The Farey Tessellation

Simon Seignourel

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Abstract

In this paper, we introduce the hyperbolic plane and define geodesics on this plane. We also define continued fractions and give several examples of real numbers expressed as continued fractions. After giving background on both these subjects, we explain how they are related to the Farey Tessellation, including the proof of how to derive a continued fraction from the Farey Tessellation. Finally, we introduce convergents and give a geometric interpretation of convergents on the Farey Tessellation.

1 Introduction to Hyperbolic Geometry

The geometry we know best, from our earliest algebra classes, is Euclidean geometry. However, other geometries exist, one of which is Hyperbolic geometry. Hyperbolic geometry can be represented using the upper half of the Euclidean plane $\mathbb{R}^2$, which can be identified with the upper half of the complex plane $\mathbb{C}$. We denote this space $\mathbb{H}$. Let us then consider the set $\{(x; y) \mid y > 0\} \subset \mathbb{R}^2$, which can be interpreted as the set of complex numbers with a positive imaginary part, or $\{z \mid \mathrm{Im}(z) > 0\}$.

We know that in the Euclidean space $\mathbb{R}^2$ the length element is

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}.$$

This means that to compute the length of a curve $\gamma(t) : [0, 1] \to \mathbb{R}^2$, we need to compute the integral

$$\int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\mathrm{d}t,$$

where the horizontal and vertical coordinates of any point in the curve are functions $x(t)$ and $y(t)$ of the parameter t.

Let us now consider a different length element:

$$\mathrm{d}s_{\mathbb{H}} = \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{y}.$$

This is the length element in the hyperbolic plane. In this plane, as we move from one point to another, we are continually dividing by the y coordinate, or renormalizing by the y coordinate. If we measure the lengths using this length element, then instead of the equation above, we will get that the hyperbolic length $l_{\mathbb{H}}(\gamma)$ of the same curve $\gamma(t) : [0, 1] \to \mathbb{R}^2$ will be

$$l_{\mathbb{H}}(\gamma) = \int_0^1 \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)}\mathrm{d}t.$$

1.1 Geodesics in Hyperbolic Geometry

In Euclidean geometry, a line is the shortest path between two points. However, in hyperbolic geometry these shortest paths, called geodesics, do not always take the shape of straight lines. Intuitively, we can see

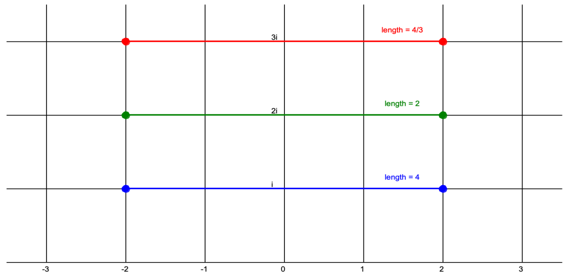

Figure 1: Segment Lengths in $\mathbb{H}$

this is true because in the hyperbolic plane (represented by the upper half of the complex plane), a horizontal straight line that is higher up would technically be shorter than one further down, since we are renormalizing by the y coordinate, as shown in Figure 1. This implies that sometimes a direct horizontal line is not the shortest path between two points; it can make paths shorter to move higher in the plane.

We can see a concrete example of this if we consider two paths connecting the points $-2+i$ and $2+i$. The first path is completely horizontal, and thus has hyperbolic length

$$\int_{-2+i}^{2+i} \frac{\sqrt{\dot{x}(t)^2+0}}{1} = 4.$$

However, suppose we have another path σ that goes diagonally up from $-2+i$ to $2i$ and then diagonally down from $2i$ to $2+i$. A parametrization of this path is given by

$$\sigma(t) = \begin{cases} (2t-2)+i(1+t) & 0 \le t \le 1 \\ (2t-2)+i(3-t) & 1 \le t \le 2 \end{cases}.$$

In $\mathbb{R}^2$, therefore, this means that

$$\begin{aligned} x(t) &= 2t-2 \\ y(t) &= \begin{cases} 1+t & 0 \le t \le 1 \\ 3-t & 1 \le t \le 2. \end{cases} \end{aligned}$$

As a result,

$$\begin{aligned} x'(t) &= 2 \\ y'(t) &= \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2. \end{cases} \end{aligned}$$

Therefore by the hyperbolic length equation

$$\begin{aligned} l_{\mathbb{H}}(\sigma) &= \int_0^1 \frac{\sqrt{5}}{1+t}\mathrm{d}t + \int_1^2 \frac{\sqrt{5}}{3-t}\mathrm{d}t \\ &= \sqrt{5}\log(1+t)\Big|_0^1 + \sqrt{5}\log(3-t)\Big|_1^2 \\ &= 2\sqrt{5}\log 2, \end{aligned}$$

which is about 3.1. Note that this path, despite being longer than the horizontal in the Euclidean plane, is shorter than a simple horizontal path in the Hyperbolic plane. It turns out that in the Hyperbolic plane, the shortest paths between two points (geodesics) are either vertical lines or semicircles with centers on the real axis.

Proposition 1. *Vertical lines are geodesics between points in the hyperbolic plane.*

Proof. Let us consider two points, a and b, which lie on the same vertical line. We can denote a path from a to b where the x coordinate is unchanging as $\eta(t)$. Let $\gamma(t)$ be an arbitrary path with the same endpoints a and b. We know that

$$\begin{aligned} l_{\mathbb{H}}(\gamma) &= \int_a^b \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)} \mathrm{d}t \\ &\geq \int_a^b \frac{\sqrt{0 + \dot{y}(t)^2}}{y(t)} \mathrm{d}t \\ &= l_{\mathbb{H}}(\eta). \end{aligned}$$

□

Proposition 2. *Semicircles with center on the real axis are geodesics between points in the hyperbolic plane.*

Proof. Let us consider two points a and b which do not lie on the same vertical line. Then there is a unique semicircle with center on the real axis which passes through both points. We assume for now that the center lies at the origin, because translating the path horizontally does not change its length in the hyperbolic plane. Suppose this semicircle has radius r_0. We denote a path from a to b on the semicircle as $\eta(t)$, and let $\gamma(t)$ be an arbitrary path with endpoints a and b. For this proof, we use polar coordinates instead of x and y coordinates, using the identity that $\dot{x}(t)^2 + \dot{y}(t)^2 = r^{-2}\dot{r}(t)^2 + \dot{\theta}(t)^2$. Then we have that

$$\begin{aligned} l_{\mathbb{H}}(\gamma) &= \int_a^b \frac{\sqrt{r^{-2}\dot{r}(t)^2 + \dot{\theta}(t)^2}}{\sin(\theta(t))} \mathrm{d}t \\ &\geq \int_a^b \frac{\sqrt{0 + \dot{\theta}(t)^2}}{\sin(\theta(t))} \mathrm{d}t \\ &= l_{\mathbb{H}}(\eta). \end{aligned}$$

□

Remark. Semicircles and vertical lines completely describe geodesics in the hyperbolic plane; there are no other kinds of shortest path.

2 Introduction to Continued Fractions

Definition 2.1. A continued fraction is defined as an expression

$$x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$$

where $x \in \mathbb{R}$ and $n_i \in \mathbb{N} \cup 0$.

We use the notation $x = [n_0; n_1, n_2, n_3 \dots]$ to express x as a continued fraction. As an example, we can look at $\frac{3}{5}$.

$$\frac{3}{5} = 0 + \frac{1}{\frac{5}{3}} = 0 + \frac{1}{1 + \frac{2}{3}} = 0 + \frac{1}{1 + \frac{1}{\frac{3}{2}}} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}$$

As a result, we have that $n_0 = 0$, $n_1 = 1$, $n_2 = 1$, and $n_3 = 2$. Thus $\frac{3}{5} = [0; 1, 1, 2]$. It is easy to see how we can use this process to represent any rational as a continued fraction.
If x is an irrational number, then this similar process produces an infinite continued fraction. For an example of an irrational number as a continued fraction, we can look at $x = \frac{1+\sqrt{5}}{2}$.

$$x = \frac{1+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}-1}{2} = 1 + \frac{1}{\frac{2}{\sqrt{5}-1}} = 1 + \frac{1}{\frac{2(\sqrt{5}+1)}{4}} = 1 + \frac{1}{\frac{1+\sqrt{5}}{2}} = 1 + \frac{1}{x}$$

This means that

$$x = 1 + \frac{1}{x} = 1 + \frac{1}{1+\frac{1}{x}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$$

and therefore that $x = [1; 1, 1, 1, \dots]$.

3 Neighbors and the Farey Tessellation

In this section, we will be building the Farey Tessellation; this is a method of dividing the hyperbolic plane that has a fascinating connection to continued fractions. However, to build the Farey Tessellation, we first need to understand neighbors.

Definition 3.1. Two rationals $\frac{p}{q}$ and $\frac{r}{s}$ are called neighbors if

$$|ps - rq| = 1.$$

As an example, note that any integer n is neighbors with $n + 1$:

$$|n \times 1 - (n+1) \times 1| = |n - n - 1| = |-1| = 1.$$

We also want to introduce a new operation, $\oplus$, which is an incorrect kind of fraction addition (what many have been told not to do in elementary school).

Definition 3.2. Given two rationals $\frac{p}{q}$ and $\frac{r}{s}$,

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

Proposition 3. *If $\frac{p}{q}$ and $\frac{r}{s}$ are neighbors, then $\frac{p}{q}$ is neighbors with $\frac{p}{q} \oplus \frac{r}{s}$.*

Proof.

$$|p(q+s) - (r+p)q| = |pq + ps - rq - pq| = |ps - rq| = 1.$$

Similarly, $\frac{r}{s}$ is also neighbors with $\frac{p}{q} \oplus \frac{r}{s}$.

$\square$

Proposition 4. *If $\frac{p}{q} < \frac{r}{s}$, then $\frac{p}{q} < \frac{p}{q} \oplus \frac{r}{s} < \frac{r}{s}$.*

Proof.

$$\begin{aligned} \frac{p}{q} &< \frac{r}{s} \\ ps &< rq \\ pq + ps &< pq + rq \\ p(q+s) &< (p+r)(q) \\ \frac{p}{q} &< \frac{p}{q} \oplus \frac{r}{s} \end{aligned}$$

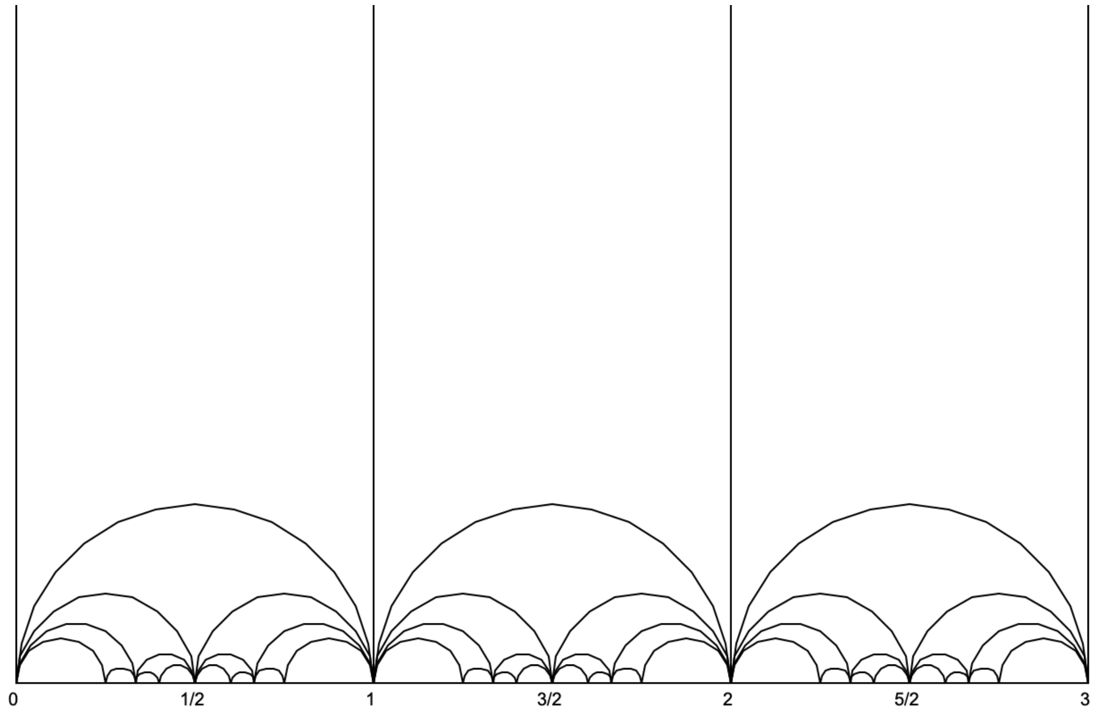

Figure 2: The Farey Tessellation

And,

$$\begin{aligned} \frac{p}{q} &< \frac{r}{s} \\ ps &< rq \\ ps + rs &< rq + rs \\ (p+r)s &< r(q+s) \\ \frac{p}{q} \oplus \frac{r}{s} &< \frac{r}{s}. \end{aligned}$$

$\square$

To build the Farey Tessellation in the complex plane, we start by drawing a vertical line from every integer $n \in \mathbb{R}$ on the real axis to ∞. We then connect each n to $n+1$ by a semicircle.
Inductively, if two rationals on the real axis $\frac{p}{q}$ and $\frac{r}{s}$ are neighbors, then we connect $\frac{p}{q}$ and $\frac{p}{q} \oplus \frac{r}{s}$ by a semicircle, and we connect $\frac{p}{q} \oplus \frac{r}{s}$ and $\frac{r}{s}$ by a semicircle. For example, given n and $n+1$, we'd connect n and $\frac{2n+1}{2}$ by a semicircle, and connect $\frac{2n+1}{2}$ and $n+1$ by a semicircle. In Figure 2, we can see an approximate representation of the Farey Tessellation after several of these iterations.

4 Farey Tessellation And Hyperbolic Geometry

It is easy to see that the tiles of the Farey Tessellation have boundaries that are geodesics in the hyperbolic plane, and therefore are hyperbolic triangles. We also include the point ∞ in this plane, which can be interpreted as the meeting point of two vertical lines. Thus we will call the triangle with vertices 0, 1 and ∞ the basic triangle and denote it by Δ.

We now want to consider the set of transformations $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$, and $T(z) = \frac{az+b}{cz+d}$ for any z in the complex plane. We will call this set $SL(2, \mathbb{Z})$; anybody familiar with linear algebra would recognize $SL(2, \mathbb{Z})$ as the set of all 2×2 matrices with integer entries and determinant 1. Note that if we have some $g \in SL(2, \mathbb{Z})$, then $g^{-1} \in SL(2, \mathbb{Z})$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & -ba + ab \\ cd - dc & -bc + ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that g^{-1} has all integer entries. Additionally, since g has determinant 1, then g^{-1} also has determinant 1, meaning $g^{-1} \in SL(2, \mathbb{Z})$. Also note that if we have g_1, g_2 in $SL(2, \mathbb{Z})$, then this means that $g_1 g_2 \in SL(2, \mathbb{Z})$ as well: the entries of this matrix are all integers, and its determinant is $\det(g_1) \times \det(g_2) = 1$.

We know that Mobius transformations, which take the form $T(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$, preserve angles and geodesics. Thus T preserves geodesics. Suppose we take some $T = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ where $ps - rq = 1$ and consider $T(\Delta)$. Then we know that $T(0) = \frac{r}{s}$, $T(1) = \frac{p+r}{q+s} = \frac{p}{q} \oplus \frac{r}{s}$, and $T(\infty) = \frac{p}{q}$. This means that the vertices of the basic triangle are brought to the vertices of the tile in the Farey Tessellation defined by $\frac{p}{q}$, $\frac{r}{s}$, and $\frac{p}{q} \oplus \frac{r}{s}$. Because T preserves geodesics, then this means that the interior of Δ is also transferred to the interior of this tile. Note that given the way we define the Farey Tessellation, each of its tiles has vertices $\frac{p}{q}$ and $\frac{r}{s}$ where $\frac{p}{q}$ and $\frac{r}{s}$ are neighbors. This means that if we assume $\frac{p}{q} > \frac{r}{s}$, $ps - rq = 1$. As a result, each tile of the Farey Tessellation is the result of some transformation $T(\Delta)$ where $T = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$.

Proposition 5. *The triangles in the Farey Tessellation cover the hyperbolic plane without overlapping each other.*

Proof. To prove this, we want to show that the tiles have no interior points in common. In other words, given the interiors I_1 and I_2 of two tiles, we want to show that $I_1 \cap I_2 = \emptyset$. We can denote the interior of Δ as Δ°; therefore, we want to show that $T_1(\Delta^\circ) \cap T_2(\Delta^\circ) = \emptyset$ since every tile in the Farey Tessellation is the result of some transformation $T \in SL(2, \mathbb{Z})$.
In fact, we only have to show that $g(\Delta^\circ) \cap \Delta^\circ = \emptyset$ for all $g \in SL(2, \mathbb{Z})$, because this implies the same for all tiles in the Farey Tessellation. As proof, suppose we let $I_1 = g_1(\Delta^\circ)$ and $I_2 = g_2(\Delta^\circ)$ where $g_1, g_2 \in SL(2, \mathbb{Z})$. This means that $g_1^{-1} g_2 \in SL(2, \mathbb{Z})$ as well. Let us assume that $\Delta^\circ \cap g_1^{-1} g_2(\Delta^\circ) = \emptyset$. We know that all members of $SL(2, \mathbb{Z})$ preserve geodesics, and therefore keep tiles of the Farey Tessellation intact, meaning that $\Delta^\circ \cap g_1^{-1} g_2(\Delta^\circ) = g_1(\Delta^\circ) \cap g_1 g_1^{-1} g_2(\Delta^\circ)$. As a result,

$$\Delta^\circ \cap g_1^{-1} g_2(\Delta^\circ) = g_1(\Delta^\circ) \cap g_1 g_1^{-1} g_2(\Delta^\circ) = g_1(\Delta^\circ) \cap g_2(\Delta^\circ) = \emptyset.$$

Therefore, if we assume by contradiction that we have some tile of the Farey Tessellation whose interior overlaps with Δ's, then show that the transformation associated with this tile is not part of $SL(2, \mathbb{Z})$, this will complete the proof. Suppose we have a tile with vertices $\frac{a}{c}$, $\frac{b}{d}$, and $\frac{a}{c} \oplus \frac{b}{d}$; assume without loss of generality that $\frac{a}{c} > \frac{b}{d}$. For this tile to have interior overlapping with Δ, either $\frac{a}{c} > 1$ and $\frac{b}{d} < 1$, or $\frac{a}{c} > 0$ and $\frac{b}{d} < 0$. If we have the first case, we can simply apply the transformation $T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, which brings 0 to 1, 1 to ∞ and ∞ to 0. This simply switches the vertices of Δ and turns the first case into the second; so we can assume that $\frac{a}{c} > 0$ and $\frac{b}{d} < 0$. However, we know this overlapping tile relates to the transformation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can assume $d > 0$ and $b < 0$, and we know a and c must have the same sign. If $a, c > 0$, then $ad - bc \leq 1 + 1 = 2$. Thus it is impossible that $ad - bc = 1$, meaning $A \notin SL(2, \mathbb{Z})$. As a result, this overlapping tile cannot be part of the Farey Tessellation. This means no tile overlaps with Δ, and therefore that no tiles in the Farey Tessellation overlap. $\square$

It is now simpler to show that given some $T \in SL(2, \mathbb{Z})$, it corresponds to a tile in the Farey Tessellation. We know by the proof above that any transformation leading to a tile that overlaps with any others cannot be part of $SL(2, \mathbb{Z})$. As a result, if $T \in SL(2, \mathbb{Z})$, then $T(\Delta)$ does not overlap interiors with any other tile. Since the Farey Tessellation covers the entire hyperbolic plane, this means that $T(\Delta)$ must be a tile in itself.

One result of this statement is that any pair of neighboring rationals must form a tile in the Farey Tessellation, since if $\frac{p}{q}$ is neighbors with $\frac{r}{s}$ and we assume that $\frac{p}{q} > \frac{r}{s}$, then $ps - rq = 1$. Since $p, q, r, s \in \mathbb{Z}$, then, the matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ must be in $SL(2, \mathbb{Z})$.
We also know that every rational number has at least one neighbor. For example, given some $\frac{p}{q}$, then we know p and q are relatively prime, meaning that by Fermat's little theorem $p^{q-1} = rq$ for some $r \in \mathbb{N}$. As a result, $\frac{p}{q}$ and $\frac{r}{p^{q-2}}$ are neighbors since $p^{q-1} - rq = 1$. This means every rational number is a vertex of the Farey Tessellation.

As a special case, consider the transformation $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We can see that $J(z) = -\frac{1}{z}$. J is the unique transformation that maps every geodesic passing through i to itself, while switching the endpoints; this is useful intuition to visualize how J acts on a geodesic.
To prove this, we first can realize that $J(i) = \frac{-1}{i} = i$. Since members of $SL(2, \mathbb{Z})$ preserve geodesics, this means that J maps any geodesic passing through i to a geodesic passing through i. If we can show that J maps the endpoints to each other, meaning that the endpoints of the geodesic are y and $-\frac{1}{y}$ given some $y \in \mathbb{R}$, this will be sufficient to complete the proof.

To show this, take some $y \in \mathbb{R}$, assuming $y > 0$ without loss of generality, and consider the geodesic passing through y and i. This geodesic is a semicircle with center c on the real axis. Since the radius connecting c to y must be equal to that connecting c and i, this means that $y - c = \sqrt{1 + c^2}$. Therefore, we can see that

$$\begin{aligned} y^2 - 2yc + c^2 &= 1 + c^2 \\ y^2 - 2yc &= 1 \\ -2yc &= 1 - y^2 \\ c &= \frac{y^2 - 1}{2y} \\ c &= \frac{y}{2} - \frac{1}{2y}. \end{aligned}$$

We can now calculate the radius:

$$r = y - c = -\frac{y}{2} - \frac{1}{2y}.$$

Therefore, the other endpoint is

$$c + r = \frac{y}{2} - \frac{1}{2y} - \frac{y}{2} - \frac{1}{2y} = -\frac{1}{y}.$$

This proves that given $y \in \mathbb{R}$, the geodesic passing through y and i has other endpoint $-\frac{1}{y}$. Since J preserves geodesics and brings i to itself, J swaps the endpoints and bring the geodesic connecting y and $-\frac{1}{y}$ to itself.

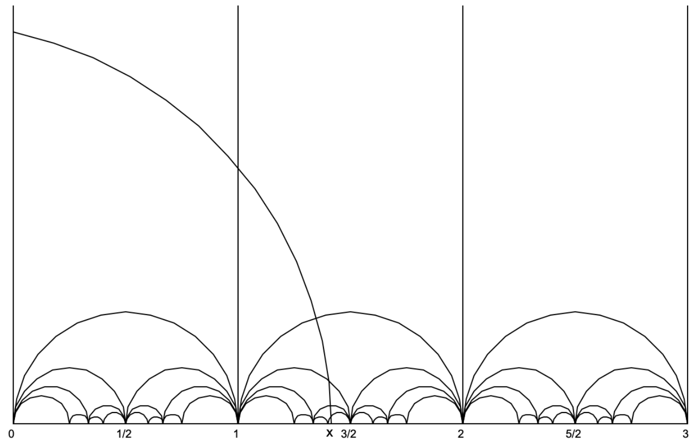

Figure 3: Geodesic to $\sqrt{2}$ on the Farey Tessellation

Figure 4: Examples of L and R Crossings

5 Farey Tessellation and Continued Fractions

The Farey Tessellation has an interesting connection to continued fractions, as introduced above. Having drawn the Farey Tessellation in the complex plane, we first take some $x \in \mathbb{R}$ on the real axis; then we draw a geodesic connecting x to any point on the imaginary axis. In Figure 3, we can see an example of this geodesic when $x = \sqrt{2}$.

Starting from the imaginary axis and working along that geodesic, this arc cuts through a series of the tiles (the hyperbolic triangles) that make up the Farey Tessellation, cutting through exactly two sides of each triangle. If the arc cuts through two sides that join at a vertex to its left, this is labeled L, and R if they meet at the arc's right. This depends on the direction of the arc; we can see some examples of L and R crossings in Figure 4. If the arc ends at a vertex of the triangle, we can label this either L or R.

By listing the series of L and R labels for the arc starting from the imaginary axis, we create a cutting sequence for x, in the format $L^{n_0}R^{n_1}L^{n_2}\ldots$. For example, take $x = \frac{3}{2}$. We start with one L, then one R, then it meets at a vertex, so $\frac{3}{2}$ has cutting sequence L^1R^2, or $L^1R^1L^1$.

As another example, take $x = \frac{3}{5}$. The arc begins cutting 1 R, then 1 L, then 1 R, then it meets the real line at a rational endpoint, so $\frac{3}{5}$ has cutting sequence $R^1L^1R^2$, or $R^1L^1R^1L^1$. Note that if $x \geq 1$, then its cutting sequence will always start with some number of L, and if $0 < x < 1$, its cutting sequence will start with an R.

Theorem 5.1. Given some $x = [n_0; n_1, n_2, \ldots]$, x has cutting sequence $L^{n_0}R^{n_1}L^{n_2}\ldots$.

Before proving this, we can first look at some examples.
We know that $\frac{3}{2}$ has cutting sequence L^1R^2 or $L^1R^1L^1$. We see that indeed

$$\frac{3}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{1 + \frac{1}{1}}.$$

Thus $\frac{3}{2} = [1; 1, 1] = [1; 2]$, and its cutting sequence is $L^1R^1L^1$ or L^1R^2.
Additionally, we saw also that $\frac{3}{5}$ had cutting sequence $R^1L^1R^2$ (which we could denote $L^0R^1L^1R^2$), and indeed

$$\frac{3}{5} = \frac{1}{\frac{5}{3}} = \frac{1}{1 + \frac{2}{3}} = \frac{1}{1 + \frac{1}{\frac{3}{2}}} = 0 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}.$$

To prove Theorem 5.1, we will need to utilize transformations in $SL(2, \mathbb{Z})$. Namely, we will use $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is easy to see that $P(z) = z + 1$ and $J(z) = -\frac{1}{z}$ for all z in the complex plane. Note that since P and J are in $SL(2, \mathbb{Z})$, they do not change the orientation of the tiles in the Farey Tessellation; this means that they do not change cutting sequences, as long as we keep track of the starting point.
Suppose we have some $x \in \mathbb{R}$ such that $x = [n_0; n_1, n_2 \ldots]$ as a continued fraction, meaning

$$x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \ldots}}.$$

We will assume without loss of generality that x is positive. Then let γ denote a geodesic from the imaginary axis $\mathbb{I}$ to x, which tracks a cutting sequence for x. Finally, we will define unit vectors $u_{-1}, u_0, u_1, \ldots$ at the points $z_{-1}, z_0, z_1, \ldots$ on γ where the cutting sequence changes from L to R or vice versa. (In this case, z_{-1} is the original starting point on the imaginary axis.)

Consider the transformation $P^{-n_0}(\gamma)$; this shifts γ n_0 leftwards and therefore brings z_0 to the imaginary axis. Now consider JP^{n_0}. This transformation flips γ into the left half of the plane, with z_0 staying on the imaginary axis and pointing to the right. This transformation brings x to $JP^{n_0}(x) = -\frac{1}{x - n_0}$. Furthermore

$$\begin{aligned} n_0 + \frac{1}{n_1 + 1} \leq x \leq n_0 + \frac{1}{n_1} \\ \frac{1}{n_1 + 1} \leq x - n_0 \leq \frac{1}{n_1} \\ n_1 + 1 \geq \frac{1}{x - n_0} \geq n_1 \\ -n_1 - 1 \geq -\frac{1}{x - n_0} \geq -n_1, \end{aligned}$$

which means that $JP^{n_0}(x)$ lands between $-n_1 - 1$ and $-n_1$ on the real axis. As a result, we know that the cutting sequence of x created by γ continues as $R^{n_1}L\ldots$ (for the same reason that a real number between natural numbers m and $m + 1$ starts with cutting sequence L^m).
We now can consider the transformation $JP^{n_1}JP^{-n_0}$. This essentially shifts $JP^{-n_0}(\gamma)$ n_1 rightwards, then flip it back into the left half of the plane. This means that afterwards, z_1 ends up on the imaginary axis pointing leftwards. We can see that $JP^{n_1}JP^{-n_0}(x) = \frac{1}{\frac{1}{x - n_0} - n_1}$ and that since $\frac{1}{x - n_0} < 1 + n_1$, then

$JP^{n_1}JP^{-n_0}(x) > 1$. Additionally,

$$\begin{aligned}
n_0 + \frac{1}{n_1 + \frac{1}{n_2+1}} \geq x \geq n_0 + \frac{1}{n_1 + \frac{1}{n_2}} \\
n_1 + \frac{1}{n_2+1} \leq \frac{1}{x-n_0} \leq n_1 + \frac{1}{n_2} \\
n_2 + 1 \geq \frac{1}{\frac{1}{x-n_0} - n_1} \geq n_2.
\end{aligned}$$

This means that $JP^{n_1}JP^{-n_0}(x)$ lands between n_2 and n_2+1, and that thus the cutting sequence continues $L^{n_2}R\ldots$. As a result, if we continue shifting and flipping γ in such a way, we would be able to find that the cutting sequence created by γ matches perfectly with the continued fraction of x. In other words, if $x = [n_0; n_1, n_2, \ldots]$, then $\gamma = L^{n_0}R^{n_1}L^{n_2}\ldots$.

Remark. It doesn't make a difference computationally whether we end with L or R. For example, the cutting sequence $L^1R^1L^1$ is effectively the same as L^1R^2, since the continued fraction $[1;1,1]$ is effectively the same as $[1;2]$. This is because $2 = 1 + \frac{1}{1}$, or $1 + \frac{1}{2} = 1 + \frac{1}{1+\frac{1}{1}}$. To generalize, some finite sequence $L^{n_0}R^{n_1}\ldots L^{n_m}R$ is the same as $L^{n_0}R^{n_1}\ldots L^{n_m+1}$ since $[n_0; n_1, \ldots, n_m, 1] = [n_0; n_1, \ldots, n_m + 1]$.

Remark. Since we know that the continued fraction of some $x \in \mathbb{R}$ is unique (except for the last value in finite continued fractions, as mentioned above), then we know that its cutting sequence must also be unique (excepting the last L or R).

6 Convergents

Definition 6.1. Given some $x = [a_0; a_1, a_2, \ldots]$, then let $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$. These $\frac{p_n}{q_n}$ are called the convergents of x.

In particular, note that $p_0 = a_0$ and $q_0 = 1$. For example, suppose that $x = \frac{1+\sqrt{5}}{2}$, meaning $x = [1;1,1,1,1\ldots]$. Then $\frac{p_0}{q_0} = 1$, $\frac{p_1}{q_1} = 1 + \frac{1}{1} = 2$, $\frac{p_2}{q_2} = 1 + \frac{1}{1+\frac{1}{1}} = \frac{3}{2}$, $\frac{p_3}{q_3} = \frac{5}{3}$, and so on.

Lemma 6.1. Take some $x = [a_0; a_1, a_2, \ldots]$ and its convergent $\frac{p_n}{q_n}$ where $n \geq 2$. Then $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-2} + q_{n-2}$.

Proof. We can prove this lemma by induction. As the base case, we know $\frac{p_0}{q_0} = a_0$ and $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0a_1+1}{a_1}$. Then

$$\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{a_2}{a_1a_2+1} = \frac{a_0a_1a_2 + a_0 + a_2}{a_1a_2+1} = \frac{a_2(a_0a_1+1) + a_0}{a_2(a_1)+1} = \frac{a_2(p_1) + p_0}{a_2(q_1) + q_0}.$$

Thus $p_2 = a_2p_1 + p_0$ and $q_2 = a_2q_1 + q_0$.
We now want to prove the inductive case. First, we assume that given some $y \in \mathbb{R} = [b_0; b_1, b_2, \ldots]$ and some m that $p_m = b_m p_{m-1} + p_{m-2}$ and $q_m = b_m q_{m-1} + q_{m-2}$. Now take some $x = [a_0; a_1, a_2, \ldots]$. Suppose we have some $\frac{r_m}{s_m} = [a_0; a_1, \ldots, a_m + \frac{1}{a_{m+1}}]$; then $\frac{r_m}{s_m} = \frac{p_{m+1}}{q_{m+1}}$. By the inductive hypothesis,

$$\frac{r_m}{s_m} = \frac{p_{m+1}}{q_{m+1}} = \frac{(a_m + \frac{1}{a_{m+1}})p_{m-1} + p_{m-2}}{(a_m + \frac{1}{a_{m+1}})q_{m-1} + q_{m-2}}$$

. This means that

$$\begin{aligned}\frac{p_{m+1}}{q_{m+1}} &= \frac{(a_m + \frac{1}{a_{m+1}})p_{m-1} + p_{m-2}}{(a_m + \frac{1}{a_{m+1}})q_{m-1} + q_{m-2}} \\ &= \frac{a_m a_{m+1} p_{m-1} + a_{m+1} p_{m-2} + p_{m-1}}{a_m a_{m+1} q_{m-1} + a_{m+1} q_{m-2} + q_{m-1}} \\ &= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}} \\ &= \frac{a_{m+1}(p_m) + p_{m-1}}{a_{m+1}(q_m) + q_{m-1}}.\end{aligned}$$

Therefore, $p_{m+1} = a_{m+1}p_m + p_{m-1}$ and $q_{m+1} = a_{m+1}q_m + q_{m-1}$, which proves the lemma by induction. □

This lemma helps prove the following corollary, as well as an important result about the placement of convergents in the Farey Tessellation.

Corollary 6.1.1. The determinant of the matrix $\begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1} \end{pmatrix} = \pm 1$ for all $n \geq 0$.

Proof. We can again use induction to prove this. For the base case, we can see that

$$\det \begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix} = p_0 q_1 - p_1 q_0 = (a_0)(a_1) - (a_0 a_1 + 1)(1) = -1.$$

To prove the inductive case, we first assume that $\det \begin{pmatrix} p_m & p_{m+1} \\ q_m & q_{m+1} \end{pmatrix} = \pm 1$, meaning that

$$p_m q_{m+1} - p_{m+1} q_m = \pm 1.$$

However, we can see that $\det \begin{pmatrix} p_{m+1} & p_{m+2} \\ q_{m+1} & q_{m+2} \end{pmatrix} = p_{m+1}q_{m+2} - p_{m+2}q_{m+1}$. From Lemma 6.1, we can see that

$$p_{m+1}q_{m+2} - p_{m+2}q_{m+1} = p_{m+1}(a_{m+2}q_{m+1} + q_m) - q_{m+1}(a_{m+2}p_{m+1} + p_m) = p_{m+1}q_m - q_{m+1}p_m.$$

Since

$$p_{m+1}q_m - q_{m+1}p_m = -\det \begin{pmatrix} p_m & p_{m+1} \\ q_m & q_{m+1} \end{pmatrix}$$

this means that

$$\det \begin{pmatrix} p_{m+1} & p_{m+2} \\ q_{m+1} & q_{m+2} \end{pmatrix} = -\det \begin{pmatrix} p_m & p_{m+1} \\ q_m & q_{m+1} \end{pmatrix}.$$

As a result, since the base matrix $\begin{pmatrix} p_0 & p_1 \\ q_0 & q_1 \end{pmatrix}$ has determinant -1, this means all such matrices have determinant ± 1. More specifically, matrices $\begin{pmatrix} p_m & p_{m+1} \\ q_m & q_{m+1} \end{pmatrix}$ have determinant -1 when m is even, and have determinant 1 when m is odd. □

Finally, we want to look at the placement of convergents in the Farey Tessellation; this will give us a convenient geometrical description of the way convergents approach a certain value x.

Corollary 6.1.2. Let $s_0, s_1, s_2, \ldots$ be the sides of the Farey Tessellation which mark the changes in the cutting sequence of a geodesic γ from L to R and vice versa, starting with s_0 being the vertical line from a_0 to ∞. Then for all $n \geq 0$, the endpoints of s_n are $\frac{p_n}{q_n}$ and $\frac{p_{n-1}}{q_{n-1}}$.

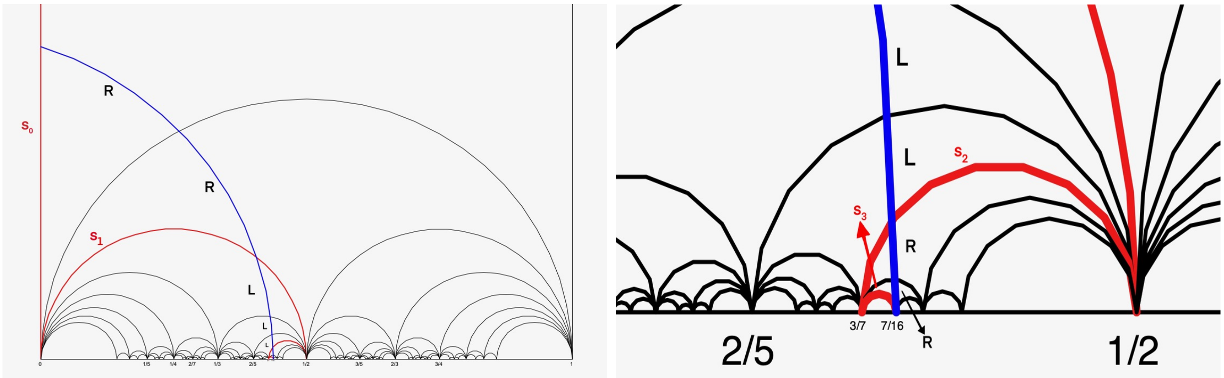

Figure 5: Geodesic to $\frac{7}{16}$ and its convergents (close-up on right)

We can first demonstrate an example of this before proving the corollary. Take the fraction $\frac{7}{16}$. Then

$$\frac{7}{16} = 0 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2}}}.$$

Thus $\frac{7}{16} = [0; 2, 3, 2]$ and has convergents $\frac{p_0}{q_0} = 0$, $\frac{p_1}{q_1} = \frac{1}{2}$, $\frac{p_2}{q_2} = \frac{3}{7}$, and $\frac{p_3}{q_3} = \frac{7}{16}$. If we draw the cutting sequence for $\frac{7}{16}$ and mark out the L and R, then we can see s_0, s_1, s_2 and s_3; their endpoints correspond exactly to these convergent values, as seen in Figure 5.

We can now prove the corollary.

Proof. Suppose we have some $x = [a_0; a_1, a_2, \ldots]$ and we draw the geodesic γ from the imaginary axis to x. Then its cutting sequence is $L^{a_0} R^{a_1} L^{a_2} \ldots$. We know s_0 has endpoints a_0 and ∞. We define $p_{-1} = 1$, and $q_{-1} = 0$ so that $\frac{p_{-1}}{q_{-1}} = \infty$ and $\frac{p_0}{q_0} = a_0$. After γ cuts s_0, there are a_1 segments of γ labelled R. We know that the left endpoint of s_1 must therefore still be $a_0 = \frac{p_0}{q_0}$. If $a_1 = 1$, then the right endpoint is $a_0 + 1$. If $a_1 = 2$, then this right endpoint is $a_0 \oplus a_0 + 1 = \frac{2a_0+1}{2}$. If $a_1 = 3$, then the endpoint is $a_0 \oplus a_0 \oplus a_0 + 1 = \frac{3a_0+1}{3}$. As a whole, the right endpoint of s_1 is

$$\frac{a_1 a_0 + 1}{a_1} = \frac{a_1 p_0 + p_{-1}}{a_1 q_0 + q_{-1}} = \frac{p_1}{q_1}$$

by Lemma 6.1. We now have a sequence of a_2 Ls. We know therefore that the right endpoint of s_2 is $\frac{p_1}{q_1}$. Similarly to the last step, the left hand endpoint moves through a_2 steps from $\frac{p_0}{q_0}$ towards $\frac{p_1}{q_1}$. If $a_2 = 1$, then the left endpoint is $\frac{p_1}{q_1} \oplus \frac{p_0}{q_0} = \frac{p_1+p_0}{q_1+q_0}$. If $a_2 = 2$, then this left endpoint is $\frac{p_1}{q_1} \oplus \frac{p_0}{q_0} \oplus \frac{p_1}{q_1} = \frac{2p_1+p_0}{2q_1+q_0}$. As a whole, the left endpoint of s_2 is $\frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} = \frac{p_2}{q_2}$ by the above lemma. We can continue in this way to show that given a step n, the endpoints are $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$, thus completing the proof. □

References

[1] Caroline Series, Continued Fractions and Hyperbolic Geometry. Loughborough LMS Summer School. July 2015.

[2] Charles Walkden, Hyperbolic Geometry. January 12, 2019.

[3] Binbin Xu, Introduction to Hyperbolic Geometry. Manuscript. August 29, 2021.