AN INTRODUCTION TO HAUSDORFF AND BOX COUNTING DIMENSION

TOMI ROSSINI

Abstract. Dimension is a way to assign a number to a set in $\mathbb{R}^d$ which captures its scaling property. In this paper we will discuss two approaches to define a notion of dimension: Hausdorff dimension and box counting dimension.

Contents

1. What is a Measure?

Measure theory seeks to generalize the notion of "area" that arises when studying subsets of Euclidean space. A measure on $\mathbb{R}^n$ is a way to define the following assignment

$$\begin{aligned} \text{subsets of } \mathbb{R}^n &\longrightarrow [0, \infty], \\ A &\longmapsto \text{area of } A. \end{aligned}$$

It is not always possible to define a notion of area which gives a consistent real number to *every* subset of euclidean space. Therefore, it becomes necessary to restrict attention to a smaller family of subsets of $\mathbb{R}^n$. This family should be closed under union and complementation - indeed, if we can measure two sets then we should be able to measure their union and complement. Again we can see some similarity to a distance function, but this time we are operating on sets. This motivates the definition of σ-algebra.

Definition 1.1. A non-empty collection of subsets $\mathcal{J}$ of $\mathbb{R}^n$ is called a *σ-algebra* if it satisfies the following properties:

(1) $\mathbb{R}^n$ is an element of $\mathcal{J}$.

(2) If A is an element of $\mathcal{J}$ then A^c is also an element of $\mathcal{J}$.

(3) If $A_1, A_2, \cdots$ are elements of $\mathcal{J}$ then $\bigcup_{i=1}^{\infty} A_i$ is also an element of $\mathcal{J}$.

Remark 1.1. As $\mathcal{J}$ is closed under complementation and countable union, it is automatically closed under countable intersection.

Thus, our measures will be assignments

$$\begin{aligned} \sigma\text{-algebra } &\longrightarrow [0,\infty], \\ A &\longmapsto \text{area of } A. \end{aligned}$$

Every topological space has a natural σ-algebra associated to it, the *Borel σ-algebra.*

Definition 1.2. The Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$.

1.1. **Defining a Measure.** We can now rigorously define a measure on a σ-algebra.

Definition 1.3. A *measure* μ on a σ-algebra $\mathcal{J}$ is a function

$$\begin{aligned} \mu : \mathcal{J} &\longrightarrow [0,\infty], \\ A &\longmapsto \mu(A) \end{aligned}$$

which satisfies the following two properties:

(1) The measure of the empty set is 0,

$$\mu(\emptyset) = 0.$$

(2) The measure μ is countable additive,

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$

for every countable sequence of disjoint sets $\{E_j\}$ in $\mathcal{J}$.

We record some easy but important consequences of this definition.

Proposition 1.4.

(1) If A, B are elements of $\mathcal{J}$ such that $B \subset A$ then

$$\mu(A \setminus B) = \mu(A) - \mu(B).$$

In particular, if $B \subset A$ then $\mu(B) \leq \mu(A)$.

(2) If $A_1, A_2, A_3 \ldots$ are elements of $\mathcal{J}$ and $A_1 \subset A_2 \subset \ldots$ then

$$\lim_{j\to\infty} \mu(A_j) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right).$$

For a proof of these facts see chapter three of [1].

1.2. **Examples of Measures.** In this section, we record some basic examples of measures.

Example 1.5. Let $\mathcal{J}$ the collection of all the subsets of $\mathbb{R}^n$. For $A \in \mathcal{J}$, define $\mu(A)$ to be the number of elements in A. This measure μ is called the *counting measure.*

Example 1.6. Let $\{x_i\}_{i=1}^{\infty}$ be a collection of points in $\mathbb{R}^n$ and let $\{a_i\}_{i=1}^{\infty}$ be a collection of real numbers.

Let $\mathcal{J}$ be the $\sigma-$algebra of all subsets of $\mathbb{R}^n$ and define μ by the following formula:

$$\mu(A) = \sum_{\{i | x_i \in A\}} a_i.$$

Then μ is a measure.

Example 1.7. Define a measure δ_x on all subsets of $\mathbb{R}^n$ by the following condition: $\delta_x(A) = 1$ if and only if $x \in A$. This measure is called the *point mass* at x.

2. Outer Measures

These examples of measures may not be that interesting. Moreover, they bear little resemblance to how we define the area of rectangles or triangles in Euclidean space. To construct a measure which generalizes this particular notion of area, it is often easiest to define an outer measure first.

Definition 2.1. An *outer measure* on $\mathbb{R}^n$ is a function $\nu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ satisfying the following three properties:

(1) The outer measure of the empty set is 0,
$$\nu(\emptyset) = 0.$$

(2) It is order preserving,
$$\text{if } B \subset A, \text{ then } \nu(B) \leq \nu(A).$$

(3) It is countably subadditive,
$$\nu\left(\bigcup_{1=j}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \nu(A_j) \text{ for any subsets } \{A_j\} \text{ of } X.$$

Outer measures are often easier to define due to their less restrictive definition. Moreover, outer measures always define a measure when we restrict to the σ-algebra of "ν-measurable subsets".

Definition 2.2. A set E is called ν *measurable* if it decomposes every subset of X additively. I.e., for all $A \subset X$,

$$\nu(A) = \nu(A \cap E) + \nu(A \setminus E).$$

Theorem 2.3. *Let ν be an outer measure. The collection $\mathcal{M}$ of ν- measurable sets forms a σ-algebra, and the restriction of ν to $\mathcal{M}$ is a measure.*

For a proof see Theorem 1.2 of [3].

Definition 2.4. We call the restriction of Hausdorff $n-$dimensional measure to $\mathcal{H}^n-$measurable sets the *Lebesgue Measure*, $\mathcal{L}^n$. From this we see that the Lebesgue measure provides an expansion of the traditional definition of volume in $\mathbb{R}^n$.

We now state and prove a key theorem:

Proposition 2.5. *Suppose $\mathcal{C}$ is a collection of subsets of $\mathbb{R}^n$ such that $\emptyset \in \mathcal{C}$ and there exists $D_1, D_2, \ldots \in \mathcal{C}$ such that $\mathbb{R}^n = \bigcup_{j=1}^{\infty} D_j$. Suppose $\ell : \mathcal{C} \to [0, \infty)$ with $l(\emptyset) = 0$. Define*

$$\nu(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) : A_i \in \mathcal{C} \text{ for each } i \text{ and } E \in \bigcup_{i=1}^{\infty} A_i \right\}.$$

Then ν is an outer measure.

The moral of this proposition is that it is possible to define an outer measure on $\mathbb{R}^n$ by stipulating the size of some "elementary sets" and then taking an appropriate infimum.

For example, we could stipulate $\mathcal{C}$ to be the collection of all rectangles in $\mathbb{R}^n$ and $\ell(C)$ to be the usual area of that rectangle. Then, by Proposition 2.5, we obtain an outer measure ν which assigns to each subset $A \subset \mathbb{R}^n$ the size of the most efficient covering of A by rectangles.

Proof of Proposition 2.5. We verify ν satisfies the definition of an outer measure.

(i) As $\ell(\emptyset) = 0$ we can see $\nu(\emptyset) = 0$. Thus, the outer measure of the empty set is 0.

(ii) Let $B \subset A$, such that $A, B \subset X$. Let $\mathcal{B}$ be the set of all covers of B, and let $\mathcal{A}$ be the set of all covers of A. As $B \subset A$ we know $\mathcal{A} \subset \mathcal{B}$ thus $\inf \mathcal{B} \leq \inf \mathcal{A}$. Hence,
$$\nu(A) \geq \nu(B).$$

(iii) Let $A_1, A_2, \cdots \subset X$, and $\epsilon > 0$. For each i there exists $C_{i1}, C_{i2}, \cdots \in \mathcal{C}$ such that $A_i \subset \bigcup_{j=1}^{\infty} C_{ij}$ such that
$$\sum_{j=1}^{\infty} \ell(C_{ij}) \leq \nu(A_i) + \frac{\epsilon}{2^i}.$$

We note that,
$$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} C_{ij}$$

and therefore,

$$\begin{aligned}\nu\left(\bigcup_{i=1}^{\infty} A_i\right) &\leq \sum_{i,j=1}^{\infty} \ell\left(C_{ij}\right) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} \ell(C_{ij})\right) \\ &\leq \sum_{i=1}^{\infty} \nu(A_i) + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \\ &= \sum_{i=1}^{\infty} \nu(A_i) + \epsilon.\end{aligned}$$

As ϵ was chosen arbitrarily we conclude that

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i).$$

□

3. Hausdorff Measure and Dimension

We now define Hausdorff measure and dimension. Hausdorff measure can be defined for any positive real parameter s. This parameter s will determine the scaling properties of the measure. Namely, if $A \subset \mathbb{R}^n$ and $\lambda > 0$ is a positive real number then

$$\mathcal{H}^s\left(\lambda A\right) = \lambda^s \mathcal{H}^s\left(A\right)$$

where $\mathcal{H}^s$ denotes the Hausdorff measure with parameter s.

In turn, Hausdorff measure allows us to define the dimension of a set. We can define an "s-dimensional set" as a subset $A \subset \mathbb{R}^n$ such that $\mathcal{H}^s\left(A\right)$ is positive and finite (this is not completely true for technical reasons but captures the important idea).

3.1. **Hausdorff Measure.**

We make a series of definitions with the end goal of defining a measure which scales like λ^s.

Definition 3.1. Let $U \subset \mathbb{R}^n, U \neq \emptyset$. The *diameter* of U is defined as

$$|U| = \sup\{|x-y| \mid x, y \in U\}.$$

Definition 3.2. Let $E \subset \mathbb{R}^n$. A δ*-cover* of E is a collection of subsets $\{U_i\}$ such that $E \subset \bigcup_{i=1}^{\infty} U_i$, and $0 < |U_i| \leq \delta$ for each i.

Definition 3.3. Letting $E \subset \mathbb{R}^n$, and let s be a non-negative number. We define the s, δ Hausdorff content of E by the following formula

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s \mid U_i \text{ is a } \delta \text{ cover of } E\right\}.$$

To get the *Hausdorff s-dimensional outer measure* of E, we let $\delta \to 0$. Note that, if $\delta_1 \leq \delta_2$ then any δ_1-cover of E is automatically a δ_2-cover of E. In particular, it follows that

$$\mathcal{H}^s_{\delta_1}(E) \geq \mathcal{H}^s_{\delta_2}(E).$$

In particular, for every $E \subset \mathbb{R}^n$ the limit $\lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$ exists. This justifies the following definition.

Definition 3.4. The s-dimensional Hausdorff measure is an outer measure on $\mathbb{R}^n$ defined by the following formula: for $E \subset \mathbb{R}^n$

$$\mathcal{H}^s\left(E\right) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

We now prove the fundamental scaling property of Hausdorff measure.

Proposition 3.5. *If $F \subset \mathbb{R}^n$ and $\lambda > 0$ then*

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}(F)$$

where $\lambda F = \{\lambda x : x \in F\}$.

Proof. We will show that

$$\mathcal{H}^s\left(\lambda F\right) \leq \lambda^s \mathcal{H}^s\left(F\right) \quad \text{ and } \quad \mathcal{H}^s\left(\lambda F\right) \geq \lambda^s \mathcal{H}^s\left(F\right).$$

(1) If $\{U_i\}$ is a δ-cover of F then λU_i is a $\lambda\delta$-cover of λF. Hence,

$$\begin{aligned} \mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \sum_{i=1}^{\infty} |\lambda U_i|^s &= \lambda^s \sum_{i=1}^{\infty} |U_i|^s \\ &\leq \lambda^s \mathcal{H}^s_\delta(F). \end{aligned}$$

This holds for any δ-cover. Letting $\delta \to 0$ gives

$$\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F).$$

(2) If we replace λ with $\frac{1}{\lambda}$ and F with λF we use the previous inequality to obtain

$$\mathcal{H}^s(F) \leq \left(\frac{1}{\lambda}\right)^s \mathcal{H}(\lambda F)$$

Rearranging we obtain the desired inequality,

$$\lambda^s \mathcal{H}^s(F) \leq \mathcal{H}^s(\lambda F).$$

$\square$

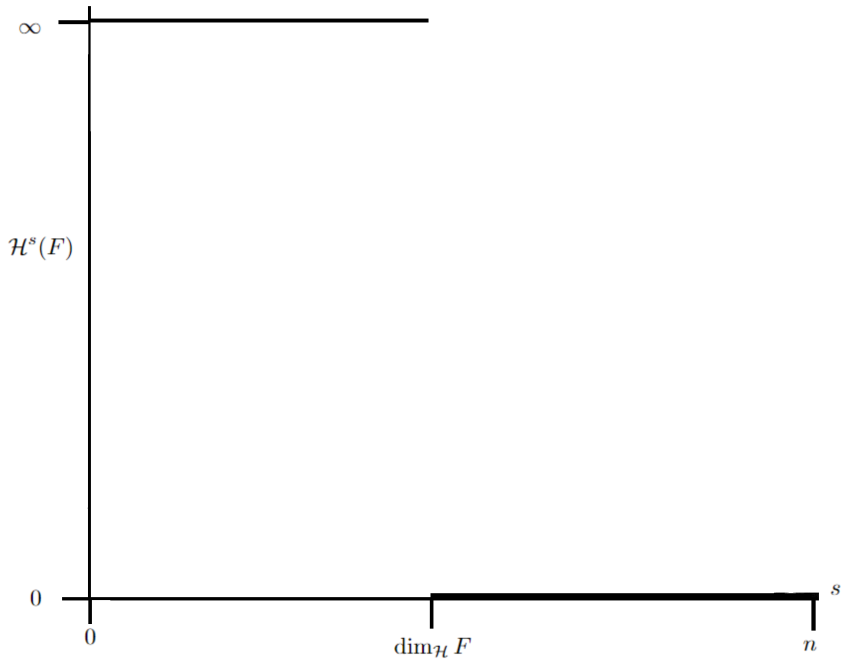


FIGURE 1. The Hausdorff dimension marks the critical transition between $\mathcal{H}^s(F) = \infty$ and $\mathcal{H}^s(F) = 0$.

3.2. **Hausdorff Dimension.** Now that we have defined the s-dimensional Hausdorff measure we can define Hausdorff dimension. The definition of Hausdorff dimension originates with the observation that for every set $F \subset \mathbb{R}^n$ there is a critical parameter s^* below which $\mathcal{H}^s(F) = \infty$ and above which $\mathcal{H}^s(F) = 0$. We call *Hausdorff dimension* $\dim_{\mathcal{H}}(F) := s^*$, where again s^* is our critical parameter. This transition and the critical parameter, is depicted in Figure 1.

The fact that there exists a critical parameter is the content of the following proposition.

Proposition 3.6. *If $\mathcal{H}^t(F) = \infty$ then for all $s < t$ $\mathcal{H}^s(F) = \infty$. Similarly, if $\mathcal{H}^s(F) < \infty$ then for all $s < t$ $\mathcal{H}^t(F) = 0$.*

Proof. Let $\{U_i\}$ be a δ-cover of F. Then,

$$\sum_{i=1}^{\infty} |U_i|^t \le \sum_{i=1}^{\infty} |U_i|^{t-s}|U_i|^s \le \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s.$$

Therefore, if $s < t$ and $\mathcal{H}^s(F) < \infty$ then letting δ to 0 gives that

$$\mathcal{H}^t(F) = 0.$$

Conversely, if $s < t$ and $\mathcal{H}^t(F) = \infty$ then for any δ-cover $\{U_i\}$ of F we have

$$\mathcal{H}^s(F) \geq \sum_{i=1}^{\infty} |U_i|^s \geq \sum_{i=1}^{\infty} |U_i|^t.$$

Sense $\mathcal{H}^t(F) = \infty$ the right-hand side can be made arbitrarily large by making δ small. Hence,

$$\mathcal{H}^s(F) = \infty.$$

$\square$

Example 3.7 (Middle thirds Cantor set)**.** A prototypical example in fractal geometry is the *middle thirds Cantor set.* This set is constructed by first removing the

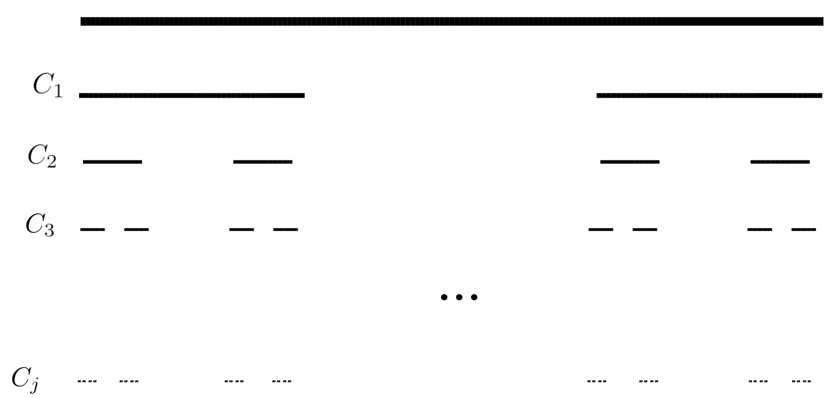


Figure 2. Middle-Thirds Cantor Set

middle third of the interval $C_0 := [0, 1]$ we get the union $E_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Now we take the intersection $C_1 := C_0 \cap E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, now as both C_0, and C_1 are closed so is the intersection. Now we remove the middle third of each interval again, getting $E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{7}\right] \cup \left[\frac{8}{9}, 0\right]$. Now again intersecting we get $C_2 := E_2 \cap C_1$, and we iterate ad infinitum. Thus, the middle thirds Cantor set $\mathcal{C}$ is defined as follows, let E_i be the sets

$$E_i = \bigcup_{i=0}^{3^{n-1}-1} \left(\left[\frac{3i}{3^n}, \frac{3i+1}{3^n}\right] \cup \left[\frac{3i+2}{3^n}, \frac{3i+3}{3^n}\right] \right).$$

Remark 3.8. In the preceding proofs we use the following notation:

$$C_j := \cap_{i=1}^{j} E_i.$$

Then define

$$\mathcal{C} \stackrel{\text{def}}{=} \bigcap_{j=1}^{\infty} E_j.$$

We state the first fact about the middle thirds Cantor set.

Proposition 3.9. *The middle third Cantor set has Lebesgue measure* 0.

Proof. Taking the Lebesgue measure of C_j

$$\mathcal{L}(C_j) = \inf \left\{ \sum_{i=1}^{\infty} \mathrm{vol}(E_i) : C_j \subset \bigcup \{U_i\} \right\}.$$

By the definition of C_j, it is the disjoint union of 2^j intervals of length 3^{-j}. Therefore,

$$\mathcal{L}(C_j) = \frac{2^j}{3^j}.$$

We see that:

$$\mathcal{L}(\mathcal{C}) = \inf_{j \in \mathbb{N}} \mathcal{L}(C_j) = \inf_{j \in \mathbb{N}} \left(\frac{2}{3} \right)^j = 0.$$

□

We can state something more precise which implies the previous proposition.

Proposition 3.10. *The middle thirds Cantor set has Hausdorff Dimension of* $s := \frac{\log(2)}{\log(3)}$, *and moreover* $\mathcal{H}^s(\mathcal{C}) = 1$.

Proof. We know that for $\epsilon > 0$ there exists $j \in \mathbb{N}$ such that:

$$\mathcal{H}^s_\delta(C_j \setminus \mathcal{C}) < \varepsilon.$$

Similar to the last proof we know by definition of C_j

$$\mathcal{H}^s_\delta(C_j) = \frac{2^j}{3^{sj}}.$$

If then $\frac{2}{3^s} < 1$ we know

$$\lim_{j \to \infty} \left(\frac{2}{3^s} \right)^j = 0.$$

As $s = \frac{\log 2}{\log 3}$ we see at once if $\delta \geq 3^{-j}$ we get $\mathcal{H}^s_\delta(\mathcal{C}) \leq 2^j 3^{-sj} = 2^j 2^{-j} = 1$. Thus, if we let $j \to \infty$ we get $\mathcal{H}^s(\mathcal{C}) \leq 1$.
We just need the opposite inequality, thus let $\mathcal{I}$ be any cover of $\mathcal{C}$. We want to show that:

$$1 \leq \sum_{I \in \mathcal{I}} |I|^s.$$

Now if we expand each interval slightly as $\mathcal{C}$ is compact we want to show that when $\mathcal{I}$ is a finite collection of closed intervals the above equation holds. Reducing further we may take each $I \in \mathcal{I}$ to be the smallest interval that contains some pair of intervals J, J' that occur in the construction of $\mathcal{C}$ (J and J' need not be intervals of the same E_j). If J, J' are the largest such intervals, then I is made up of J, and an interval K in the complement of $\mathcal{C}$, and then J', i.e:

$$I = J \cup J' \cup K.$$

From the construction of E_j we can see that:

$$|J|, |J'| \leq |K|.$$

Then we see by the additivity of measure:

$$\begin{aligned} |I|^s &= (|J| + |K| + |J'|)^s \\ &\geq \left(\frac{2}{3}(|J| + |J'|)^s\right) \\ &= 2\left(\frac{1}{2}|J|^s + \frac{1}{2}|J'|^s\right) \\ &\geq |J|^s + |J'|^s, \end{aligned}$$

as the function x^s is concave, and $3^s = 2$. And so we can see replacing I with the two subintervals J, J' does not increase the sum. Now we may proceed as so finitly until we have achieved a covering of $\mathcal{C}$ by equal intervals of length, say, 3^{-j}. This must include all the intervals of E_j, and so as $1 \leq \sum_{J \in \mathcal{J}} |J|^s$ is true for this cover it must be true for the original cover: $\mathcal{I}$. $\square$

4. Box Dimension

Box-counting dimension is a way for defining dimension without a measure. For $F \subset \mathbb{R}^n$ we calculate the box-counting dimension by, for a given radius δ, finding the smallest number of balls that cover F. We call this quantity N_δ. Then taking $\frac{\log(N_\delta)}{-\log \delta}$ as $\delta \to 0$ gives the box counting dimension. More formally we define the lower and upper box dimension of $F \subset \mathbb{R}^n$ the following:

Definition 4.1.

$$\begin{aligned} \underline{\dim}_B F &:= \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \\ \overline{\dim}_B F &:= \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \end{aligned}$$

(And if this limit exists), We call the *box-counting dimension* of F:

$$\dim_B F := \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

And we call N_δ any of the following:

(i) the smallest number of closed balls of radius δ that cover F;
The rest of these are likewise equivilent to the first notion.

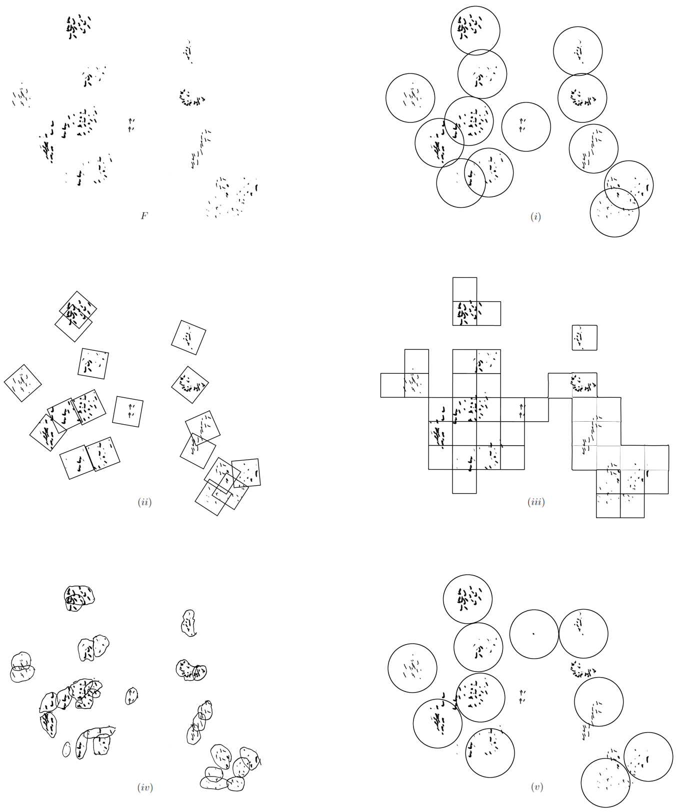
(ii) the smallest number of cubes of side δ that cover F;

(iii) the number of $\delta-$mesh cubes that intersect F;

(iv) the smallest number of diameter at most δ that cover F;

(v) the largest number of disjoint balls of radius δ with centers in F.

Remark 4.2. The following is an image of each of the different notions of N_δ



Definition 4.3. For some set $F \subset \mathbb{R}^n$ we call F_δ the $\delta-$*neigborhood* if

$$F_\delta = \{x \in \mathbb{R}^n | |x - y| \leq \delta, y \in F\}.$$

In other words the set of points within distance δ of F.

Proposition 4.4. *If* $F \subset \mathbb{R}^n$, *then*

$$\underline{\dim}_B F = n - \limsup_{\delta \to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta}$$
$$\overline{\dim}_B F = n - \liminf_{\delta \to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{\log \delta}.$$

Where F_δ *is the* δ*-neigborhood of* F

Proof. If F can be covered by $N_\delta(F)$ balls of radius $\delta < 1$ then F_δ can be covered by the concentric balls of radius 2δ. this is because each point in F_δ is with in a δ of some point in F. Hence,

$$\mathrm{vol}^n(F_\delta) \le N_\delta c_n (2\delta)^n$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$. Now taking logarithms of each side,

$$\frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \le \frac{\log 2^n c_n + n\log\delta + \log N_\delta(F)}{-\log\delta}$$
$$\frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \le \frac{\log 2^n c_n}{-\log\delta} + \frac{n\log\delta + \log N_\delta(F)}{-\log\delta}.$$

Hence,

$$\liminf_{\delta\to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \le -n + \liminf_{\delta\to 0} \frac{n\log\delta + \log N_\delta(F)}{-\log\delta}.$$

Hence,

$$\liminf_{\delta\to 0} \frac{\log \mathrm{vol}^n(F_\delta)}{-\log\delta} \le -n + \underline{\dim}_B F. \tag{4.5}$$

We can construct a similar inequality for the upper limits in the same way. On the other hand if there are $N_\delta(F)$ disjoint balls of radius δ and centers in F, then by adding the volumes

$$N_\delta(F) c_n \delta^n \le \mathrm{vol}^n(F_\delta).$$

Once again by taking logarithms and letting $\delta \to 0$ we get the opposite inequality of (4.5) when using the equivalent definition given in definition 5.1 (v). □

Now we will do the calculation of box dimension for a few interesting sets. For these examples we can see that they are all countable, implying that their Hausdorff dimension is zero, however as we shall see the same is not true for box dimension.

Example 4.6. Find the box dimension of

(1) $F = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$
(2) $F_e = \{e^{-n} | n \in \mathbb{N}\}$.

The set F

Proof. (i) Let $\delta \in (0, 1]$ if $0 < \delta < \frac{1}{2}$, then let k be the natural number that satisfies the following:

$$\frac{1}{k(k-1)} > \delta \ge \frac{1}{k(k+1)}.$$

If ball U has $|U| \leq \delta$, then as

$$\frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)} > \delta$$

we know U can cover at most $\left\{1, \frac{1}{2}, \ldots, \frac{1}{k}\right\}$. Thus, At least k sets of diameter δ are required to cover F, so $N_\delta(F) \geq k$ giving:

$$\frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log k}{\log(k(k-1))}.$$

Now as we take $\delta \to 0$ (note this makes $k \to \infty$ by its definition)

$$\liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \geq \frac{\log k}{\log(k^2+k)}$$
$$\underline{\dim}_B(F) \geq \frac{1}{2}.$$

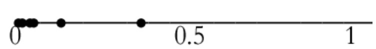
But if instead $\frac{1}{2} > \delta > 0$ take $k \in \mathbb{N}$ that satifies the following:

$$\frac{1}{k(k-1)} > \delta \geq \frac{1}{k(k+1)}.$$

Then $k+1$ balls of diameter δ cover $[0, \frac{1}{k}]$, leaving $k-1$ points of F that can be covered by $k-1$ balls. Thus, $N_\delta(F) \leq 2k$, so

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log(2k)}{\log(k^2-k)}$$
$$\limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log(2k)}{\log(k^2-k)}$$
$$\overline{\dim}_B(F) \leq 1/2.$$

And so, $\dim_B(F) = \frac{1}{2}$.



The set F_e

(ii) Let $\delta \in (0,1]$, and let $k \in \mathbb{N}$ satisfy the following equation:

$$e^{-(k+1)} < \delta \leq e^{-(k-1)}.$$

Thus if ball U has $|U| \leq \delta$ at least k balls of diameter δ are required to cover F_e. Hence, $N_\delta \geq k$ implying:

$$\frac{\log N_\delta(F_e)}{-\log \delta} \geq \frac{\log k}{\log(e^{-(k+1)})}$$
$$= \frac{\log k}{-(k+1)}$$
$$= \frac{\log k}{\log \delta}.$$

Taking $\delta \to 0$ we get

$$\lim_{\delta \to 0} \frac{\log k}{\log \delta} = 0.$$

Thus, $\underline{\dim}_B(F_e) \geq 0$. Now to upper box dimension. Let $\delta \in (0, 1]$, and let $k \in \mathbb{N}$ satisfy the following equation:

$$e^{-(k+1)} < \delta \leq e^{-(k-1)}.$$

Thus, if we cover $\left[0, e^{-(k+1)}\right]$ with $k + 1$ balls of diameter δ and each remaining point with one such ball we will have at most $2k$ balls covering F_e, thus $N_\delta(F_e) \leq 2k$. Hence,

$$\begin{aligned} \frac{\log N_\delta(F_e)}{-\log \delta} &\leq \frac{\log(2k)}{-(k-1)} \\ &= \frac{\log 2}{\log \delta} + \frac{\log k}{\log \delta} \end{aligned}$$

Taking $\delta \to 0$ we get: $\overline{\dim}_B(F_e) = 0$.

$\square$

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References

[1] Bass, Richard F. *Analysis for Graduate Students Analysis for Graduate Students: Measure and Integration Theory.* 2014.
[2] Falconer, Kenneth J., and Kenneth J. Falconer. *The Geometry of Fractal Sets.* Cambridge University Press, 1986.
[3] Falconer, Kenneth J. *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley et Sons, 2014.