LIE GROUPS AND RATNER'S ORBIT CLOSURE THEOREM

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Abstract. Lie groups are a type of group distinguished by having a nice, smooth topological structure. Each Lie group has a corresponding Lie algebra, and there is a rich relationship between the two objects. One important theorem in the theory of Lie groups is the Borel Density theorem, which is used to prove Ratner's Orbit Closure theorem, a similarly significant result. This paper assumes some basic algebraic and topological concepts and results, and will build up to the statements of the two theorems and some basic examples.

Contents

1. Lie Groups

A Lie group is a group G that is also a differentiable manifold. The following properties must also be satisfied:

(1) The product map $p : G \times G \to G$ defined by $p(a, b) = ab$ is smooth.
(2) The inverse map $i : G \to G$ defined by $i(a) = a^{-1}$ is smooth.

Many of the most interesting Lie groups are matrix Lie groups, though not every Lie group is isomorphic to a matrix Lie group.

Definition 1.1. The **general linear group** over $\mathbb{R}$ of $n \times n$ matrices is the set of all $n \times n$ invertible matrices with real entries under the group operation of matrix multiplication. We typically denote this set $GL(n, \mathbb{R})$.

Definition 1.2. The **standard linear group** over $\mathbb{R}$, denoted $SL(n, \mathbb{R})$, is the set of all $n \times n$ matrices with determinant 1. This set forms a group under matrix multiplication.

Both $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ are Lie groups. Another common Lie group is $SO(n, \mathbb{R})$, which is the collection of orientation preserving rotation matrices in $\mathbb{R}^n$.

Definition 1.3. The **exponential** of an $n \times n$ matrix X is given by

$$e^X = \sum_{m=1}^{\infty} \frac{X^m}{m!}$$

Theorem 1.4. *For any $X \in M_n(\mathbb{C})$, e^X converges. Moreover, e^X is continuous in X.*

Proof. See Proposition 2.1 in [1]. $\square$

The exponential function is the key function relating a Lie group to to its corresponding Lie algebra, which will be discussed later. In fact, the exponential function can be generalized to all Lie groups, not just matrix Lie groups. This abstraction requires the notion of a tangent vector, which we do not introduce until a little later.

One interesting result is as follows.

Theorem 1.5. *For any $n \times n$ real matrix X,*

$$\det(e^X) = e^{\text{trace}(X)}$$

Proof. We first consider the case when X is diagonalizable. Say X has eigenvalues $\lambda_1, \lambda_2, ...\lambda_n$ with corresponding eigenvectors $v_1, v_2, ...v_n$. Then denote B as the matrix with columns given by the eigenvectors $v_1, v_2, ...v_n$, and C as the matrix given by

$$\begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & & & \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$$

We must consider $e^{BCB^{-1}}$. The claim is $e^{BCB^{-1}} = Be^C B^{-1}$. To see this, observe:

$$\begin{aligned} e^{BCB^{-1}} &= \sum_{m=1}^{\infty} \frac{(BCB^{-1})^m}{m!} \\ &= \sum_{m=1}^{\infty} \frac{BC^m B^{-1}}{m!} \\ &= B(\sum_{m=1}^{\infty} \frac{C^m}{m!})B^{-1} \end{aligned}$$

As desired. But C is a diagonal matrix, so we see in the sum $\sum_{m=1}^{\infty} \frac{C^m}{m!}$ that each eigenvalue λ_i is given by $\lambda_i = \sum_{m=1}^{\infty} \frac{\lambda_i^m}{m!} = e^{\lambda_i}$. But recall the determinant is the product of the eigenvalues, so

$$\begin{aligned} \det(e^X) &= e^{\lambda_1} e^{\lambda_2} ... e^{\lambda_n} \\ &= e^{\lambda_1 + \lambda_2 + ... + \lambda_n} \\ &= e^{\text{trace(X)}} \end{aligned}$$

Now we are left with the case when X is not diagonalizable. However, it is a linear algebra result that any matrix is the limit of a sequence of diagonalizable matrices, reducing to the diagonalizable case above. Informally, this is because a matrix A is diagonalizable when A has distinct eigenvalues. We can write some non-diagonalizable matrix B as an upper triangular matrix in some basis with the eigenvalues on the diagonal. Then, we can approximate B by changing the eigenvalues to make them distinct – this approximation is diagonalizable. Using this method, we can get arbitrarily close to our original matrix B. $\square$

2. Ratner's Orbit Closure Theorem

We will now discuss Ratner's Orbit Closure Theorem. We will not discuss the proof, which is quite involved, but will instead walk through several examples.

Definition 2.1. A **unipotent** element g of a group G is an element such that $(g-e)^n = 0$ for some n where e is the identity element of G.

Example 2.2. We will mostly consider unipotent matrices, as we are mostly concerned with matrix Lie groups. A square matrix A is unipotent when $(A-I)^n = 0$ for some n. So, $A - I$ is nilpotent. It is well known that nilpotent matrices have only 0 as an eigenvalue, or $(A-I)v = 0$ for each eigenvector v. But then $Av = v$ for all eigenvectors v, so A only has 1 as an eigenvalue (with multiplicity n).

Definition 2.3. A **discrete subgroup** is a subgroup where there exists an open neighborhood U about the identity element e of G with $U \cap G = \{e\}$. A **lattice** Γ in a group G is a discrete subgroup of G such that there exists a Borel measure on G/Γ which is finite and G - invariant.

The second part of definition 2.3 can be reworded as the following: if we divide G into equivalence classes based on the lattice, the collection of equivalence classes has finite measure and the measure does not change under the group operation.

Example 2.4. One common lattice in the group $\mathbb{R}^2$ under vector addition is $\mathbb{Z}^2$. We can identify the quotient $\mathbb{R}^2/\mathbb{Z}^2$ with the unit square with vertices $(0,0), (0,1), (1,0)$ and $(1,1)$. Each point in $\mathbb{R}^2$ can be represented as a point in this unit square shifted by some integer valued vector. Notice the quotient $\mathbb{R}^2/\mathbb{Z}^2$ has finite area (a finite Borel measure). Additionally, the identity, (0,0), has an open region that does not include any points with integer values (except (0,0) itself). To form this open region, draw a circle of radius less then one about $(0,0)$. Furthermore, shifting any region by a vector with integer coordinates has no affect on the measure (in this case, area).

Example 2.5. We will now consider a simple example which can be used to introduce Ratner's orbit closure theorem. Let x be a point in $\mathbb{R}^2$ and let v be a vector in $\mathbb{R}^2$. Consider the one parameter subgroup of $\mathbb{R}^2$ given by the mapping $t \to vt$ where $t \in \mathbb{R}$.

If v is zero we are left with the trivial case where $x + vt$ is just the point x. So long as v is nonzero, $x + vt$ represents a line in $\mathbb{R}^2$. We can mod out by $\mathbb{Z}^2$, considering the image of $x+vt$ in $\mathbb{R}^2/\mathbb{Z}^2$, denoted $[x+vt]$. The angle v makes with the x-axis determines the behavior of $[x+vt]$. If this angle is rational, $[x+vt]$ is closed and periodic in t. However, if the angle v makes with the x-axis is irrational, it turns out $[x+vt]$ is dense in $\mathbb{R}^2/\mathbb{Z}^2$. The key idea here is that the closure of $[x+vt]$ forms a nice, clean subset of $\mathbb{R}^2/\mathbb{Z}^2$ - if v is the zero vector, we are left with a point, if v is nonzero we get either a nice, discrete line, or the whole space.

A similar, more general property forms Ratner's orbit closure theorem. In particular, $\mathbb{R}^2$ can be generalized to a connected Lie group, $\mathbb{Z}^2$ can be generalized to an arbitrary lattice, and the one parameter subgroup $[x+vt]$ is replaced with a unipotent subgroup.

Ratner's Orbit Closure Theorem can be stated several ways. We will adopt the version used in [3].

Theorem 2.6. ***Ratner's Orbit Closure Theorem.*** *Let G be a connected Lie group and let U be a unipotent subgroup of G. For all lattices Γ in G and for all $x \in G$, there exists a connected, closed subgroup $L < G$ containing U where $xLx^{-1} \cap \Gamma$ is a lattice in L with $\overline{Ux\Gamma} = Lx\Gamma$.*

Informally, Ratner's Orbit Closure theorem says the closure of the orbit of any point around a unipotent subgroup forms a nice, clean subset of the group. In fact, the statement that $xLx^{-1} \cap \Gamma$ is a lattice in L is equivalent to the statement that there exists a measure μ under which $Lx\Gamma$ has finite U-invariant measure. In other words, the orbit Lx when modded out by Γ has a finite 'area'.

Definition 2.7. For a group G, a **one parameter subgroup** is a continuous mapping $\gamma : \mathbb{R} \to G$ that preserves the group operation. Equivalently, $\gamma(a + b) = \gamma(a)\gamma(b)$ for all $a, b \in \mathbb{R}$.

Example 2.8. We will consider the example of $SL(2, \mathbb{R})$. We will consider the unipotent one parameter subgroup $U = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}$. Any matrix of this form is indeed unipotent, and we can check U forms a subgroup by checking U is closed by the following calculation:

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t+s \\ 0 & 1 \end{bmatrix}$$

We denote $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, as is typical, although the notation is clumsy. We will consider a lattice given by $\Gamma = SL(2, \mathbb{Z})$. It is a known fact in algebra that there are only three connected subgroups of $SL(2, \mathbb{R})$ that contain the subgroup U. Those subgroups are:

(1) U itself.
(2) The collection of upper triangular 2 by 2 matrices.
(3) $SL(2, \mathbb{R})$ itself.

However, it is an algebraic result that there are no lattices in the upper triangular matrices. Since part of Theorem 4.5 guarantees L has a lattice, we can safely exclude the collection of upper triangular matrices. We can now use Ratner's orbit closure theorem and conclude for all $x \in SL(2, \mathbb{R})$ either $\overline{Ux\Gamma} = SL(2, \mathbb{R})x\Gamma$ or $\overline{Ux\Gamma} = Ux\Gamma$. In the first case, Ux is dense in $SL(2, \mathbb{R})$; in the second case, Ux is closed.

In this fashion, Theorem 2.6 can be used to find properties of orbits of points in a Lie group.

Remark 2.9. We will not prove the theorem. Ratner's orbit closure theorem is typically proved as a corollary of Ratner's genericity theorem, which is in turn a result derived from Ratner's measure classification theorem. A nice proof of the measure classification theorem can be found in [2], and a proof for the orbit closure theorem can be found in Ratner's original paper [4].

3. Lie Algebras

Each Lie group has a corresponding Lie algebra. Many properties of the Lie group can be studied through the Lie algebra. We will first discuss properties of tangent vectors in an arbitrary smooth manifold M before applying this to the Lie group setting.

Informally, for a point $x \in M$, the tangent space is the space of directions one can move from x. For example, given a sphere, the tangent space at any point x is the tangent plane through x. Each direction one can move in the tangent space is a tangent vector. More formally, consider a curve $f : \mathbb{R} \to M$ that passes through some point $x \in M$. We can map M locally to $\mathbb{R}^n$, allowing us to fix coordinates. So we can treat f near x as function of $\mathbb{R}$ to $\mathbb{R}^n$, denoted $f(\lambda) = (x_1, x_2, ...x_n)$. We can then consider the following derivative:

$$\frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{dx_i}{d\lambda} \frac{\partial f}{\partial x_i} \tag{3.1}$$

Definition 3.2. If M is a smooth manifold and a curve $f : \mathbb{R} \to M$ passes through a point λ, then we say $\frac{d}{d\lambda}$ is the **tangent vector** to the curve f at λ.

Note that by fixing a point λ and a tangent vector $\frac{d}{d\lambda}$, we can form an equivalence relation by considering all the curves that yield a tangent vector of $\frac{d}{d\lambda}$ at λ to be equivalent. That is, we don't care about what the curve does anywhere except at λ. The collection of all possible tangent vectors $\frac{d}{d\lambda}$ forms a vector space, as derivatives behave nicely under addition and scalar multiplication.

Definition 3.3. The **tangent space** of a smooth manifold M at a point λ is the vector space formed by the collection of all tangent vectors passing through λ.

Definition 3.4. We can define a **Lie algebra** $\mathfrak{g}$ as a vector space over a field of characteristic 0 paired with an operation, called the bracket operation, $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfing the following properties.

(1) The bracket operation $[X, Y]$ is bilinear, or $[kX, Y] = k[X, Y]$
(2) The bracket operation $[X, Y]$ is skew symmetric, or $[X, Y] = -[Y, X]$.
(3) The Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Remark 3.5. We will primarily consider real or complex Lie algebras, which are Lie algebras over $\mathbb{R}$ or $\mathbb{C}$. The above definition can be generalized to fields of nonzero characteristic.

Note for matrix Lie algebras, this bracket operation is the same as the commutator defined by $[X, Y] = XY - YX$.

Definition 3.6. Each Lie group has an associated Lie algebra. The **Lie algebra $\mathfrak{g}$ of a Lie group** G is defined as the tangent space of G at the identity element e, denoted T_eG.

Using the matrix exponential, we can draw a further relationship between a matrix Lie group G and the Lie algebra $\mathfrak{g}$ of G. Fix a curve f through a point $\lambda_0 \in \mathbb{R}$. Consider another point on the curve $\lambda_0 + \epsilon$. We can use a Taylor series expansion to calculate $f(\lambda_0 + \epsilon)$.

$$f(\lambda_0 + \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \frac{d^n f}{d\lambda^n}\Big|_{\lambda_0} \tag{3.7}$$

$$= e^{f(\lambda_0 + \epsilon)} \tag{3.8}$$

In this way the matrix exponential allows us to generate elements of the Lie group from only elements of the Lie algebra. The Lie algebra of a matrix Lie group

G is, in fact, the set of all matrices X such that $e^{tX} \in G$ for real t. Several questions follow naturally. Does a Lie algebra include all relevant information about a Lie group? The answer, unfortunately, is no, although many important properties are encoded by the Lie algebra. A later result, Theorem 3.21, gives that any finite dimensional Lie algebra is the Lie algebra of some Lie group.

Theorem 3.9. *The Lie algebra $\mathfrak{g}$ of a matrix Lie group G is a Lie algebra, that is, T_eG satisfies the the axioms of definition 3.4 under the commutator bracket given by $[X, Y] = XY - YX$.*

Proof. See section 3.3 of [1]. $\square$

Example 3.10. Consider the cross product in $\mathbb{R}^3$. Notice if $\mathfrak{g} = \mathbb{R}^3$, then the cross product satisfies the above axioms of the bracket operation. Recall for vectors $v, w \in \mathbb{R}^3$, $v \times w = ||v|| * ||w|| \sin(\theta) n$ where n is perpendicular to both v and w and follows the right hand rule. Notice $(\alpha v) \times w = \alpha(v \times w)$, and $w \times v = -v \times w$ as the orientation of n flips. The Jacobi identity is the most involved to verify, but it can be done. So, $\mathbb{R}^3$ paired with the cross product is a Lie algebra.

Example 3.11. We will show the Lie algebra $sl(n, \mathbb{R})$ of the special linear group $SL(n, \mathbb{R})$ is given by the collection of $n \times n$ matrices with trace 0. Recall we stated earlier the Lie algebra $\mathfrak{g}$ of a matrix Lie group is given by all matrices X such that $e^{tX} \in G$ for $t \in \mathbb{R}$.

In this case, we are looking for X with $e^{tX} \in SL(n, \mathbb{R})$. Equivalently, $\det(e^{tX}) = 1$. By Theorem 3.3, $e^{\text{trace}(tX)} = 1$, so $e^{t * \text{trace}(X)} = 1$ for all $t \in \mathbb{R}$. This is true exactly when $\text{trace}(X) = 0$.

So $sl(n, \mathbb{R})$ is precisely the collection of $n \times n$ matrices with trace 0.

The following definitions are important for the setup of the Borel Density theorem.

Definition 3.12. A **subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{h} \subset \mathfrak{g}$ and is closed under the bracket operation, so $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.

Definition 3.13. A subalgebra $\mathfrak{h}$ is an **ideal** in $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and all $Y \in \mathfrak{h}$.

Definition 3.14. A Lie algebra $\mathfrak{g}$ with $\dim(\mathfrak{g}) \geq 2$ is **simple** when the only ideals in $\mathfrak{g}$ are $\mathfrak{g}$ and $\{0\}$. A Lie algebra $\mathfrak{g}$ is semisimple if the Lie algebra is the direct sum of simple Lie algebras. We call a Lie group **semisimple** if the corresponding Lie algebra is semisimple.

Simple Lie algebras can be classified with so-called Dynkin diagrams. More details can be found in [6].

Example 3.15. We show the Lie algebra $sl(2, \mathbb{C})$ is simple. Recall $sl(n, \mathbb{R})$ is the set of real matrices with trace 0, and similarly $sl(n, \mathbb{C})$ is the set of complex matrices with trace 0. Note the bracket operation is simply the commutator $[X, Y] = XY - YX$. We first notice any matrix in $sl(2, \mathbb{C})$ can be represented as some linear combination of the following matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So we suppose $\mathfrak{h}$ is some subalgebra of $sl(2,\mathbb{C})$. Say $H = aX + bY + cZ$ and $H \in \mathfrak{h}$. We will show if either a, b, or c is nonzero then $\mathfrak{h} = sl(n,\mathbb{C})$ which is sufficient to show $sl(2,\mathbb{C})$ is simple.

We will need a couple identities. First, notice

$$[X,Y] = XY - YX = Z$$

Notice also

$$[Z,X] = 2X$$

and

$$[Z,Y] = -2Y$$

We first will assume $b \neq 0$ and consider the following:

$$[X,[X,H]]$$

We first calculate $[X,H]$, which is in $\mathfrak{h}$ by definition.

$$\begin{aligned}[X,H] &= XH - HX \\ &= X(aX + bY + cZ) - (aX + bY + cZ)X \\ &= bXY - bYX + cXZ - cZX \\ &= b[X,Y] + c[X,Z]\end{aligned}$$

By the identities established above, $b[X,Y]+c[X,Z] = bZ-2cX$. Now we calculate

$$\begin{aligned}[X,[X,H]] &= [X, bZ - 2cX] \\ &= X(bZ - 2cX) - (bZ - 2cX)X \\ &= bXZ - bZX \\ &= -b2X\end{aligned}$$

This means, should $b \neq 0$, some nonzero multiple of X is in $\mathfrak{h}$. But $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ by definition, so $X \in \mathfrak{h}$ as well. But then the identities above guarantee $Z \in \mathfrak{h}$ and hence $Y \in \mathfrak{h}$. Since $\mathfrak{h}$ contains the entire basis for $\mathfrak{g}$, $\mathfrak{h} = \mathfrak{g}$.

So we can safely suppose $b = 0$. We next suppose $c \neq 0$. Considering $[X,H]$, which we calculated above, we see $[X,H]$ is just some multiple of X, reducing to the argument from the previous paragraph.

Now if $a \neq 0$ and $b = c = 0$, we just get $H = aX$ and again $X \in \mathfrak{h}$, reducing to the previous argument.

We will now generalize the notion of the matrix exponential to non-matrix Lie groups. In particular, we can define the exponential function on an arbitrary Lie group as follows.

Definition 3.16. Say G is a Lie group with identity e and Lie algebra $\mathfrak{g}$. Let $\gamma : \mathbb{R} \to \mathfrak{g}$ be the unique one parameter subgroup of G where the tangent vector at e equal to X. Define the exponential map $e : \mathfrak{g} \to G$ by $e^X = \gamma(1)$.

Remark 3.17. It is not immediately clear why there exists a unique γ as described in the above definition. In fact, γ is a so called **integral curve**. Given a manifold M, a vector field is a function assigning each point $m \in M$ a vector from the tangent space T_m of m. Denote the vector associated with m by the vector field v_m. When we have a vector field and select a point $m \in M$, if we wish to draw a curve through m with the derivative of the curve at m equal to the v_m, there is precisely one curve we can draw. This curve is given by the differential equation $\frac{dm}{d\lambda} = v_m$.

There are a series of theorems relating Lie groups to their Lie algebras, which we will now present.

Theorem 3.18. *Isomorphic Lie groups have isomorphic Lie algebras.*

Proof. A proof can be found in [10]. □

The converse of Theorem 3.18 is not true. Two Lie groups with isomorphic Lie algebras are not necessarily isomorphic. However, we do get the following slightly weaker result.

Theorem 3.19. *If $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic Lie algebras, then their corresponding Lie groups are locally isomorphic.*

Proof. A proof can be found in [10]. □

Although two different Lie groups may have the same Lie algebra, there is still a special Lie group that is unique to any given Lie algebra. This is described in the following result.

The next two theorems are major results in the theory of Lie groups. Both proofs are quite involved, but the results are worth mentioning as they are both foundational in the field.

Theorem 3.20. ***Ado's Theorem.*** *Any Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is isomorphic to a Lie algebra of a matrix Lie group.*

Proof. A proof can be found in [11]. □

Theorem 3.21. ***Lie's third theorem.*** *For any finite dimensional Lie algebra $\mathfrak{g}$, there exists some unique simply connected Lie group G whose Lie algebra is $\mathfrak{g}$.*

Proof. This theorem is typically proved using Theorem 3.20, though several proofs exist. A proof can be found in [11]. □

We now pivot to the adjoint representation. The adjoint representation of a Lie group G is a representation of G as a collection of linear transformations of the Lie algebra $\mathfrak{g}$ of G.

Definition 3.22. For a Lie group G with a Lie algebra $\mathfrak{g}$, we define the following linear map. For $A \in G$, define $\mathrm{Ad}_A(X) : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{Ad}_A(X) = AXA^{-1}$.

Remark 3.23. We can verify Ad_A is linear in the following fashion.

$$\begin{aligned}\mathrm{Ad}_A(X)\mathrm{Ad}_A(Y) &= (AXA^{-1})(AYA^{-1})\\ &= AXYA^{-1}\\ &= \mathrm{Ad}_A(XY)\end{aligned}$$

There is some sense in which this is an odd definition. After all, $A \in G$ but $X \in \mathfrak{g}$, so a natural question is whether or not it makes sense at all to consider the product AXA^{-1}. In the case of an $n \times n$ matrix Lie group, elements of the tangent space at any point are $n \times n$ matrices, so the product AXA^{-1} is well defined. An analogous statement can be made for non-matrix Lie groups.

Definition 3.24. For a Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$, we define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_X(Y) = [X, Y]$.

We will denote the collection of all invertible linear transformations over a Lie algebra $\mathfrak{g}$ by $GL(\mathfrak{g})$. We are now prepared to define the adjoint map.

Definition 3.25. The **adjoint map** or **adjoint representation** of a Lie group G with Lie algebra $\mathfrak{g}$ is defined as $\mathrm{Ad} : G \to GL(\mathfrak{g})$ given by $A \mapsto \mathrm{Ad}_A$.

Ad is a homomorphism, as shown in proposition 3.33 of [1]. So, we can treat $GL(\mathfrak{g})$ as a Lie group. We denote the Lie algebra of $GL(\mathfrak{g})$ as $gl(\mathfrak{g})$.

Definition 3.26. Analogously, the **adjoint map** or **adjoint representation** of a Lie algebra $\mathfrak{g}$ is the map $\mathrm{ad} : \mathfrak{g} \to gl(\mathfrak{g})$ given by $X \mapsto \mathrm{ad}_X$.

In fact, these definitions have a natural relation: it turns out that differentiating Ad at the identity gives ad, or $x \to \mathrm{ad}_x$ is given by $d(\mathrm{Ad}_X)_e(X)$.

The adjoint representation of both a Lie group or Lie algebra represents elements of that Lie group or Lie algebra as linear transformations. This can be quite useful, as linear transformations can be easy to work with. Much of Lie group theory is devoted to the representation theory of Lie groups, and representing Lie groups in different ways. This method is used often in physics, as Lie groups like $SO(3, \mathbb{R})$ are used to model the rotations of particles and different representations can be applied to simplify certain problems.

4. Adjoint Representation Generalization of Ratner's Theorem

We will now introduce some concepts needed for the generalization of Ratner's theorem. We will speak briefly about the Zariski topology, which is a rich topic in its own right, but we only need a couple of basic properties. Recall topological spaces are typically defined by a collection of open sets whose complements are closed. The Zariski topology, on the other hand, is defined by a collection of closed sets whose complements are open. The Zariski closure is denoted Zcl and is a closure in the traditional sense under the Zariski topology. For our purposes it is enough to present a much simplified view of the Zariski topology.

Definition 4.1. Let S be a set of polynomials over $\mathbb{R}^n$. Define the set $V(S) = \{x \in \mathbb{R}^n \mid f(x) = 0 \text{ for all f } \in S\}$. A set C is closed in the Zariski topology if and only if $C = V(S)$ for some set of polynomials S.

Remark 4.2. It is worthwhile to note $G = \mathrm{Zcl}(\Gamma)$ if and only if every polynomial that vanishes on Γ also vanishes identically on G.

Remark 4.3. A worthwhile exercise is showing the Zariski topology indeed forms a topology, or the following properties are satisfied.

(1) The empty set and $\mathbb{R}^n$ are both open under the Zariski topology.

Proof. Note if S contains only the zero polynomial, then $V(S) = \mathbb{R}^n$, so $\mathbb{R}^n$ is closed and $\emptyset$ is open. However, $\emptyset$ is closed as well, if S consists of one constant nonzero polynomial then $V(S)$ is empty. This means $\mathbb{R}^n$ is open. □

(2) The Zariski topology is closed under arbitrary union.

Proof. Suppose we have some collection of Zariski open sets in $\mathbb{R}^n$. We can denote this set $\{\mathbb{R}^n \setminus V(S_\lambda)\}_{\lambda\in\Lambda}$ for some index set Λ. Consider the union $\bigcup_{\lambda\in\Lambda} \mathbb{R}^n \setminus V(S_\lambda)$. But this is just $\mathbb{R}^n \setminus \bigcap_{\lambda\in\Lambda} V(S_\lambda)$. Notice that

$$\begin{aligned}\bigcap_{\lambda\in\Lambda} V(S_\lambda) &= \{x \in \mathbb{R}^n \mid f(x) = 0 \text{ for all } f \in \bigcup_{\lambda\in\Lambda} S_\lambda\} \\ &= V(\bigcup_{\lambda\in\Lambda} S_\lambda)\end{aligned}$$

So we have $\bigcup_{\lambda\in\Lambda} \mathbb{R}^n \setminus V(S_\lambda) = \mathbb{R}^n \setminus V(\bigcup_{\lambda\in\Lambda} S_\lambda)$ which is open and we are done. □

(3) The Zariski topology is closed under finite intersection.

Proof. We can deal with finite intersections in much the same fashion. Suppose we have a finite collection of Zariski open sets in $\mathbb{R}^n$ denoted $\{\mathbb{R}^n \setminus V(S_k)\}_{k\in N}$ for some index set N. Consider the intersection $\bigcap_{k\in N} \mathbb{R}^n \setminus V(S_k)$. Recall this is $\mathbb{R}^n \setminus \bigcup_{k\in N} V(S_k)$. But now

$$\bigcup_{k\in N} V(S_n) = \{x \in \mathbb{R}^n \mid \ f(x) = 0 \text{ for all } f \in S_k \text{ for some } k \in N\}$$

We will show the special case with the union of only two closed sets, which can quickly be expanded to the case of finite union. Call our two closed sets $V(S_1)$ and $V(S_2)$. Define $S_3 = \{fg \mid f \in S_1 \text{ and } g \in S_2\}$. Now, if $x \in V(S_1)$, then x vanishes for all polynomials in S_1 and hence all polynomials in S_3. Similarly, for all $x \in V(S_2)$, we see $x \in V(S_3)$. So $V(S_1) \cup V(S_2) \subset V(S_3)$.

Let $x \in V(S_3)$. We see $fg(x) = 0$ for all $f \in S_1$ and all $g \in S_2$. Should $f(x) \neq 0$ and $g(x) \neq 0$ for some $f \in S_1$ and $g \in S_2$, we see immediately $fg(x) \neq 0$, an immediate contradiction. So $V(S_3) \subset V(S_1) \cup V(S_2)$. We conclude that $V(S_1) \cup V(S_2) = V(S_3)$, exactly what we wished to show. This result can be quickly expanded to the case of finite union. the set $\bigcup_{n\in N} V(S_n)$ is closed for finite N and hence $\bigcap_{k\in N} \mathbb{R}^n \setminus V(S_k)$ is open. □

Here are a series of definitions needed for Borel's Density theorem.

Definition 4.4. An **algebraic** group over some field is an algebraic variety with a group structure. Algebraic varieties are a complex topic, largely beyond the scope of this paper. It suffices for us to recognize $GL(n, \mathbb{R})$ and $SL(n, \mathbb{R})$ are both $\mathbb{R}$-algebraic groups. Further reading is found in [8].

Now we will introduce Borel's density theorem, an important result in the study of Lie groups and an ingredient in the proof of Ratner's Orbit Closure Theorem.

Theorem 4.5. ***The Borel Density Theorem.*** *Let G be a connected semisimple $\mathbb{R}$-algebraic group such that all connected, normal, compact subgroups of G are trivial. Suppose $\Gamma < G$ is a lattice. Then Γ is Zariski dense in G.*

Proof. A proof can be found in section 4.7 of [2]. □

We will now consider the simple example of $SL(n, \mathbb{R})$ with the lattice $SL(n, \mathbb{Z})$. To build up to the case of $SL(n, \mathbb{R})$ we first look at the case of $\mathbb{R}$ under the lattice $\mathbb{Z}$.

In order to show $\mathrm{Zcl}(\mathbb{Z}) = \mathbb{R}$, we must show any polynomial f with $f(x) = 0$ when $x \in \mathbb{Z}$ is the 0 polynomial. Since any nonzero polynomial has a finite number of roots, this statement is vacuously true.

Next we show $\mathrm{Zcl}(\mathbb{Z}^n) = \mathbb{R}^n$. So, suppose f is a function with $f(x) = 0$ for all $x \in \mathbb{Z}^n$. We note that it is sufficient to show $f(q) = 0$ for all $q \in \mathbb{Q}^n$. This is because $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, and polynomials are continuous. That is, if by contradiction $f(q) = 0$ for all $q \in \mathbb{Q}^n$ and $f(r) \neq 0$ for some $r \in \mathbb{R}^n$, the continuity of f is immediately violated.

Fix some $q \in \mathbb{Q}^n$ with $q = (\frac{q_1}{p_1}, \frac{q_2}{p_2}, ..., \frac{q_n}{p_n})$ where $\frac{q_i}{p_i} \in \mathbb{Q}$. Now define $g(t) = f(tq)$. Notice $f(tq) = 0$ when $t\frac{q_1}{p_1}, t\frac{q_2}{p_2}, ...t\frac{q_n}{p_n}$ are all integers. This happens when t is an integer multiple of the least common multiple of $p_1, p_2, ..., p_n$. But there are infinite integer multiples of t, and a nonzero single variable polynomial has finitely many roots, so g is identically 0. This means (setting $t = 1$) we see $f(q) = g(1) = 0$. This holds for all $q \in \mathbb{Q}^n$, so f is identically 0 and hence $\mathrm{Zcl}(\mathbb{Z}^n) = \mathbb{R}^n$.

We finally arrive at the case of $SL(n, \mathbb{R})$ with the lattice $SL(n, \mathbb{Z})$. We consider a polynomial f in n^2 variables. Suppose $f(X) = 0$ for all $X \in SL(n, \mathbb{Z})$. Next, recall that elementary matrices representing row addition generate $SL(2, \mathbb{R})$ as described in [7]. Denote the elementary matrix with 1 along the diagonal, a real number r in row i and column j, and 0 everywhere else by $e_{ij}(r)$. We can decompose any matrix $X \in SL(n, \mathbb{R})$ by $X = e_{i_1j_1}(r_1)e_{i_2j_2}(r_2)...e_{i_nj_n}(r_n)$. In a similar fashion to above, we can restrict to the rationals - it suffices to show for any $Q \in SL(n, \mathbb{Q})$ we have $f(Q) = 0$. We fix some $Q \in SL(n, \mathbb{Q})$ and denote $Q = (\frac{q_1}{p_1}, \frac{q_2}{p_2}, ...\frac{q_{n^2}}{p_{n^2}})$. We now consider $Q(t) = (t\frac{q_1}{p_1}, t\frac{q_2}{p_2}, ...t\frac{q_{n^2}}{p_{n^2}})$. Now $f(Q(t))$ is a single variable polynomial, and $f(Q(t))$ is 0 when t is an integer multiple of the least common multiple of $p_1, p_2, ...p_{n^2}$. There are infinitely many such t values, so f is identically 0. This holds for all $Q \in SL(2, \mathbb{Q})$ and this is all we must show to conclude $\mathrm{Zcl}(SL(n, \mathbb{Z})) = SL(n, \mathbb{R})$.

In the following theorem the notation $\langle U \rangle$ denotes the subgroup generated by elements of U. This generalization of Ratner's orbit closure theorem is further discussed in [5].

Theorem 4.6. *Let G be a Lie group and Γ be a closed subgroup (not necessarily a lattice). Let $W \leq G$ and $U \subset W$ such that U consists of Ad_G-unipotent elements and $\mathrm{Ad}_G(W) \subset \mathrm{Zcl}(Ad_G(\langle U \rangle))$. Suppose G/Γ has a finite G-invariant measure. Then for all $x \in G/\Gamma$, there exists a closed subgroup $L < G$ containing W with $\overline{Wx} = Lx$.*

Remark 4.7. There is another part to this theorem that serves as an analog to the second part of Ratner's Orbit Closure theorem - the bit where $x\Gamma x^{-1} \cap \Gamma$ forms a lattice in L. Recall this is equivalent to the existence of a finite measure on Lx/Γ. In this case, instead of L we look at the connected component of L about the identity denoted L^0.

5. Acknowledgments

I am incredibly grateful for my mentor, Elizaveta Shuvaeva, and I would like to thank her for her hard work and guidance in writing this paper and teaching me all about Lie groups and related topics. I would also like to thank Daniil Rudenko and László Babai for their excellent lectures. Finally, I would like to thank Peter May for organizing the program and reading all of our papers.

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