FRACTAL INTERSECTIONS AND PRODUCTS VIA TRADITIONAL AND ALGORITHMIC METHODS

JINGWEI (ARIANA) QIN

Abstract. Bounds on classical Hausdorff dimension can be proven through traditional methods in geometric measure theory or through algorithmic dimension. Regarding the latter approach, for a point, the algorithmic dimension describes its information density, defined through Kolmogorov complexity. The point-to-set principle relates the Hausdorff dimension of a set to the algorithmic dimension of the points in the set. This allows bounding of set dimension, a global property, through point dimension, a local property. The algorithmic method extends results from Borel sets (as in traditional proofs) to arbitrary sets. This paper applies methods in both geometric measure theory and algorithmic to derive bounds for dimension of fractal intersections and Cartesian products in Euclidean spaces.

Contents

1. Introduction

One of the central goals in classical geometric measure theory is to rigorously define dimension and measure the dimension of fractal sets. In 1918, Felix Hausdorff introduced the Hausdorff dimension [1]. Yet, in general, calculating and bounding

Date: September 1st, 2023.

fractal dimension remain difficult and the search for relevant techniques continues. The development of a seemingly unrelated field, complexity theory, offers a new perspective on proofs regarding fractal dimensions. In the mid 1960s, Kolmogorov algorithmic complexity was invented to quantify the amount of information in individual data objects [2]. By adopting the algorithmic perspective, results previously restricted to Borel sets can be extended to arbitrary sets.

This paper contrasts the algorithmic perspective with the traditional methods. Section 2 reviews the definition of the Hausdorff dimension and packing dimension. Section 3 covers the definition of Kolmogorov complexity, allowing us to understand algorithmic dimension of point objects. Section 4 looks at the point-to-set principle, the bridge between point-wise algorithmic dimension and set-wise dimension. Section 5 introduces two theorems that bound the intersection and product of fractal sets. For each theorem, we offer a traditional proof and an algorithmic proof, illuminating the difference between the two approaches.

2. Fractal dimension of sets

This section introduces Hausdorff dimension and packing dimension, two classical concepts in fractal dimension. We may construct proofs directly from the definitions, but the task is difficult. In the following sections, we introduce equivalent algorithmic characterizations.

2.1. Hausdorff dimension.

[3] First, we define the *Hausdorff dimension*. The Hausdorff dimension is constructed through first defining an *s-dimensional Hausdorff measure*. The s-dimensional Hausdorff measure is an outer measure.

Definition 2.1. For $E \subseteq \mathbb{R}^n$, let $\mathcal{U}_\delta(E)$ be the collection of all countable covers of E by sets of positive diameter at most δ. For all $s \geq 0$, let

$$H^s_\delta(E) = \inf\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(U_i)^s : \{U_i\}_{i\in\mathbb{N}} \in \mathcal{U}_\delta(E)\right\}.$$

The *s-dimensional Hausdorff (outer) measure* of E is

$$H^s(E) = \lim_{\delta\to 0^+} H^s_\delta(E),$$

The *Hausdorff dimension* of E is

$$\dim_{\mathrm{H}}(E) = \inf\{s > 0 : H^s(E) = 0\}.$$

Remark 2.2. Notice that for all $s > dim_H(E)$, $H^s(E) = 0$.

2.2. Packing dimension.

[3] In addition to the Hausdorff dimension, the theorems in Section 5 also concerns with *packing dimension*.

Definition 2.3. For all $x \in \mathbb{R}^n$ and $\rho > 0$, let $B_\rho(x)$ denote the open ball of radius ρ and center x. For all $E \subseteq \mathbb{R}^n$, let $\mathcal{V}_\delta(E)$ be the class of all countable collections of pairwise disjoint open balls with centers in E and diameters at most δ. That is, for every $i \in \mathbb{N}$, we have $V_i = B_{\rho_i}(x_i)$ for some $x_i \in E$ and $\rho_i \in [0, \delta/2]$, and for every $j \neq i, V_i \cap V_j = \emptyset$. For all $s \geq 0$, define

$$P^s_\delta(E) = \sup\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(V_i)^s : \{V_i\}_{i\in\mathbb{N}} \in \mathcal{V}_\delta(E)\right\},$$

and let

$$P_0^s(E) = \lim_{\delta \to 0^+} P_\delta^s(E).$$

The *s-dimensional packing (outer) measure* of E is

$$P^s(E) = \inf \left\{ \sum_{i \in \mathbb{N}} P_0^s(E_i) : E \subseteq \bigcup_{i \in \mathbb{N}} E_i \right\}$$

and the *packing dimension* of E is

$$\dim_{\mathrm{P}}(E) = \inf \{ s : P^s(E) = 0 \}.$$

Remark 2.4. The packing dimension and the Hausdorff dimension are similar in that they involve two steps: first, defining the s-dimensional outer measure; then, the dimension is the infimum of s such that the s-dimensional outer measure is zero.

However, constructing the s-dimensional packing measure requires one more step, where a premeasure is explicitly defined. In generating s-dimensional Hausdorff measure, the premeasure μ_0 is defined implicitly:

$$\mu_0 = diam(U_i)^s.$$

In generating s-dimensional packing measure, the premeasure μ_0 is defined explicitly:

$$\mu_0 = P_0^s.$$

(From the premeasures, the outer measure μ^s is generated:

$$\mu^s(E) = inf\{\sum_{i \in \mathbb{N}} \mu_0(E_i) : E \subseteq \bigcup_{i \in \mathbb{N}} E_i\}.)$$

Why is there this difference between the two measures? Hausdorff measure is generated through covers, while the packing measure is created by fitting disjoint balls. Hence, in generating packing measure, there is an additional step to extend the premeasure to outer measure through covers. (The difference is clear on the surface too: $\lim_{\delta \to 0^+} H_\delta^s$ is a measure, $\lim_{\delta \to 0^+} P_\delta^s$ is a premeasure.)

3. Algorithmic dimension of points

The goal of this section is to introduce point-wise algorithmic dimension, both in terms of intuition and rigorous formulation. This serves as preparation for understanding set-wise fractal dimension from an algorithmic perspective.

Section 3.1 and 3.2 introduce some background and concepts in complexity theory and computability theory that help us understand algorithmic dimension.

3.1. Preliminaries: Communication.

To understand the proceeding section on algorithm complexity, it is helpful to first familiarize ourselves with the scenario of communication [2].

Imagine the following scenario of communication: Sender (A) wants to communicate some information, say an element $x \in X$, to receiver (B). A encodes the information as a binary string, known as the *message*. Upon receiving this message, B wants to decode the message and reconstruct the element x.

This process of communication requires an agreed coding/description method beforehand, namely a mapping between *source words* and *code words* known to

both A and B. This mapping is the *decoding function* $D : \{0,1\}^* \to X$, where $\{0,1\}^*$ is the set of binary codes, namely the set of finite strings over $\{0,1\}$. The domain of the function is the *source words* and the range of the function is the *code words*.

Hence, the inverse of D can be seen as the *encoding* E of the function:

$$E := D^{-1}, where D^{-1}(x) = \{y : D(y) = x\}$$

although E is not necessarily a function, namely that there might be multiple ways of encoding the source code.

One issue emerges: we cannot uniquely recover source words x and y from their encoding $E(xy)$. To illustrate, for example, if E is the identity mapping, then $E(00)E(00) = 0000 = E(0)(000)$. To resolve this issue, we introduce *prefix* codes.

Definition 3.1. [2] A binary string x is a *proper prefix* of a binary string y if we can write $y = xz$ for $z \neq \epsilon$ where ϵ is the empty word.

A set $x, y, ... \subseteq \{0,1\}^*$ is *prefix-free* is for any pair of distinct elements in the set neither is a proper prefix of the other.

A function $D : \{0,1\}^* \to X$ is a *prefix-code* if its domain is prefix free.

All prefix codes are uniquely decodable, namely that the source words can be uniquely reconstructed from the encoding. This is because we can decode the message word by word. There is no ambiguity of when a previous word ends and when the next word begins, since no code word is the prefix of another code word. In the next section discussing Kolmogorov complexity, we require the encoding to be prefix-free.

3.2. **Preliminaries: Computable function.** This section introduces the concept of *computable functions*. In later sections, we will limit our concern to computable functions. Informally, a function is computable if there exists a Turing machine that executes this function. (For readers who are unfamiliar, a Turing machine can be thought of as an algorithm written in a general purpose language.)

Definition 3.2. [2] A function $f : \mathbb{N} \to \mathbb{N}$ is computable if there exists a Turing Machine T that implements f. This means that for all input x, T outputs $f(x)$ and halts.

Definition 3.3. [2] We call a given function $f : \mathbb{N} \to \mathbb{R}$ computable if there exists a Turing machine that, when input $< x, y >$ with $x \in \mathbb{N}$ and $y \in \mathbb{N}$, outputs $f(x)$ to precision $1/y$. In other words, the Turing machine outputs a pair $< p, q >$ such that $|p/q - |f(x)|| < 1/y$, and an additional bit to indicate whether $f(x)$ is larger or smaller than 0.

3.3. **Kolmogorov complexity.** Complexity theory studies the amount of resources (time and storage) required to solve different kinds of problems. The theory of Kolmogorov complexity is concerned with understanding the computational resources needed to specify an object, such as a piece of string. Kolmogorov complexity is also known as algorithmic complexity because here the computational resource is studied as the length of the shortest algorithm that generates the object.

Definition 3.4. [4] The (prefix-free) *conditional Kolmogorov complexity* $K(\sigma|\tau)$ of a string $\sigma \in \{0,1\}^*$ given another string $\tau \in \{0,1\}^*$ is the length of the shortest

binary program that outputs σ when given τ as an input.

$$K(\sigma|\tau) = \min_{\pi\in\{0,1\}^*}\{|\pi| : U(\pi,\tau) = \sigma\}$$

where U is a fixed universal Turing machine that is prefix-free in its first input and $|\pi|$ denotes the length of the binary program π.

Definition 3.5. The *Kolmogorov complexity* of $\sigma \in \{0,1\}^*$ is conditional Kolmogorov complexity of $\sigma \in \{0,1\}^*$ given λ, where λ is the empty string:

$$K(\sigma) = K(\sigma|\lambda)$$

We introduce the *chain rule* of Kolmogorov complexity, which decomposes the complexity of a long string into two shorter constituent strings through conditional complexity. The right hand side of the equation describes a process where τ is first produced, and then σ is produced given τ.

Lemma 3.6. [5] *For all* $\sigma, \tau \in \{0,1\}^*$,

$$K(\sigma\tau) = K(\sigma|\tau) + K(\tau) + O(log|\sigma\tau|)$$

3.4. **Algorithmic dimension.** The algorithmic dimension is closely related to the information density of a point, which requires us to define the Kolmogorov complexity at a precision.

Definition 3.7. For $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$, the Kolmogorov complexity of x *at precision* r is

$$K_r(x) = \min\{K(q) : q \in \mathbb{Q}^n \cap B_{2^{-r}}(x)\}$$

In other words, the Kolmogorov complexity of x at precision r gives a rational point that is in the 2^{-r} neighborhood of x. The base of the power is 2 because the string is binary.

Definition 3.8. $\frac{K_r(x)}{r}$ is the *algorithmic information density* of x at precision r.

The limits of algorithmic information density are used to define algorithmic dimensions.

Definition 3.9. 1. The *lower algorithmic dimension* of x is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}$$

2. The *upper algorithmic dimension* of x is

$$\mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}$$

One established result is that these dimensions are preserved by well-behaved functions (bi-Lipschitz computable functions).

Definition 3.10. A function $f : U \to \mathbb{R}^m, U \in \mathbb{R}^n$ is *bi-Liptchitz* if it is Lipchitz, injective and its inverse function is also Lipchitz.

In other words, the function is bi-Lipchitz if there exists a constant $K \geq 1$ such that for all $x_1, x_2 \in U$,

$$\frac{1}{K}|x_1 - x_2| \leq |f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

.

Lemma 3.11. *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is computable and bi-Lipschitz, then $dim(x) = dim(f(x))$ and $Dim(x) = Dim(f(x))$ for all $x \in \mathbb{R}^m$.*

An example implication is that $\dim(x, x - z) = \dim(x, z)$ for $x, z \in \mathbb{R}^n$ because the mapping $f : (x, x - z) \to (x, z)$ is bi-lipchitz and computable. This result will be directly used in bounding product sets.

3.5. **Conditional Dimension.** Recall we have previously introduced the concept of conditional Kolmogorov complexity. Now, we introduce conditional Kolmogorov complexity at a precision. This is defined via two steps.

Definition 3.12. For $x \in \mathbb{R}^m$ and $q \in \mathbb{Q}^n$ and $r \in \mathbb{N}$, the *conditional Kolmogorov complexity of x at precision r given q* is

$$\hat{K}_r(x|q) = \min\{K(p|q) : p \in \mathbb{Q}^n \cap B_{2^{-r}}(x)\}$$

In other words, $\hat{K}_r(x|q)$ gives the Kolmogorov complexity of the point in the neighborhood of x that can be most easily described when given the $q \in \mathbb{Q}$. This can be seen as a minimizing process. The second step is a maximizing process that seeks the least helpful rational point in a neighborhood of a real point.

Definition 3.13. For $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ and $r, s \in \mathbb{N}$, the *conditional Kolmogorov complexity of x at precision r given y at precision s* is

$$K_{r,s}(x|y) = \max\{\hat{K}_r(x|q) : q \in \mathbb{Q}^n \cap B_{2^{-s}}(y)\}$$

Combining the two steps, we looked at the rational point in the neighborhood of y that gives the least information about some rational point in the neighborhood of x.

Finally, we introduce a notation.

Notation 3.14. Let $K_r(x|y)$ denote $K_{r,r}(x|y)$.

Defining the conditional Kolmogorov complexity allows us to define the conditional dimension of points.

Definition 3.15. Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. The *lower and upper conditional dimensions* of x given y are

$$dim(x|y) = \liminf_{r\to\infty} \frac{K_r(x|y)}{r}$$

and

$$Dim(x|y) = \limsup_{r\to\infty} \frac{K_r(x|y)}{r}$$

Now, we introduce the *chain rule for conditional dimension*, which links the dimension of a tuple of points to the dimension of its constituents points via conditional dimension. This theorem results from Lemma 3.6.

Theorem 3.16. *(Chain rule for conditional dimension) For all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$,*

$$\begin{aligned} dim(x|y) + dim(y) &\leq dim(x, y) \\ &\leq Dim(x|y) + dim(y) \\ &\leq Dim(x, y) \\ &\leq Dim(x) + Dim(x|y) \end{aligned}$$

3.6. **Relative dimension and oracle.** The concept of relativization (through an oracle) closely resembles the concept of conditioning. In conditioning, say $K_r(x|y)$, the Turing machine has access to *finite* information – a finite amount of digits that resembles y (bounded-precision access). In relativization, the Turing machine is given access to an oracle $A \subseteq \mathbb{N}$, which gives *countably infinite* information (arbitrary access).

For example, we might define an oracle $A_y \in \mathbb{N}$ that gives the information of $y \in \mathbb{R}^n$ to arbitrary precision. Consider the binary expansion of y, $\dot{y}$, which is a sequence of $\{0, 1\}$. We let $i \in A_y$ iff the i^{th} digit of $\dot{y}$ is 1. This way, the oracle A_y encodes the countable information of y.

We can hence define the complexity and dimension of points relative to an oracle A. The definition of $K^A(\sigma), K^A(\sigma|\tau), K^A_r(x), K^A_r(x|y), dim^A(x), Dim^A(x), dim^A(x|y)$ and $Dim^A(x|y)$ are the same as their unrelativized counterpart, except that the Turing machine U can access A.

Now, we introduce an established lemma that describes the relationship between relativization and conditioning in terms of Kolmogorov complexity and dimension.

Lemma 3.17. [3] *If $x \in \mathbb{R}^m, y \in \mathbb{R}^n$ and $r \in \mathbb{N}$, then $K^y_r(x) \leq K_r(x|y) + O(logn)$, and therefore $dim^y(x) \leq dim(x|y)$ and $Dim^y(x) \leq Dim(x|y)$.*

This lemma intuitively makes sense. If we are given countably infinite information of y in the oracle, one possible program is to access y up to a bounded precision r and ignore y beyond this precision. The Kolmogorov complexity to access the y up to precision r is $K_r(x|y)$ and the complexity of the instruction to ignore is $O(logn)$. This gives an upper bound of $K^y_r(x)$, since it is the length of the *shortest* program with arbitrary access to y. Dividing by r and taking $\liminf_{r\to\infty}$ and $\limsup_{r\to\infty}$, we get the conclusion regarding lower and upper algorithmic dimension respectively.

We can generalize this intuition to other applications too. Since given more information cannot increase the dimension, adding oracles and conditioning to a point cannot increase its dimension. This is a useful technique used in proofs in Section 5.

4. Point-to-set principle

Proving bounds on fractal intersections and products using algorithmic dimension can be seen as a process of concluding set-wise results from point-wise knowledge. Hence, we need a theorem that connects dimension of sets to dimension of its constituent points. This is given by the point-to-set principle.

Theorem 4.1. *(Point-to-set principle)* [3] *For every $E \subseteq \mathbb{R}^n$, the Hausdorff and Packing dimension of E are*

$$1. dim_H(E) = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} dim^A(x)$$

$$2. dim_P(E) = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} Dim^A(x)$$

In other words, the dimension of E is determined by the point in E with the largest algorithmic dimension under relativization.

The point-to-set principle can be applied as the following collorary:

Corollary 4.2. *1. There exists an oracle A s.t. $dim_H(E) = \sup_{x\in E} dim^A(x)$*

2. For every $A \subseteq \mathbb{N}$ and every $\epsilon > 0$, there is a point $x \in E$ such that $dim^A(x) > dim_H(E) - \epsilon$.

3. There exists an oracle A s.t. $dim_P(E) = \sup_{x \in E} Dim^A(x)$

4. For every $A \subseteq \mathbb{N}$ and every $\epsilon > 0$, there is a point $x \in E$ such that $Dim^A(x) > dim_P(E) - \epsilon$.

Corollary 4.2.1 and 4.2.3 hold because the theorem is over the *minimum* of all oracles, not just the infimum. Corollary 4.2.2 and 4.2.4 hold by the approximation characterization of supremum.

These corollaries are helpful because they allow us to apply known point-wise theorems in proving set-wise results. Corollary 4.2.1 and 4.2.3 allows us to work with points rather than sets, where we can apply known results of algorithmic dimension of points. After getting the desired results with points, Corollary 4.2.2 and 4.2.4 allows us to convert these point-wise results set-wise. This process can be seen in the proofs in Section 5.

5. Bounding fractal intersections and products

5.1. **Bounding fractal intersection.** In this section, we bound the dimension of fractal intersection with two methods. We first prove the inequality for Borel sets using traditional methods from geometric measure theory. Then, we prove the theorem for arbitrary sets through an algorithmic perspective.

5.1.1. *Geometric measure theoretic method.*

Theorem 5.1. [1] *If E, F are Borel subsets of $\mathbb{R}^n$ then,*

$$dim_H(E \cap (F + x)) \leq max\{0, dim_H(E \times F) - n\}$$

for almost all $x \in \mathbb{R}^n$

The intuition for the theorem is that, for a set of x of measure zero, E and $F + x$ would not intersect at all. When they do intersect, they intersect at a dimension $max\{0, dim_H(E \times F) - n\}$. For example, consider two planes in $\mathbb{R}^3$. For a set of measure 0, namely when they are parallel, the sets do not intersect. When they do intersect, they form a line. The dimension is 1, which is equal to $dim_H(E \times F) - 3$.

For simplicity, we limit our proof to $E, F \subset \mathbb{R}$. But the proof can be generalized to higher dimensions.

First, we prove a lemma relating the dimension of a set to the total dimensions of its parallel sections. We work in the (x, y) plane and let L_x denote the line parallel to the y-axis through $(x, 0)$.

Lemma 5.2. *Let F be a Borel subset of $\mathbb{R}^2$. If $1 \leq s \leq 2$ then,*

$$\int_{-\infty}^{\infty} H^{s-1}(F \cap L_x)dx \leq H^s(F)$$

Proof: Given $\epsilon > 0$, let $\{U_i\}$ be a δ-cover of F such that

$$\sum_i |U_i|^s \leq H^s_\delta(F) + \epsilon$$

We know such a cover exists because the H^s_δ is an infimum. Hence, $\sum_i |U_i|^s$ can get arbitrarily close to it.

Each U_i is contained in a square S_i of side-length $|U_i|$ parallel to the coordinate axes.

Let χ_i be the indicator function of S_i. (In other words, $\chi_i(x,y) = 1$ if $(x,y) \in S_i$. $\chi_i(x,y) = 0$ if $(x,y) \notin S_i$.)

Since $\{S_i\}$ is also a cover of F, for each x, the sets $\{S_i \cap L_x\}$ form a δ-cover of $F \cap L_x$. Hence,

$$\begin{aligned} H_\delta^{s-1}(F \cap L_x) &\leq \sum |S_i \cap L_x|^{s-1} \\ &= \sum |U_i|^{s-2} |S_i \cap L_x| (\text{because} |U_i| = |S_i| \text{at any vertical cross section}) \\ &= \sum_i |U_i|^{s-2} \int \chi_i(x,y) dy \end{aligned}$$

Hence, by integrating with respect to x,

$$\begin{aligned} \int H_\delta^{s-1}(F \cap L_x) dx &\leq \sum_i |U_i|^{s-2} \int \int \chi_i(x,y) dx dy \\ &= \sum_i |U_i|^s \\ &\leq H_\delta^s(F) + \epsilon \end{aligned}$$

Since the choice of ϵ is arbitrary, we have

$$\int H_\delta^{s-1}(F \cap L_x) dx \leq H_\delta^s(F)$$

Taking $\delta \to 0$, we have the desired result. ■

This leads to the following lemma:

Lemma 5.3. *Let F be a Borel subset of $\mathbb{R}^2$. Then, for L^1-almost all x,*

$$dim_H(F \cap L_x) \leq max\{0, dim_H(F) - 1\}.$$

Proof: Take $s > dim_H(F)$. Hence, $H^s(F) = 0$. By Lemma 5.2, $H_\delta^{s-1}(F \cap L_x) = 0$ for Lebesgue almost all x. Hence, if $s > 1$, $dim_H(F \cap L_x) \leq s - 1$ for Lebesgue almost all x. ■

Now, we are ready to prove Theorem 5.1, by constructing a line and applying Lemma 5.3:

Proof: We prove the theorem for when $n = 1$.

Let L_c be the line in the (x,y)-plane with the equation $x = y + c$. Assuming $dim_H(E \times F) > 1$, we may apply Lemma 5.3:

$$dim_H((E \times F) \cap L_x) \leq dim(E \times F - 1) \tag{5.4}$$

(Although here the line in Lemma 5.3 is rotated by 45 degrees, the same principle holds.)

Now, we show that $dim_H((E \times F) \cap L_x) = dim_H(E \cap (F + x))$. If we see E as points on the x-axis and F as points on the y-axis, then points in $E \times F$ lie on the (x,y) plane. The projection of $(E \times F) \cap L_x$ onto the x-axis is $E \cap (F + c)$. Intuitively, the process of constructing the former can be thought of as projecting E and F both onto L_x and measuring the dimension of their intersection. We get the same dimension by projecting F onto L_c, then onto the x-axis and finally taking the dimension of intersection with E. Hence, by constructing L_c and projecting twice, we are able to shift F by c and take its intersection with E. It follows from (5.4) that our desired result holds.

We proved the theorem for when $n = 1$ and the proof for higher dimensions is similar, using an higher-dimensional analogue for Lemma 5.3 resulting from an analogue for Lemma 5.2. ■

5.1.2. *Algorithmic method.* In this section, we extend the bound on fractal intersection to arbitrary sets.

Theorem 5.5. [4] *If E, F are arbitrary subsets of $\mathbb{R}^n$ then,*

$$dim_H(E \cap (F + x)) \leq max\{0, dim_H(E \times F) - n\}$$

for almost all $x \in \mathbb{R}^n$.

Beforehand, we prove a relevant lemma.

Lemma 5.6. *Let A be an oracle and $s \in \mathbb{N}$. Let $E = \{z \in \mathbb{R}^n : dim^A(z) < s\}$. Then $H^s(E) = 0$.*

Proof: Let

$$E_i = \{x : dim^A(x) < s - \frac{1}{i}\}$$

Then, $E = \bigcup_{i=1}^{\infty} E_i$. We show that $H^s(E_i) = 0$ for all i. (Hence, H^s of the countable union would also be 0.)

For an arbitrary i, consider $x \in E_i$. Since $dim^A(x) \leq s - \frac{1}{i}$, we know $K_r^A(x) \leq k$, where $k = r(s - \frac{1}{i})$, for selected large r. There exists a program π whose length is less than k. There are 2^k many such binary programs.

For a point $x_0 \in E_i$, consider its neighborhood $B_{2^{-r}}(x_0)$. All points $y \in B_{2^{-r}}(x_0) \cap E_i$ are generated by the same program to precision r, since the first r digits of all these points coincide.

We can construct a cover for E_i. We cover E_i with balls of radius 2^{-r}. Each ball corresponds to a program, so there are at-most 2^k many balls.

Hence,

$$H_{-r}^{s-\frac{1}{i}}(E_i) = 2^{-r(s-\frac{1}{i})} \cdot 2^{r(s-\frac{1}{i})} = 1.$$

This implies $H^s(E_i) = 0$. We take countable union on E_i: $H^s(E) = 0$. ■

Lemma 5.7. *Let E be as defined in Lemma 5.6. Then, $L^n(E) = 0$, where L^n is the n-dimensional Lebesgue measure.*

Proof: Because $L^n(E) = H^n(E)$, we have $L^n(E) = 0$. ■

Now we are ready to prove Theorem 5.5. In proving the theorem, we first use Corollary 4.2.1 to establish lower bound for $dim_H(E \times F)$. This also allows us to work with points rather than sets, where we can apply point-wise results such as Lemma 3.11 and Lemma 3.17. Afterwards, we apply Corollary 4.2.2 to convert point-wise result to the desired set-wise result.

Proof: Let $E, F \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^n$. If $E \cap (F + z) = \emptyset$, then the inequality holds trivially as $dim_H(E \cap F) = 0$. Hence, we assume that $E \cap F$ is non-empty. We apply Corollary 4.2.1: There exists an oracle $A \subseteq \mathbb{N}$ such that

$$dim_H(E \times F) = \sup_{(x,y) \in E \times F} dim^A(x, y)$$

Since $E \cap F$ is non-empty, there exists a point $(x, x - z) \in E \times F$.

$$dim_H(E \times F) \geq dim^A(x, x - z)$$

By Lemma 3.11 on computable and bi-Lipchitz mapping,

$$dim^A(x, x - z) = dim^A(x, z)$$

By (3.16) in the chain rule for conditional dimension,

$$dim^A(x, z) \geq dim^A(x|z) + dim^A(z)$$

By Lemma 3.17,

$$dim^A(x|z) + dim^A(z) \geq dim^{A,z}(x) + dim^A(z)$$

By Corollary 4.2.2,

$$dim^{A,z}(x) + dim^A(z) \geq dim^A(z) + dim_H(E \cap (F + z)) - \epsilon$$

Let $\epsilon \to 0$, we have

$$dim_H(E \cap (F + z)) \leq dim_H(E \times F) - dim^A(z)$$

Thus, Theorem 5.5 holds whenever $dim^A(z) = n$. By Lemma 5.7, it holds for Lebesgue almost all z. ■

The algorithmic approach allow us to quickly conclude the corresponding theorem for packing dimension, since the proof is similar.

Theorem 5.8. *For all $E, F \subseteq \mathbb{R}^n$, and for (Lebesgue) almost all $z \in \mathbb{R}^n$*

$$dim_P(E \cap (F + z)) \leq max\{0, dim_P(E \times F - n)\}$$

where $F + z = \{x + z : x \in F\}$

In the proof, we apply Corollary 4.2.3, Lemma 3.11 on bi-Lipchitz and computable mapping, (3.16) in the chain rule for conditional dimension, Lemma 3.17 and Corollary 4.2.4.

5.2. **Fractal product inequalities.** In this section, we prove the fractal product inequalities using two methods. First, the traditional method from geometric measure theory is used to prove the inequalities for Borel sets. Then, the algorithmic method offers an alternative proof that extends the inequalities to arbitrary sets.

5.2.1. *Geometric measure theoretic proof.* We first prove a lower bound on fractal products for Borel sets:

Theorem 5.9. [1] *For $E \subseteq \mathbb{R}^m$, $F \subseteq \mathbb{R}^n$ Borel,*

$$dim_H(E) + dim_H(F) \leq dim_H(E \times F)$$

The proof requires established technical results in geometric measure theory. We introduce three lemmas:

Lemma 5.10. *Let μ be a measure on $\mathbb{R}^n$. Let $F \subset \mathbb{R}^n$ be Borel and let $0 < c < \infty$ be a constant.*

If for all $x \in F$, $\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^s} < c$, then $H^s(F) \geq \mu(F)/c$.

Lemma 5.10 provides a lower bound for $H^s(F)$. We may construct appropriate μ for results of interest in subsequent proofs.

Lemma 5.11. *Let $F \subset \mathbb{R}^n$ and $0 < H^s(F) < \infty$. Then, for H^s-almost all $x \in F$,*

$$\bar{D}^s(F, x) \leq 1.$$

where $\bar{D}^s(F, x) = \limsup_{r \to 0} \frac{H^s(F \cap B(x,r))}{(2r)^s}$ is called the upper density of F at x.

Lemma 5.12. *Let F be a Borel subset of $\mathbb{R}^n$ with $0 < H^s(F) \leq \infty$. Then, there is a subset $E \subset F$ such that $0 < H^s(E) < \infty$.*

To conclude the result on Hausdorff dimension, we first establish a proposition on Hausdorff measure.

Proposition 5.13. *If $E \subseteq \mathbb{R}^m, F \subseteq \mathbb{R}^n$ are Borel sets with $H^s(E), H^t(F) < \infty$, then*

$$H^{s+t}(E \times F) \geq cH^s(E)H^t(F),$$

where $c > 0$ depends only on s and t.

Proof: For simplicity, assume that $E, F \subset \mathbb{R}$, so that $E \times F \subset \mathbb{R}^2$.

If $H^s(E)$ or $H^t(F) = 0$, then the inequality is trivial. Hence, we may assume that $0 < H^s(E), H^t(F)$.

For $I, J \subset \mathbb{R}$, we define a measure μ:

$$\mu(I \times J) = H^s(E \cap I)H^t(F \cap J)$$

Indeed, μ satisfies the properties of a measure.

Notice that μ resembles the right hand side of the desired inequality. Hence, we now seek to bound $\mu(E \times F)$ with the desired Hausdorff measure. We do so using Lemma 5.10, which connects the local properties of a constructed measure to Hausdorff measure. But first, we establish a local property regarding $\mu(B(x, y), r)$ using Lemma 5.11.

Apply Lemma 5.11 to E and F. For H^s-almost all x,

$$\bar{D}^s(E, x) = \limsup_{r \to 0} \frac{H^s(E \cap B(x, r))}{(2r)^s} \leq 1. \tag{5.14}$$

For H^t-almost all y,

$$\bar{D}^s(F, y) = \limsup_{r \to 0} \frac{H^t(F \cap B(y, r))}{(2r)^t} \leq 1. \tag{5.15}$$

(5.14) and (5.15) hold for μ-almost all $(x, y) \in E \times F$. Consider the set $E' \subset \mathbb{R}$ for which (5.14) fails to hold. Since $H^s(E') = 0$, we have $H^s(E' \cap I) = 0$. The same idea applies for (5.15).

Since $B((x, y), r) \subseteq B(x, r) \times B(y, r)$, we have, by definition of μ,

$$\mu(B((x, y), r)) \leq \mu(B(x, r) \times B(y, r)) = H^s(E \cap B(x, r))H^t(F \cap B(y, r)),$$

Divide by $(2r)^{s+t}$,

$$\frac{\mu(B((x, y), r))}{(2r)^{s+t}} \leq \frac{H^s(E \cap B(x, r))}{(2r)^s} \frac{H^t(F \cap B(y, r))}{(2r)^t}$$

By (5.14) and (5.15), for μ-almost all $(x, y) \subseteq E \times F$,

$$\limsup_{r \to 0} \frac{\mu(B((x, y), r))}{(2r)^{s+t}} \leq 1$$

By Lemma 5.10,

$$H^{s+t}(E \times F) \geq 2^{-(s+t)}\mu(E \times F) = 2^{-(s+t)}H^s(E)H^t(F) \blacksquare$$

We are now ready to prove Theorem 5.9.

Proof: Suppose s and t are arbitrary numbers such that $s < dim_H(E)$ and $t < dim_H(F)$. To prove the desired result, since the choice of s and t is arbitrary,

it is sufficient to show $H^{s+t}(E \times F) > 0$, as the k-dimensional Hausdorff measure is greater than 0 for any k that is too small.

We apply Lemma 5.12 to $E \times F$. Since $s < dim_H(E)$ and $t < dim_H(F)$, we have $H^s(E) = H^t(F) = \infty$. Hence, by lemma, there exists $E_0 \subset E$ and $F_0 \subset F$ such that $0 < H^s(E_0), H^t(F_0) < \infty$.

By Proposition 5.13, we have that

$$H^{s+t}(E_0 \times F_0) \geq cH^s(E_0)H^t(F_0) > 0$$

Since $E_0 \times F_0 \subset E \times F$,

$$H^{s+t}(E \times F) > H^{s+t}(E_0 \times F_0) > 0$$

We can conclude that $dim_H(E \times F) \geq s + t$. By choosing s arbitrarily close to $dim_H(E)$ and t arbitrarily close to $dim_H(F)$, we get the desired result. ■

5.2.2. *Algorithmic proof.* Using an algorithmic approach, the same bound may be proven for arbitrary sets.

Theorem 5.16. [4] *For any $E \subseteq \mathbb{R}^m$, $F \subseteq \mathbb{R}^n$,*

$$dim_H(E) + dim_H(F) \leq dim_H(E \times F)$$

Notice that the theorem formally resembles 3.16. Indeed, we use 3.16 to prove the result.

Proof: We want to show

$$dim_H(E \times F) \geq dim_H(E) + dim_H(F).$$

By Corollary 4.2.1, there exists an oracle A such that

$$dim_H(E \times F) = \sup_{(x,y) \in E \times F} dim^A(x, y) \tag{5.17}$$

For all ϵ, there exists Corollary 4.2.2 gives points $x \in E$ and $y \in F$ such that

$$dim^A(x) \geq dim_H(E) - \epsilon \tag{5.18}$$

and

$$dim^A(y) \geq dim_H(F) - \epsilon \tag{5.19}$$

Hence, by (5.17)

$$dim_H(E \times F) \geq dim^A(x, y)$$

By the chain rule for conditional dimension,

$$dim^A(x, y) \geq dim^A(x) + dim^A(y|x)$$

By Lemma 3.17,

$$dim^A(x) + dim^A(y|x) \geq dim^A(x) + dim^{A,x}(y)$$

By (5.18) and (5.19),

$$dim^A(x) + dim^{A,x}(y) \geq dim_H(E) + dim_H(F) - 2\epsilon$$

Take $\epsilon \to 0$, we conclude desired result. ■

In addition, we want to prove an upper bound for $dim_H(E \times F)$. The upper bound for fractal product also allows us to bound fractal intersection (5.5).

Notice that the theorem formally resembles 3.16. Indeed, we use 3.16 to prove the result.

Theorem 5.20. [4] *If E, F are arbitrary subsets of $\mathbb{R}^n$, then*

$$dim_H(E \times F) \leq dim_P(E) + dim_H(F).$$

Proof: By Corollary 4.2.3, there exists an oracle B such that

$$dim_P(E) = \sup_{z \in E} Dim^B(z) \tag{5.21}$$

By Corollary 4.2.1, there exists an oracle C such that

$$dim_H(F) = \sup_{z \in F} dim^C(z) \tag{5.22}$$

By Corollary 4.2.2, for arbitrarily $\epsilon \geq 0$, there exists $(u, v) \in E \times F$ such that

$$dim^{B,C}(u, v) \geq dim_H(E \times F) - \epsilon \tag{5.23}$$

Hence, for the point (u, v), by (5.21) and (5.22),

$$dim_P(E) + dim_H(F) \geq Dim^B(u) + dim^C(v)$$

Because relativization and conditioning cannot increase dimension,

$$Dim^B(u) + dim^C(v) \geq Dim^{B,C}(u) + dim^{B,C}(v) \geq Dim^{B,C}(u|v) + dim^{B,C}(v)$$

By (3.16) in the chain rule for conditional dimension,

$$Dim^{B,C}(u|v) + dim^{B,C}(v) \geq dim^{B,C}(u, v)$$

By (5.23),

$$dim^{B,C}(u, v) \geq dim_H(E \times F) - \epsilon$$

Take $\epsilon \to 0$, we proved the desired result. ■

The idea of the proofs is as follows: We start with the set on the greater side of the inequality. For the dimension of this set, we use the point-to-set principle to establish a point-wise lower bound. Since we can now shift our attention to points, we can apply point wise results, as well as construct appropriate relativizations and conditioning. This allows us to establish inequality between sum of dimensions of individual points and the dimension of a corresponding point of a product set. As we have the desired point-wise inequality, finally, we apply the point-to-set principle to the point(s) to land on the desired set-wise lower bound, concluding the proof.

Acknowledgments

I would like to thank my mentor Iqra Altaf for introducing me to this fascinating interdisciplinary field between geometric measure theory and complexity theory, as well as her support and guidance along the process. I am grateful for Professor May for organizing the REU and giving me helpful feedback on the paper. I am also thankful for the opportunity to participate in the REU.

References

[1] Kenneth Falconer. Fractal Geometry: Mathematical Foundations and Application. John Wiley & Sons, Ltd. 2003.
[2] Peter Grünwald and Paul Vitányi. Shannon Information and Kolmogorov Complexity. CoRR, cs.IT/0410002. 2004.
[3] Jack H. Lutz and Neil Lutz. Algorithmic information and plane Kakeya sets. Association for Computing Machinery, 10.1145/3201783. 2018.
[4] Neil Lutz. Fractal Intersections and Products via Algorithmic Dimension. Association for Computing Machinery, 10.1145/3460948. 2021.

[5] A. K. Zvonkin and L. A. Levin. The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. Russian Mathematical Surveys, 25:83–124, 1970.