that $T \in \mathcal{K}(H)$ iff $T^* \in \mathcal{K}(H)$, and thus all results that hold for T, will also hold for T^*. Our first result concerns the position of 0 in the spectrum.

Proposition A.22. *If H is an infinite-dimensional Hilbert space, and $T \in \mathcal{K}(H)$, then $0 \in \sigma(T)$. Furthermore, if H is not seperable, then $0 \in \sigma_p(T)$.*

If H is not seperable, there are examples where $0 \in \sigma(T) \setminus \sigma_{\mathrm{p}}(T)$. The first statement follows from the fact that T is not invertible when H is infinite dimensional. The second statement is a consequence of the fact that $\overline{T(H)}$ is seperable, and thus $T(H)^\perp \neq \emptyset$. Our next result shows that the eigenspace of a nonzero eigenvalue of a compact operator $T \in \mathcal{B}(H)$ is finite dimensional.

Proposition A.23. *If H is an infinite-dimensional Hilbert space, $T \in \mathcal{K}(H)$, and $\lambda \neq 0$, then $Ker(T - \lambda)$ is finite dimensional.*

This result follows from the fact that the kernel of a bounded operator is closed, and thus if it is infinite-dimensional, it must contain an orthonormal sequence (see [2] Theorem 3.40). Since orthogonal sequences do not contain convergent subsequences, we get a contradiction of the compactness of T. We now give our main result which describes the spectrum of a compact operator on an infinite-dimensional Hilbert space.

Theorem A.24. *Let H be an infinite dimensional Hilbert space and let $T \in \mathcal{K}(H)$. Then $0 \in \sigma(T)$, and $\sigma(T)$ is either finite or has the form $\{0, \lambda_1, \lambda_2, ...\}$, where (λ_n) is a sequence of distinct complex numbers converging to 0. For each non-zero $\lambda \in \sigma(T)$, we have that $\lambda \in \sigma_p(T)$, and the eigenspace corresponding to λ is finite-dimensional.*

Of particular interest are the self-adjoint compact operators for which a lot more can be said. The following result summarizes properties of these which will be useful for our purposes.

Theorem A.25. *Let A be a self-adjoint compact operator on a Hilbert space H. Then A has pure point spectrum in H, i.e. $\sigma(A) = \sigma_p(A)$. The set of non-zero eigenvalues of A is non-empty and is either finite or consists of a sequence which tends to zero. Each non-zero eigenvalue is real and has finite multiplicity. Eigenvectors corresponding to different eigenvalues are orthogonal. The range of A is spanned by eigenvectors of A. Any maximal orthonormal system $\{e_n\}$ of eigenvectors of A is an orthonormal basis for the Hilbert space $\overline{Im(A)}$, and the operator A has the representation*

$$Ax = \sum_{n \geq 1} \lambda_n \langle x, e_n \rangle e_n,$$

where $\{\lambda_n\}$ is the set of eigenvalues corresponding to $\{e_n\}$.

We now turn our attention to Hilbert-Schmidt operators which are defined on a seperable Hilbert spaces.

Definition A.26 (Hilbert-Schmidt Operator)**.** Let H be a seperable Hilbert space, let $\{e_n\}$ be an orthonormal basis for H. A **Hilbert-Schmidt operator** is an operator $A : H \to A$ which satisfies

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty. \tag{A.3}$$

For such an operator we denote the sum in (A.3) by $\|A\|_{\mathrm{HS}}^2$.

Note A.27. The above definition does not depend on the choice of orthonormal basis. Indeed by Parseval's Theorem, for any operator A, and orthonormal bases $\{e_i\}, \{\tilde{e}_j\}$, we have

$$\sum_i \|Ae_i\|^2 = \sum_{i,j} |\langle Ae_i, \tilde{e}_j\rangle|^2 = \sum_{i,j} |\langle e_i, A^*\tilde{e}_j\rangle|^2 = \sum_j \|A^*\tilde{e}_i\|^2 = \sum_{j,k} |\langle A^*\tilde{e}_j, \tilde{e}_k\rangle|^2$$
$$= \sum_{j,k} |\langle \tilde{e}_j, A\tilde{e}_k\rangle|^2 = \sum_k \|A\tilde{e}_k\|^2 .$$

As it turns out, Hilbert-Schmidt operators are not only bounded, but they are compact.

Proposition A.28. *Every Hilbert-Schmidt operator A is compact, and satisfies $\|A\| \leqslant \|A\|_{HS}$.*

The estimate follows from considering $\|Ae\|$ for unit vectors e, and using the fact that one can always complete $\{e\}$ to an get an orthonormal basis for H. For compactness, one can fix an orthonormal basis $\{e_n\}$, and use (A.3) to show that A is the limit of the sequence of finite rank operators $F_n = AP_n$, where P_n is the projection on to the subspace span $\{e_1, ..., e_n\}$.

There is a class of Hilbert-Schmidt operators of particular interest.

Definition A.29 (Hilbert-Schmidt integral operator)**.** Let $(X, \mathcal{A}, \mu)$ be a seperable σ-finite measure space. An operator $A_k : L^2(X, \mathcal{A}, \mu) \to L^2(X, \mathcal{A}, \mu)$ of the form

$$(A_k f)(x) = \int_X k(x, y) f(y)\, d\mu(y)$$

where $k \in L^2(X \times X, \mathcal{A} \otimes \mathcal{A}, \mu \otimes \mu)$, is called a **Hilbert-Schmidt integral operator.**

It turns out that these operators are well-defined, and they are Hilbert-Schmidt operators on the seperable Hilbert-Space $L^2(X, \mathcal{A}, \mu)$. It turns out that these are all of the Hilbert-Schmidt operators on $L^2(X, \mathcal{A}, \mu)$ (see [14] Proposition 2.8.6). One can also show by direct computation that $A^*_{k(x,y)} = A_{\overline{k(y,x)}}$. In particular, Theorem A.25 applies for the operators A_k whose kernels $k(x, y)$ satisfy $k(x, y) = \overline{k(y, x)}$. From here we get the Hilbert-Schmidt Theorem for integral operators, and which we summarize in Theorem 3.18.

A.4. **Resolvent Operators.** In this section, we give a brief discussion of the resolvent operators of a densely defined symmetric closed operator on a Hilbert space. These arise naturally from the definition of the spectrum. Note that in this section, our operators will be unbounded (see the brief discussion on unbounded operators A.2)

Definition A.30 (Resolvent operator, regular point)**.** Let A be densely defined symmetric closed operator on a Hilbert space H. For $\lambda \in \mathbb{C}\backslash\mathbb{R}$, define the **resolvent** operator R_λ on $\mathrm{Im}(A - \lambda)$ by $R_\lambda = (A - \lambda)^{-1}$.

We have a first interesting result.

Proposition A.31. *Let A be densely defined symmetric closed operator on a Hilbert space H, and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then R_λ is well-defined on $D(R_\lambda) = Im(A - \lambda)$.*

Moreover, R_λ is a bounded operator on its domain with

$$\|R_\lambda\| \leq \frac{1}{|Im(\lambda)|}. \tag{A.4}$$

Moreover, if A is also assumed to be positive, and $Re(z) < 0$, we have

$$\|R_\lambda\| \leq \frac{1}{|Re(\lambda)|}.$$

The proof of the first part follows by expanding $\mathrm{Im}\langle (A-\lambda)x, x\rangle$, for $x \in D(A)$, (use the symmetry of A to get that $\langle Ax, x\rangle \in \mathbb{R}$), and applying the Cauchy-Schwartz inequality, to get an expression from which we can deduce injectivity of $(A-\lambda)$ and (A.4). The second part follows by expanding $\mathrm{Re}\langle (A-\lambda)x, x\rangle$.

Note A.32. We can rewrite A.4 as

$$\|(A-\lambda)x\| \geq |\mathrm{Im}(\lambda)| \, \|x\| \, , \qquad \forall x \in D(A).$$

From this it follows that if for some sequence $(x_n) \subseteq D(A)$, the sequence $((A-\lambda)x_n)$ is convergent, then the sequence (x_n) is also convergent, and thus so is (Ax_n). Hence, since A is closed, it follows that (x_n) converges to some $x \in D(A)$, and (Ax_n) converges to Ax. Hence, $((A-\lambda)x_n)$ converges to $(A-\lambda)x \in \mathrm{Im}(A-\lambda)$, and thus $\mathrm{Im}(A-\lambda)$ is closed.

Of particular interest are the resolvent operators of densely defined self-adjoint closed operators. A first reason for this is that they are defined on the entire Hilbert space H.

Proposition A.33. *Let A be a densely defined self-adjoint on a Hilbert space H, and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then, R_λ is defined on the entire space H.*

This follows from the fact that $(A-\lambda)$ is injective. Hence, if the closed set $\mathrm{Im}(A-\lambda)$ were not the entire space H, it would have a non-empty orthogonal complement. One can then show that this orthogonal complement lies in $D(A^*) = D(A)$, and moreover using the fact that A is self-adjoint and densely defined, one can show that this orthogonal complement lies in the kernel of $(A-\lambda)$, which is a contradiction.

Another interesting property coming from the self-adjoint operators is the following formula.

Proposition A.34 (Hilbert Formula)**.** *Let A be a densely defined self-adjoint on a Hilbert space H, and let $\lambda, \gamma \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$R_\lambda - R_\gamma = (\lambda - \gamma) R_\lambda R_\gamma.$$

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References

[1] Alan F. Beardon. *The Geometry of Discrete Groups*. Graduate Texts in Mathematics. Springer New York, NY, 1983.

[2] Martin A. Youngson Bryan P. Rynne. *Linear Functional Analysis*. Springer Undergraduate Mathematics Series. Springer London, 2nd edition, 2007.

[3] John B. Conway. *A course in functional analysis*. Graduate texts in mathematics ; 96. Springer Science+Business Media, New York, 2nd edition, 2010.

[4] B. Sz¨okefalvi.-Nagy F. Riesz. *Functional Analysis, Translated from the 2nd French ed. by Leo F. Boron.* F. Ungar Pub. Co., New York, 1955.

[5] Jerry Shurman Fred Diamond. *A First Course in Modular Forms*. Graduate Texts in Mathematics. Springer New York, NY, 2005.

[6] D. J. H. Garling. A 'short' proof of the riesz representation theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, 73(3):459–460, 1973.

[7] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, volume 222 of *Graduate Texts in Mathematics*. Springer, New York, 2010.

[8] S. Helgason. Differential geometry and symmetric spaces. *Academic Press, New York*, 1962.

[9] Henryk Iwaniec. *Spectral methods of automorphic forms*, volume 53. American Mathematical Society, Revista Matemática Iberoamericana (RMI), 2021.

[10] Anthony W. Knapp. *Lie Groups Beyond an Introduction*. Volume 140 of Progress in Mathematics. Springer Science & Business Media, 2002.

[11] John M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, NY, 2012.

[12] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. IV Analysis of Operators*. Academic Press, New York, 1978.

[13] John Voight. *Quaternion Algebras*. Graduate Texts in Mathematics. Springer International Publishing, 2021.

[14] W.Arveson. *A Short Course on Spectral Theory*. Springer Science & Business Media, 2001.