CLIFFORD ALGEBRAS AND BOTT PERIODICITY

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Abstract. In this expository paper, we present Max Karoubi's 1968 proof of the real Bott Periodicity Theorem. This argument connects the eightfold periodicity in real K-theory with an analogous eightfold periodicity in the structures of the Clifford Algebras $C^{k,0}$ (observed in a 1964 paper by Michael Atiyah, Raoul Bott, and Arnold Shapiro). We first introduce the Clifford algebras and present their periodicity properties. We then explain Karoubi's argument, which directly connects the Clifford algebras to real K-theory, and thereby offers an alternative proof of real Bott Periodicity. Along the way, we note possible footholds for an analogous proof of real equivariant Bott Periodicity.

Contents

1. Introduction

Bott Periodicity is a fundamental result in topological K-theory. One way to view it is as a pattern in the real K theory of a space X, namely that

$$KO^{n+8}(X) = KO^n(X)$$

Equivalently, one can, as Raoul Bott originally did in the 1959 paper [3], take the direct limit $O(\infty)$ of the orthogonal groups $O(n)$, with appropriate inclusions

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$O(n) \to O(n+1)$ defined, and compute the homotopy groups of O as follows:

$$\begin{aligned}
\pi_0(O(\infty)) &= \mathbb{Z}_2 \\
\pi_1(O(\infty)) &= \mathbb{Z}_2 \\
\pi_2(O(\infty)) &= 0 \\
\pi_3(O(\infty)) &= \mathbb{Z} \\
\pi_4(O(\infty)) &= 0 \\
\pi_5(O(\infty)) &= 0 \\
\pi_6(O(\infty)) &= 0 \\
\pi_7(O(\infty)) &= \mathbb{Z} \\
\pi_{n+8}(O(\infty)) &= \pi_n(O(\infty))
\end{aligned}$$

One notes that $KO^{-n}(*)$, where $*$ is the space consisting of a single point, is isomorphic to $\pi_n(O(\infty))$.

In the 1964 paper [2], Bott, working with Michael Atiyah and Arnold Shapiro, found a similar periodicity in the Clifford algebras $C^{k,0}$.[1] Furthermore, they found Abelian groups A_k associated with these Clifford algebras with the curious property that $A_k \cong \pi_K(O(\infty)) \cong KO^{-k}(*)$. In fact, Atiyah, Bott, and Shapiro found an isomorphism between the graded ring composed of the groups A_k and the graded ring $\sum_{k\geq 0} KO^{-k}(*)$, connecting the algebraic and K-theoretic periodicity results [2][Theorem (11.5)]. However, in order to show that this map was indeed an isomorphism, they needed to use Bott Periodicity; they were not able to prove this theorem from the structure of the Clifford algebras.

In the 1965 paper [8], Regina Wood used Atiyah, Bott, and Shapiro's results on Clifford algebras to prove an eightfold periodicity result about the loop spaces of the general linear group $GL(\infty, \mathbb{Z})$. In particular, Wood showed that $GL(\infty, \mathbb{R})_*$, the path component of $GL(\infty)$ containing the basepoint, is weak homotopy equivalent to the eightfold loop space $\Omega^8(GL(\infty, \mathbb{R}))_*$. The result is presented as a special case of Proposition 4.7 in [8].

Max Karoubi provided a more abstract proof of real Bott Periodicity in 1968, in [5]. Karoubi defined the K-theory of a certain type of category, a so-called Banach category. His definition of this K-theory is built upon the theory of modules of Clifford algebras, and so for a Banach category $\mathcal{C}$, one immediately gets $K^{n+8}(\mathcal{C}) \cong K^n(\mathcal{C})$ from Atiyah, Bott, and Shapiro's work. Karoubi then shows how to relate the real topological K-theory of a space X to K-theory of a certain category, from which he is able to derive Bott periodicity more generally.

In this paper, we are specifically interested in presenting Karoubi's proof. Karoubi's audacious approach to the problem is to define several new notions of K-theory in a more categorical setting, define groups K^n whose periodicity is an immediate consequence of the periodicity of Clifford Algebras, and then show that these abstract K-groups are isomorphic to the KO-theory of familiar topological spaces.

We begin by introducing the Clifford algebras $C^{k,0}$ and studying the modules over them, restricting ourselves to the most part to the ingredients needed for Karoubi's proof of Bott Periodicity. Conspicuously missing from this exposition are the Spin groups, which play no part in Karoubi's proof. Then, we build up the

[1] They referred to these algebras simply as C_k; we will often use this notation when we only need to consider a single Clifford algebra

mountain of definitions underlying Karoubi's proof: we introduce Banach categories and the group $K(\mathcal{C})$; the notion of a $\mathcal{C}$-bundle; the K-theory of functors and of certain pullback diagrams called Banach Squares; and the K-theory of categories, functors, and Banach squares with respect to Clifford algebras, which is where periodicity originates. With these definitions in place, we obtain a general theorem about these categorical notions of K-theory, which has real Bott Periodicity as a corollary.

We will focus on describing the large-scale arguments made in these papers; many details will be left to their original sources. Along the way, we will allude to places where proofs may be adapted to account for equivariance; the only known proof of real equivariant Bott Periodicity, due to Atiyah in [1], relies on tools from analysis, and a proof of the equivariant case using the more algebraic tools of Karoubi would be quite desirable.

2. Clifford Algebras

Clifford Algebras can be defined quite generally, as follows. Let k be a field, V a vector space over k, and Q a quadratic form on V. Let $T(V)$ be the tensor algebra over V. That is,

$$T(V) = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

There is a natural ($\mathbb{Z}$-)grading on $T(V)$, where we view elements in the $V^{\otimes k}$ summand to be in the grade-k subspace, and consider the negative-graded components to be zero.

Now, we take the quotient of $T(V)$ by the ideal I generated by $v \otimes v - Q(v) \cdot 1$ for each $v \in V$, and call this quotient $C(V,Q)$. $C(V,Q)$ is known as a Clifford algebra. Notice that this quotient identifies elements of the grade-n space with elements of the grade-$(n-2)$ space. For instance, we identify the grade-2 element $v \otimes v$ with the grade-0 element $Q(v) \cdot 1$. Thus, we can naturally view $C(V,Q)$ as being $\mathbb{Z}_2$ graded: elements of $T(V)$ which were $2k$-graded are 0-graded in $C(V,Q)$, and those which were $(2k+1)$-graded are 1-graded in $C(V,Q)$. We have an injection $i_Q : V \to C(V,Q)$ given by composing the natural injection of V into $T(V)$ with the quotient map $V \to C(V,Q)$; that this map is indeed an injection is seen in [2, Proposition (1.1)].

The Clifford algebra $C(V,Q)$ enjoys the following property, as noted in [2, Propositions (1.2)-(1.3)].

Proposition 2.1. *Let A be a k-algebra and $\phi : V \to A$ a linear map, such that for all $x \in V$ we have $\phi(x)^2 = Q(x) \cdot \phi(1)$. Then there exists a unique map $\tilde{\phi} : C(V,Q) \to A$ such that $\tilde{\phi} \circ i_Q = \phi$.*

In particular, this extension is defined on tensor products of elements of V, such as $x \otimes y$, by $\tilde{\phi}(x \otimes y) = \phi(x)\phi(y)$, using the algebra structure of A. The condition that $\phi(x)^2 = Q(x) \cdot \phi(1)$ ensures that the homomorphism is compatible with the fact that in $C(V,Q)$, we have $x \otimes x = Q(x) \otimes 1$.

We can naturally view $C(V,Q)$ as a vector space over k.

Proposition 2.2. *The dimension of $C(V,Q)$ as a k-vector space is $2^{\dim_k(V)}$. If $e_1, \ldots, e_n$ is a basis for V, then a basis of $C(V,Q)$ is given by products of the form $e_{i_1} \cdots e_{i_r}$, $i_1 < i_2 < \cdots < i_r$.*

A proof is given in [2, Proposition (1.4)] by comparing $C(V,Q)$ to the exterior algebra on V. In fact, if we let Q be the quadratic form $Q(v)=0$, then $C(V,Q)$ actually is the exterior algebra.

We also have the following general property, which allows us to combine or decompose Clifford algebras.

Lemma 2.3. *Let $V=V_1\oplus V_2$ be an orthogonal decomposition of V (with respect to the inner product $\langle x, Ay\rangle$ corresponding to the quadratic form $Q(x)=x^TAx$). Let Q_1 be the restriction of Q to V_1, and likewise for V_2. Then we have an isomorphism*

$$\psi : C(V,Q)\cong C(V_1,Q_1)\hat{\otimes}_k C(V_2,Q_2)$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product over k.

Specifically, the algebra $C(V_1,Q_1)\hat{\otimes}_k C(V_2,Q_2)$ has the same underlying set as the ordinary tensor product over k. If x is in the i-grade of $C(V_2,Q_2)$ and y is in the j-grade of $C(V_1,Q_1)$, then we have

$$(u\otimes x)\cdot(y\otimes v)=(-1)^{ij}uy\otimes xv$$

where u and v can be any elements of the appropriate algebras.

Proof of Lemma 2.3. Consider a vector $v\in V$; let v_1 and v_2 be the projections of v onto V_1 and V_2. Define $\phi : V\to C(V_1,Q_1)\hat{\otimes}_k C(V_2,Q_2)$ by $\phi(v)=(v_1\otimes 1)+(1\otimes v_2)$. This map is linear, and we observe that

$$\begin{aligned}
\phi(v)^2 &= ((v_1\otimes 1)+(1\otimes v_2))^2 \\
&= Q(v_1)(1\otimes 1)+Q(v_2)(1\otimes 1)+(v_1\otimes 1)(1\otimes v_2)+(1\otimes v_2)(v_1\otimes 1) && (2.4)\\
&= Q(v_1)(1\otimes 1)+Q(v_2)(1\otimes 1)+(v_1\otimes v_2)-(v_1\otimes v_2) && (2.5)\\
&= (Q(v_1)+Q(v_2))(1\otimes 1) && (2.6)\\
&= Q(v)(1\otimes 1), && (2.7)
\end{aligned}$$

where we have arrived at 2.5 by using the multiplication in the given tensor product; in particular, the minus sign has appeared as a result of the 1-grading on v_1 and v_2. It follows that the map ϕ satisfies the conditions of 2.1, and so there is a homomorphism extending ϕ,

$$\hat{\phi} : C(V,Q)\to C(V_1,Q_1)\hat{\otimes}_k C(V_2,Q_2).$$

That $\hat{\phi}$ is an isomorphism can be seen by checking that a basis of $C(V,Q)$ is mapped to a basis of $C(V_1,Q_1)\hat{\otimes}_k C(V_2,Q_2)$; see [2, Proposition (1.6)]. □

We now restrict our attention to some specific Clifford algebras. We focus on the real vector spaces $\mathbb{R}^k$, with orthonormal basis $e_1,e_2,\dots,e_k$. Let $Q_{p,q}$, for $p+q=k$, be the quadratic form on $\mathbb{R}^k$ defined by

$$Q_{p,q}(e_i)=\begin{cases}-1, & i\le p\\ 1, & p+1\le i\le p+q\end{cases} \tag{2.8}$$

In other words, $Q_{p,q}$ is negative on the first p basis elements and positive on the remaining q basis elements. We will denote by $C^{p,q}$ the Clifford algebra $C(\mathbb{R}^k,Q_{p,q})$. The Clifford algebras $C^{k,0}$ and $C^{0,k}$ are of particular interest, and so we will also refer to them as C_k and C'_k, respectively.

The algebras $C^{p,q}$ are related by the following special case of Lemma 2.3:

Corollary 2.9. *The Clifford algebra $C^{p+p',q+q'}$ is isomorphic to*

$$C^{p,q}\hat{\otimes}_{\mathbb{R}}C^{p',q'}$$

In particular, C_k is isomorphic to the k-fold graded tensor product of C_1 with itself.

The structure of the algebras C_k is elucidated by the following result.

Proposition 2.10. *For e_i, e_j two basis vectors of $\mathbb{R}^k$, we find that in C_k, we have*

$$e_i^2 = -1 \text{for } i \neq j,\ e_i e_j = -e_j e_i$$

Proof. That $e_i^2 = -1$ follows immediately from viewing C_k as the k-fold graded tensor product of C_1 with itself. To see that $e_i e_j = -e_j e_i$, one can write

$$\begin{aligned}
e_i e_j &= e_i(e_i + e_j - e_i) \\
&= e_i(e_i + e_j) - e_i^2 \\
&= (e_i + e_j - e_j)(e_i + e_j) - e_i^2 \\
&= (e_i + e_j)^2 - e_j(e_i + e_j) - e_i^2 \\
&= (e_i + e_j)^2 - e_j e_i - e_j^2 - e_i^2 \\
&= -2 - e_j e_i + 1 + 1 \\
&= -e_j e_i
\end{aligned}$$

where we have used the fact that for our quadratic form Q with $Q(e_s) = -1$ for all s, we have $Q(e_i + e_j) = -2$. ☐

Note that the proof that $e_i e_j = -e_j e_i$ holds just the same in C'_k, except that we will write

$$(e_i + e_j)^2 - e_j e_i - e_j^2 - e_i^2 = 2 - e_j e_i - 1 - 1$$

instead.

In fact, there are even more relations between the various algebras $C^{p,q}$. Before exploring them further, however, let us consider the first few algebras C_k.

Example 2.11. Let us attempt to identify the Clifford algebra $C_1 = C^{1,0}$. This algebra is built upon the vector space $\mathbb{R}^1$, generated by the single basis element e_1. The tensor algebra $T(\mathbb{R}^1)$ can be viewed as consisting as polynomials of the form

$$a_0 + a_1 e_1 + a_2(e_1 \otimes e_1) + \cdots + a_n e_1^{\otimes n}$$

Now, we quotient out by the relation $e_1 \otimes e_1 = Q(e_1) \cdot 1 = -1$. The result is that all elements of $C(V, Q)$ can be written in the form $a + be_1$, with the multiplication

$$(a + be_1)(c + de_1) = (ac - bd) + (bc + ad)e_1$$

This algebra, then, is isomorphic to the complex numbers $\mathbb{C}$.

By Proposition 2.2, we could have immediately seen that the the dimension (over $\mathbb{R}$) of C_1 is 2 (since the underlying vector space is 1-dimensional); of course, the dimension of $\mathbb{C}$ as a real vector space is also 2, as we would expect.

Example 2.12. Let us now study the Clifford algebra C_2. By Proposition 2.2 and Proposition 2.10, we can express C_2 as a 4-dimensional real vector space with the basis $1, e_1, e_2, e_1e_2$ and the following multiplication relations:

$$e_1 \cdot e_2 = e_1e_2e_2 \cdot e_1 = -e_1e_2$$
$$e_2 \cdot (e_1e_2) = e_1(e_1e_2) \cdot e_2 = -e_1$$
$$(e_1e_2) \cdot e_1 = e_2e_1 \cdot (e_1e_2) = -e_2$$
$$e_1^2 = e_2^2 = (e_1e_2)^2 = -1e_1 \cdot e_2 \cdot (e_1e_2) = -1$$

It is now apparent that C_2 is isomorphic to the quaternions, $\mathbb{H}$, with e_1, e_2, and e_1e_2 filling the roles of i, j, and k, respectively. One notes that in this isomorphism, we should think of the quaternions i and j as being 1-graded, since they are expressed as products of odd numbers of basis elements of $\mathbb{R}^2$; likewise, 1 and k are both 0-graded.

Example 2.13. Finally, let us consider C_3. By Proposition 2.2 and Proposition 2.10, we write C_3 as an 8-dimensional real vector space with basis

$$e_1e_2$$
$$e_3e_1e_2$$
$$e_2e_3e_1e_3$$
$$1e_1e_2e_3$$

As an algebra, there is an anti-commutative multiplication, and the square of any of these basis elements is -1.

There is a non-obvious isomorphism between C_3 and $\mathbb{H} \oplus \mathbb{H}$. In particular, one can take $1 = (1 + e_1e_2e_3)/\sqrt{2}$ and $\bar{1} = (1 - e_1e_2e_3)/\sqrt{2}$ to be the identity elements of the first and second copies of $\mathbb{H}$. In the first copy, we let

$$i = (e_1e_3 + e_2)/(\sqrt{2})$$
$$j = (e_2e_3 + e_1)/\sqrt{2}$$
$$k = (e_1e_2 - e_3)/(\sqrt{2})$$

and it is a straightforward exercise to verify the quaternion relationships, recalling that the "-1" which these elements should square to is $-1 - e_1e_2e_3$. Similarly, we define

$$\bar{i} = (e_1e_3 - e_2)/\sqrt{2}$$
$$\bar{j} = (e_2e_3 - e_1)/\sqrt{2}$$
$$\bar{k} = (e_1e_2 + e_3)/\sqrt{2}$$

Notice that the elements $1, i, j, k$ in the first copy of $\mathbb{H}$, and the elements $\bar{1}, \bar{i}, \bar{j}, \bar{k}$ in the second copy of $\mathbb{H}$, are all "mixed-grading," that is, they are written as the sums of grade-0 elements of C_3 and grade-1 elements. Thus, the isomorphism $C_3 \cong \mathbb{H} \oplus \mathbb{H}$ in some sense "respects the $\mathbb{Z}_2$ grading of C_3" less than the isomorphisms $C_2 \cong \mathbb{H}$ or $C_1 \cong \mathbb{C}$ do.

As this third example shows, identifying the Clifford algebras C_k by hand quickly becomes quite difficult as the number of basis elements grows. First, we leave the following fact, corroborated in [2, page 11], as an elementary exercise.

Proposition 2.14. *The algebras C_1' and C_2' (i.e., $C^{0,1}$ and $C^{0,2}$) are isomorphic to $\mathbb{R} \oplus \mathbb{R}$ and $\mathbb{R}(2)$, the ring of all 2×2 matrices, respectively.*

As it turns out, knowing C_1, C_2, C_1', and C_2' is enough to compute, with relatively little computation, the rest of the algebras C_k and C_k'. The following result allows us to bootstrap our way up.

Lemma 2.15. *We have the following isomorphisms:*

$$C'_{k+2} \cong C_k \otimes_{\mathbb{R}} C'_2$$
$$C_{k+2} \cong C_2 \otimes_{\mathbb{R}} C'_k$$

Note that these are ordinary, ungraded tensor products. Thus, this lemma is more useful than Lemma 2.3 when it comes to identifying the Clifford algebras; while the ungraded structures of the algebras C_k and C_k' turn out to be quite familiar, the grading is often quite awkward, as the $C_3 \cong \mathbb{H} \oplus \mathbb{H}$ example shows.

Proof of Lemma 2.15. We shall prove the first isomorphism; the proof of the second is exactly analogous. Let $e_1, \ldots, e_k$ be the generators for C_k and e_1', e_2' the generators for C_2'. Let $v_1, \ldots, v_{k+2}$ be an orthonormal basis of $\mathbb{R}^{k+2}$ with respect to the negative-definite quadratic form used to define C_{k+2}. Consider the linear map $\psi : \mathbb{R}^{k+2} \to C_k \otimes_{\mathbb{R}} C_2'$ defined on the basis $v_1, \ldots, v_k$ by

$$\psi(v_i) = \begin{cases} e_{i-2} \otimes_{\mathbb{R}} (e_1' e_2'), & 2 < i \leq k+2 \\ 1 \otimes_{\mathbb{R}} e_i, & 1 \leq i \leq 2\psi \end{cases}$$

Note that in C_2', we have $(e_1'e_2')^2 = -1$; since multiplication in C_2' is anticommutative, we have

$$e_1'e_2'e_1'e_2' = -e_1'e_2'e_2'e_1' = -1$$

Thus, for $2 < i \leq k+2$,

$$\begin{aligned} \psi(v_i)^2 &= e_{i-2}^2 \otimes_{\mathbb{R}} (e_1'e_2')^2 \\ &= (-1) \otimes_{\mathbb{R}} (-1) \\ &= 1 \otimes 1 \end{aligned}$$

and for $i = 1$ or $i = 2$, we have $\psi(v_i)^2 == 1 \otimes_{\mathbb{R}} (e_i')^2 = 1 \otimes_{\mathbb{R}} 1$. Thus, the map ψ satisfies the property described in Proposition 2.1, so that ψ extends to a map $\tilde{\psi} : C'_{k+2} \to C_k \otimes_{\mathbb{R}} C_2'$. To see that the map is an isomorphism, it is enough to check that the dimensions as real vector spaces are equal and that ψ maps the basis of C'_{k+2} to a linearly independent set in $C_k \otimes_{\mathbb{R}} C_2'$, which follows from the fact that product $v_{i_1} \cdots v_{i_r}$ in the basis of C'_{k+2} is mapped to a unique product in $C_k \otimes_{\mathbb{R}} C_2'$, and these products must be linearly independent. Thus, the map $\tilde{\psi}$ is bijective, and thus an isomorphism of $\mathbb{R}$-algebras. $\square$

We have a few corollaries which are both easy and extremely useful;.

Corollary 2.16. $C_4 \cong C_4'$.

Proof. Using the isomorphisms of Lemma 2.15, we write

$$C_4 \cong C_2 \otimes_{\mathbb{R}} C_2'$$
$$C_4' \cong C_2 \otimes_{\mathbb{R}} C_2'$$

from which the result immediately follows. $\square$

In fact, one can verify directly that C_4 and C_4' are isomorphic as *graded* algebras. Another less direct proof appears in [5, page 172] as a consequence of [5, Proposition (1.1.14)]. We will make use of a corollary of this fact

Corollary 2.17. *There is an isomorphism of graded algebras between $C^{p,q+4}$ and $C^{p+4,q}$*

Proof. By Lemma 2.3, we write $C^{p,q+4}$ as $C^{p,q}\hat{\otimes}C^{0,4}$, apply the isomorphism mentioned above to show that this tensor product is isomorphic to $C^{p,q}\hat{\otimes}C^{4,0}$, and then apply Lemma 2.3 again to get an isomorphism to $C^{p,q+4}$ □

We have a similar result about ungraded tensor products, which indicates a four-fold periodicity in the algebras C^k.

Corollary 2.18. $C_{k+4} \cong C_k \otimes_{\mathbb{R}} C_4$.

Proof. Using Lemma 2.15, we write

$$C_{k+4} = C_{k+2+2} \cong C_2 \otimes_{\mathbb{R}} C'_{k+2} C'_{k+2} \cong C'_2 \otimes_{\mathbb{R}} C_k$$

so that

$$C_{k+4} \cong C_k \otimes_{\mathbb{R}} C'_2 \otimes_{\mathbb{R}} C_2$$

But using Lemma 2.15 again, $C'_2 \otimes_{\mathbb{R}} C_2 \cong C_4$. Thus, $C_{k+4} \cong C_k \otimes_{\mathbb{R}} C_4$. □

It is this four-fold periodicity which will play a key role in Karoubi's proof of Bott Periodicity. And yet of more immediate interest is the following corollary, which exhibits the first hint that the Clifford algebras may have something to do with the eightfold periodicity phenomenon.

Corollary 2.19. *There is an isomorphism $C_{k+8} \cong C_k \otimes_{\mathbb{R}} C_8$. Likewise, there is a isomorphism $C_{k+8'} \cong C'_k \otimes_{\mathbb{R}} C'_8$*

Proof. Using Corollary 2.18, we write $C_{k+8} \cong C_{k+4} \otimes_{\mathbb{R}} C_4 \cong C_k \otimes_{\mathbb{R}} C_4 \otimes_{\mathbb{R}} C_4$. We apply the result again to get $C_8 \cong C_4 \otimes_{\mathbb{R}} C_4$, and conclude that $C_{k+8} \cong C_k \otimes_{\mathbb{R}} C_8$. The same argument gives the result for the algebras C'_k. □

Notice that $C_8 \cong C_4 \otimes_{\mathbb{R}} C_4$ and $C'_8 \cong C'_4 \otimes_{\mathbb{R}} C'_4$. Since C_4 and C'_4 are isomorphic, this implies that $C_8 \cong C'_8$. In fact, both are isomorphic to the full algebra of 16×16 real matrices.

One can now calculate every algebra C_k. The results, as seen in [2, Table 1], are as follows; here, $F(n)$ denotes the ring of $n \times n$ matrices with entries in the ring F. We also include the results of the tensor products $C_k \otimes_{\mathbb{R}} \mathbb{C}$, which correspond to the two-fold complex Bott Periodicity

k	C_k	C'_k	$C_k \otimes_{\mathbb{R}} \mathbb{C}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$

Note that since $C_8 = \mathbb{R}(16)$, $C_k \otimes_{\mathbb{R}} C_8 = C_k(16)$, the algebra of 16×16 matrices with entries in C_k. Thus, the algebras after C_8 can all be easily computed using the table above.

The pattern amongst the Clifford algebras C_k is striking enough in isolation, but the true beauty emerges when we consider the modules over these algebras. Let $M(C_k)$ be the Grothendieck completion of the commutative monoid generated by the irreducible $\mathbb{Z}_2$-graded C_k modules (i.e. those which contain no submodules) under the operation of direct sum.[2] We define the Abelian group $N(C_k^0)$ analogously, using the ungraded modules over the 0-graded component of C_k; henceforth, we denote by C_k^i the i-graded component of C_k. These two constructions turn out to be equivalent:

Proposition 2.20. *The map R which maps a $\mathbb{Z}_2$-graded module $M = M^0 \oplus M^1$ to the 0-graded component M^0 induces an isomorphism, $M(C_k) \cong N(C_k^0)$.*

This result is proven in [2, Proposition (5.1)]. The proof is short, although not especially enlightening. It proceeds by defining the map S, which sends M^0 to $C_k \otimes_{C_k^0} M^0$, and then verifying that $S \circ R$ and $R \circ S$ are both naturally isomorphic to the identity. The next result can be proven somewhat more directly.

Proposition 2.21. *There is an isomorphism $C_k \cong C_{k+1}^0$.*

Proof. Let $e_1, \ldots, e_{k+1}$ be an orthonormal basis of $\mathbb{R}^{k+1}$ yielding generators for C_{k+1}^0 and, abusing notation slightly, let $e_1, \ldots, e_k$ be an orthonormal basis of $\mathbb{R}^k$ yielding generators for C_k. We consider the map $\phi : \mathbb{R}^k \to C_{k+1}^0$ which maps each basis element e_i to $e_i e_{k+1}$ Then since $\phi(e_i)^2 = e_i e_{k+1} e_i e_{k+1} = -1$, we use Proposition 2.1 to get an extension $\tilde{\phi} : C_k \to C_{k+1}$. This map sends the basis element $e_{i_1} \cdots e_{i_r}$ to the same basis element of C_{k+1} if r is even, and sends it to $e_{i_1} \cdots e_{i_r} e_{k+1}$ if r is odd. Since every basis element of C_{k+1}^0 is either a 0-graded basis element of C_k or a 1-graded basis element of C_k times e_{k+1}, the map $\tilde{\phi}$ maps a basis of C_k to a basis of C_{k+1}^0; thus, it is bijective, and therefore an isomorphism of algebras. □

These two propositions directly yield the following corollary:

Corollary 2.22. *The Abelian group $M(C_k)$ is isomorphic to $N(C_{k-1})$, the Grothendieck completion of the commutative monoid generated by the ungraded C_{k-1}-modules under direct sum.*

Fortunately, the algebras C_k do not admit many irreducible modules. For R a division ring or field, the matrix algebra $R(n)$ is simple (see [4, Theorem (3.8), p. 368]), and since irreducible modules over a ring correspond to maximal ideals,[3] the only irreducible modules over all C_k algebras of this form are the algebras themselves. The remaining algebras C_k are of the form $R(n) \oplus R(n)$ for R a division ring. This implies that the only irreducible modules over these algebras are given

[2] The reader unfamiliar with this construction may view $M(C_k)$ as being the free Abelian group generated by the irreducible $\mathbb{Z}_2$-graded modules over C_k, although we will soon need to conflate the formal expression $S + S$ and the C_k-module $S \oplus S$, for S an irreducible C_k module.

[3] This can be shown by taking an R-module M and considering the R-module generated by a nonzero element $m \in M$. If M is irreducible, then the R-module generated by m must be all of M, and one can then show that the module generated by m is isomorphic to R/I for I maximal by using, for instance, the first isomorphism theorem

by $R(n) \oplus 1$ and $1 \oplus R(n)$. Thus, every $N(C_k)$ is either $\mathbb{Z} \oplus \mathbb{Z}$, when $k = -1 \mod 4$, or $\mathbb{Z}$ otherwise; it follows that $M(C_k)$ is $\mathbb{Z} \oplus \mathbb{Z}$ when $k = 0 \mod 4$ and $\mathbb{Z}$ otherwise.

To make things more interesting, and to coax out a property of C_k which is merely 8-periodic and not 4-periodic, we consider certain maps $M(C_{k+1}) \to M(C_k)$, defined as follows. We have natural injections $i_k : C_k \to C_{k+1}$ defined by sending the generators e_i in C_k to the corresponding generators in C_{k+1}. These induce maps $i_k^* : M(C_{k+1}) \to M(C_k)$. Let A_k be the cokernel of the map i_k^*.

Example 2.23. $A_1 = \mathbb{Z}_2$. To see this, we first recall that $C_2 = \mathbb{H}$, $C_1 = \mathbb{C}$, and thus $M(C_2)$ and $M(C_1)$ are generated by $\mathbb{H}$ and $\mathbb{C}$, respectively. Viewing $\mathbb{H}$ as being generated by $1, e_1, e_2, e_1e_2$ and $\mathbb{C}$ as being generated by $1, e_1$, we observe that i_1^* maps the $\mathbb{H}$-module $\mathbb{H}$ to the $\mathbb{C}$-module $\mathbb{C} \oplus \mathbb{C}$; that is, when we only consider the action of $C_1 \subset C_2$ on the C_2-module $\mathbb{H}$, the submodule generated by 1 and e_1 becomes "detached" from the submodule generated by e_2 and e_1e_2. Multiplication by 1 is still the identity, and multiplication by e_1 switches the two generators of each submodule and multiplies by -1. Thus, i_1^* maps the generator of $M(C_2)$ to twice the generator of $M(C_1)$, so that the cokernel of this map is $\mathbb{Z}_2$

If one computes the rest of these cokernels, as in [2], one obtains the following results (see Table 2 of [2]).

k	C_k	$M(C_k)$	A_k
1	$\mathbb{C}$	$\mathbb{Z}$	$\mathbb{Z}_2$
2	$\mathbb{H}$	$\mathbb{Z}$	$\mathbb{Z}_2$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{Z}$	0
4	$\mathbb{H}(2)$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$
5	$\mathbb{C}(4)$	$\mathbb{Z}$	0
6	$\mathbb{R}(8)$	$\mathbb{Z}$	0
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{Z}$	0
8	$\mathbb{R}(16)$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$

Furthermore, since $C_{k+8} \cong C_8 \otimes_{\mathbb{R}} C_k$, we have $A_k \cong A_{k+8}$. For details, see [2, §6] and in particular [2, Proposition (6.8)], in which this isomorphism is given explicitly by forming a $\mathbb{Z}$-graded ring from the groups $M(C_k)$, with multiplication given by graded tensor products, and using this structure to show that multiplication by the unique irreducible C_8-module induces an isomorphism between A_k and A_{k+8}

We now have an eightfold periodicity in the cokernels A_k. Furthermore, these cokernels correspond exactly to the homotopy groups of O. This suggests a deep connection between the Clifford algebras C_k and real K-theory. Before moving on to Karoubi's proof of Bott Periodicity, which elucidates that connection, we make a quick note for those interested in the equivariant case.

Remark 2.24. Let G be a group and let $\mathbb{Z}[G]$ be the corresponding group ring over $\mathbb{Z}$. Suppose we consider the groups $M(C_k) \otimes_{\mathbb{Z}} \mathbb{Z}[G]$. Then the right exactness of the tensor product turns the right exact sequence

$$M(C_{k+1}) \to M(C_k) \to A_k \to 0$$

into the right exact sequence

$$M(C_{k+1}) \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to M(C_k) \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to A_k \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to 0$$

In other words, not only do we see the eightfold periodicity when we introduce a group action, we see it in the exact same way. This perhaps suggests that the equivariant scenario may not be all that different, algebraically, from the nonequivariant case.

3. Karoubi's Proof

We now present Karoubi's 1968 proof of real Bott Periodicity, as described in [5]. The proof requires quite a bit of preliminary development first, so we begin by defining and studying the K-Theory of Banach Categories. We will then define the category of $\mathcal{C}$-bundles over a space X, where $\mathcal{C}$ is a Banach category; the motivating example for this is when $\mathcal{C}$ is the category of finite dimensional real vector spaces, in which case a $\mathcal{C}$-bundle over X is just a real vector bundle over X. These bundles motivate the definitions of the K-theory of functors and of certain pullback diagrams, which we call Banach Squares.

We will then incorporate Clifford algebras, defining the group K^n of a Banach category $\mathcal{C}$ to be, more or less, "the K-theory of the C_n-modules in $\mathcal{C}$." The four- and eight-fold periodicity of the modules over Clifford algebras, which we studied in the previous section, thus induce by definition an eightfold periodicity on $K^n(\mathcal{C})$. These K-theory groups are, in fact, not topological at all; each one is just the Grothendieck completion of a certain monoid. In the last part of this section, we put all this machinery together; we outline Karoubi's proof of a theorem which connects the new groups K^n with the familiar KO-groups of certain pairs of spaces. In fact, we will prove a more general result pertaining to certain K^V groups, for V a vector bundle of X, which are the "K-theory of the category of $\mathcal{C}$-bundles over X which are also $C(V)$-modules.

Most of this section consists of definitions; the miracle of Karoubi's proof is that they turn out to be the right definitions.

3.1. **Banach Categories and their K-theory.**

First of all, we need to develop some basic theory of Banach categories. We begin with the basic definitions and some examples relevant to K-theory. The next main goal will be to define the K-theory of a Banach category, and more generally the K-theory of an additive category, which is defined like topological K-theory, by way of Grothendieck completions. Once we have those basic definitions out of the way, we will be in a position to study more complicated structures.

First, we recall the definition of an additive category.

Definition 3.1. A category $\mathcal{C}$ is called preadditive if for all objects X and Y in $\mathcal{C}$, the hom set $\hom_{\mathcal{C}}(X, Y)$ has the structure of an Abelian group, and the for all objects X, Y, and Z in $\mathcal{C}$, the composition map

$$\hom_{\mathcal{C}}(X, Y) \times \hom_{\mathcal{C}}(Y, Z) \to \hom_{\mathcal{C}}(X, Z)$$

is bilinear.

Definition 3.2. A preadditive category $\mathcal{C}$ is called additive if it has all finite products.

Of particular note for our purposes is that an additive category has a zero object and direct sums, see [7, Definition (12.3.8)].

We need several more definitions before we work our way up to the definition of a Banach category. First, we introduce the preliminary notion of a "k-prebanach category:"

Definition 3.3. Let $\mathcal{C}$ be an additive category, and let k be a field (either $\mathbb{R}$ or $\mathbb{C}$). We call $\mathcal{C}$ a k-prebanach category if:

(1) For each pair of objects M, N in $\mathcal{C}$, the hom-set $\hom_{\mathcal{C}}(M, N)$ has the structure of a Banach space over k, with the same additive structure as when $\mathcal{C}$ is simply viewed as an additive group.
(2) For all triples of objects M, N, P in $\mathcal{C}$, the composition map
$$\hom_{\mathcal{C}}(M, N) \times \hom_{\mathcal{C}}(N, P) \to \hom_{\mathcal{C}}(M, P)$$
is both bilinear and continuous.

The property that makes a prebanach category into a Banach category has to do with idempotent endomorphisms.

Definition 3.4. Let $\mathcal{C}$ be a category and X an object in $\mathcal{C}$. An endomorphism $p : X \to X$ is idempotent if $p \circ p = p$. An idempotent endomorphism is also called a projection.

Notice that in the category of vector spaces, an idempotent endomorphism is just a projection in the familiar sense.

Definition 3.5. A k-prebanach category $\mathcal{C}$ is called k-Banach if every idempotent endomorphism $p : X \to X$ admits a kernel.

In essence, a Banach category is a category whose hom-sets are Banach spaces, where composition is both bilinear and continuous, and with kernels for all idempotent endomorphisms. The existence of direct sums is a crucial property of Banach categories, as we use it to define a vast generalization of the familiar group $K(X)$ of vector bundles over X.

Definition 3.6. Let $\mathcal{C}$ be a small additive category. We define the K-theory of $\mathcal{C}$, written $K(\mathcal{C})$, to be the Grothendieck completion of the Abelian monoid of isomorphism classes of objects in $\mathcal{C}$ under direct sum.

Remark 3.7. Definition 3.6 does not make use of the full structure of Banach categories. The condition that the hom-sets of a Banach category $\mathcal{C}$ must be normed topological spaces will be used in the construction of $\mathcal{C}$-bundles in the following section. The condition that these hom-sets must also be real vector spaces is more subtle, and will be useful when we consider the K-theory of functors, as in the proof of Proposition 3.19.

The similarity of Definition 3.6 to the definition of topological K-theory is no mere coincidence; our next goal will be to construct from a space X a Banach category whose K-theory is $K(X)$. Of course, we will actually construct something more general.

3.2. **Banach Category Bundles.** The following construction, given as Lemma and Definition (1.2.2) in [5], allows us to obtain a Banach category from a prebanach one; this construction is known as the Karoubi envelope.[4] Let $\mathcal{C}$ be a prebanach

[4]More generally, this construction turns a preadditive category into a pseudo-abelian one

category. Let $\mathcal{C}'$ be the category whose objects are pairs (E, p), where E is an object in $\mathcal{C}$ and $p : E \to E$ is an idempotent endomorphism. The morphisms in $\mathcal{C}'$ from (E, p) to (E', p') will be given by maps $f : E \to E'$ such that $p' \circ f = f \circ p$. That is, the following diagram must commute:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ E \arrow[r, "f"'] & E' \end{tikzcd}$$

Now, we introduce an equivalence relation between maps $(E, p) \to (E', p')$ in $\mathcal{C}'$, namely that f and g are equivalent if $p' \circ f = p' \circ g$. We denote by f' the equivalence class of f under this relation. Now, let $\tilde{\mathcal{C}}$ be the category with the same objects as $\mathcal{C}'$, and with morphisms given by the equivalence classes of morphisms in $\mathcal{C}'$. Intuitively, one should think of the object (E', p') as representing the image of the projection p; two maps into the pair (E', p') are then equivalent if they have the same images in the image of p'.

Proposition 3.8. *$\tilde{\mathcal{C}}$ is a Banach category.*

We offer a sketch of the proof below; the remainder of the details can be verified through a straightforward diagram chase, see [5, Definition and Lemma (1.2.2)].

Proof. Most of the Banach category structure of $\tilde{\mathcal{C}}$ is inherited from $\mathcal{C}$; we need only check that if (E, p) is an object in $\tilde{\mathcal{C}}$ and $f : (E, p) \to (E, p)$ is an idempotent endomorphism of that object, then we can find a kernel of f. This kernel is given by the map

$$(1 - f') : (E, p \circ (1 - f')) \to (E, p)$$

where 1 is the equivalence class of the identity map $E \to E$. In particular, let $g' : (F, q) \to (E, p)$ be a map such that $f' \circ g' = 0$. Then one can show, by a diagram chase, that $g' : (F, q) \to (E, p)$ factors uniquely through the map $g' : (F, q) \to (E, p(1 - f'))$. □

The idea behind this construction is that the kernel of a projection $f : E \to E$ is represented by E together with a projection onto the orthogonal complement of the image of f, that is, a projection onto the kernel of f; the vector-space structure of the hom-sets of $\mathcal{C}$ guarantees the existence of a map $(1 - f)$ given f.

We will make extensive use of this construction, which is sometimes called the idempotent completion or the Karoubi envelope construction. Its usefulness, broadly speaking, is that it allows us to access "subobjects" of the objects in an abstract Banach category.

The following examples, which can be found in [5, page 178] describe important examples of Banach categories.

Example 3.9. Let $\mathcal{V}_{\mathbb{R}}$ be the category of finite dimensional real vector spaces. The hom-sets in $\mathcal{V}_{\mathbb{R}}$ are described by finite dimensional real vector spaces, so in particular they are Banach spaces; the composition maps are given by matrix multiplication, which is both bilinear and continuous, since we are in the finite dimensional case. Since all the objects in $\mathcal{V}_{\mathbb{R}}$ are finite dimensional vector spaces, this category certainly has all finite products. Finally, the kernels of projection maps are just the classical kernels with injections; since these will always be finite dimensional vector spaces with linear maps, these kernels always exist in $\mathcal{V}_{\mathbb{R}}$. Thus, $\mathcal{V}_{\mathbb{R}}$ is a Banach category.

The next example suggests a connection between Banach categories and topological K-theory.

Example 3.10. Let A be a Banach algebra, i.e. an algebra over $\mathbb{R}$ or $\mathbb{C}$ which is also a Banach space. Let $\mathcal{C}$ be the category of free A-modules of finite type. One can check that $\mathcal{C}$ is a prebanach category; the associated Banach category $\tilde{\mathcal{C}}$ can be identified with the category $\mathcal{L}(A)$ of projective A-modules of finite type. If X is a compact space and A is the $\mathbb{R}$-algebra of continuous functions $X \to \mathbb{R}$, then $\mathcal{A}$ is, in turn, equivalent to the category $\mathcal{E}_K(X)$ of finite-dimensional vector bundles over X. This equivalence is the Serre-Swan theorem; see, for instance, [6].

It is worth noting that Banach algebras played a central role in Wood's proof of the Bott periodicity theorem.

Our final two examples (also found in [5, page 178]), for now, describe Banach subcategories of $\mathcal{E}_K(X)$. The first will play a role in the proof of Bott periodicity, while the second will not be used here, but explains one way to bring equivariance into the picture.

Example 3.11. Let V be a vector bundle over X, equipped with a non-degenerate quadratic form Q. Then we define $C(V)$ to be the algebra bundle over X whose fibers are the Clifford algebras $C(V, Q)$ over the fibers of V. Let $\mathcal{E}^V(X)$ be the subcategory of $\mathcal{E}_K(X)$ consisting of vector bundles which are also $C(V)$ modules and maps which are also module homomorphisms. The prebanach category structure is then inherited from the Banach category structure of $\mathcal{E}_K(X)$. Furthermore, we constructed $\mathcal{E}_K(X)$ as a Banach category through the construction described in Proposition 3.8, and so the kernel of a projection $f : E \to E$ appears as a pair $(E, p(1 - f'))$; in particular, if E is a $C(V)$ module, this kernel is as well. Thus, $\mathcal{E}^V(X)$ is a Banach category. Notice that among the objects in this category is the bundle $C(V)$ itself, which gains a $C(V)$ action by the usual multiplication of the Clifford algebra; in particular, $\mathcal{E}^V(X)$ is not empty.

Example 3.12. Let G be a topological group which acts on a compact space X. Let $\mathcal{E}_G(X)$ be the category of finite dimensional vector bundles with an action by G compatible with projection onto X. The hom-sets in $\mathcal{E}_G(X)$ can be viewed as closed subspaces of the corresponding hom-sets in $\mathcal{E}_K(X)$, making $\mathcal{E}_G(X)$ into a Banach category as well.

The most obvious feature of Banach categories is the Banach space structure of the hom-sets. Of special interest, then, are the functors between Banach categories which preserve this structure.

Definition 3.13. Let $\mathcal{C}$ and $\mathcal{C}'$ be two Banach categories. A functor $\phi : \mathcal{C} \to \mathcal{C}'$ is called linear-continuous if for all objects M, N in $\mathcal{C}$, the natural map from $\hom_{\mathcal{C}}(M, N)$ to $\hom_{\mathcal{C}'}(\phi(M), \phi(N))$ is linear and continuous. A linear continuous functor which is also an equivalence of categories is called an equivalence of Banach categories.

Two particularly important classes of linear-continuous functors are the Serre functors and quasi-surjective functors, defined as follows.

Definition 3.14. Let $\mathcal{C}$ and $\mathcal{C}'$ be Banach categories and $\phi : \mathcal{C} \to \mathcal{C}'$ a linear-continuous functor. We say that ϕ is a Serre functor if it satisfies the following equivalent conditions:

(1) For all objects M in $\mathcal{C}$, the the natural map $A_m : Aut_{\mathcal{C}}(M) \to Aut_{\mathcal{C}'}(\phi(M))$ is a Serre fibration.
(2) For all objects M in $\mathcal{C}$, the natural map $E_m : End_{\mathcal{C}}(M) \to End_{\mathcal{C}'}(\phi(M))$ is surjective.
(3) For all pairs of objects M, N in $\mathcal{C}$, the natural map $E_{m,n} : \hom_{\mathcal{C}}(M, N) \to \hom_{\mathcal{C}'}(\phi(M), \phi(N))$ is surjective.

A proof of the equivalence of these conditions is given in [5, Proposition (1.2.7)].

Definition 3.15. Let $\mathcal{C}$ and $\mathcal{C}'$ be two Banach categories and $\phi : \mathcal{C} \to \mathcal{C}'$ a functor between them. The functor ϕ is called quasi-surjective if every object in $\mathcal{C}'$ is a direct factor of an object in the image of $\mathcal{C}'$.

The fact that the hom-sets in Banach categories are required to be Banach spaces, and not merely vector spaces, hints at the importance of the topological structure of Banach categories, as does the use of Serre fibrations in defining Serre functors. Another important feature of this structure is that it allows us to define a notion of "$\mathcal{C}$-bundles over topological spaces" when $\mathcal{C}$ is a Banach space.

Let $\mathcal{C}$ be a Banach category and X a topological space. Let us construct the category $\mathcal{C}_T(X)$, called the "category of trivial $\mathcal{C}$-bundles over X," as follows. The objects of $\mathcal{C}_T(X)$ are the objects of $\mathcal{C}$. A map $M \to N$ in $\mathcal{C}_T$ is a continuous map from X to $\hom_{\mathcal{C}}(M, N)$; that is, it is a choice of map $f_x \in \hom_{\mathcal{C}}(M, N)$ for each point $x \in X$, such that these maps vary continuously with X.

Example 3.16. If we take $\mathcal{C}$ to be $\mathcal{V}_{\mathbb{R}}$, then $\mathcal{C}_T(X)$ is the category of trivial real vector bundles over X; in particular, the objects in this category represent the fiber $\mathbb{R}^k$ and the morphisms are continuous maps which are linear on each fiber.

The hom-sets of $\mathcal{C}_T(X)$ inherit a vector space structure from the vector space structure of the hom-sets of $\mathcal{C}$. If we use the norm of the hom-sets of $\mathcal{C}$, we can assign the supremum norm to each map $X \to \hom_{\mathcal{C}}(M, N)$, that is, to each map in $\hom_{\mathcal{C}_T(X)}(M, N)))$, making the hom-sets of $\mathcal{C}_T(X)$ into Banach spaces. In particular, this choice of norm ensures that the hom-sets are complete. Bilinearity and continuity of composition can be shown using the fact that these properties hold at each point; $\mathcal{C}_T(X)$ has the same finitary products as $\mathcal{C}$. Thus, $\mathcal{C}_T(X)$ is a prebanach category, although not necessarily a Banach category.

Using Proposition 3.8, we define $\mathcal{C}(X)$ to be the Karoubi envelope of $\mathcal{C}_T(X)$, and call $\mathcal{C}(X)$ the category of locally-trivial $\mathcal{C}$-bundles, or simply the category of $\mathcal{C}$-bundles. Formally, an object in $\mathcal{C}(X)$ is a pair (E, p_x), where E is an object in $\mathcal{C}$ and p_x is a family of projections $E \to E$, continuously parametrized by the points of X. One should think of the objects in $\mathcal{C}(X)$ as being an object E of $\mathcal{C}$ and a continuous assignment of subobjects of E to points of x by considering the images of the family p_x of projection maps. The maps in $\mathcal{C}(X)$ from (E, p) to (E', p') are families of maps $\phi_x : E \to E'$, considered up to equivalence at each point x after composition with p'_x.

Given a map $f : Y \to X$, we get pullback maps $f^*_T : \mathcal{C}_T(X) \to \mathcal{C}_T(Y)$ and $f^* : \mathcal{C}(X) \to \mathcal{C}(Y)$. Both $\mathcal{C}_T(X)$ and $\mathcal{C}_T(Y)$ have the same objects as $\mathcal{C}$, by construction, so f^*_T maps an object E in $\mathcal{C}_T(X)$ to the corresponding object in $\mathcal{C}_T(Y)$. The behavior of f_T^* on morphisms is as follows. Recall that a morphism in $\mathcal{C}_T(X)$ from M to N is a family ϕ_x of morphisms from M to N continuously parametrized by points in X. The functor f_T^* maps the family ϕ_x in $\hom_{\mathcal{C}_T(X)}(M,N)$ to the family $\phi_{f(y)}$, which is continuously parametrized by points in Y, and is therefore an element of the hom-set $\hom_{\mathcal{C}_T(Y)}(M,N)$.

The functor $f^* : \mathcal{C}(X) \to \mathcal{C}(Y)$ is defined similarly. On objects, f^* maps a pair (E, p_x), where E is an object of $\mathcal{C}$ and p_x is a family of projections $E \to E$ continuously parametrized by points in X, to the pair $(E, p_{f(x)})$ in $\mathcal{C}(Y)$. The behavior of f^* on the morphisms in $\mathcal{C}(X)$ is as follows. Recall that a map in $\mathcal{C}(X)$ from (E, p_x) to (E', p'_x) is an equivalence class of the maps in $\mathcal{C}_T(X)$ from E to E', with ϕ_x equivalent to ψ_x if $p'_x \circ \phi_x = p'_x \circ \psi_x$. In other words, the maps, ϕ_x and ψ_x must agree on the image of p'_x at each point x in X. If this is the case, then we must also have, at each point y of Y,

$$p'_{f(y)} \circ \phi_{f(y)} = p'_{f(y)} \circ \psi_{f(y)}$$

Therefore, we define f^* as taking the equivalence class of the morphism ϕ_x in $\mathcal{C}(X)$ to the equivalence class of $\phi_{f(y)}$ in $\mathcal{C}(Y)$.

The fundamental example of a locally-trivial $\mathcal{C}$-bundle, and the one which will be important to the proof of Bott periodicity, is the case where $\mathcal{C}$ is the category $\mathcal{L}(k)$ of finite dimensional k-vector spaces and X is compact. In this case, $\mathcal{C}(X)$ is equivalent to the category $\mathcal{E}_k(X)$ of k-vector bundles over X. For more details on this equivalence, see the remark on page 184 of [Kar]. Intuitively, and in a sense made precise by Karoubi in [5] in the results from Proposition (1.2.8) to Lemma (1.2.15), for any $\mathcal{C}$-bundle (E,p) in $\mathcal{C}(X)$ and any point $x \in X$, we can find a neighborhood U of x such that the restriction of (E,p) to U is a trivial bundle.[5] In other words, a $\mathcal{C}$-bundle can actually be thought of as a bundle whose fibers are objects in $\mathcal{C}$.

Since $\mathcal{C}(X)$ is a Banach category, we can use Definition 3.6 to define $K(\mathcal{C}(X))$.

Example 3.17. If $\mathcal{C}$ is the category $\mathcal{L}_{\mathbb{R}}$, then using the equivalence between $\mathcal{C}(X)$ and $\mathcal{E}_{\mathbb{R}}(X)$, we see that $K(\mathcal{C}(X))$ is isomorphic to the Grothendieck completion of the vector bundles over X under direct sum; that is, $K(\mathcal{C}(X))$ is isomorphic to the familiar group $KO(X)$.

There are a couple of advantages of Karoubi's theoretical approach to the K-theory of a space, as opposed to the classical one. The first is that it situates $KO(X)$ within a broader theory, providing us with a standard way of generalizing results in K-theory. The second is that it opens the door to alternative definitions of the groups $K^n(\mathcal{C})$; we can choose a definition that makes the eightfold periodicity of real K-theory a complete triviality, and we then are left with showing that these general definitions are actually related to the classical groups $KO^n(X)$.

3.3. **K-Theory of Functors.** Beyond the group $K(\mathcal{C})$, we will need to define two other types of categorical K-theory: the K-theory of a functor, and K-theory of a category or functor with respect to a vector bundle.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two Banach categories, and let $\phi : \mathcal{C} \to \mathcal{C}'$ be a linear-continuous functor. Let $\Gamma(\phi)$ be the set of pairs of objects in $\mathcal{C}$ which are isomorphic in $\mathcal{C}'$; we describe elements of $\Gamma(\phi)$ as triples (E, F, α), where $\alpha : \phi(E) \to \phi(F)$ is

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[5] The restriction is induced as the pullback functor from $\mathcal{C}(X)$ to $\mathcal{C}(Y)$ induced by the inclusion $U \to X$

an isomorphism. An elementary triple is one of the form (E, E, Id); two triples (E, F, α) and (E', F', α') are called homotopic if there is a triple $(e, f, \aleph)$ in the set $\Gamma(\phi[0,1])$ whose restriction to $\{0\}$ is isomorphic to (E, F, α) at and whose restriction to $\{1\}$ is (E', F', α'). Here, $\phi[0,1]$ is the functor from $\mathcal{C}([0,1])$, the category of $\mathcal{C}$-bundles over the space $[0,1]$, to $\mathcal{C}'([0,1])$; this functor is induced by ϕ. Notice that e and f are both pairs $(\bar{E}, p_t)$ and $(\bar{F}, q_t)$, with the families of projections p_t and q_t parametrized by the interval $[0,1]$. Thus, a homotopy of triples consists of an object e containing E and E' as subobjects, usually direct factors, with a homotopy from projection onto E to projection onto E'; an object f containing F and F' as subobjects with a similar homotopy of projections; and a homotopy of maps $\aleph_t : e \to f$ such that $\aleph_0$ is α and $\aleph_1$ is α'.

We take $K(\phi)$ to be the monoid we get if we quotient $\Gamma(\phi)$ by the equivalence relation generated by taking homotopy equivalences and adding elementary triples; we will refer to the image of (E, F, α) under this quotient by $d(E, F, \alpha)$. It may seem odd to call this monoid $K(\phi)$, since we expect the K theory of an object to be a group. The following proposition clarifies this choice of name.

Remark 3.18. The reader familiar with the topological results of [2, §7] should have in mind for this section the characterization of the relative K-theory $K(X, Y)$ as being given by vector bundles E_1 and E_2 such that $0 \to E_1 \to E_2 \to 0$ is exact on Y, i.e. such that the restrictions of E_1 and E_2 to Y are isomorphic. In particular, if $\mathcal{C}$ is the category of real finite dimensional vector spaces and ϕ is the restriction functor $\mathcal{C}(X) \to \mathcal{C}(Y)$, then $K(\phi)$ agrees with $KO(X, Y)$ in the sense described in [2].

Proposition 3.19. *$K(\phi)$ is an Abelian group.*

We present this relatively simple proof in order to give some idea of how one works with the K-theory of a functor, since we will omit most of the other proofs of this form. It is also notable because it uses the vector space structure of the hom-sets of the Banach categories. This proof appears in [5, Proposition (1.3.5)]

Proof. We want to show that $d(E, F, \alpha) + d(F, G, \beta) = d(E, G, \beta \circ \alpha)$. This will imply that (E, F, α^{-1}) is an inverse for (E, F, α) (since α is an isomorphism, it is always invertible). We know that

$$d(E, F, \alpha) + d(F, G, \beta) = d(E \oplus F, F \oplus G, \alpha \oplus \beta)$$

The isomorphism $\alpha \oplus \beta$ can be factored:

$$E \oplus F \xrightarrow{\alpha \oplus 1} F \oplus F \xrightarrow{1 \oplus \beta} F \oplus G$$

There is an automorphism $\gamma : F \oplus F \to F \oplus F$ given by the block matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; it is homotopic to the identity by the rotation $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Likewise, there is an isomorphism $\delta : F \oplus G \to G \oplus F$ given by the block matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We then have a composition:

$$E \oplus F \xrightarrow{\alpha \oplus 1} F \oplus F \xrightarrow{\gamma} F \oplus F \xrightarrow{1 \oplus \beta} F \oplus G \xrightarrow{\delta} G \oplus F$$

This composition applies $\beta \circ \alpha$ to the first component and the identity to the second component. Thus, we have that

$$d(E, F, \alpha) + d(F, G, \beta) = d(E \oplus F, G \oplus F, \beta \circ \alpha \oplus 1) = d(E, G, \beta \circ \alpha)$$

where the last equality comes from the equivalence of triples which differ by the elementary triple $(F, F, 1)$. Thus, we have verified the identity which guarantees inverses, and so we conclude that $K(\phi)$ is a monoid. □

This general strategy of "rotating a direct sum" is one of the main computational techniques used by Karoubi to study the K-theory of functors. Most of the calculations we omit use this technique to characterize equivalence classes within K-theory of functors.

In fact, even the K-theory of a functor is not enough to prove the result we are interested in. We instead need to consider the K-theory of a "Banach square"[6] defined as follows.

Definition 3.20. A Banach square is a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi_2"] \arrow[d, "\phi_1"'] & \mathcal{C}_2 \arrow[d, "\psi_2"] \\ \mathcal{C}_1 \arrow[r, "\psi_1"'] & \mathcal{C}_{12} \end{tikzcd}$$

where the objects are all Banach categories, the morphisms are all linear-continuous functors, and there is a natural isomorphism $c_{21} : \psi_1\phi_1 \to \psi_2\phi_2$.

In fact, the K-theory of a Banach square $\mathfrak{D}$ is just the K-theory of a certain linear-continuous functor. Let $\mathfrak{D}$ be a Banach square as in Definition 3.20; let $\mathcal{C}'$ be the category whose objects are triples (E_1, E_2, ϵ), where E_i is an object in $\mathcal{C}_i$ and $\epsilon : \psi_1 E_1 \to \psi_2 E_2$ is an isomorphism in $\mathcal{C}_{12}$. The morphisms of $\mathcal{C}'$ are given by pairs of maps $f_1 : E_1 \to E_1'$ and $f_2 : E_2 \to E_2'$ such that the following diagram in $\mathcal{C}_{12}$ commutes:

$$\begin{tikzcd} \psi_1 E_1 \arrow[r, "\psi_1 f_2"] \arrow[d, "\epsilon"'] & \psi_1 E_1' \arrow[d, "\epsilon'"] \\ \psi_2 E_2 \arrow[r, "\psi_2 f_2"] & \psi_2 E_2' \end{tikzcd}$$

Notice that because $\mathfrak{D}$ is made up of Banach categories and linear-continuous functors, $\mathcal{C}'$ will also be a prebanach category. We take $\phi_{\mathfrak{D}} : \mathcal{C} \to \mathcal{C}'$ to be the functor mapping an object $E \in \mathcal{C}$ to the trio $(\phi_1 E, \phi_2 E, c_{12}(E))$ and mapping a morphism in E to the two image morphisms in E_1 and E_2. It can be verified directly that this function is linear and continuous, which follows from the fact that ϕ_1 and ϕ_2 are.

We can now define $K(\mathfrak{D})$, the K-theory of the Banach square, as the K-theory of the functor $\phi_{\mathfrak{D}}$. The elements of this group can be thought of as quadruples $(E, F, \alpha_1, \alpha_2)$, where E and F are objects in $\mathcal{C}$ and α_i is an isomorphism $\phi_i E \to \phi_i F$. We consider an elementary quadruple to be (E, E, Id, Id), and consider $K(\mathfrak{D})$ to be the group of equivalence classes of these quadruples up to homotopy and addition of elementary quadruples; as in the K-theory of a functor, the equivalence class of $(E, F, \alpha_1, \alpha_2)$ is denoted $d(E, F, \alpha_1, \alpha_2)$. Like in the K-theory of a functor, homotopy is defined in terms of quadruples relative to the induced map

$$\phi_{\mathfrak{D}}([0,1]) : \mathcal{C}([0,1]) \to \mathcal{C}'([0,1])$$

---

[6] Called a *grille carrée* by Karoubi, but we take the liberty of renaming it to highlight the important Banach-category properties

For our purposes, it is enough to consider the K-theory of a special type of Banach square. Let X be a compact space and $Y \subset X$ a closed subspace. Let $\mathcal{C}, \mathcal{C}'$ be two Banach categories and $\phi : \mathcal{C} \to \mathcal{C}'$ a linear continuous functor. We now take $\mathfrak{D}$ to be the Banach square

$$\begin{tikzcd} \mathcal{C}(X) \arrow[r, "\phi(X)"] \arrow[d, "r"'] & \mathcal{C}'(X) \arrow[d, "r'"] \\ \mathcal{C}(Y) \arrow[r, "\phi(Y)"] & \mathcal{C}'(Y) \end{tikzcd}$$

where $\phi(X)$ and $\phi(Y)$ are the functors induced by ϕ, and r and r' are the restriction functors induced by the inclusion of Y into X. The natural isomorphism c_{21} represents the fact that it does not matter if we restrict a $\mathcal{C}$-bundle over X to one over Y and then map via ϕ to a $\mathcal{C}'$-bundle, or if we map a $\mathcal{C}$-bundle over X to a $\mathcal{C}'$-bundle over X, and then restrict this bundle to Y.

In a square like this, we will write $K(X, Y; \phi)$ to mean $K(\mathfrak{D})$.

Let us take a moment to unpack the definitions at work here. An object in $K(X, Y; \phi)$ is given by:

(1) Two objects, E and F, in the category $\mathcal{C}$.
(2) Two continuous families of projections: $p_x : E \to E$ and $q_x : F \to F$, parametrized by points $x \in X$.
(3) A continuous family of isomorphisms $\alpha_{1,y} : (E, p) \to (F, q)$, parametrized by points $y \in Y$, and a family of isomorphisms $\alpha_{2,x} : \phi(X)(E, p) \to \phi(X)(F, p)$, which are compatible with the natural transformation c_{21}.
(4) The equivalence class of this data.

We will make special use of the case where X is D^n, the n-ball in $\mathbb{R}^n$, and Y is S^{n-1}, viewed as the boundary of D^n. The result [5, Lemma (1.3.8)] characterizes a convenient set of representatives for $K(D^n, S^{n-1}; \phi)$; because our goal is to present the ideas of Karoubi's proof, and not the calculations involved, this result will not be presented here, although the weaker fact that the elements $K(D^n, S^{n-1}; \phi)$ can all be expressed in the form $d(E, E, \alpha_1, \alpha_2)$ is interesting.

3.4. **Cliffordian K-Theory: K^n and $K^{p,q}$.** It is finally time for Clifford algebras to make their return. We will use them to construct groups $K^n(\mathcal{C})$ for a Banach category $\mathcal{C}$ so that the eightfold periodicity of these groups is an immediate consequence of the periodicity of modules over Clifford algebras. From there, proving real Bott Periodicity is merely a matter of showing that this definition has a genuine interpretation in terms of topological K-theory.

Let $\mathcal{C}$ be a Banach category and $C^{p,q}$ the Clifford algebra $C(\mathbb{R}^{p+q}, Q_{p,q})$, where $Q_{p,q}$ is the quadratic form from 2.8. Let $\mathcal{C}^{p,q}$ be the subcategory of $\mathcal{C}$ of objects in $\mathcal{C}$ which are also modules over $C^{p,q}$, as in Example 3.11. Notice that isomorphisms between the various Clifford algebras $C^{p,q}$ induce equivalences between the categories $\mathcal{C}^{p,q}$. If $C^{p,q}$ is isomorphic to $C^{p',q'}$, then this isomorphism gives an action of $C^{p,q}$ on the objects of $\mathcal{C}^{p',q'}$, and vice versa. In Corollary 2.16, we showed that $C^{4,0}$ is isomorphic to $C^{0,4}$, and as we mentioned then, they are in fact isomorphic as graded algebras. It follows that $\mathcal{C}^{4,0}$ is equivalent to $\mathcal{C}^{0,4}$.

There are two particular equivalences of categories of the form $\mathcal{C}^{p,q}$ which we should mention; the first is a direct consequence of the structure of the Clifford algebras; the second is somewhat more difficult to verify.

Proposition 3.21. *Let $\mathcal{C}$ be a Banach category. Then $\mathcal{C}^{p+4,q}$ is isomorphic to $\mathcal{C}^{p,q+4}$.*

Proof. It suffices to show that $C^{p+4,q} \cong C^{p,q+4}$; Using Lemma 2.3 and Corrolaries 2.16 and 2.18, we write

$$\begin{aligned} C^{p+4,q} &\cong C^{p+4,0} \hat{\otimes}_{\mathbb{R}} C^{0,q} \\ &\cong C^{4,0} \otimes_{\mathbb{R}} C^{p,0} \hat{\otimes}_{\mathbb{R}} C^{0,q} \\ &\cong C^{0,4} \otimes_{\mathbb{R}} C^{0,q} \hat{\otimes}_{\mathbb{R}} C^{p,0} \\ &\cong C^{p,q+4} \end{aligned}$$

□

The next equivalence is on the level of categories, not Clifford algebras, and so the proof is somewhat less straightforward.

Proposition 3.22. *There exists a linear-continuous functor χ from $\mathcal{C}^{p,q}$ to $\mathcal{C}^{p+1,q+1}$ which gives an equivalence of Banach categories.*

Proof. Let $\{e_1, \ldots, e_p, \varepsilon_1, \ldots, \varepsilon_q\}$ be the basis for $\mathbb{R}^{p+q}$ which gives a set of generators for $C^{p,q}$; likewise, let $\{e_1, \ldots, e_{p+1}, \varepsilon_1, \ldots, \varepsilon_{q+1}\}$ be a basis for $\mathbb{R}^{p+q+2}$ giving a basis for $C^{p+1,q+1}$.

We define χ as follows. For an object E in $\mathcal{C}^{p,q}$, we define $\chi(E)$ to be $E \oplus E$; the actions of the generators e_1 through e_p and ε_1 through ε_q in $C^{p+1,q+1}$ on each component of $\chi(E)$ are the same as the actions of the corresponding generators in $C^{p,q}$ on E. The action of the new generators e_{p+1} and ε_{q+1} on $\chi(E)$ are given by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. A morphism $f : E \to F$ in $\mathcal{C}^{p,q}$ is mapped by χ to the morphism $\chi(f) : E \oplus E \to F \oplus F$ whose action on the components is given by the matrix

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

The map from f to $\left(\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix}\right)$ is both linear and continuous as a function of f, so χ is a linear-continuous functor.

We define an inverse functor $\chi' : \mathcal{C}^{p+1,q+1} \to \mathcal{C}^{p,q}$ by selectively "forgetting the action of e_{p+1} and ε_{q+1}," as follows. The action of the element $\eta = e_{p+1}\varepsilon_{q+1}$ on an object E in $\mathcal{C}^{p+1,q+1}$ is a $C^{p,q}$-module automorphism, and the composition $\eta \circ \eta$ is given by the action of $(e_{p+1}\varepsilon_{q+1})^2$. Now, the anticommutativity of multiplication in a Clifford algebra, along with the knowledge that $Q(e_{p+1}) = -1$ and $Q(\varepsilon_{q+1}) = 1$, shows that the element $(e_{p+1}\varepsilon_{q+1})^2$ of $C^{p+1,q+1}$ is equal to the unit 1. Thus, η is in fact an involution. We now let E_0 be the kernel of the action of $\frac{1-\eta}{2}$ and let E_1 be the kernel of the action of $\frac{1+\eta}{2}$; since η is an involution, the composite of these two maps is zero and E can be written as $E_0 \oplus E_1$. We define the functor χ' by $\chi'(E) = E_0$ and $\chi'(f) = f|_{E_0}$; the restriction of f is both linear and continuous, so χ' is a linear-continuous functor.

One now must check that $\chi' \circ \chi$ is naturally isomorphic to the identity on $\mathcal{C}^{p,q}$ and that $\chi \circ \chi'$ is naturally isomorphic to the identity on $\mathcal{C}^{p+1,q+1}$. For the first identity, we compute that the automorphism η on $\chi(E) = E \oplus E$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It follows that $\frac{1-\eta}{2}$ is just projection of $E \oplus E$ onto the second component; then the kernel of this map is the first component of $E \oplus E$, which is naturally isomorphic to E. On the other hand, if $E = E_0 \oplus E_1$ is an object of $\mathcal{C}^{p+1,q+1}$, the map $\chi \circ \chi'$ sends E to $E_0 \oplus E_0$. We therefore need to check that $E_0 \oplus E_1$ is naturally isomorphic to $E_0 \oplus E_0$; this isomorphism is given as follows. By the definition of the decomposition $E = E_0 \oplus E_1$, the action of e_{p+1} on E is given by a matrix of the form

$$\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix}$$

and it follows that the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

gives an isomorphism from $E^0 \oplus E^0$ to $E^0 \oplus E^1$, which preserves the first component of $E^0 \oplus E^0$ and sends an element x in the second component to $-e_{p+1}x$. $\square$

The decomposition $E = E_0 \oplus E_1$ using elements $\frac{1-\eta}{2}$ and $\frac{1+\eta}{2}$ is reminiscent of the isomorphism between C_3 and $\mathbb{H} \oplus \mathbb{H}$ that we gave in Example 2.13. Given that the two components are related by multiplication by the element e_{p+1}, which is 1-graded in the algebra $C^{p+1,q+1}$, one might now ask whether the categories $\mathcal{C}^{p+q}$ have a $\mathbb{Z}_2$-grading. The answer is yes.

Definition 3.23. A $\mathbb{Z}_2$-graded prebanach category is a prebanach category $\mathcal{C}$ such that every hom-set $\hom_{\mathcal{C}}(M, N)$ can be decomposed into the direct sum of vector spaces $\hom^0_{\mathcal{C}}(M, N) \oplus \hom^1_{\mathcal{C}}(M, N)$, such that this grading is compatible with the direct sum of objects and the composition of morphisms. That is, the grade of the composite of two morphisms is the sum of the grades of each morphism, and the i-grade of the hom-set of direct sums is the direct sum of the i-grades of the hom-sets of the summands. For example, $\hom^i(M \oplus N, P) \cong \hom^i(M, P) \oplus \hom^i(N, P)$, and likewise if the target of the hom-set is a direct sum of objects. A graded Banach category is a graded prebanach category in which every grade-zero projection admits a kernel

We can define the graded Clifford category $\hat{\mathcal{C}}^{p,q}$ as the subcategory of $\mathcal{C}$ consisting of objects which can be viewed as graded $C^{p,q}$-modules and graded $C^{p,q}$-module homomorphisms. The graded-algebra isomorphism between $C^{p+4,q}$ and $C^{p,q+4}$, which follows from Lemma 2.3 and the graded algebra isomorphism between $C^{4,0}$ and $C^{0,4}$, then induces an equivalence of graded Banach categories between $\hat{\mathcal{C}}^{p+4,q}$ and $\hat{\mathcal{C}}^{p,q+4}$. Likewise, we have an equivalence of graded Banach categories between $\hat{\mathcal{C}}^{p,q}$ and $\hat{\mathcal{C}}^{p+1,q+1}$.

Remark 3.24. By letting the generator ϵ_{q+1} stand in for grading, we get an isomorphism between $\mathcal{C}^{p,q+1}$ and $(\hat{\mathcal{C}}^{p,q})^0$. (Recall that $Q(\epsilon_{q+1}) = +1$). To view objects E and F of the ungraded category of $\mathcal{C}^{p,q+1}$ as objects of the graded category $\mathcal{C}^{p,q}$, we say that a map $f : E \to F$ of $C^{p,q}$ modules is of 0-graded if it commutes with multiplication by ε_{q+1} and 1-graded if it anticommutes, that is, if $f \circ \varepsilon_{q+1} = -\varepsilon_{q+1} \circ f$, where the map ε_{q+1} is given by the $C^{p,q+1}$-module action. Notice that if f is 0-graded, then it is also a map of $C^{p,q+1}$ modules, so that the zero-grade of $\hat{\mathcal{C}}^{p,q}$ consists of those elements of $\mathcal{C}$ with an action by $C^{p,q+1}$, and is therefore equivalent to $\mathcal{C}^{p,q+1}$.

To simplify future notation, we define $\mathcal{C}^n$ to be the category $\mathcal{C}^{n,0}$ when n is non-negative and $\mathcal{C}^{0,-n}$ when n is negative; we define $\hat{\mathcal{C}}^n$ analogously.

For any graded Banach category $\hat{\mathcal{C}}$, let $U_{\hat{\mathcal{C}}} : \hat{\mathcal{C}}^0 \to \mathcal{C}$ be the forgetful functor which simply forgets the grading on the zero-graded component of $\hat{\mathcal{C}}$; this functor U is quasi-surjective. This definition allows us to define the Cliffordian K-theory of a category.

Definition 3.25. Let $\mathcal{C}$ be a Banach category. The group $K^{p,q}(\mathcal{C})$ is the K-theory of the forgetful functor $U_{\hat{\mathcal{C}}^{p,q}})$. The group $K^n(\mathcal{C})$ is K-theory of the forgetful functor $U_{\hat{\mathcal{C}}^n}$.

In other words, an element of $K^{p,q}$ is the equivalence class of a pair of graded objects (E, F) in $\mathcal{C}$ together with a map α which makes E and F isomorphic as ungraded objects. One can view $K^{p,q}(\mathcal{C})$ as a kind of "graded" Grothendieck completion of the monoid of isomorphism classes of objects in $\hat{\mathcal{C}}^{p,q}$ under direct sum.

At last, we are ready to prove something which at least resembles Bott periodicity.

Theorem 3.26. *The group $K^{n+8}(\mathcal{C})$ is isomorphic to the group $K^n(\mathcal{C})$.*

Proof. We already know that $C^{p,q}$ is isomorphic to $C^{p+1,q+1}$ as a graded algebra, and that $C^{p+4,q}$ is isomorphic to $C^{p,q+4}$ as a graded algebra. Then the graded algebra $C^{p,q}$ is isomorphic to the graded algebra $C^{p+4,q+4}$, which in turn is isomorphic to the graded algebra $C^{p+8,q}$, as well as to the graded algebra $C^{p,q+8}$. These isomorphisms of graded algebras induce isomorphisms between the graded Banach categories $\mathcal{C}^{p+8,q}$, $\mathcal{C}^{p,q+8}$, and $\mathcal{C}^{p,q}$. These give isomorphisms between $K^{p+8,q}(\mathcal{C})$, $K^{p,q+8}(\mathcal{C})$, and $K^{p,q}(\mathcal{C})$. Now, choosing either $p = n$ and $q = 0$, or $p = 0$ and $q = -n$ when n is negative, yields the desired isomorphism between $K^n(\mathcal{C})$ and $K^{n+8}(\mathcal{C})$. □

This theorem looks a lot like Bott Periodicity, but so far it is not clear whether or not it means anything from the perspective of topological K-theory. We have seen, for instance in Example 3.17, a way to connect the group $K(\mathcal{C})$ where $\mathcal{C}$ is a a Banach category with the topological group $KO(X)$, but the groups $K^n(\mathcal{C})$ are, so far, totally formal. The remainder of our work will be to connect the groups $K^n(\mathcal{C})$, in certain cases, with the real topological K-theory of certain spaces, and to use this connection to derive a genuine periodicity in the topological setting.

When $n = 0$, we have a relatively simple connection between $K^n(\mathcal{C})$ and the group $K(\mathcal{C})$; as we discussed in Example 3.17, the right choice of category $\mathcal{C}$ gives an isomorphism between $K(\mathcal{C})$ and $KO(X)$ for a space X. Notice that the elements of $K^0(\mathcal{C})$ are triples (E, F, α), where E and F are two graded objects in $\mathcal{C}$ and $\alpha : E \to F$ is an isomorphism. We have the following equivalence

Proposition 3.27. *The group $K^0(\mathcal{C})$ is isomorphic to $K(\mathcal{C})$, by the homomorphism sending a triple $d(E, F, \alpha)$ in $K^0(\mathcal{C})$ to the equivalence class $[E^0] - [F^0]$ in $K(\mathcal{C})$*

We offer a sketch of the proof below; it appears in [5, Proposition (2.1.7)], and is proven as a special case of [5, Proposition (2.1.10)].

Sketch of Proof. What must be shown is that the map is injective and surjective. To show injectivity, we consider an object $d(E,F,\alpha)$ such that $E^0 \cong F^0$; we must show that $d(E,F,\alpha) = 0$. To do so, we might write $E = H \oplus E^1$ and $F = H \oplus F^1$, express both E and F as "subobjects" of $H \oplus E^1 \oplus F^1$ (i.e. images of projections), and then show that the endomorphism of $H \oplus E^1 \oplus F^2$ induced by α is homotopic to the identity. Along the way, we might add or subtract elementary triples. To show surjectivity, one must produce, for each object E in $\mathcal{C}$, an object in $\hat{\mathcal{C}}^0$ whose 0-graded component is isomorphic to E, and an object F in $\hat{\mathcal{C}}^0$ which is isomorphic to E in $\mathcal{C}$ but whose 0-graded component is isomorphic by a map α to the zero object; then (E,F,α) in $K^0(\mathcal{C})$ will map to $[E^0] - [0] = [E]$ in $K(\mathcal{C})$. For instance, one can either put all of E in grade 0 or all of it in grade 1. The only elements in C^0 are of grade-0, so the grading of objects and hom-sets is more-or-less arbitrary. □

Before moving on, we will state, but in the interest of time not prove, a relevant technical result, which allows us to easily find Serre functors and quasi-surjective functors between categories $\mathcal{C}^{p,q}$ and $\mathcal{C}'^{p,q}$.

Proposition 3.28. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two Banach categories and $\phi : \mathcal{C} \to \mathcal{C}'$ a Serre functor. Then the restricted functor $\phi^{p,q} : \mathcal{C}^{p,q} \to \mathcal{C}'^{p,q}$ is a Serre functor. Likewise, if ϕ is quasi-surjective, then so is $\phi^{p,q}$*

This statement is proven in [5, Proposition (2.1.8)]. This statement allows us to pass from Banach square of categories to Banach squares of the subcategories with action by a Clifford algebra $C^{p,q}$. We now extend our Cliffordian K-theory to functors.

Definition 3.29. Let $\mathcal{C}$ and $\mathcal{C}'$ be two Banach categories, and $\phi : \mathcal{C} \to \mathcal{C}'$ a graded, quasi-surjective Serre functor. The graded K-theory of ϕ, written $\hat{K}(\phi)$, is the K-group of the Banach square

$$\begin{tikzcd}
\hat{\mathcal{C}}^0 \arrow[r, "\phi^0"] \arrow[d, "U_{\mathcal{C}}"'] & \hat{\mathcal{C}}'^0 \arrow[d, "U_{\mathcal{C}'}"] \\
\mathcal{C} \arrow[r, "\phi"] & \mathcal{C}'
\end{tikzcd}$$

In particular, we can use the isomorphism between $\mathcal{C}^{p,q+1}$ and $\left(\hat{\mathcal{C}}^{p,q}\right)^0$ mentioned in Remark 3.24 to define, for a quasi-surjective Serre functor $\psi : \mathcal{C} \to \mathcal{C}'$, the K-groups $K^{p,q}(\psi)$. We define $K^{p,q}(\psi)$ to be the K-group of the Banach square

$$\begin{tikzcd}
C^{p,q+1} \arrow[r, "\psi^{p,q+1}"] \arrow[d] & \mathcal{C}'^{p,q+1} \arrow[d, "U_{\mathcal{C}'}"] \\
\mathcal{C} \arrow[r, "\phi"] & \mathcal{C}'
\end{tikzcd}$$

.

For the purposes of notation, we will suggestively define $K^n(\psi)$ to be $K^{n,0}(\psi)$ when n is positive and $K^{0,-n}(\psi)$ when n is negative.

Ordinarily, the K-theory of a Banach square would be formed by equivalence classes of certain quadruples. A simpler characterization of $K^{p,q}(\psi)$ is given in [5, page 205], and goes as follows. Consider the set $\Gamma^{p,q}(\psi)$ of triples (E,F,α), where E and F are objects of $\mathcal{C}^{p,q+1}$ and $\alpha : E \to F$ is an isomorphism of $C^{p,q}$ modules.

We call (E, F, α) elementary if α has degree 0; a homotopy between two triples (E, F, α) and (E', F', α') will be a triple $(e, f, \aleph)$ in $\Gamma^{p,q}(\psi[0,1])$ whose restriction to 0 is (E, F, α) and whose restriction to 1 is (E', F', α'). We can express $K^{p,q}(\psi)$ as the quotient of $\Gamma^{p,q}(\psi)$ by the equivalence relation generated by homotopy and addition of elementary triples.

Like with the Cliffordian K-groups of categories, $K^0(\psi)$ is isomorphic to $K(\psi)$, with the triple $d(E, F, \alpha)$ mapping to the triple $d(E^0, F^0, \alpha^0)$, as seen in [5, Proposition (2.1.10)]. There is another characterization of $K^{p,q}(\psi)$ which is sometimes useful. First, we introduce a shorthand term.

Definition 3.30. A grader ε on an object E of $\mathcal{C}^{p,q}$ is an involution $\varepsilon : E \to E$ which anticommutes with the action of the generators of $C^{p,q}$. A grader induces a grading on E by letting E_0 be the kernel of the map $\frac{1-\varepsilon}{2}$.

A grader therefore plays the role of η in the proof of Proposition 3.22.

Let $\Delta^{p,q}(\psi)$ be the additive category whose objects are triples $(E, \varepsilon_1, \varepsilon_2)$, where E is an object of $\mathcal{C}^{p,q}$ and ε_1 and ε_2 are two graders of E such that $\psi(\varepsilon_1) = \psi(\varepsilon_2)$. A morphism in $\Delta^{p,q}(\psi)$ from $(E, \varepsilon_1, \varepsilon_2)$ to (F, η_1, η_2) is given by a map $f : E \to F$ of $C^{p,q}$ modules such that $\eta_i f = f \varepsilon_i$. We call a triple $(E, \varepsilon_1, \varepsilon_2)$ elementary if $\varepsilon_1 = \varepsilon_2$. We call two triples $\sigma_1 = (E, \varepsilon_1, \varepsilon_2)$ and $\sigma_2 = (F, \eta_1, \eta_2)$ homotopic if there is an object (e, ξ_1, ξ_2) of the category $\Delta^{p,q}(\psi([0,1]))$ whose restriction to 0 is σ_1 and whose restriction to 1 is σ_2.

In [5, Proposition (2.1.12)], it is shown that the group $K^{p,q}(\psi)$ is isomorphic to the quotient of the monoid Γ, consisting of the objects of $\Delta^{p,q}(\psi)$ under direct sum, by the equivalence relations generated by homotopy and addition of elementary triples.

We will introduce two new pieces of notation, both of which will be important to the general statements of Bott Periodicity, and both of which are just special cases of constructions we have already seen.

The first piece of notation is for a special case of the Cliffordian K-theory of a functor. Let X be a compact space, Y a closed subspace of X, and $\rho : \mathcal{C}(X) \to \mathcal{C}(Y)$ the restriction functor. Then we denote by $K^{p,q}(X, Y; \mathcal{C})$ the group $K^{p,q}(\rho)$; likewise, $K^n(X, Y; \mathcal{C})$ will denote the group $K^n(\rho)$. In particular, if $\mathcal{C}$ is the category of real finite dimensional vector spaces, then $K^0(X, Y; \mathcal{C})$ is the relative K-theory of the pair (X, Y), as discussed in Remark 3.18.

The second piece of notation is a slight generalization of the notation $\mathcal{C}^{p,q}$ and $K^{p,q}$. Suppose we have a compact space X, a closed subspace Y of X, a finite dimensional vector bundle W over X whose fibers are equipped with a non-degenerate quadratic form Q, and a sub-bundle V of W such that the restriction of Q to V is non-degenerate. For a Banach category $\mathcal{C}$ and a vector bundle V over X with non-degenerate quadratic form Q, we denote by $\mathcal{C}^V(X)$ the subcategory of $\mathcal{C}$ consisting of objects which are $C(V; Q)$-modules and morphisms which are $C(V; Q)$-module homomorphisms, where $C(V; Q)$ is the algebra bundle over X.

We define $K^{W;V}(X, Y; \mathcal{C})$ to be the K-theory of the Banach square

$$\begin{tikzcd} \mathcal{C}^W(X) \arrow[r] \arrow[d] & \mathcal{C}^V(X) \arrow[d] \\ \mathcal{C}^W(Y) \arrow[r] & \mathcal{C}^V(Y) \end{tikzcd}$$

where the vertical arrows are given by restriction and the horizontal arrows are given by inclusion of $\mathcal{C}^W$ into $\mathcal{C}^V$, since V is a sub-bundle of W, every $C(W,Q)$ bundle is a $C(V,Q|_V)$ bundle by forgetting the action of points in W which are not in V. More explicitly, the elements of $K^{W;V}(X,Y;\mathcal{C})$ can be viewed as triples (E,w_1,w_2), where E is an object in $\mathcal{C}$ and w_1 and w_2 are both structures of $C(W)$-modules on E which restrict to the same $C(V)$-module structures over X and the same $C(W)$-module structures over Y; these triples are taken modulo homotopy and addition of elementary triples (E,w,w). Specifically, the "structure of a $C(W)$ module on E" specifies the action of $C(W)$ on E by specifying a map from $C(W)$ to the Banach space of endomorphisms on E.

In the special case where V is any vector bundle over X with a non-degenerate quadratic form Q, and $T^{0,1}$ is a trivial line bundle $X \times \mathbb{R}^1$ spanned by a basis element e_t, with the quadratic form $R(e_t) = 1$, we use $K^V(X,Y;\mathcal{C})$ to refer to $K^{W;V}(X,Y;\mathcal{C})$. As in Remark 3.24, the group $K^V(X,Y;\mathcal{C})$ is associated with the subcategory of $\mathcal{C}$ of graded $C(V;Q)$ modules.

The important case to us is when V is the trivial bundle $T^{p,q} = X \times \mathbb{R}^{p+q}$ with the quadratic form $Q_{p,q}$, given by

$$Q_{p,q}(e_i) = \begin{cases} -1, & 1 \le i \le p \\ 1, & p < i \le p+q \end{cases}$$

In this case, a $C(T^{p,q};Q_{p,q})$-module is just a $C^{p,q}$-module, and so $K^V(X,Y;C)$ is just $K^{p,q}(X,Y;\mathcal{C})$. If V is the zero vector bundle, that is $V = X \times \mathbb{R}^0$, then $K^V(X,Y;\mathcal{C})$ is just $K^0(X,Y;\mathcal{C})$, and in particular if Y is the empty set, then $K^V(X,Y;\mathcal{C})$ will be denoted $K^V(X;\mathcal{C})$.

3.5. **Bott Periodicity.** At long last, it is time for the theory of Banach categories to pay off; we will connect the notions of Cliffordian K-theory with actual topological K-theory, allowing us to connect the periodicity of Clifford modules, which we have already exploited, with the periodicity of K-theory. We will show that for any compact space X, we have an isomorphism between $K^n(X)$ and $K^{n+8}(X)$.

Let X be a compact space and V a vector bundle over X equipped with a non-degenerate form Q. Let $V = V' \oplus V''$ be an orthogonal decomposition of V with respect to Q such that the restriction of Q to V' is positive definite. Let 1 be the trivial line bundle over X. If $S(V \oplus 1)$ is the unit sphere bundle associated with $V \oplus 1$, and $S^+(V \oplus 1)$ is the bundle of upper hemispheres, with "upper" defined relative to the trivial line bundle component, we have a projection p from $S^+(V \oplus 1)$ onto the ball bundle $B(V')$. For example, we can view $B(V')$ as being the unit disk passing through the equator of the unit sphere $S(V' \oplus 1)$. Note that the projection p is a homeomorphism which fixes $S(V')$, the boundary of $B(V')$. Let π be the canonical projection of $S^+(V \oplus 1)$ onto X; let $\bar{\pi} : B(V') \to X$ be the result of composing π with the inverse of p.

Theorem 3.31. *There is an isomorphism*

$$K^{V' \oplus V''}(X;\mathcal{C}) \cong K^{\bar{\pi}^* V''}(B(V'), S(V');\mathcal{C}),$$

where $\bar{\pi}^* V''$ is the pullback bundle over $B(V')$.

The proof of Theorem 3.31 is rather arduous; we will give a sketch of it shortly, along with directions to the details in [5]. First, however, let us present the main attraction.

Corollary 3.32 (Real Bott Periodicity Theorem). *The real topological K-theory of a compact space X has period* 8. *That is, $KO^{-n}(X) \cong KO^{-n-8}(X)$.*

Proof. Let V' be the bundle $T^{0,8}$ and let V'' be the zero bundle, so that $\bar{\pi}^*(V'')$ will be zero as well. Then $V = V' \oplus V''$ is just the bundle $T^{0,8}$. Furthermore, $K^{V'}$ of any pair of spaces in any Banach category is just $K^{0,8} = K^{-8}$. By Theorem 3.26, $K^{-8}(X;\mathcal{C})$ is isomorphic to $K^0(X;\mathcal{C})$. This isomorphism, together with Theorem 3.31 tells us that

$$K^0(X;\mathcal{C}) \cong K^{V'\oplus V''}(X;\mathcal{C}) \cong K^{\pi^*(0)}(B(V'), S(V');\mathcal{C}) = K^0(B(V'), S(V'))$$

Let us take $\mathcal{C}$ to be the category of finite dimensional real vector spaces, so that for a compact space Y and closed subspace Z, the group $K^0(Y,Z;\mathcal{C})$ is simply the real K-theory $KO(Y,Z)$. Then we have the isomorphism

$$KO(X) \cong KO(B(V'), S(V'))$$

since V' is the trivial $\mathbb{R}^8$ bundle over X, $B(V')$ and $S(V')$ are the trivial D^8 and S^7 bundles over X, respectively. Thus, we have that

$$KO(X) \cong KO(X \times D^8, X \times S^7)$$

We now need only appeal to basic properties of topological K-theory. First, we rewrite $KO(X \times D^8, X \times S^7)$ as the reduced group $\tilde{KO}((X \times D^8)/(X \times S^7))$, which is just the reduced group $\tilde{KO}(X \times S^8)$. We rewrite $X \times S^8$ as the smash product $X_+ \wedge S^8$, which is just $\Sigma^8 X_+$. Now, we apply the suspension axiom to get $\tilde{KO}(\Sigma^8 X_+) \cong \tilde{KO}^{-8}(X_+)$, and then use the definition of reduced K-theory to show that this reduced group is isomorphic to $KO^{-8}(X)$. Thus, we have shown that $KO^0(X) \cong KO^{-8}(X)$. To get the full result, that is, to show that $KO^{-n}(X) \cong KO^{-n-8}(X)$, we apply the above isomorphism to the suspension $\Sigma^n(X)$ to get $KO(\Sigma^n(X)) \cong KO^{-8}(\Sigma^n(X))$. The suspension isomorphism then tells us that

$$KO^{-n}(X) \cong KO^{-n-8}(X)$$

□

All that remains is to sketch the proof of Theorem 3.26; there is a homomorphism t between $K^{V'\oplus V''}(X;\mathcal{C})$ and $K^{\pi^*V''}(B(V'), S(V');\mathcal{C})$, which Karoubi proves is an isomorphism. The map is actually defined from $K^{V'\oplus V''}(X;\mathcal{C})$ to $K^{\pi^*V''}(S^+(V' \oplus 1), S(V');\mathcal{C})$, and defined as follows. An element $d(E, w_1, w_2)$ of $K^{V'\oplus V''}(X;\mathcal{C})$ is mapped to the triple $d(\pi^*E, \varepsilon(w_1), \varepsilon(w_2))$. Here $\pi^*(E)$ is the pullback of E. We view E as an object of $\mathcal{C}_T(X)$, that is, it is an object of $\mathcal{C}$ whose morphisms to an object F are continuous maps from X to $\hom_{\mathcal{C}}(E,F)$; the pullback of E lies in the category $\mathcal{C}_T(S^+(V'\oplus 1)$, and corresponds to the same object of $\mathcal{C}$ as E does (but has different morphisms into and out of it). The map ε maps a $C(V' \oplus V''; Q)$-module structure on E to a $C(V'' \oplus 1)$-module structure on $\pi^*(E)$ as follows. The action of the vector $(v'', \mu) \in (V'' \oplus 1)$ on the object $\pi^*(E)$ above the point[7] (ν', λ) in the module structure $\varepsilon(w_i)$ is given by

$$\varepsilon(w_i)(\nu'', \mu) = w_i(0, \nu'', 0) + \mu w_i(\nu', 0, \lambda)$$

[7]The object which can be thought of as "the fiber of $\pi^*(E)$ above a point x" is the pair (F, p_x), that is, it is the object in $\mathcal{C}$ associated with E paired with a single projection map from the family of projection maps given as an object of E.

where w_i maps a point p in $V' \oplus V'' \oplus 1$ to the element of $\hom_{\mathcal{C}_T(S^+(V''\oplus 1))}(\pi^* E, \pi^* E)$, which is the pullback by π of the element $w_i(p)$ in $\hom_{\mathcal{C}_T(X)}(E, E)$ originally specified.

With the map $t : K^{V' \oplus V''}(X; \mathcal{C}) \to K^{\pi^* V''}(S^+(V' \oplus 1), S(V'); \mathcal{C})$ defined, it remains to show that it is an isomorphism. Let us sketch Karoubi's proof of this fact; the gory details are found in [5, Section 2.2], with specific results cited as they arise.

Proof of Theorem 3.26. The basic strategy is to work by induction on the dimension of V'. This uses the following special case of the base case, namely the case where the space X is just a point.

Theorem 3.33. *Let $\mathcal{C}$ be a Banach category. Then there is a map $T : K^{p,q+1}(\mathcal{C}) \to K^{p,q}(D^1, S^0; \mathcal{C})$, which is both a homomorphism and an isomorphism.*

This result is given in [5, Theorem (2.2.2)], and needs several steps; the description of the homomorphism T can be found there, and uses the fact, given in [5, Lemmas (2.2.3) and (2.2.4)], that the elements of $K^{p,q}(D^1, S^0; \mathcal{C})$ are parametrized in terms of the cosines and sines of an angle θ between 0 and π, a special grading ε_{q+1}, and a family α of $C^{p,q}$ module isomorphisms parametrized by θ, with $\alpha(0)$ the identity and $\alpha(\pi)$ anticommuting with ε_{q+1}. All of this is proven by carefully studying the K-groups $K^{p,q}(D^1, S^0)$. The choices of families $\alpha(\theta)$ break down into certain types ("Laurentian," "quasi-polynomial," and "quasi-affine"), based on how they relate to θ and ε_{q+1}; restricting α to each of these types of families of automorphisms gives new K-theories $K_L^{p,q}, K_P^{p,q}$, and $K_A^{p,q}$. Karoubi next shows, in [5, Lemmas (2.2.5)-(2.2.9)] that each map in the following sequence is an isomorphism, and we take T to be its composite:

$$K^{p,q+1}(\mathcal{C}) \xrightarrow{t_4} K_A^{p,q}(D^1, S^0) \xrightarrow{t_3} K_P^{p,q}(D^1, S^0) \xrightarrow{t_2} K_L^{p,q} \xrightarrow{t_1} K^{p,q}(D^1, S^0)$$

The proof that t_1 is an isomorphism, in [5, Lemma (2.2.5)], uses the existence of Fourier series to show that any family of isomorphisms $\alpha(\theta)$ can be viewed as a "Laurentian" family. The proof that t_2 is an isomorphism, in [5, Lemma (2.2.6)], uses explicit rotation homotopies between elements of the various K-groups in order to show that t_2 is injective and surjective; likewise, the proof that t_4 is an isomorphism, in [5, Lemma (2.2.9)], also proceeds by explicit construction of relevant homotopies. The proof that t_3 is an isomorphism, in [5, Lemma (2.2.8)], uses an explicit description of inverses of elements in each of these modified K-groups, given in [5, Lemma (2.2.7)]. The culmination of these efforts is an isomorphism T between $K^{p,q+1}(\mathcal{C})$ and $K^{p,q}(D^1, S^0; \mathcal{C})$.

Next, we prove the case where V' and V'' are both trivial bundles, by induction on p. We suppose that Theorem 3.26 holds when the dimension of V' is less than n. We construct a homomorphism

$$s : K^{V' \oplus V''}(X, Y) \to K^{V''}(X \times B(V'), X \times S(V') \cup Y \times B(V')) \tag{3.34}$$

which is equal to T when Y is the empty set, and assume by induction that this map is an isomorphism when the dimension of V' is less than n. Now, we take V' to be a trivial n-dimensional real vector bundle, with V_1' the first $n-1$ dimensions of this bundle and V_2' the nth dimension. The inductive hypothesis gives us an isomorphism

$$t : K^{V_1' \oplus (V_2' \oplus V'')}(X) \to K^{V_2' \oplus V''}(S^+(V_1' \oplus 1), S(V_1')) \tag{3.35}$$

Applying s now gives us an isomorphism to the rather monstrous looking group

$$K^{V''}(S^+(V_1'\oplus 1)\times_X S^+(V_2'\oplus 1), S^+(V_1'\oplus 1)\times_X S(V_2')\cup S(V_1')\times_X S^+(V_2'\oplus 1) \tag{3.36}$$

Fortunately, Karoubi constructs, in [5, page 219], a homeomorphism h between the pair in (3.36) and the pair $(S^+(V_1'\oplus V_2'\oplus 1), S(V_1'\oplus V_2'))$. This homeomorphism induces an isomorphism in the K-groups, and so by reassembling the components of V', we have that $K^{V'\oplus V''}(X)$ is isomorphic to $K^{V''}(S^+(V'\oplus 1), S(V'))$.

It remains to prove the case where the vector bundles are nontrivial. For B a subspace of A, we use the abbreviation $K_1^V(A,B)$ to mean the group $K^V(A\times D^1, A\times S^0\cup B\times D^1)$; notice that the pair in (3.36) is of this form. For a single space A, the abbreviation $K_1^V(A)$ denotes the group $K^V(A\times D^1, A\times S^0)$, that is, it is the special case where B is the empty set. The isomorphism s from (3.34) shows that $K_1^V(A,B)$ is isomorphic to $K^{V\oplus 1}(A,B)$, where 1 is the trivial line bundle over A.

Karoubi shows ([5, Lemma (2.2.11)]) that given subsets U_1 and U_2 of X, we have a Mayer-Vietoris exact sequence

(3.37)

$$K_1^V(U_1)\oplus K_1^V(U_2)\longrightarrow K_1^V(U_1\cap U_2)\xrightarrow{d} K^V(U_1\cup U_2)\longrightarrow K^V(U_1)\oplus K^V(U_2)\longrightarrow K^V(U_1\cap U_2)$$

One then takes a closed cover $\{T_\lambda\}$ of X where the bundles V' and V'' are trivial on each set and uses compactness of X to make this cover finite; we can now prove the result on X by proving it over induction on the finite set, using the Mayer-Vietoris sequence to show that if the theorem holds on $U_1=\bigcup_{i=1}^{n-1}T_i$, $U_2=T_n$, and $U_1\cap U_2$, then it holds on $U_1\cup U_2$. To do this, one adds to the diagram in (3.37) an analogous exact sequence, using the maps

$$\begin{aligned} t :&K_1^V(A)\to K_1^{V''}((B(V'),S(V'))_A)\\ t :&K^V(A)\to K^{V''}((B(V'),S(V'))|_A)\end{aligned}$$

to get a diagram like

$$\begin{tikzcd} K_1^V(U_1)\oplus K_1^V(U_2) \arrow[r] \arrow[d, "t"] & K_1^V(U_1\cap U_2) \arrow[r, "d"] \arrow[d, "t"] & K^V(U_1\cup U_2) \arrow[r] \arrow[d, "t"] & K^V(U_1)\oplus K^V(U_2) \arrow[r] \arrow[d, "t"] & K^V(U_1\cap U_2) \arrow[d, "t"] \\ K_1^{V''}(B(V'),S(V'))^{\oplus 2} \arrow[r] & K_1^{V''}(B(V'),S(V')) \arrow[r] & K^{V''}(B(V'),S(V')) \arrow[r] & K^{V''}(B(V'),S(V')) \arrow[r] & K^{V''}(B(V'),S(V')) \end{tikzcd}$$

In the above diagram, all ball and sphere bundles should be viewed as being restricted to the appropriate subspaces U_1, U_2, $U_1\cap U_2$, and $U_1\cup U_2$. Karoubi shows that the diagram commutes in [5, Corollary (2.1.13)]. In the base case, where both U_1 and U_2 are from the cover by subspaces on which V is trivial, the four outer vertical maps are just the map from (3.35); in particular, since the bundle V is trivial on U_1, U_2, $U_1\cap U_2$, these arrows are isomorphisms. Since the diagram commutes, the Five Lemma implies that the middle vertical map is an isomorphism. Thus, in this step, $K^V(U_1\cup U_2)$ is isomorphic to $K^{V''}(B(V'),S(V'))$. In subsequent steps of the induction, the inductive hypothesis is exactly that the four outer arrows commute. Thus, we also get that $K^V(U_1\cup U_2)$ is isomorphic to $K^{V''}(B(V'),S(V'))$. By completing the induction on the finite family $\{T_i\}$, we conclude that $K^V(X)$ is isomorphic to $K^{V''}(B(V'),S(V'))$. $\square$

In the proof of real Bott Periodicity, all our vector bundles were trivial. We presented the full result because it is a useful tool in its own right. For example, the following "Generalized Bott Periodicity Theorem" appears in [5, Theorem (2.3.2)].

Theorem 3.38 (Generalized Bott Periodicity Theorem)**.** *Let $\mathcal{C}$ be a real Banach category. Then there exists an isomorphism*

$$\beta_{\mathbb{R}} : K(\mathcal{C}) \to K(D^8, S^7; \mathcal{C}),$$

In [5, Remark 2, page 222], Karoubi explains how one can obtain a cohomology theory $h^n(X, Y)$ from the groups $K^n(X)$, $K^n(Y)$, and $K^n(\phi)$, where ϕ is the restriction map $\mathcal{C}(X) \to \mathcal{C}(Y)$. This allows one to obtain Theorem 3.38 by applying the suspension axiom repeatedly. On the other hand, if one takes $\mathcal{C}$ to be the category of finite dimensional real vector bundles over X, then one recovers the classical real Bott Periodicity Theorem. On the other hand, if we take $\mathcal{C}$ to be the category of real G-vector bundles over a compact G space X, Theorem 3.38 yields an isomorphism between $K_G(X)$ and $K_G(X \times D^8, X \times S^7$, an isomorphism noted by Karoubi in [5, Page 224].

We conclude with a final remark on the equivariant case. Karoubi's proof of Bott Periodicity uses the periodicity of the groups $M(C_k)$ (and a similar periodicity of the groups $M(C'_k)$). We previously mentioned, in Remark 2.24, a periodicity in the groups $M(C_k) \otimes \mathbb{Z}[G]$. One might try, then, to define subcategories $\mathcal{C}_G^{p,q}$ that capture this tensor product. One could also try to define this subcategory to be the subcategory of objects of $\mathcal{C}$ which are $(C_k \otimes \mathbb{Z}[G])$-modules, although the difficulty here is that such a definition requires one to understand a great deal about representations of G.

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