MARTINGALES IN GAMBLING AND FINANCE

PRAKHAR SAXENA

Abstract. This paper explores the concepts of Martingales in Probability Theory. We first begin with some definitions and concepts to make the paper more accessible for a general audience. Then we see how Martingales can be used to analyze simple betting games and examine the famous Martingale betting strategy. We end the paper by seeing some applications of Martingales in Financial Mathematics, specifically with the Fundamental Theorem of Asset Pricing and the Black-Scholes options pricing formula.

Contents

1. Measure, Probability and Random Variables

We begin with some familiar definitions from probability theory cast in a measure-theoretic framework.

Definition 1.1. A **probability space** is a triple $(\Omega, \mathcal{F}, P)$ where:

- Ω is a set of outcomes
- $\mathcal{F}$ is a set of events
- $P : \mathcal{F} \to [0, 1]$ is a function that assigns probabilities

We assume that $\mathcal{F}$ is a σ-algebra, that is a nonempty collection of subsets of Ω. Note that a **measurable space**, $(\Omega, \mathcal{F})$ is the same as a probability space without a function, P that assigns probabilities.

Definition 1.2. A **measure** is a function, $\mu : \mathcal{F} \to \mathbb{R}$ with

(1) $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$
(2) $\mu(\cup_i A_i) = \sum_{\forall i} \mu(A_i)$, where $A_i \in \mathcal{F}$ is a countable sequence of disjoint sets

Definition 1.3. A **probability measure** is a measure with $\mu(\Omega) = 1$

Definition 1.4. A **measurable function** is a function $X : \Omega \to E$ from a measurable space, $(\Omega, \mathcal{F})$, to another measurable space, $(E, \mathcal{S})$,
if $X^{-1}(B) \equiv \{\omega : X(\omega) \in B\} \in \mathcal{F}$ for all $B \in \mathcal{S}$, where B is any Borel set.[1]

We now introduce and discuss basic and familiar concepts about random variables which will be central throughout the discussion of Martingales.

Definition 1.5. A **random variable** is a real valued measurable function,
X : $\Omega \to E$, where Ω is a set of outcomes and E is a measurable subset of $\mathbb{R}$. When we want to emphasize the σ-algebra, $\mathcal{F}$, we say $X \in \mathcal{F}$ or we say X is $\mathcal{F}$-measurable.

Definition 1.6. The **distribution** of a random variable X is a probability measure, μ, on $\mathbb{R}$ such that $\mu(A) = P(X \in A)$ where A is any Borel set.

Definition 1.7. The **distribution function**, $F(x) = P(X \leq x)$, of a random variable X is a function which is generally used to describe the distribution of X.

Definition 1.8. The **expected value** of a random variable X is defined as
$E[X] = \int_{-\infty}^{\infty} X dP$ where P is a probability measure.

Definition 1.9. The **variance** of a random variable X is defined as
$Var(X) = E[X^2] - (E[X])^2$

2. Martingales and a Simple Game

Consider a simple game with a fair coin, which has a probability of $\frac{1}{2}$ of landing on its head and an equal probability of $\frac{1}{2}$ of landing on its tail. Suppose you start out with \$0 and win \$1 if you land a heads and lose \$1 if you land a tails. To study games like this one it is useful to understand the concept of a Martingale, which we build up to in this section.

Definition 2.1. Assume we have a probability space $(\Omega, \mathcal{F}_0, P)$, a σ-algebra $\mathcal{F} \subset \mathcal{F}_0$ and a random variable $X \in \mathcal{F}_0$ with $E[X] < \infty$. The **conditional expectation of X given $\mathcal{F}$**, $E[X|\mathcal{F}]$, is defined to be any random variable Y such that:

(1) $Y \in \mathcal{F}$, that is Y is $\mathcal{F}$-measurable.
(2) for all $A \in \mathcal{F}$, $\int_A X dP = \int_A Y dP$

Any such Y is said to be a **version** of $E[X|F]$.

I assert that the conditional expectation exists and is unique without providing the proof as that would be beyond the focus of this paper and does not add much to our analysis later on. Instead we will look at some useful properties of conditional expectations.

Theorem 2.2. *Properties of Conditional Expectation:*

(1) Linearity: $E[aX + Y|\mathcal{F}] = aE[X|\mathcal{F}] + E[Y|\mathcal{F}]$
(2) Monotonocity: If $X \leq Y$*, then* $E[X|\mathcal{F}] \leq E[Y|\mathcal{F}]$

Proof. For (1): First we note that the right hand side of the equality is a version of the left hand side, which means it is $\mathcal{F}$-measurable as well. If $A \in \mathcal{F}$, then by

[1] A Borel set is any set that can be formed from open sets through the operations of countable union, countable intersection, and relative complement.

linearity of the integral we have:

$$\int_A (aX+Y)dP = a\int_A XdP + \int_A YdP$$
$$= a\int_A E[X|\mathcal{F}]dP + \int_A E[Y|\mathcal{F}]dP \quad = \int_A (aE[X|\mathcal{F}] + E[Y|\mathcal{F}])dP$$

which proves (1).

For (2):

$$\int_A E[X|\mathcal{F}]dP = \int_A XdP \leq \int_A YdP = \int_A E[Y|\mathcal{F}]dP$$

Let $A = \{E[X|\mathcal{F}] - E[Y|\mathcal{F}] \geq \epsilon > 0\}$. Now, we see the set A has probability 0 for all $\epsilon > 0$, which proves (2). □

Definition 2.3. A **filtration** $\mathcal{F}_n$ is an increasing sequence of σ-algebras.

Definition 2.4. A sequence X_n is said to be **adapted** to $\mathcal{F}_n$ if $X_n \in \mathcal{F}_n$, for all n.

Definition 2.5. If X_n is a sequence that satisfies:

(1) $E[|X_n|] < \infty$
(2) X_n is adapted to $\mathcal{F}_n$
(3) $E[X_{n+1}|\mathcal{F}_n] = X_n$

for all n, then X is a **Martingale** with respect to $\mathcal{F}_n$

Going back to the game mentioned at the start of this section, it is evident that if you were to play this game for a long time, the expected value of your winnings would be \$0. The winnings here are in fact an example of a Martingale, as I will now prove:

Proposition 2.6. *Let X_n represent your wealth after the nth round of the fair coin flip game has been played. Then X_n is a martingale.*

Let $\xi_n = 1$ if you land a head and $\xi_n = -1$ if you land a tail on the nth toss. Let X_n denote your winnings at time n, that is $X_n = \xi_1 + \xi_2 + ... + \xi_n$ and $X_0 = 0$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(\xi_1, ..., \xi_n)$, which is the σ-algebra generated by $(\xi_1, ..., \xi_n)$. Firstly, we can observe that $X_n \in \mathcal{F}_n$ for all $n \geq 0$ which means X_n is adapted to $\mathcal{F}_n$. Further, we have $E[|X|] < \infty$. This satisfies the first two properties in Definition 2.5. As each coin flip is independent of the result of the previous flips, we have:

$$E[X_{n+1}|\mathcal{F}_n] = E[X_n + \xi_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[\xi_{n+1}|\mathcal{F}_n]$$

by the first property in Theorem 2.2.

As $X_n \in \mathcal{F}_n$ and $\xi_{n+1} \in \mathcal{F}_n$,

$$E[\xi_{n+1}|\mathcal{F}_n] = E[\xi_{n+1}] = 0 \quad \text{and} \quad E[X_n|\mathcal{F}_n] = X_n$$
$$\Rightarrow E[X_{n+1}|\mathcal{F}_n] = X_n$$

satisfying the third property in Definition 2.5 and proving that the earnings, X_n, in this game are a Martingale.

This shows that when playing this game your expected winnings do not change in any round and always remain \$0, which was the initial amount of money you started with. This can be expressed by $E[X_n] = X_0$. Games with this property are called **fair games**. In the next section we will look at some games which are not fair.

3. Unfair Games

We will now examine a betting strategy for the game of Roulette, which is commonly found in casinos. We will look at the American Roulette wheel which has 38 coloured slots: 18 red, 18 black and 2 green slots. Although it is possible to make a variety of bets in a regular game of Roulette, we will consider a simple strategy where you only bet on a single colour in each round the game is played. If you bet on red (black) and the ball lands in a red (black) slot, then you win the same amount that you bet. If you bet on green and the ball lands in a green slot, then you end up winning 17 times the amount you bet.

Looking at this game it is evident that if you were to play the same strategy (picking the same colour) repeatedly then you would end up with a net loss over a long period of time. This is an example of a **supermartingale**, which is defined in the same way as a Martingale except (3) in Definition 2.5 is changed to:
$E[X_n|\mathcal{F}_m] \leq X_m$ for any $n > m$.

Proposition 3.1. *Let X_n represent your wealth after the nth round of roulette has been played. Then X_n is a supermartingale.*

We begin with $X_0 > 0$. Suppose B_n represents the amount of money you bet and W_n represents the amount of money you won in round n. Further, let $\mathcal{F}_n = \sigma(B_1(W_1 - 1), ..., B_n(W_n - 1))$ for $n \geq 1$. This means:

$$X_{n+1} = X_n + B_n(W_n - 1)$$

$$\begin{aligned} \Rightarrow E[X_{n+1}|\mathcal{F}_n] &= E[X_n + B_n(W_n - 1)|\mathcal{F}_n] \\ &= E[X_n|\mathcal{F}_n] + E[B_n(W_n - 1)|\mathcal{F}_n] \\ &= X_n + E[B_n|\mathcal{F}_n]E[(W_n - 1)|\mathcal{F}_n] \end{aligned}$$

As $B_n \in \mathcal{F}_n$, $E[B_n|\mathcal{F}_n] = B_n$ and as W_n is constant, $E[(W_n-1)|\mathcal{F}_n] = E[(W_n-1)]$. Now we can see that if the strategy picked is betting on red/black and you 'win' in a round then $W_n = 2$ and $W_n = 0$ otherwise. As the probability of 'winning' is $\frac{18}{38}$

$$\Rightarrow E[(W_n - 1)] = \frac{36}{38} - 1 = -\frac{1}{19}$$

Similarly, if the strategy is betting on green: $W_n = 18$ if you win,

$$\Rightarrow E[(W_n - 1)] = \frac{36}{38} - 1 = -\frac{1}{19}$$

$$\Rightarrow E[X_{n+1}|\mathcal{F}_n] = X_n - \frac{B_n}{19} < X_n$$

as $B_n > 0$

which satisfies (3) from Definition 2.5 and (1) and (2) are also clearly true. This shows that your winnings in this game are a supermartingale.

An 'unfair' game may also may be one that is favourable towards the players. A simple example of this can be seen by amending the game in Proposition 2.6, by using an unfair coin that has a higher chance of landing on heads. Then, $P(\xi_n = 1) > \frac{1}{2}$ for all n. It is evident that if you were to play this game repeatedly you would have $E[X_n] > X_0$, where X_n represents your winnings. This is an example of **submartingale**, which is defined in the same way as a Martingale except (3) in Definition 2.5 is changed to: $E[X_n|\mathcal{F}_m] \geq X_m$ for any $n > m$.

As the proof that winnings actually are a submartingale in this unfair coin flip game is very similar to Proposition 2.6 we will skip that and move on to assess a famous betting strategy known as the Martingale strategy, which can theoretically be used to make a profit in fair games.

4. Martingale Betting Strategy and Stopping Times

We first state some definitions which will allow us to define and analyze the betting strategy more formally

Definition 4.1. Let $\mathcal{F}_n, n \geq 0$ be a filtration. H_n, for $n \geq 1$, is called a **predictable sequence** if $H_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$

This essentially means that the value of H_n can be predicted with certainty, using only the information available at time $n - 1$.

Definition 4.2. A random variable N is called a **stopping time** if $\{N = n\} \in \mathcal{F}_n$ for all $n < \infty$

In our context a stopping time can be thought of as the time you decide to stop gambling and walk away from the game with your winnings (losses). It is important to note that the stopping time must be determined only using the history that a gambler has already seen, including and up to the current round of play. For example, in the following strategy discussed for the fair coin flip game, deciding to stop playing as soon as you make a profit once is an example of a stopping time. This can be changed to other types of stopping times as well, for example deciding to stop playing only when you win 3 times in a row.

Suppose you are playing the fair coin flip game from Proposition 2.6, but now begin with $X_0 > 0$. We also stop playing this game as soon as we make a profit. Let H_n be a betting strategy such that $H_1 = 1$ and $H_n = 2H_{n-1}, n \geq 2$ if $\xi_{n-1} = -1$. This means if we lose money in round n we double our bet in round $n + 1$. This strategy allows you to always leave with a profit of \$1, as if you play this game for a large number of rounds you would almost surely land a head at least once:
P(Flip = Tails) for a large number of rounds consecutively $= (\frac{1}{2})^N \to 0$ as $N \to \infty$
For example, suppose you land a tails on the first k successive rounds and a heads on the $(k + 1)$th round. Then your winnings on the $(k + 1)$th round are given by: $-1 - 2 - 3... - 2^k + 2^{k+1} = 1$, which is a profit.

However, this strategy is only theoretically effective as in reality your wealth is bounded by some finite number and you cannot keep on doubling your bets forever on a losing streak. The following theorem states the conditions under which you cannot beat a fair game:

Theorem 4.3. ***Discrete Martingale Stopping Theorem**: If X_n is a martingale with respect to $\mathcal{F}_n$, and if N is a stopping time for $\mathcal{F}_n$ then*

$$E[X_N] = X_0$$

whenever one of the following holds:

(1) There is a constant b such that, $|X_n| \leq b$ *for all* $n \leq N$
(2) N is bounded, almost surely
(3) $E[N] < \infty$ *and there exists a constant c such that,* $E[X_{i+1} - X_i|\mathcal{F}_i] < c$ *for all i*

We will only prove the theorem for the first condition, as that is the one which is most relevant to the gambling example as you only have finite wealth (X_n is bounded). Before proving this we need to state another important theorem in the study of Martingales:

Theorem 4.4. ***Martingale Convergence Theorem*** *If* X_n *is a submartingale with* $sup\{E[{X_n}^+]\} < \infty$ *then as* $n \to \infty$*,* X_n *converges almost surely to a limit* X *with* $E[X] < \infty$

Using this we can now prove the first condition.

Proof. Suppose (1) in Theorem 4.3 is true. As X_n is bounded, using Theorem 4.4, we have X_n converging pointwise to a random variable, which we will call X_N. As $X_n < b$ for all n, we have $|X_{N\wedge n}| < b$. The wedge, $\wedge$, here represents min{N,n}. As $|X_{N\wedge n}|$ is bounded by an integrable function we can use the dominated convergence theorem. This gives us

$$\lim_{n\to\infty} \int_\Omega X_{N\wedge n} dP = \int X_N dP$$

Therefore, $\lim_{n\to\infty} X_{N\wedge n} = X_N$ and $E[X_T] = E[X_0]$ □

This shows that under 'realistic' conditions there is no strategy that will allow you to make a certain profit/loss in a fair game, and you will end up with the same amount of money that you started with.

Interestingly, there is another theorem which shows that there is no betting strategy that will allow you to make money with certainty, in a game where your winnings are a supermartingale. For example, in the roulette game in Proposition 3.1. First, I state a Lemma that we need to prove this theorem.

Lemma 4.5. *If* $X \in \mathcal{F}$ *and* $E[Y], E[XY] < \infty$*, then* $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$

I do not prove this Lemma as it involves some deeper understanding of measure theory. One can find it in [4]. Now for the supermartingale theorem:

Theorem 4.6. *Let* $X_n, n \geq 0$*, be a supermartingale. If* $H_n \geq 0$ *is predictable and each* H_n *is bounded then* $(H \cdot X)_n$ *is a supermartingale. Here* $(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1})$*.*

Proof.

$$\begin{aligned} E[(H \cdot X)_{n+1}|\mathcal{F}_n] &= E[(H \cdot X)_n|\mathcal{F}_n] + E[H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n] \\ &= (H \cdot X)_n + H_{n+1}E[(X_{n+1} - X_n)|\mathcal{F}_n] \end{aligned}$$

as $(H \cdot X)_n \in \mathcal{F}_n$, $H_n \in \mathcal{F}_{n-1}$ and by property (1) in Theorem 2.2. Now we have:

$$E[(H \cdot X)_{n+1}|\mathcal{F}_n] = (H \cdot X)_n + H_{n+1}E[(X_{n+1} - X_n)|\mathcal{F}_n] \leq (H \cdot X)_n$$

as $E[(X_{n+1} - X_n)|\mathcal{F}_n] \leq 0$ as X_n is a supermartingale, and $H_{n+1} \geq 0$

□

This shows that there is no strategy that will allow you to make a certain profit or even take back the amount of money you started with in a game that follows a supermartingale process. This means there is no 'winning' strategy for the game of roulette and other similar games found in casinos, making sure 'The House' always wins!

In the next section we see another application of Martingales in the field of finance.

5. Martingales in Finance

One of the main motivations in modelling financial markets has been to determine what the 'right' or 'fair' price is for an asset. Martingales are very useful in describing the process that the price of a fair asset follows. In this section we will state the Fundamental Theorem of Asset Pricing, but first we introduce some assumptions and definitions we need to fully understand the theorem.

We will start off with the **No Arbitrage** assumption. This means that you cannot make a risk-free profit without any initial investment. An example of arbitrage is if the same security was selling for different prices on different markets, then you could buy it at the cheaper price in one market and sell it at the higher price in the other market making a risk free profit. We assume no such opportunities exist.

We also assume that a risk free savings instrument exists where you can invest money for a certain time and receive a fixed rate of return. We call this the **risk free rate**. A good real world example of this is a US Treasury bond, where you can almost certainly make money at the issued interest rate without any risk. We assume the risk free rate is constant and equal to r.

Definition 5.1. The **Risk Neutral Probability** of an event A:

$$P_{RN}(A) = \frac{\text{Price}\{\text{Contract paying 1 dollar at time T if A occurs}\}}{\text{Price}\{\text{Contract paying 1 dollar at time T no matter what }\}}$$

For a general case we can say that the risk neutral probability represents the market's expectation of the probability that event A will occur.

The denominator here represent the price of buying the risk free rate instrument (treasury bond). Compounding at rate r continuously for time T means that the price of such a risk free contract would be given by e^{-rT}, which will therefore be the denominator in Definition 5.1.

Using this definition, we can state and interpret 2 versions of the Fundamental Theorem of Asset Pricing

Theorem 5.2. ***Fundamental Theorem of Asset Pricing*** *If the assumption of No Arbitrage is satisfied then there exists a risk neutral probability, P_{RN}, on the set of outcomes, Ω, such that $P_{RN}(\omega) > 0$, for all $\omega \in \Omega$. Let $S(n)$ denote the price of the stock at time n, and let $\tilde{S}(n) = \frac{S(n)}{A(n)}$ where A is a risk free asset. Further $\tilde{S}(n)$ is a martingale with respect to the risk neutral probability, P_{RN}. Namely,*

$$E[\tilde{S}(n+1)|S(n)] = \tilde{S}(n)$$

This is true only under the assumption of no arbitrage as stated above, which establishes the existence of the risk neutral probability P_{RN}. $\tilde{S}(n)$ is known as the **discounted price** of the stock. This theorem essentially states that the current discounted price of a stock reflects the future expectations of the stock's price. Using this theorem, we can say that if we have a stock (which does not pay dividends) worth $\$X$ at time T, then its price today should be $E_{RN}[X]e^{-rT}$.

A similar version of this theorem can be stated for derivatives of financial instrument. A financial derivative is a contract that derives its value from the performance of an underlying asset/security. An example of this is an options contract on a stock or a futures contract on crude oil. Investors can use these derivatives to speculate on the underlying security's price movements. We will look at options in further detail but first we state the Fundamental Theorem of Asset Pricing for derivatives.

Theorem 5.3. ***Fundamental Theorem of Asset Pricing (Derivatives)*** *Consider the same setup as Theorem 5.2. Let D be a derivative with S as its underlying security, where $D(n)$ represents the price of the derivative at time n. Let $\tilde{D}(n) = \frac{D(n)}{A(n)}$ be the discounted price of the derivative. Then $\tilde{S}(n)$ and $\tilde{D}(n)$ are martingales with respect to P_{RN}. Namely,*

$$E[\tilde{S}(n+1)|S(n)] = \tilde{S}(n) \quad \textit{and} \quad E[\tilde{D}(n+1)|S(n)] = \tilde{D}(n)$$

Again this is true only under the assumption of no arbitrage. This theorem expands on Theorem 5.2 by stating that the derivative's current price reflects the future expectations of its price, based on the underlying security's current price. In the next section we learn a bit more about what options are and how they are priced.

6. Martingales and Options Pricing

An option is a financial contract with a specified strike price and expiry date that gives you the right, but not the obligation, to purchase the underlying security at the specified strike price. We will consider European options, which can only be exercised on the expiry date (American options can be exercised anytime before or on the expiry date). There are 2 types of options: call and put. A call option allows you to buy the underlying at the strike price and a put option allows you to sell at the strike price. Options are attractive to investors as a hedging and speculative tool.

We now build up to and examine the Black-Scholes formula, which is widely used to price European options. We will do this without getting into a lot of technical details and just to understand the formula and how Martingales play a role in it. First, we describe the concept of a Wiener Process.

Definition 6.1. The **Wiener Process**, W_t, is characterized as follows:

1. $W_0 = 0$ almost surely
2. For every $t > 0$, the future increments $(W_{t+u} - W_t, u \geq 0)$ are independent of the past values of $W_s, s < t$
3. The increments, $(W_{t+u} - W_t)$, are normally distributed with mean 0 and variance u
4. W_t is almost surely continuous in t

This Wiener process is an example of a continuous time Martingale. In our previous examples wealth was an example of a discrete time Martingale, as the wealth was only updated after each round of play. However, with stock market conditions we receive price updates at rates faster than a millisecond. This makes a continuous time Martingale better suited in this situation. The Black-Scholes formula assumes that the log of an asset's price follows a Wiener process adjusted with a drift term, which means the mean of the process is now going to be non zero. Writing this assumption precisely we have: the log of an asset price, X, at a fixed future time, T, is a normal random variable, N, with mean, μ, and variance, $T\sigma^2$ with respect to the risk neutral probability. Here, σ^2 is a measure of volatility. We now build up to the Black-Scholes formula:

(1) As X is log normally distributed we have $E[e^N] = e^{\mu + T\sigma^2/2}$
(2) Let X_0 be the current price of the asset. From the Martingale property of Theorem 5.2 we have

$$X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu+(\sigma^2/2-r)T}$$

(3) This means $\mu = \ln(X_0) + T(r - \frac{\sigma^2}{2})$
(4) Let g be a function. Then the price of a contract that pays $g(X)$ at time T is $E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$

Consider a European call option. Here we define $g(X) = max\{0, X - K\}$, where K is the strike price. This is the profit you can make from the option, as you will only exercise it if $X > K$.

Theorem 6.2. *The Black-Scholes formula states that the price of contract, with $g(X)$ defined as above is given by:*

$$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

where N is a normal variable with mean, $\mu = ln(X_0) + T(r - \frac{\sigma^2}{2})$, and variance, $T\sigma^2$.

As N is a normal variable we can use the formula for its cumulative distribution, Φ, and write the whole form for the price of the call option. Without going into the technical derivation we have the final price of the option as:

$$\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$$

where

$$d_1 = \frac{ln(\frac{X_0}{K}) + T(r + \frac{\sigma^2}{2})}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{ln(\frac{X_0}{K}) + T(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T}}$$

This price can also be derived from the Black-Scholes equation which is a partial differential equation that describes how the price of an option varies over time. Solving that equation to find the correct discounted price gives a Martingale as the answer. This means the option's price is a martingale which is equal to the expected value of the discounted payoff of the option, as stated by the Fundamental Theorem of Asset Pricing in Theorem 5.3.

7. Acknowledgments

I want to thank my mentor, Angela Wang, for helping me with the concepts in this paper. I would also like to thank Professor Peter May for organizing the REU, which showed me what mathematics research is like and exposed me to different areas of the subject I had never encountered before.

8. Bibliography

References

[1] Michael Downes. Short Math Guide for LaTeX. http://tex.loria.fr/general/downes-short-math-guide.pdf
[2] Richard Durett. *Probability: Theory and Examples, Fifth Edition (1990)*
[3] Wikipedia - Borel Set. https://en.wikipedia.org/borel_set
[4] Wikipedia - Optional Stopping Theorem. https://en.wikipedia.org/wiki/Optional_stopping_theorem
[5] Math StackExchange - https://math.stackexchange.com/questions/3269065/gambling-systems-and-strong-convergence
[6] Dr. Salim El Rouayheb, Rutgers University - http://eceweb1.rutgers.edu/~csi/ECE541/Chapter10.pdf
[7] Martingales, risk neutral probability, and Black-Scholes option pricing, Lecture Notes - https://math.mit.edu/~sheffield/2019600/martingalenotes.pdf
[8] Marek Capinski, Tomasz Zastawniak *Mathematics for Finance: An Introduction to Financial Engineering (2003)*
[9] Wikipedia - Derivative (Finance). https://en.wikipedia.org/wiki/Derivative_(finance)
[10] Wikipedia - Wiener Process. https://en.wikipedia.org/wiki/Wiener_process
[11] Wikipedia - Black-Scholes Model. https://en.wikipedia.org/wiki/BlackScholes_model#BlackScholes_equation