A COURSE IN COMPLEX ANALYSIS VIA BROWNIAN MOTION

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ABSTRACT. Brownian motion is a central object in probability theory, with connections to several disparate parts of mathematics. One such area is complex analysis, with recent work exploring random surfaces in complex spaces. In this paper, we present striking proofs via Brownian motion of several central theorems in basic complex analysis. These proofs are rather satisfying and offer insight not afforded by their analytic counterparts.

CONTENTS

1. INTRODUCTION

Despite originating from observations in biology and physics, Brownian motion has found itself in a central position in modern probability theory, with applications to diverse areas within and without mathematics. The purpose of this paper is to describe how Brownian motion can cast new light on basic results in complex analysis. In particular, we present probabilistic proofs of the maximum modulus principle, Picard's little theorem and the fundamental theorem of algebra. The first and last of these are covered in any standard course on complex analysis and Picard's little theorem is often mentioned as a curiosity.

However, their analytic proofs (especially of Picard's theorem) can sometimes feel obfuscating, and require some doing. Brownian motion offers satisfying proofs of these results. The proofs of Picard's theorem and the fundamental theorem of algebra rely on the so-called conformal invariance of Brownian, which states that a sufficiently nice map sends Brownian motion to Brownian motion.

This paper assumes a knowledge of the basic definitions and results in complex analysis as well as a familiarity with probabilistic reasoning. For the purposes of the paper, f will be a holomorphic function unless stated otherwise, U will denote an open, connected region and all functions will be assumed to be from and to the complex plane (or a subset). The letters E and P denote the standard expectation and probability operators.

2. Brownian motion and harmonic functions

Much of this content is derived from [5].

Definition 2.1. A stochastic process $B : [0, \infty) \to \mathbb{R}$ is called **Brownian motion** if it satisfies the following conditions

1. (Normal increments) For any $s < t$, $B_t - B_s \sim N(0, t-s)$.
2. (Independent increments) For all $0 \leq t_1 \leq ... \leq t_N$, the increments $B_{t_1} - B_0, ..., B_{t_N} - B_{t_{N-1}}$ are independent.
3. (Continuity) B_t is continuous with probability 1.

With $B_0 = 0$, this is usually referred to as standard 1-dimensional Brownian motion. For the purposes of this paper, however, we concern ourselves with 2-dimensional complex-valued Brownian motion. In order to define it, we establish the following notation. Fix $\epsilon > 0$. Then, letting $\tau_0 = \tau_0(\epsilon) = 0$, define $\tau_i(\epsilon) = \inf\{t > \tau_{i-1} : |Z_t - Z_{\tau_{i-1}}| =\}$. Then, let $_i = Z_{\tau_i} - Z_{\tau_{i-1}}$. Note that $|_i| = \epsilon$.

The following definition is due to Davis [3].

Definition 2.2. A stochastic process $Z : [0, \infty) \to \mathbb{C}$ is called **complex Brownian motion** if it satisfies the following conditions

1. The path Z_t is continuous and unbounded.
2. The complex-valued random variables $_i$ are uniformly distributed on $\{z : |z| = \epsilon\}$.

If complex Brownian motion satisfies the following the condition, it is called standard complex Brownian motion.

3. The real-valued random variables $\tau_i - \tau_{i-1}$ are independent and identically distributed. Moreover, $E(\tau_i(1) - \tau_{i-1}(1)) = \frac{1}{2}$.

Remark 2.3. Let X_t, Y_t be standard (real-valued) Brownian motion. Then, it is a standard fact in probability that $X_t + iY_t$ is standard complex Brownian motion. However, in this paper, we will only require complex Brownian motion (both standard and otherwise). So, henceforth (standard) Brownian motion will refer to (standard) complex Brownian motion. Unless indicated otherwise, Brownian motion will start at 0.

We now discuss some of the theory of harmonic functions, which are intricately linked both to analytic functions in complex variable theory and Brownian motion. First, as always, we define these central objects.

Definition 2.4. A function $u : U \subset \mathbb{C} \to \mathbb{R}$ on a neighborhood U is called **harmonic** if it satisfies Laplace's equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Harmonic functions also have the following equivalent definition. That they are equivalent is a standard exercise. We will primarily use the second definition.

Definition 2.5. A function $u : U \subset \mathbb{C} \to \mathbb{R}$ on a neighborhood U is **harmonic** if it obeys the following mean-value property at all $z \in U$ such that $\{z : |z - z_0| \leq$

$\epsilon\} \subset U$.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \epsilon e^{i\theta})d\theta.$$

Harmonic functions are therefore those functions that are take on the average value of the sphere at its centre. Harmonic functions and Brownian motion are linked by the following theorem of Kakutani. The presentation is the same as Davis's [3].

Theorem 2.6. *Let u be a harmonic function on the region U and u continuous and bounded on $\overline{U}$. Let $\tau_U = \inf\{t > 0 : Z_t \in \partial U\}$. Then, if $z_0 \in U$ and $P(\tau_U < \infty) = 1$,*

$$E(u(Z_{\tau_U})) = u(z_0),$$

for Brownian motion Z_t such that $Z_0 = z_0$.

Proof. Fix $\epsilon > 0$. Then, let $N = \min\{k : B(Z_{\tau_k(\epsilon)}, \epsilon) \cap \partial U = \emptyset\}$. We do not consider the case $N = 0$, by simply taking ϵ to be sufficiently small, so that $P(N \geq 1) = 1$. Then,

$$E(u(Z_{\tau_1})) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \epsilon e^{i\theta})d\theta = u(z_0),$$

by the definition of expectation and the fact that u is harmonic.

Now, conditioned on N taking on the appropriate value, we see (by the tower rule) that

$$\begin{aligned} E\left(u(Z_{\tau_k}) - u(Z_{\tau_{k-1}})\right) &= E(E(u(Z_{\tau_k}) - u(Z_{\tau_{k-1}})|Z_{\tau_{k-1}})) \\ &= E\left(\frac{1}{2\pi}\int_0^{2\pi} u(Z_{\tau_{k-1}} + \epsilon e^{i\theta})d\theta - u(Z_{\tau_{k-1}})\right) \\ &= E(0) \\ &= 0. \end{aligned}$$

So,

$$E(u(Z_{\tau_{k\wedge N}})) = E(u(Z_{\tau_1})) + E(u(Z_{\tau_2}) - u(Z_{\tau_1})) + ... + E(u(Z_{\tau_N}) - u(Z_{\tau_{N-1}})) = u(z_0).$$

Taking $k \to \infty$, we get $E(u(Z_{\tau_N})) = u(z_0)$ by the dominated convergence theorem. We have defined Brownian motion to be unbounded, so as $\epsilon \to 0$, $Z_{\tau_N} \to Z_{\tau_R}$ with probability 1. So, $u(Z_{\tau_N}) \to u(Z_{\tau_R})$ by continuity. Then, another application of the dominated convergence theorem completes the proof.

□

In the two-dimensional (complex) case, Brownian motion visits every neighborhood infinitely often. To formalize this notion, we define the following.

Definition 2.7. A random continuous function $X : [0, \infty) \to \mathbb{C}$ is **neighborhood recurrent** if for all $z \in \mathbb{C}, T > 0$ and $\epsilon > 0$,

$$P(\text{there is a } t > T \text{ such that } |X_t - z| < \epsilon) = 1.$$

The following theorem is the first application of Kakutani's formula in this paper.

Lemma 2.8. *Complex Brownian motion is neighborhood recurrent.*

Proof. Fix $z_0 \in \mathbb{C}$. Let $0 < \epsilon < |z_0| < R$. Then, define

$$u(z) = \log |z - z_0|.$$

On $U = \{z : \epsilon < |z| < R\}$, u is harmonic. This can be verified by a computation of its derivatives. By Kakutani's formula (2.6)

$$u(0) = E(Z_{\tau_U}) = u(\epsilon)P(|Z_{\tau_U} - z_0| = \epsilon) + u(R)P(|Z_{\tau_U} - z_0| = R).$$

Since the Brownian motion started at 0 can only exit (and must exit) U from $\{z : |z - z_0| =\}$ or $\{z : |z - z_0| = R\}$, this is the same as

$$u(\epsilon)P(|Z_{\tau_U} - z_0| = \epsilon) + u(R)(1 - P(|Z_{\tau_U} - z_0| = \epsilon)).$$

Rearranging,

$$P(|Z_{\tau_U} - z_0| = \epsilon) = \frac{u(0) - u(R)}{u(\epsilon) - u(R)} = \frac{\log |z_0| - \log |R - z_0|}{\log |\epsilon - z_0| - \log |R - z_0|}.$$

This approaches 1 as $R \to \infty$. So, in the limiting case, the probability that Brownian motion gets ϵ-close to z_0 is 1.

□

Remark 2.9. In 2 dimensions, Brownian motion is neighborhood recurrent. This is unique to 2 dimensions in some sense, since 1-dimensional Brownian motion is pointwise recurrent (hits every point infinitely often) and d-dimensional Brownian motion is not neighborhood recurrent for $d > 2$. These results can be found in any standard treatment of Brownian motion.

3. Whetting the appetite: the maximum modulus principle

Before presenting the central results of this paper (which rely on a property of 2-dimensional Brownian motion called conformal invariance), we demonstrate a short consequence of probabilistic methods in complex analysis, namely, the maximum modulus principle. This follows by a fairly quick application of Kakutani's formula and the Cauchy-Riemann equations, which are reviewed below.

Fact 3.1. *The Cauchy-Riemann equation say that if $f = u + iv$ is a holomorphic function,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

In particular, u, v are both harmonic. Now we prove the maximum modulus principle. This is a result I learned of from [2].

Proposition 3.1. *Let $f : \overline{U} \to \mathbb{C}$ be a function, holomorphic on U and continuous and bounded on $\overline{U}$, where U is open and connected. Then,*

$$|f(z)| \leq \sup_{w \in \partial U} |f(w)|,$$

for $z \in U$.

Proof. Let $f = u + iv$, where $u, v : \overline{U} \to \mathbb{C}$ are harmonic by the Cauchy-Riemann equations. Then, by Kakutani's formula (Theorem 2.6),

$$f(z) = u(z) + iv(z) = E(u(Z_{\tau_U})) + iE(v(Z_{\tau_U})) = E(f(Z_{\tau_U})).$$

Then,

$$|f(z)| \leq E(|f(Z_{\tau_U})|) \leq \sup_{w \in \partial U} |f(w)|.$$

□

4. Conformal invariance of Brownian motion

In this section we prove that the image of Brownian motion under an non-constant entire function is also Brownian motion. For the purposes of this paper, we will not require the following definition, but it simplifies the discussion in the remarks that follow.

Definition 4.1. A map $f : U \to \mathbb{C}$ is **conformal** if it is holomorphic and $f'(z) \neq 0$ anywhere on U.

A map $f : U \to V \subset \mathbb{C}$ is a conformal isomorphism if it is conformal, bijective and its inverse is conformal (the third condition is redundant).

The title of this section is thus a slight misnomer, since we will not prove the result, due to Lévy, that Brownian motion under a conformal isomorphism is a time change of Brownian motion.

At first glance, it may seem that the result we are to prove is not just a simpler case, but a disjoint one, since not all entire maps are conformal (sin, for example). However, all non-constant entire maps are locally conformal on all but a countable set without limit points. Since 2-dimensional Brownian motion avoids such sets with probability 1 our case is indeed specific.

Now, we prove the following (striking!) result due to Davis [3].

Theorem 4.2. *If $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire function and Z_t is Brownian motion started at z_0, then $f(Z_t)$ is Brownian motion started at $f(z_0)$.*

Proof. Continuity of $f(Z_t)$ follows by continuity of f. Furthermore, since $\{z : |f(z)| \geq n\}$ is a neighborhood, and Brownian motion is neighborhood recurrent, Z_t visits it infinitely often with probability 1. So $f(Z_t)$ is almost surely unbounded.

Now, let $\tau_1 = \inf\{t : |Z_t - z_0| = \epsilon\}$. Then, we want to show that Z_{τ_1} is uniformly distributed on $\{z : |z - z_0| = \epsilon\}$. To do so, we claim that for all functions u harmonic on $S = \{z : |z - z_0| < \epsilon\}$ and continuous on $\overline{S}$, $E(u(f(Z_{\tau_1}))) = u(f(z_0))$. This follows by Theorem 2.6, letting R be the connected component of $f(S)$ containing $f(z_0)$ and noting that $f \circ u$ is harmonic on R.

Now, let $\tau_2 = \inf\{t : |Z_t - Z_{\tau_1}| = \epsilon\}$. It follows that $f(Z_{\tau_2}) - f(Z_{\tau_1})$ is distributed independetly like $f(Z_{\tau_1}) - f(z_0)$ by the memorylessness of Brownian motion. Continuing inductively, we see that $f(Z_{\tau_1}) - f(z_0), ..., f(Z_{\tau_n}) - f(Z_{\tau_{n-1}})$ are independently uniformly distributed on $\{z : |z| = \epsilon\}$.

Therefore, we see that $f(Z_t)$ satisfies 1. and 2. in the definition of complex Brownian motion, giving us the result.

□

5. Picard's little theorem

We have finally developed the tools to prove Picard's little theorem probabilistically. The theorem is given below. The proof is outlined in [3], but we follow [1].

Theorem 5.1. *If f is a non-constant, entire function then its image is all of $\mathbb{C}$, except possible one number.*

The proof of this statement proceeds by contradiction, for which we assume that $f(\mathbb{C})$ does not contain a or $b \in \mathbb{C}$. To simplify the proof, without loss of generality we let $a = 1$, $b = -1$ and $f(0) = 0$.

Let ϵ be small enough that for any $|z| < \epsilon$, $f(z)$ can be connected to 0 by a curve in $f(\mathbb{C}) \cap B_{1/2}(0)$. By neighborhood recurrence (Lemma 2.8), Z_t (a Brownian motion started at 0) visits $\{z : |z| < \epsilon\}$ infinitely often. Then, when $Z_t \in \{z : |z| < \epsilon\}$, let L_t be the curve that connects 0 to Z_t. Since the plane is not punctured, the curve formed by the path of Z_t and L_t must be a homotopic to a point. Thus, $f(\text{curve})$ must be homotopic to a point. In proposition 5.2, we show that this is not case, a contradiction!

Proposition 5.2. *Let L_t be as above. Then, with probability 1, there is s such that if $t > s$, the curve formed by $f(L_t)$ and $f(Z_t)$ is not homotopic to a single point.*

The idea of this proof is to observe that the Brownian motion gets "tangled" around 1 and -1 after enough time. This is because, upon returning to $\{z : |z| < \epsilon\}$ the Brownian motion is roughly equally likely to go above and around 1, below and around 1, above and around -1 and below and around -1. Only one of these untangles the Brownian motion, whereas the other three tangle it up further. Then, by the law of large numbers, the Brownian motion is going to get tangled at some time. This probabilistic element is the essential idea and there is no way to replicate such a proof without a probabilistic construction.

The complex analytic aspect of the hypothesis is no longer necessary, so we simply identify it with $\mathbb{R}^2$ in the following.

Proof. Let

$$A_0 = \{-1 < x < 1, y = 0\},$$
$$A_1 = \{x < -1, y = 0\},$$
$$A_2 = \{x > 1, y = 0\},$$
$$A_3 = \{x = -1 \text{ or } x = 1, y > 0\},$$
$$A_4 = \{x = -1 \text{ or } x = 1, y < 0\}.$$

Let $T_1 = 0$. let $b_i \in \{0, 1, 2, 3, 4\}$ be such that $f(Z_{T_i}) \in A_{b_i}$ and

$$T_{i+1} = \inf\{t > T_i : f(Z_t) \in \bigcup_{j=0}^{4} A_j - A_{b_i}\}.$$

The $T_i's$ are the times to hit a different A_i than the last one hit. We form the following sequence that allow us to encode the information of the Brownian motion's traversal. Consider the sequence $0b_1b_2...b_n$ and then reduce it according to the following rules.

1. If the sequence ends in 030 or 040, delete the last two entries.
2. If b_{n-2} and b_n are the same and $b_{n-1} \neq 0$, delete the last two entries.

3. If the sequence ends in 0 and has the form $...0b_{i_1}...b_{i_j}0b_{i_j}...b_{i_1}0$, delete the string $b_{i_1}...b_{i_j}0b_{i_j}...b_{i_1}0$.

Apply the rules until they cannot be applied anymore. Then, the resulting sequence is our reduced sequence.

Note that this is a complete description of the homotopy class of $f(Z_t)$ with $f(L_t)$ added whenever $Z_t \in \{z : |z| < \epsilon\}$. Specifically, if this curve is homotopic to 0, the reduced sequence must be 0.

We will show that the number of 0's in the reduced sequence goes to ∞ with probability 1.

Now, suppose the last segment of the current reduced sequence between two 0's is $0b_{i_1}...b_{i_j}0$. To reduce the number of 0's, the Brownian motion must travel so that the next segment is $b_{i_j}...b_{i_1}0$. By symmetry, however, we are equally likely to get $b_{i_1}...b_{i_j}0$, which would add one more 0. Moreover, there are always at least two other sequences that would add zeros, and a non-zero probability of attaining these two sequences next. So,

$$P(\text{number of 0's increases by 1}) \geq \frac{1}{2} + \delta,$$

$$P(\text{number of 0's decreases by 1}) \leq \frac{1}{2} + \delta.$$

By the law of large numbers, the number of 0's goes to ∞ as the Brownian motion keeps travelling. So, we have the result.

□

We have thus proved Picard's little theorem!

6. Fundamental theorem of algebra

Having built this machinery to do probability in the complex plane, we can give another (also striking) proof of a central theorem in mathematics, the fundamental theorem of algebra. The central idea of the proof is that is if a polynomial avoids 0, it must avoid a neighborhood of 0. But since polynomials are entire, the image of Brownian motion under a polynomial must be Brownian motion and so must hit every neighborhood, which gives us a contradiction. We formalize this below, following [1].

Theorem 6.1. *Let $p(z)$ be a non-constant polynomial with degree n. Then, $0 \in p(\mathbb{C})$.*

Proof. As $|z| \to \infty$, $|p(z)| \to \infty$ (this is the only part of the proof in which we use the fact that p is a polynomial, and not just an entire function). So, it is possible to pick R such that $|p(z)| \geq 1$ if $|z| \geq R$.

Suppose, for a contradiction that $p(z) \neq 0$. Note that $p(\overline{B_R(0)})$ is compact and thus closed, and does not contain 0. Thus there is a neighborhood U of 0 such that $U \cap p(\overline{B_R(0)}) = \emptyset$. Without loss of generality let $U \subset B_{1/2}(0)$. Then, $U \cap p(\mathbb{C}) = \emptyset$.

Now consider a Brownian motion Z_t started at 0. By conformal invariance, $p(Z_t)$ is Brownian motion as well and thus neighborhood recurrent. So, it must hit U, a contradiction.

Therefore, $0 \in p(\mathbb{C})$!

□

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