WEYL INTEGRATION FORMULA

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Abstract. The Weyl Integration Formula describes a procedure to integrate over compact connected Lie groups. In this paper, we derive this formula.

Contents

1. Introduction

In the paper, we will introduce the notions of (compact connected) Lie groups, weight-space decomposition, maximal tori, differential forms on manifolds, Haar measures, and integrations on compact connected Lie groups. All these concepts culminate in the derivation of the Weyl Integration Formula (Theorem 10.20).

In Section 2 we introduce Lie groups as topological groups with the structure of a manifold. We also discuss the special class of closed linear groups, which are closed subgroups of the general linear groups over $\mathbb{R}$ and $\mathbb{C}$. Each of these groups admit a linear algebra in terms of the tangent space at the identity. In Section 3, we discuss the correspondence between closed linear groups and their linear Lie algebras through the matrix exponential map. In Section 4, we focus on substructure of Lie algbras, namely subalgebras, ideals and their properties. Some important results in this section pertain to semisimple Lie algebras and their decompositions into simple ideals.

In Section 6, we give a brief background on finite representations of topological groups. We introduce the definition of Radon measure on locally compact space and prove the Hurwitz's unitarian trick. Section 7 discusses the importance of maximal

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tori in compact connected Lie groups. The main result is that every element is conjugate to an element in some maximal tori.

We lay the analytical foundation in Section 8 and 9. These two sections allow us to talk about integrals over Lie groups as well-defined concepts. They also provide tools to compute the Jacobian determinant in terms of Lie algebraic operations. Everything comes together in Section 10 with the Weyl Integration Formula. We first introduce a naive formula and derive the main result by fixing problems with the former formula. This formula is the gateway into the analytic treatment of representation theory of Lie groups through the Weyl Character Formula (Chapter 8 of Knapp [1]).

2. Closed Linear Groups and Linear Lie Algebras

Definition 2.1. A **topological group** G is a topological space with continuous group operations, i.e.

$$\cdot : G \times G \to G,\ (x, y) \to xy,$$

$$^{-1} : G \to G,\ x \to x^{-1}$$

are continuous maps.

Definition 2.2. The **real general linear group**, denoted $GL(n, \mathbb{R})$, is the group of invertible n-by-n real matrices. The **complex general linear group**, denoted $GL(n, \mathbb{C})$, is defined analogously.

We can impose topologies on $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ by identifying them with subsets of $\mathbb{R}^{n^2}$ and $\mathbb{R}^{2n^2}$, respectively. Multiplications and inversions in $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are continuous because these operations are given by polynomials in the entries of the matrices.

Definition 2.3. A **closed linear group** is a topologically closed subgroup of $GL(n, \mathbb{C})$. Any closed linear group inherits a topology from $GL(n, \mathbb{C})$ and becomes a topological group. Some examples of closed linear groups are:

$$SO(n) = \{x \in GL(n, \mathbb{R}) \mid xx^T = 1 \text{ and } \det x = 1\}, \tag{2.4}$$

$$SU(n) = \{x \in GL(n, \mathbb{C}) \mid xx^* = 1 \text{ and } \det x = 1\}, \tag{2.5}$$

$$SL(n, \mathbb{R}) = \{x \in GL(n, \mathbb{R}) \mid \det x = 1\}, \tag{2.6}$$

$$SL(n, \mathbb{C}) = \{x \in GL(n, \mathbb{C}) \mid \det x = 1\}. \tag{2.7}$$

These subgroups are closed because they are preimages of continuous functions on closed sets.

We now turn our attention to defining general Lie groups, starting with a quick review of manifolds.

Definition 2.8. A **separable metric space** is a space with a countable dense subset.

Definition 2.9. Let M be a separable metric space, not necessarily connected, with a well-specified dimension m. We specify a system of charts (U, ψ), where U is an open subset of M and ψ is a homeomorphism of U onto an open subset $\psi(U)$ of $\mathbb{R}^m$. These charts satisfy the following two properties:

(a) each pair of charts (U_1, ψ_1) and (U_2, ψ_2) is **smoothly compatible**, meaning

$$\psi_2 \circ \psi_1^{-1} : \psi_1(U_1 \cap U_2) \to \psi_2(U_1 \cap U_2)$$

is diffeomorphic, and

(b) the system of compatible charts (U, ψ) is a C^∞**-atlas** in that the sets U together cover M.

The topological space M with a C^∞-atlas as above is called a **smooth manifold** of dimension n.

Definition 2.10. Let E be an open subset of a smooth n-manifold. Let $\{(U_i, \psi_i)\}$ be a subsystem of charts that covers E. A function $f : E \to \mathbb{R}$ is called a **smooth function** if $f \circ \psi_i^{-1} : \psi_i(U_i) \to \mathbb{R}$ is C^∞ for all charts in the subsystem.

Definition 2.11. A **Lie group** G is a topological group with an additional structure of a smooth manifold such that multiplication and inversion are smooth. An **analytic Lie group** is a Lie group that is connected.

Remark 2.12. We can check that $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ and the closed linear groups in Definition 2.3 are Lie groups by verifying that the underlying topology is that of a smooth manifold and that the group operations are smooth. However, the proof for general closed linear groups is more involved. Theorem 0.15 of Knapp's book [1] states that each closed linear group becomes a Lie group "uniquely" under certain conditions. Since G is a topological space, we can meaningfully discuss topological properties of G such as closedness, compactness and connectedness.

Theorem 2.13 (Closed subgroup theorem)**.** *If G is a Lie group with a closed subgroup H, then there exists a unique smooth manifold structure on H such that H becomes a Lie group.*

Proof. The proof for this theorem was published by Élie Cartan in 1930 [2]. □

Now that we have defined Lie groups, we turn our discussion to their Lie algebras.

Definition 2.14. A **smooth curve** in a Lie group G is a C^∞ function $c : \mathbb{R} \to G$. We say that c is a **smooth curve at the identity** if $c(0) = I$ as well.

Definition 2.15. Let G be a closed linear group. The **(linear) Lie algebra** of G is the set

$$\mathfrak{g} = \{c'(0) \mid c : \mathbb{R} \to G \text{ is a smooth curve with } c(0) = I\}. \tag{2.16}$$

Since G is a matrix group, members of $\mathfrak{g}$ are also matrices of the same dimension, although not necessarily invertible.

We refer the discussion of basic linear Lie algebras and its Lie brackets to section 0.1 of Knapp [1].

3. The Exponential Map and Lie Algebra Homomorphisms

In the last section, we see that the Lie algebra $\mathfrak{g}$ of a closed linear group G is the space spanned by tangent vectors of smooth curves at the identity. The exponential map gives us the tool to go from $\mathfrak{g}$ to G.

Definition 3.1. If A is an n-by-n complex matrix, the **exponential** of A is given by the Taylor series expansion

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{3.2}$$

Proposition 3.3. *For any n-by-n matrix A, e^A converges (entry-wise) to an n-by-n matrix.*

Proof. We evaluate the operator norm on the Cauchy sequence of partial sums. Then the convergence of the matrix exponentiation is bounded above by the convergence of exponentiation in $\mathbb{R}$. For proof, see Proposition 0.11a of Knapp [1] □

We now discuss some properties of the matrix exponential.

Proposition 3.4. *For n-by-n matrices X and Y, we have the following set of properties:*

(a) $e^X e^Y = e^{X+Y}$ *if X and Y commute,*
(b) e^X *is invertible,*
(c) $\frac{d}{dt}(e^{tX}) = Xe^{tX}$,
(d) $t \mapsto e^{tX}$ *is a smooth curve at the identity into $GL(n, \mathbb{C})$,*
(e) $\det e^X = e^{Tr(X)}$.
(f) $X \mapsto e^X$ *is a C^∞ map from the space of n-by-n matrices to itself.*

Proof. See Proposition 0.11 of Knapp [1]. □

We now demonstrate the connection between the linear Lie algebra $\mathfrak{g}$ and the closed linear group G through the exponential map.

Proposition 3.5. *If G is a closed linear group and X is an element of its linear Lie algebra $\mathfrak{g}$, then $\exp X$ is an element of G and*

$$\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \exp tX \textit{ is in } G \textit{ for all real } t\}.$$

Proof. See Proposition 0.14 of Knapp [1]. □

Proposition 3.6. *If G is a closed linear group with its linear Lie algebra $\mathfrak{g}$, then $\exp \mathfrak{g}$ generates the identity component G_0.*

Proof. Since exp is continuous, $\exp \mathfrak{g}$ is a connected subset of G. Then $\exp \mathfrak{g} \subseteq G_0$. To show equality, we need to use Theorem 0.15 from Knapp's book [1], which states that $\exp \mathfrak{g}$ contains a nonempty neighborhood of the identity in G_0. This neighborhood, being an open subgroup, is also closed. Since G_0 is a connected component, the only nonempty clopen subset of G_0 is G_0 itself. □

Until further notice, let G and H be closed linear groups and let $\mathfrak{g}$ and $\mathfrak{h}$ be their respective Lie algebras. Suppose $\pi : G \to H$ is a smooth homomorphism between G and H. We want to explore the differential map $d\pi : \mathfrak{g} \to \mathfrak{h}$ between these tangent spaces.

Definition 3.7. For a given $X \in \mathfrak{g}$, let $c(t)$ be a smooth curve at the identity in G such that $c'(0) = X$. Then $\pi(c(t))$ is a smooth curve at the identity in H. We define the differential on the tangent vector X as

$$d\pi(X) = (\pi \circ c)'(0).$$

In fact, this definition is independent of the choice of smooth curve c (see pg. 16 of Knapp's book [1]). The differential has two important properties, namely it is a linear map and furthermore a Lie algebra homomorphism. We refer discussion of these properties to section 0.5 of Knapp [1]. The following theorem relates π, $d\pi$ and the exponential map.

Theorem 3.8. *If $\pi : G \to H$ is a smooth homomorphism between closed linear groups, then $\pi \circ \exp = \exp \circ d\pi$.*

Proof. See Theorem 0.23 of Knapp [1]. □

For the special case of Ad and ad, we have

$$\exp(\mathrm{ad}X) = \mathrm{Ad}(\exp X). \tag{3.9}$$

4. Ideals, Solvability, Nilpotency and Semisimplicity

Building off the construction of linear Lie algebras in Section 2, we now give more abstract and general definitions of Lie algebras and their homomorphisms.

Definition 4.1. Let $\Bbbk$ be a field. A **Lie algebra** $\mathfrak{g}$ is a vector space over $\Bbbk$ with a billinear form called the Lie bracket that also satisfies

(a) $[X, X] = 0$ for all $X \in \mathfrak{g}$ (and thus $[X, Y] = -[Y, X]$) and
(b) the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Remark 4.2. We denote $\mathrm{End}_{\Bbbk} V$ as the associative algebra of all $\Bbbk$ linear maps from the $\Bbbk$ vector space V itself. If we define a bracket by $[X, Y] = XY - YX$ then $\mathfrak{g}$ becomes a Lie algebra.

Definition 4.3. For a Lie algebra $\mathfrak{g}$, we have the linear map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\Bbbk}\mathfrak{g}$ such that

$$(\mathrm{ad}X)(Y) = [X, Y].$$

Definition 4.4. Since the Lie bracket is linear, we can show that ad $\mathfrak{g}$ is a subspace of $\mathrm{End}_{\Bbbk}\mathfrak{g}$. If we define the Lie bracket on ad $\mathfrak{g}$ as

$$[\mathrm{ad}\ X, \mathrm{ad}\ Y] = \mathrm{ad}[X, Y], \tag{4.5}$$

then ad $\mathfrak{g}$ is also a Lie algebra.

Definition 4.6. If $\Bbbk = \mathbb{R}$, we call $\mathfrak{g}$ a **real Lie algebra**. Otherwise if $\Bbbk = \mathbb{C}$, we call $\mathfrak{g}$ a **complex Lie algebra**.

Remark 4.7. The linear Lie algebras of Definition 2.15 are Lie algebra under Definition 4.1

Definition 4.8. A **Lie algebra homomorphism** is a linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\psi([X, Y]) = [\psi(X), \psi(Y)].$$

for all elements X and Y of $\mathfrak{g}$. A **Lie algebra isomorphism** is a bijective homomorphism between Lie algebras.

Definition 4.9. If $\mathfrak{a}$ and $\mathfrak{b}$ are subsets of $\mathfrak{g}$, then we define

$$[\mathfrak{a}, \mathfrak{b}] = \text{span}\{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{b}\}.$$

Definition 4.10. A **Lie subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace that is closed under the Lie bracket, that is $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. An **ideal** $\mathfrak{h}$ in $\mathfrak{g}$ is a special subalgebra in that it satisfies $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. A Lie algebra is **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Proposition 4.11. *If $\mathfrak{a}, \mathfrak{b}$ are ideals of a Lie algebra, then $[\mathfrak{a}, \mathfrak{b}]$ is also an ideal.*

Proof. This result follows from the Jacobi identity. □

Definition 4.12. Let $\mathfrak{g}$ denotes a finite-dimensional Lie algebra. Recursively define

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i].$$

Each $\mathfrak{g}^i$ is an ideal in $\mathfrak{g}$ by applying induction to Proposition 4.11. This allows us to put these ideals in a decreasing sequence

$$\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$$

called the **commutator series** of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called **solvable** if its commutator series contains a $\mathfrak{g}^i = 0$ for some i.

Proposition 4.13. *For every finite-dimensional Lie algebra $\mathfrak{g}$, there exists a unique maximally solvable ideal $\mathfrak{s}$ that contains all other solvable ideals of $\mathfrak{g}$. We call $\mathfrak{s}$ the radical of $\mathfrak{g}$, denoted rad $\mathfrak{g}$.*

Proof. See Proposition 1.12 of Knapp's book [1]. First, we prove that the sum of two solvable ideals $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$ is solvable. Then, we use a variant of the Second Isomorphism Theorem for Lie algebra to arrive at a solvable quotient of Lie algebra $\mathfrak{h}/\mathfrak{a}$. We now show that since $\mathfrak{a}$ is solvable, $\mathfrak{h}$ must also be solvable. □

Definition 4.14. Similarly, we can recursively define

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i].$$

Once again, each $\mathfrak{g}^i$ is an ideal in $\mathfrak{g}$ by applying induction to Proposition 4.11. This allows us to put these ideals in another decreasing sequence

$$\mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$$

called the **lower central series** for $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called **nilpotent** if its lower central series contains a $\mathfrak{g}_i = 0$ for some i.

Remark 4.15. For a nonzero nilpotent $\mathfrak{g}$, the last nonzero $\mathfrak{g}_i$ is an abelian ideal and thus a subset of the center.

Proposition 4.16. *Nilpotent Lie algebras are solvable.*

Proof. Let $\mathfrak{g}$ be a nilpotent Lie algbera. Obviously, $\mathfrak{g}^0 \subseteq \mathfrak{g}_0$. Inductively, suppose $\mathfrak{g}^i \subseteq \mathfrak{g}_i$, then

$$\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i] \subseteq [\mathfrak{g}_i, \mathfrak{g}_i] \subseteq [\mathfrak{g}^i, \mathfrak{g}] = \mathfrak{g}_{i+1}.$$

Suppose $\mathfrak{g}_j = 0$ for some j, then $\mathfrak{g}^j = 0$ as well and thus $\mathfrak{g}$ is solvable. □

Proposition 4.17. *The Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if the associated Lie algebra ad* $\mathfrak{g}$ *is nilpotent*

Proof. See Proposition 1.32 of Knapp [1]. $\square$

Definition 4.18. A finite-dimensional Lie algebra $\mathfrak{g}$ is **simple** if $\mathfrak{g}$ is nonabelian and $\mathfrak{g}$ has no proper nonzero ideals. In other words, $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$ imply that $\mathfrak{h} = 0$ or $\mathfrak{h} = \mathfrak{g}$.

This definition is reminiscent of that of simple groups. Just as finite groups are built from simple groups, **some** finite-dimensional Lie algebra are built from simple Lie algebras. These are called semisimple Lie algebras. There also exists a classification of these simple components, discussed further in Section 2.8 of Knapp's book [1].

Definition 4.19. A finite-dimensional Lie algebra $\mathfrak{g}$ is **semisimple** if $\mathfrak{g}$ has no nonzero solvable ideals, that is if rad $\mathfrak{g} = 0$.

Theorem 4.20. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if it can be written as a unique direct sum of simple Lie algebras* $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_m$.

Proof. See Theorem 1.54 of Knapp's book [1]. $\square$

Lemma 4.21. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are ideals such that* $\mathfrak{a} \cap \mathfrak{b} = 0$*, then* $[\mathfrak{a}, \mathfrak{b}] = 0$.

Proof. Since $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, we have $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{a}$ and $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{b}$. Thus

$$[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{a} \cap \mathfrak{b} = 0.$$

$\square$

Lemma 4.22. *In a simple Lie algebra,* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Proof. For simple $\mathfrak{g}$, the commutator $[\mathfrak{g}, \mathfrak{g}]$ is an ideal so it is 0 or $\mathfrak{g}$. However, it is not 0 because $\mathfrak{g}$ is nonabelian, so it is $\mathfrak{g}$. $\square$

Corollary 4.23. *A semisimple Lie algebra* $\mathfrak{g}$ *is such that* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Proof. Recall that $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where each $\mathfrak{g}_i$ is simple. We have

$$\begin{aligned} [\mathfrak{g}, \mathfrak{g}] = [\bigoplus \mathfrak{g}_i, \bigoplus \mathfrak{g}_j] &= \bigoplus_{i,j} [\mathfrak{g}_i, \mathfrak{g}_j] \\ &= \bigoplus_{i,j} \delta_{ij} [\mathfrak{g}_i, \mathfrak{g}_j] = \bigoplus [\mathfrak{g}_i, \mathfrak{g}_i] \\ &= \bigoplus \mathfrak{g}_i = \mathfrak{g}. \end{aligned}$$

The third equality follows from Lemma 4.21 and the fifth equality comes from Lemma 4.22. $\square$

Proposition 4.24. *Let* $\mathfrak{g}_0$ *be a real Lie algebra with its complexification being* $\mathfrak{g}$. *Then* $\mathfrak{g}_0$ *is semisimple if and only if* $\mathfrak{g}$ *is semisimple.*

Proof. See Corollary 1.53 of Knapp's book. $\square$

Definition 4.25. A Lie algebra $\mathfrak{g}$ is **reductive** if for each ideal $\mathfrak{a}$ in $\mathfrak{g}$, we have that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ for some ideal $\mathfrak{b}$ in $\mathfrak{g}$.

Corollary 4.26. *The Lie algebra $\mathfrak{g}$ is reductive if and only if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z_{\mathfrak{g}}$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple and $Z_{\mathfrak{g}}$ being the center of $\mathfrak{g}$.*

Proof. See Corollary 1.56 of Knapp's book [1]. □

Definition 4.27. A Lie group is said to be **solvable**, **nilpotent**, or **semisimple** if it is connected and if its Lie algebra is solvable, nilpotent, or semisimple, respectively.

5. Weight-Space Decomposition and Cartan Subalgebras

Definition 5.1. Let $\mathfrak{h}$ be a finite-dimensional complex Lie algebra. A **representation** π of $\mathfrak{h}$ on a complex vector space V is a complex-linear Lie algebra homomorphism of $\mathfrak{h}$ into $\mathrm{End}_{\mathbb{C}}(V)$. Given π and V, let $\alpha \in \mathfrak{h}^*$ be a linear functional on $\mathfrak{h}$. We define

$$V_\alpha = \{v \in V \mid (\pi(H) - \alpha(H)I)^n v = 0 \text{ for all } H \in \mathfrak{h} \text{ and } n = \dim V\}$$

If $V_\alpha \neq 0$, then V_α is the **generalized weight space** of the **weight** α. Elements of V_α are called **generalized weight vectors**. From the theory of Jordan normal form, V_α is maximal when $n = \dim V$. When π is a

Proposition 5.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. If $\mathfrak{h}$ is a nilpotent Lie subalgebra, then generalized weight spaces of $\mathfrak{g}$ relative to $ad_{\mathfrak{g}}\mathfrak{h}$ satisfy*

(a) $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$, *with*

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (ad\ H - \alpha(H)I)^n X = 0 \textit{ for all } H \in \mathfrak{h} \textit{ and } n \in \mathbb{N}\},$$

(b) $\mathfrak{h} \subseteq \mathfrak{g}_0$,
(c) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$
(d) $\mathfrak{g}_0$ *is a subalgebra,*

Proof. (a) See Proposition 2.4 of Knapp's book [1]. This gives the weight-space decomposition for any finite dimensional representations.

(b) Since $\mathfrak{h}$ is nilpotent, ad $\mathfrak{h}$ is nilpotent by Proposition 4.17. Then $\mathfrak{h} \subseteq \mathfrak{g}_0$.

(c) Let $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$, and $H \in \mathfrak{h}$. Then

$$\begin{aligned}(\mathrm{ad}\ H - (\alpha(H) + \beta(H))I)[X, Y] &= [H, [X, Y]] - \alpha(H)[X, Y] - \beta(H)[X, Y] \\ &= [(\mathrm{ad}\ H - \alpha(H)I)X, Y] + [X, (\mathrm{ad}\ H - \beta(H)I)Y].\end{aligned}$$

The last line follows from expanding $[H, [X, Y]]$ using the Jacobi identity. By induction, we can then show that

$$\begin{aligned}&(\mathrm{ad}\ H - (\alpha(H) + \beta(H))I)^n[X, Y] \\ &\qquad = \sum_{k=0}^{n} \binom{n}{k} [(\mathrm{ad}\ H - \alpha(H)I)^k X, (\mathrm{ad}\ H - \beta(H)I)^{n-k} Y].\end{aligned}$$

For $n \geq 2 \dim \mathfrak{g}$, we have $k \geq \dim \mathfrak{g}$ or $n - k \geq \dim \mathfrak{g}$, which means that nilpotency applies and the right hand side vanishes.

(d) From part (c), we can have that $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{g}_0$.

□

Definition 5.3. We say $\mathfrak{g}_0$ is the Cartan subalgebra of $\mathfrak{g}$.

6. Finite Representations of Topological Groups

Definition 6.1. A finite-dimensional **representation** of a topological group G is a continuous homomorphism

$$\Phi : G \to GL_{\mathbb{C}}(V),$$

where $GL_{\mathbb{C}}(V)$ is the group of complex invertible linear maps on the finite-dimensional complex vector space V.

Definition 6.2. An **invariant subspace** of Φ is a vector subspace U of V such that $\Phi(g)U \subseteq U$ for all $g \in G$.

Definition 6.3. Given a finite-dimensional complex vector space V, we can specify a Hermitian inner product $\langle \cdot, \cdot \rangle$. A representation Φ on V is **unitary** if

$$\langle \Phi(g)u, \Phi(g)v \rangle = \langle u, v \rangle$$

for all $g \in G$ and $u, v \in V$.

Proposition 6.4. *For a unitary representation, the orthogonal complement $U^\perp$ of the invariant subspace U is an invariant subspace.*

Proof. Observe that

$$\langle \Phi(g)u^\perp, u \rangle = \langle \Phi(g^{-1})\Phi(g)u^\perp, \Phi(g^{-1})u \rangle = \langle u^\perp, \Phi(g^{-1})u \rangle \in \langle u^\perp, U \rangle = 0 \tag{6.5}$$

for all $u^\perp \in U^\perp$, $u \in U$.

∎

Definition 6.6. A topological space X is **locally compact** if for all $x \in X$, there exists an open set U containing x and a compact set K such that $U \subseteq K$.

Definition 6.7. Let m be a measure on the σ-algebra of Borel sets of a locally compact Haussdorff space X.

(a) The measure m is **inner regular** if for any open set U,

$$m(U) = \sup_{\substack{K \subseteq U \\ K \text{ compact}}} m(K).$$

(b) The measure m is **outer regular** if for any Borel set B,

$$m(B) = \inf_{\substack{U \subseteq B \\ U \text{ open}}} m(U).$$

(c) The measure m is **regular** if it is both inner regular and outer regular.

Definition 6.8. The measure m is a **Radon measure** if it is regular and finite on compact sets.

Definition 6.9. Suppose G is a compact topological group. A **left Haar measure** $d\mu_l$ is a nonzero Radon measure that is invariant under left translation, i.e., $d\mu_l(gS) = d\mu_l(S)$ for all Borel sets S. A **right Haar measure** is defined similarly with right translation invariance.

Remark 6.10. A result from representation theory is that every compact topological group admits a unique normalized two-sided invariant Haar measure. Therefore whenever G is compact, we can write integrals with respect to this normalized Haar measure by the expression $\int_G f(x)dx$ without denoting the measure.

Proposition 6.11 (Hurwitz's Unitarian Trick)**.** *If Φ is a representation of G on a finite-dimensional V, then V admits a Hermitian inner product such that Φ is unitary.*

Proof. Let $\langle \cdot, \cdot \rangle$ be any Hermitian inner product on V. Define

$$(u, v) = \int_G \langle \Phi(x)u, \Phi(x)v \rangle dx.$$

Verify that

$$\begin{aligned}(\Phi(g)u, \Phi(g)v) &= \int_G \langle \Phi(x)\Phi(g)u, \Phi(x)\Phi(g)v \rangle \ dx \\ &= \int_G \langle \Phi(xg)u, \Phi(xg)v \rangle \ dx \\ &= \int_G \langle \Phi(x)u, \Phi(x)v \rangle \ dx \\ &= (u, v).\end{aligned}$$

The third equality follows from the right-invariance of the Haar measure. □

7. Tori of Compact Connected Lie Groups

We start this section with a summary of some important results about compact Lie groups.

Proposition 7.1. *Let G be a compact Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then the real vector space $\mathfrak{g}$ admits an inner product $(\cdot, \cdot)$ that is invariant under $Ad(G) : (Ad(g)u, Ad(g)v) = (u, v)$. Relative to this inner product the members of $Ad(G)$ act by orthogonal transformations, and the members of ad $\mathfrak{g}$ act by skew-symmetric transformations.*

Proof. See Proposition 4.24 of Knapp [1]. □

Corollary 7.2. *Let G be a compact Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then $\mathfrak{g}$ is reductive, and hence $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $Z_{\mathfrak{g}}$ is the center and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.*

Proof. Define the inner product $(\cdot, \cdot)$ as Proposition 7.1. Observe that the ideals of $\mathfrak{g}$ are invariant subspaces of ad $\mathfrak{g}$. Let $\mathfrak{a}$ be an ideal. By Proposition 6.4, $\mathfrak{a}^{\perp}$ is also an invariant subspace, and thus is an ideal in $\mathfrak{g}$. With $(\cdot, \cdot)$ a definite inner product, $\mathfrak{g}$ becomes a Hilbert space with $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$. Thus $\mathfrak{g}$ is reductive and the decomposition of $\mathfrak{g}$ into semisimple and abelian components follows from Corollary 4.26. □

To employ previous sections of the paper, we use a corollary of the powerful Peter-Weyl theorem. The proof of Peter Weyl's theorem combines fundamental results in representation theory, operator theory and analysis of the $L_2(G)$ function space.

Theorem 7.3 (Compact Lie groups - closed linear group correspondence)**.** *Any compact Lie group G has a faithful finite-dimensional representation and thus is isomorphic to a closed linear group.*

Proof. See Corollary 4.22 of Knapp's book [1]. $\square$

Remark 7.4. With the previous theorem, all the results about closed linear groups from Section 2 now apply to our discussion of compact Lie groups.

Definition 7.5. Let $\mathfrak{g}_0$ be a real Lie algebra. The **complexification** of $\mathfrak{g}_0$ is a complex Lie algebra $(\mathfrak{g}_0)^{\mathbb{C}}$ that is the tensor product

$$(\mathfrak{g}_0)^{\mathbb{C}} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}.$$

Elements of $(\mathfrak{g}_0)^{\mathbb{C}}$ has the form (X_1, X_2) where $X_1, X_2 \in \mathfrak{g}_0$. We define Lie algebraic operations on $(\mathfrak{g}_0)^{\mathbb{C}}$ by the formulas:

(a) $(X_1, X_2) + (Y_1, Y_2) = (X_1 + Y_1, X_2 + Y_2)$
(b) $(\alpha + i\beta) \cdot (X, Y) = (\alpha X - \beta Y, \alpha Y + \beta X)$ for all $\alpha, \beta \in \mathbb{R}$.
(c) $[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1] - [X_2, Y_2], [X_1, Y_2] + [X_2, Y_1])$

From this point on on, let G be a compact connected Lie group, with its Lie algebra being $\mathfrak{g}_0$, and let $(\mathfrak{g}_0)^{\mathbb{C}}$ denotes its complexification.

Definition 7.6. A **torus** is a product of circle groups S^1.

Remark 7.7. As a result of Theorem 2.13, every Lie subgroup of G is closed and thus is compact. By Corollary 1.103 of Knapp's book [1], we have that every compact connected abelian Lie group is a torus. The tori within G are ordered by inclusion. Since G is finite dimensional, every torus is contained in a maximal torus.

Proposition 7.8. *The maximal tori in G are exactly the analytic groups corresponding to the maximal abelian subalgebras $\mathfrak{g}_0$.*

Proof. See Proposition 4.30 of Knapp [1]. $\square$

Remark 7.9. Let T be a maximal torus in G, and let $\mathfrak{t}_0$ be its Lie algebra. By Corollary 7.2, $\mathfrak{g}_0$ is reductive, so $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$ semisimple. By Proposition 7.8, $\mathfrak{t}_0$ is maximal abelian in $\mathfrak{g}_0$, and $\mathfrak{t}_0$ is of the form $\mathfrak{t}_0 = Z_{\mathfrak{g}_0} \oplus \mathfrak{t}_0'$, where $\mathfrak{t}_0'$ is maximal abelian in $[\mathfrak{g}_0, \mathfrak{g}_0]$. Since complexification distributes over direct sum, we get $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$, with $[\mathfrak{g}, \mathfrak{g}]$ semisimple following from Proposition 4.24.

Remark 7.10. The complexification $\mathfrak{t}$ of $\mathfrak{t}_0$ has the direct sum decomposition $\mathfrak{t} = Z_{\mathfrak{g}} \oplus \mathfrak{t}'$ with $\mathfrak{t}'$ maximal abelian in $[\mathfrak{g}, \mathfrak{g}]$.

Notation 7.11. We write $\mathrm{ad}_{\mathfrak{g}} X$ instead of ad X to denote that $X \in \mathfrak{g}$ and ad X acts on members of $\mathfrak{g}$.

Recall from Proposition 7.1 that elements of $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{t}_0)$ are real skew-symmetric linear maps. Mapping to matrices using Theorem 7.3, we see that these elements are diagonalizable over $\mathbb{C}$. Repeating the proof of Proposition 7.1 for the complex vector space $\mathfrak{g}$, we find that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{t})$ is isomorphic to some space of skew-Hermitian matrices, which are also diagonalizable over $\mathbb{C}$. It follows that elements of $\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]}(\mathfrak{t}')$ are also diagonalizable. By Proposition 2.13 of Knapp's book [1], $\mathfrak{t}'$ is a Cartan subalgebra of the complex semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Using the weight-space decomposition described in Proposition 5.2(a) on $[\mathfrak{g}, \mathfrak{g}]$, we can decompose

$$\mathfrak{g} = Z_{\mathfrak{g}} \oplus \mathfrak{t}' \oplus \bigoplus_{\alpha \in \Delta([\mathfrak{g},\mathfrak{g}],\mathfrak{t}')} [\mathfrak{g},\mathfrak{g}]_\alpha.$$

Observing the decomposition and recalling definition of Cartan subalgebra, we can see that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Extending the weights α in $\Delta([\mathfrak{g},\mathfrak{g}],\mathfrak{t}')$ to $\mathfrak{t}$ by defining them to be 0 on $Z_{\mathfrak{g}}$, we have the new weight-space decomposition of $\mathfrak{g}$ relative to ad $\mathfrak{t}$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{t})} \mathfrak{g}_\alpha. \tag{7.12}$$

Here, $\mathfrak{g}_\alpha$ is simply the eigen-weight space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X \text{ for all } H \in \mathfrak{t}\}.$$

Definition 7.13. Referring to (7.12), we say that α is a **root** if $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$. Members of $\mathfrak{g}_\alpha$ are called **root vectors** of α, $\Delta(\mathfrak{g},\mathfrak{t})$ is called the **root system** of $\mathfrak{g}$ relative to $\mathfrak{t}$ and (7.12) is the **root-space decomposition** of $\mathfrak{g}$ relative to $\mathfrak{t}$.

Definition 7.14. We can introduce a notion of **positivity** on the root system $\Delta(\mathfrak{g},\mathfrak{t})$ such that:

(i) For any nonzero $\alpha \in \Delta(\mathfrak{g},\mathfrak{t})$, exactly one of α or $-\alpha$ is positive.
(ii) The sum of positive elements is positive, and positive multiple of a positive element is positive.

We call the set of positive elements of Δ a **positive root system**, denoted Δ^+.

Remark 7.15. It is also possible to introduce the notion of positivity on a root system. One way that this is done is by means of a **lexicographic ordering**. Further discussion of this is given in Chapter 2 of Knapp's book [1].

Definition 7.16. For a real reductive Lie algebra $\mathfrak{g}_0'$, we call a Lie subalgebra of $\mathfrak{g}_0'$ a **Cartan subalgebra** if its complexification is a Cartan subalgebra of $\mathfrak{g}'$. The dimension of the Cartan subalgebra is called the **rank** of $\mathfrak{g}_0'$ and of the corresponding analytic group.

Remark 7.17. The Lie algebra $\mathfrak{t}_0$ of the maximal torus T of the compact connected Lie group G is a Cartan subalgebra of $\mathfrak{g}_0$, and the rank of $\mathfrak{g}_0$ and of G is the dimension of $\mathfrak{t}_0$.

Definition 7.18. Extending the inner product on $\mathfrak{g}_0$ in Proposition 7.1 to a Hermitian inner product on $\mathfrak{g}$, we see that $\mathrm{Ad}(T)$ is unitary on $\mathfrak{g}$. As such, the simultaneous eigenspace decomposition is given by (7.12). The action of $\mathrm{Ad}(T)$ on the 1-dimensional $\mathfrak{g}_\alpha$ is a 1-dimensional representation of T of the form.

$$\mathrm{Ad}(t)X = \xi_\alpha(t)X \quad \text{for } t \in T. \tag{7.19}$$

where $\xi_\alpha : T \to S^1$ is a continuous homomorphism of T into the circle group. This homomorphism is called the **multiplicative character**. Taking the differential of both sides, we get

$$\mathrm{ad}(H)X = \alpha(H)X \quad \text{for } H \in \mathfrak{t}_0.$$

Since $\mathfrak{t}_0$ is skew-symmetric, $\alpha(\mathfrak{t}_0)$ is imaginary valued and thus the roots are real valued on $i\mathfrak{t}_0$.

Notation 7.20. We introduce the following notation when talking about group conjugation:

$$\begin{aligned} y^x &= xyx^{-1}, \\ T^g &= \{t^g \mid t \in T\}, \\ T^G &= \bigcup_{g \in G} T^g. \end{aligned}$$

We now state without proof the two theorems that describe the importance of maximal tori in the context of compact connected Lie groups. Then, we will give some useful corollaries.

Lemma 7.21. *For a compact connected Lie group, any two maximal tori are conjugate.*

Notation 7.22. Let M be a smooth manifold. For $p \in M$, we denote $T_p(M)$ as the tangent space at p.

Theorem 7.23. *If G is a compact connected Lie group and T is a maximal torus, then each element of G is conjugate to a member of T. Following the notation above, the theorem claims that $T^G = G$.*

Proof. We omit the full proof (Lemma 4.35 and Theorem 4.36 [1]) of these two theorems as they require other unproven lemmas. However, we note one important equation that comes up in the argument. Let $\psi : G \times T \to G$ be given by $\psi(y, x) = x^y$. Then the differential is

$$d\psi(Y, X) = \mathrm{Ad}(y)((\mathrm{Ad}(x^{-1}) - 1)Y + X), \tag{7.24}$$

where $Y \in T_y(G)$ and $X \in T_x(T)$. ☐

Corollary 7.25. *Every element of a compact connected Lie group G lies in some maximal torus.*

Proof. Let T be a maximal torus. For $g \in G$, we have $g = xyx^{-1}$ for $x \in G$ and $y \in T$ by Theorem 7.23. Then g is in T^x, which is a maximal torus by Lemma 7.21. ☐

We note one other useful proposition without proof.

Proposition 7.26. *In a compact connected Lie group G, a maximal torus T is its own centralizer. In other words, $Z_G(T) = T$.*

Proof. See Corollary 4.52 of Knapp's book [1] ☐

8. Differential Forms and Measure Zero

For this section, let M be an m-dimensional manifold as in Definition 2.9.

Definition 8.1. The manifold M is **oriented** if for an atlas of compatible charts (U_α, ψ_α), the m-by-m derivative matrices of all coordinate changes

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta) \tag{8.2}$$

have everywhere positive determinants.

Definition 8.3. Let M be oriented. A compatible chart (U, ψ) is **positive** relative to the atlas (U_α, ψ_α) if the derivative matrix of $\psi \circ \psi_\alpha^{-1}$ has everywhere positive determinant for all α.

Remark 8.4. Adjoining compatible charts (U, ψ) that are positive relative to (U_α, ψ_α) will keep M oriented. Integration over smooth m forms is well-defined on an oriented manifold M using the notion of pullbacks of differential forms.

Definition 8.5. Let N also be an oriented k-dimensional manifold and let $\Phi : M \to N$ be a smooth map between manifolds. If ω is a smooth k form on N, then the **pullback** $\Phi^*\omega$ is the smooth k form on M given by

$$(\Phi^*\omega)_p(\xi_1, ..., \xi_k) = \omega_{\Phi(p)}(d\Phi_p(\xi_1), ...d\Phi_p(\xi_k)) \tag{8.6}$$

with p in M, $\xi_1, ..., \xi_k$ in the tangent space $T_p(M)$ and $d\Phi_p$ the differential of Φ at p. In the case that M and N are open subsets of $\mathbb{R}^m$ with the smooth m form $F(y_1, ..., y_m)dy_1 \wedge \cdots \wedge dy_m$ on N, the pullback $\Phi^*\omega$ on M is

$$\Phi^*\omega = (F \circ \Phi)(x_1, ..., x_m) \det(\Phi'(x_1, ..., x_m))dx_1 \wedge \cdots \wedge dx_m, \tag{8.7}$$

where Φ has m components depending on $x_1, ..., x_m$ and Φ' denotes the Jacobian matrix $\left(\frac{\partial y_i}{\partial x_j}\right)$.

Definition 8.8. Suppose we have a smooth m form ω on M, then the integration $\int_M f\omega$ is well-defined for all f in the space $C_{\text{com}}(M)$ of continuous functions of compact support on M. First, suppose that f is compactly supported in the coordinate neighborhood U_α. The pullback of ω under ψ_α^{-1} in $\psi_\alpha(U_\alpha)$ is given by the smooth m form

$$(\psi_\alpha^{-1})^*\omega = F_\alpha(x_1, ..., x_m)dx_1 \wedge \cdots \wedge dx_m \tag{8.9}$$

where $F_\alpha : \psi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth. With $(f \circ \psi_\alpha^{-1})$ compactly supported in $\psi_\alpha(U_\alpha)$, we can define

$$\int_M f\omega = \int_{\psi_\alpha(U_\alpha)} (f \circ \psi_\alpha^{-1})(x_1, ..., x_m)F_\alpha(x_1, ..., x_m)dx_1 \cdots dx_m. \tag{8.10}$$

Proposition 8.11. *Suppose that f is also compactly supported in the intersection $U_\alpha \cap U_\beta$, then*

$$\int_M f\omega = \int_{\psi_\alpha(U_\alpha)} (f \circ \psi_\alpha^{-1})(x_1, ..., x_m)F_\alpha(x_1, ..., x_m)dx_1 \cdots dx_m \tag{8.12}$$

$$= \int_{\psi_\beta(U_\beta)} (f \circ \psi_\beta^{-1})(y_1, ..., y_m)F_\beta(y_1, ..., y_m)dy_1 \cdots dy_m. \tag{8.13}$$

Proof. See page 525 of Knapp [1]. □

Remark 8.14. Recall the change of variables formula in multivariable calculus

$$F_\beta(y_1, ...y_m) = F_\alpha(x_1, ..., x_m) \det\left(\frac{\partial y_i}{\partial x_j}\right)^{-1}. \tag{8.15}$$

Before, we can discuss integration for general continuous functions of compact support, we need to introduce the notion of partition of unity.

Definition 8.16. Let X be a topological space. A **partition of unity** is the set P of continuous functions from M to $[0, 1]$ such that for every point $x \in M$:

(a) there is a neighbourhood of x where only a finite number of functions ρ of P are nonzero, and
(b) $\sum_{\rho \in P} \rho(x) = 1$.

Proposition 8.17. *The notion of integration $\int_M f\omega$ for general f in $C_{com}(M)$ is well-defined.*

Proof. Using the atlas (U_α, ψ_α), we can create an open cover of supp f which we can reduce to a finite subcover $\{U_i\}_{i \in I}$. Recall that a manifold is locally compact. Additionally, since M is a metric space, it is also Hausdorff. We generate a partition of unity $\{\rho_i\}_{i \in I}$. Then $f = \sum_{i \in I} \rho_i f$ is a finite sum, so we can define

$$\int_M f\omega = \sum_{i \in I} \int_M (\rho_i f)\omega. \tag{8.18}$$

Observe that $(\rho_i f)$ is locally compact on the coordinate neighborhood U_i and thus integration is well-defined by Definition 8.8.

□

Definition 8.19. We call a smooth m form ω **positive** relative to the given atlas if each local expression in (8.9) has F_α everywhere positive on $\psi_\alpha(U_\alpha)$.

The next two propositions outline a method for creating and recognizing positive k forms.

Proposition 8.20. *For an m-dimensional manifold M that admits an everywhere nonzero m form ω, M can be oriented so that ω is positive.*

Proof. Let $\{(U_\alpha, \psi_\alpha)\}$ be an atlas for M. Each U_α is possibly made up of several components U^α_β, each of which is open. To each open set U^α_β, we associate the restricted homeomorphism $\psi^\alpha_\beta = \psi_\alpha|_{U^\alpha_\beta}$. Then $\{(U^\alpha_\beta, \psi^\alpha_\beta)\}$ is another atlas of M. Without loss of generality, we can work with this new atlas. For each U^α_β, let F^α_β be the function in (8.9) in the local expression of ω in $\psi^\alpha_\beta(U^\alpha_\beta)$. Since ω is everywhere nonzero and U^α_β is connected, F^α_β has the same sign throughout by the intermediate value theorem.

Let us define the map $\phi : (x_1, ..., x_2, ..., x_m) \mapsto (-x_1, ..., x_2, ..., x_m)$. For a negative F^α_β, we redefine ψ^α_β to $\phi \circ \psi^\alpha_\beta$. Then the associated F^α_β is everywhere positive. Following this process, we can make all the F^α_β positive on their respective domains. By (8.15), this forces $\det\left(\frac{\partial y_i}{\partial x_j}\right)$ to be positive for all pairs of compatible charts. This makes M an oriented manifold of Definition 8.3. Since F^α_β are all positive, ω is positive. □

Proposition 8.21. *Let M be an oriented and connected manifold. If ω is a everywhere nonzero smooth m form on M, then either ω is positive or $-\omega$ is positive.*

Proof. See Proposition 8.10 of Knapp's book [1]. Let S be a set such that at each point p of M, all the functions F_a representing ω locally are positive. We show that S is a nonempty clopen set. Since M is connected, $S = M$. ☐

Definition 8.22. Fixing a smooth m form ω, from (8.10) and (8.18) we see that the map $f \mapsto \int_M f\omega$ is a linear functional on $C_{\text{com}}(M)$. Then $f \geq 0$ implies that $\int_M f\omega \geq 0$. In this case we say that the linear functional $f \mapsto \int_M f\omega$ is **positive**.

Remark 8.23. By Riesz–Markov–Kakutani Representation Theorem, for a positive linear functional in $C_{\text{com}}(M)$, there exists a Radon measure $d\mu_\omega$ on M such that $\int_M f\omega = \int_M f(x)d\mu_\omega(x)$ for all $f \in C_{\text{com}}(M)$. Propositions 8.20 and 8.21 allow us to use everywhere nonzero smooth m forms to define measures on manifolds. However, there is an equivalent way of defining sets of measure zero independent of smooth m forms and orientation.

Proposition 8.24. *Suppose that M is an oriented manifold with a postive m form ω. Let $d\mu_\omega$ be the associated Radon measure from Remark 8.23 . For a subset S of M, S has measure zero with respect to $d\mu_\omega(S)$ if and only if $\psi_\alpha(S \cap U_\alpha)$ has m-dimensional Lebesgue measure zero for all α.*

Proof. From (8.9) and (8.10), we have

$$d\mu_\omega(S \cap U_\alpha) = \int_{S\cap U_\alpha} d\mu_\omega = \int_{\psi_\alpha(S\cap U_\alpha)} F_\alpha(x_1, ..., x_m)dx_1 \cdots dx_m. \tag{8.25}$$

In the forward direction $d\mu_\omega(S) = 0$ implies that $d\mu_\omega(S \cap U_\alpha) = 0$. Since ω is positive, F_α is positive. However, we have that the left side of (8.25) is zero but the integrand of the right side is positive everywhere. Thus $\psi_\alpha(S \cap U_\alpha)$ has Lebesgue measure zero.

Conversely, let $S \cap U_\alpha$ have Lebesgue measure zero for all α. Consequently, the integral on the right hand side is zero and so is $d\mu_\omega(S \cap U_\alpha)$. Since M is a separable metric space, S has a countable open cover. Therefore, S can be covered with countably many U_α. Then

$$d\mu_\omega(S) = d\mu_\omega\left(\bigcup_\alpha (S \cap U_\alpha)\right) \leq \sum_\alpha d\mu_\omega(S \cap U_\alpha) = 0.$$

☐

Definition 8.26. Let $\Phi : M \to N$ be a smooth map between m-dimensional manifolds. A **critical point** p of Φ is a point where $d\Phi_p$ has rank $< m$. We call $\Phi(p)$ a **critical value**.

We introduce an important theorem related to measure zero sets of smooth maps between manifolds.

Theorem 8.27 (Sard's Theorem)**.** *If $\Phi : M \to N$ is a smooth map between m-dimensional manifolds, then the set of critical values of Φ has measure zero in N.*

Proof. See Theorem 8.12 of Knapp's book. ☐

Corollary 8.28. *If $\Phi : M \to N$ is a smooth map between manifolds with dim $M <$ dim N, then the image of Φ has measure zero in N.*

Proof. Let dim $M = m < n =$ dim N. Let (U_α, ψ_α) and (V_β, ϕ_β) be atlases for M and N respectively. Let $d\mu_N$ be the measure on N. For each pair of U_α and V_β such that $V_\beta \cap \Phi(U_\alpha) \neq \varnothing$, consider the following composition of maps

$$\Psi_{\alpha,\beta} : \mathbb{R}^n \xrightarrow{\text{proj}} \mathbb{R}^m \xrightarrow{\psi_\alpha^{-1}} U_\alpha \cap \Phi^{-1}(V_\beta) \xrightarrow{\Phi} V_\beta \xrightarrow{\phi_\beta} \mathbb{R}^n$$

where proj is the projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$. The differential $d\Psi$ is a composition of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $\mathbb{R}^m$ to $\mathbb{R}^n$. Recall that $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$ for any two compatible matrices A and B. Then $\text{rank}(d\Psi) \leq m < n$ everywhere on the domain. Then every point in the domain is a critical point, and therefore every point in the image is a critical value. By Sard's Theorem, this set of critical values has measure zero. In other words, the set $\phi_\beta(\Phi(U_\alpha \cap \Phi^{-1}(V_\beta)))$ has Lebesgue measure zero. By Proposition 8.24, $\Phi(U_\alpha \cap \Phi^{-1}(V_\beta))$ has measure zero relative to $d\mu_N$. Since countable V_β covers $\Phi(U_\alpha)$, we have

$$d\mu_N(\Phi(U_\alpha)) = d\mu_N\big(\bigcup \Phi(U_\alpha \cap \Phi^{-1}(V_\beta))\big) \leq \sum d\mu_N(\Phi(U_\alpha \cap \Phi^{-1}(V_\beta))) = 0$$

We also have that a countable collection of U_α covers M. Then

$$d\mu_N(\Phi(M)) = d\mu_N\big(\bigcup \Phi(U_\alpha)\big) \leq \sum d\mu_N(\Phi(U_\alpha)) = 0$$

□

Definition 8.29. A **lower-dimensional set** in N is a set that is contained in the countable union of smooth images of manifolds M with dim $M <$ dim N.

Remark 8.30. By Corollary 8.28, lower-dimensional sets in N have measure zero.

Definition 8.31. Let M and N be oriented m-dimensional manifolds and let $\Phi : M \to N$ be a diffeomorphism. The map Φ is **orientation preserving** if, for every chart (U_α, ψ_α) in the atlas of M, the chart $(\Phi(U_\alpha), \psi_\alpha \circ \Phi^{-1})$ is positive relative to the atlas of N.

Remark 8.32. We can take $\{(\Phi(U_\alpha), \psi_\alpha \circ \Phi^{-1})\}$ to be the atlas for N. Then, the change of variables formula for multiple integrals can be expressed using pullbacks.

Proposition 8.33. *Let M and N be oriented m-dimensional manifolds with an orientation-preserving diffeomorphism $\Phi : M \to N$. If ω is a smooth m form on N, then*

$$\int_N f\omega = \int_M (f \circ \Phi)\Phi^*\omega$$

for all f in $C_{com}(N)$.

Proof. Let the atlases for M and N be $\{(U_\alpha, \psi_\alpha)\}$ and $\{(\Phi(U_\alpha), \psi_\alpha \circ \Phi^{-1})\}$ respectively. It suffices to prove the proposition for f of compact support in $\Phi(U_\alpha)$. Then, we can apply the method of partition of unity to extend integration to the rest of N. On N, (8.10) gives

$$(8.34)\quad \int_N f\omega = \int_{(\psi_\alpha\circ\Phi^{-1}(\Phi(U_\alpha))} f\circ(\Phi\circ\psi_\alpha^{-1})(x_1,..,x_m)F_\alpha(x_1,...,x_m)dx_1\cdots dx_m,$$

where F_α is the smooth function in the pullback

$$((\psi_\alpha\circ\Phi^{-1})^{-1})^*\omega = F_\alpha(x_1,...,x_m)dx_1\wedge\cdots\wedge dx_m.$$

Incidentally,

$$((\psi_\alpha\circ\Phi^{-1})^{-1})^*\omega = (\psi_\alpha^{-1})^*\Phi^*\omega.$$

Since $f\circ\Phi$ has compact support in U_α, we also have by (8.10) that

$$(8.35)\quad \int_M (f\circ\Phi)\Phi^*\omega = \int_{\psi_\alpha(U_\alpha)} (f\circ\Phi)\circ\psi_\alpha^{-1}(x_1,...,x_m)F_\alpha(x_1,...,x_m)dx_1\cdots dx_m.$$

With right sides of (8.34) and (8.35) being equal, we proved our proposition. □

9. Haar Measure on Lie Groups

To be consistent with the material from previous sections, let G be a closed linear group with $\mathfrak{g}$ its Lie algebra. However, the theory in this section works with some modifications for general Lie groups and Lie algebras.

Definition 9.1. For $g\in G$, define $L_g : G\to G$ and $R_g : G\to G$ to be the **left translation** and **right translation** $L_g(x) = gx$ and $R_g(x) = xg$.

Definition 9.2. A vector field $\widetilde{X}$ on G is **left invariant** if $(dL_g)_x(\widetilde{X}_x) = \widetilde{X}_{gx}$. Similarly, $\widetilde{X}$ is **right invariant** if $(dR_g)_x(\widetilde{X}_x) = \widetilde{X}_{xg}$

Proposition 9.3. *A left-invariant vector field is uniquely determined by its tangent vector at the identity.*

Proof. Let $\widetilde{X}$ be a left-invariant vector field and let $\widetilde{X}_e$ be its tangent vector at the identity. Then $(dL_g)_e(\widetilde{X}_e) = \widetilde{X}_g$ gives the value of the vector field for all $g\in G$. □

Definition 9.4. A smooth k form ω on G is **left invariant** if $L_g^*\omega = \omega$ for all $g\in G$. Similarly, it is **right invariant** if $R_g^*\omega = \omega$ for all $g\in G$.

Remark 9.5. We can identify $\mathfrak{g}$ with the tangent space at the identity of G. Let $X_1,...,X_m$ be a basis of $\mathfrak{g}$. These tangent vectors correspond uniquely to left-invariant vector fields $\widetilde{X}_1,...,\widetilde{X}_m$ on G by Proposition 9.3. We can define smooth 1 forms $\omega_1,...,\omega_m$ on G such that $(\omega_i)_p\big((\widetilde{X}_j)_p\big) = \delta_{ij}$ for all $p\in G$.

Proposition 9.6. *The smooth 1 forms $\omega_1,...,\omega_m$ on G are left invariant and they form a basis of the dual of the tangent space at each point of G.*

Proof. To show left invariance, we have that

$$\begin{aligned}(L_g^*\omega_i)_p\big((\widetilde{X}_j)_p\big) &= (\omega_i)_{L_g(p)}\big((dL_g)_p((\widetilde{X}_j)_p)\big)\\ &= (\omega_i)_{L_g(p)}\big((\widetilde{X}_j)_{L_g(p)}\big)\\ &= \delta_{ij} = (\omega_i)_p((\widetilde{X}_j)_p).\end{aligned}$$

The first equality comes from the definition of pullbacks (8.6) and the second comes from the definition of left-invariant vector fields (Definition 9.2). Since $(L_g^*\omega_i)_p$ and

$(\omega_i)_p$ are both linear maps and equal on all basis tangent vector for each point $p \in G$, it must be that $(L_g^*\omega_i)_p$ and $(\omega_i)_p$ are equal as 1 forms. Thus ω_i is left invariant.

Now, we show that these 1 forms form a basis of the dual. At each point $p \in G$, by construction $(\omega_i)_p\big((\widetilde{X}_j)_p\big) = \delta_{ij}$. So the $(\omega_i)_p$'s are linearly independent. Since the number of dual vectors $(\omega_i)_p$ coincides with the number of basis vectors $(\widetilde{X}_j)_p$, $(\omega_i)_p$ form a basis of the dual of the tangent space at p. $\square$

Remark 9.7. Taking the wedge product, we can construct a smooth m form $\omega = \omega_1 \wedge \cdots \wedge \omega_m$ that is everywhere nonzero on G. Since pullback commutes with the wedge product, we have

$$L_g^*\big(\bigwedge \omega_i\big) = \bigwedge (L_g^*\omega_i) = \bigwedge \omega_i.$$

And thus ω is left invariant as well.

Theorem 9.8. *If G is a Lie group of dimension m, then G admits an everywhere nonzero left-invariant smooth m form ω. We can orient G so that ω is positive, and ω defines a nontrivial Radon measure $d\mu_l$ on G such that $d\mu_l(L_gE) = d\mu_l(E)$ for all $g \in G$ and for every Borel set E in G.*

Proof. With Remark 9.5 and Proposition 9.6, we can construct an everywhere nonzero left-invariant smooth m form ω. From Proposition 8.20, ω can be oriented positively. Now let $d\mu_l$ be the associated measure from Remark 8.23 such that $\int_G f\omega = \int_G f(x)d\mu_l(x)$ for all $f \in C_{\text{com}}(G)$. From Proposition 8.33 and $L_g^*\omega = \omega$, we have

$$\begin{aligned}
\int_G f(x)d\mu_l(x) &= \int_G f\omega \\
&= \int_G (f \circ L_g)(L_g^*\omega) \\
&= \int_G (f \circ L_g)\omega \\
&= \int_G f(gx)d\mu_l(x) \qquad (9.9)
\end{aligned}$$

for all $f \in C_{\text{com}}(G)$. For a compact set K in G, consider all the functions f that are bounded from below by the characteristic function I_K of K. We have

$$d\mu_l(K) = \inf_{f \geq I_K} \int_G f(x)d\mu_l(x). \qquad (9.10)$$

Similarly, since the set $L_{g^{-1}}K$ is also compact, we can use (9.10) to get that

$$d\mu_l(L_{g^{-1}}K) = \inf_{f \geq I_K} \int_G f(gx)d\mu(x) \qquad (9.11)$$

since $f(x) \geq I_K(x)$ implies $f(gx) \geq I_K(gx) = I_{L_{g^{-1}}K}(x)$. Applying (9.9) to the right side of (9.10) and (9.11), we get $d\mu_l(K) = d\mu_l(L_{g^{-1}}K)$. Since $d\mu_l$ is a Radon measure, we have $d\mu_l(E) = d\mu_l(L_{g^{-1}}E)$ for all Borel sets E. $\square$

Definition 9.12. A nonzero Radon measure on G that is invariant under left translation is called a **left Haar measure** on G.

Remark 9.13. By Theorem 9.8, a left Haar measure exists for any Lie group G. This left Haar measure is a special case of the left Haar measure of Definition 6.9 for topological groups.

Theorem 9.14. *If G is a Lie group, then any two left Haar measures on G are proportional.*

Proof. See Theorem 8.23 of Knapp's book [1] $\square$

Definition 9.15. A nonzero Radon measure on G invariant under right translation is called a **right Haar measure** on G.

Remark 9.16. Similarly to a left Haar measure, we can construct a right Haar measure from right-invariant 1 forms and then m forms. Analogously, any two right Haar measures are proportional.

Notation 9.17. We simplify the notation for left and right Haar measure on G by writing $d_l x$ and $d_r x$.

Proposition 9.18. *Left and right Haar measures have a few important properties:*

(a) Any nonempty open set has nonzero Haar measure .
(b) Any lower-dimensional set in G has Haar measure zero .

Proof. For part (a), let S be a nonempty open set and K be a nonempty compact set. Pick a point $s \in S$. For each point $k \in K$, a left translation by ks^{-1} overlays S onto K. Then $\{L_{ks^{-1}}S\}_{k\in K}$ is an open cover of K. By compactness of K, we can reduce this to a finite subcover $\{L_{k_i s^{-1}}S\}_{i=1}^n$. We have that

$$d_l(K) \leq d_l\big(\bigcup_{i=1}^n L_{k_i s^{-1}}S\big) \leq n \cdot d_l(L_{k_i s^{-1}}S) = n \cdot d_l(S).$$

Then $\frac{1}{n}d_l(K) \leq d_l(S)$. Since $d_l(K)$ is nonzero, so is $d_l(S)$.

For part (b), Theorem 9.8 establishes that both the left and right Haar measures satisfy the conditions of Proposition 8.24. Then the notion of measure zero under the Haar measure is equivalent to that of Definition 8.29. Thus, our proposition follows from Remark 8.30.

$\square$

Notation 9.19. For $t \in G$, we define the measure $d_l(\cdot t)$ on $x \in G$ which is given by $d_l(xt)$. Similarly, $d_r(t\cdot)$ on $x \in G$ is given by $d_r(tx)$.

Proposition 9.20. *Let $d_l(\cdot)$ be an arbitrary left Haar measure. Then $\tilde{d}_l = d_l(\cdot t)$ is also a left Haar measure.*

Proof. We have $\tilde{d}_l(L_g E) = d_l(L_g(Et)) = d_l(Et) = \tilde{d}_l(E)$. $\square$

Definition 9.21. By Theorem 9.14, any two left Haar measures are proportional. Consider d_l and $d_l(\cdot t)$. Observe that the proportionality constants of these two measures only depend on t. We define the **modular function** $\Delta : G \to \mathbb{R}^+$ of G by

$$d_l(\cdot t) = \Delta(t)^{-1} d_l(\cdot) \tag{9.22}$$

As we will see later, the modular function gives us the Jacobian determinant for when we integrate over the maximal torus.

Proposition 9.23. *If G is a Lie group, then the modular function for G is given by $\Delta(t) = |\det Ad(t)|$.*

Proof. See Proposition 8.27 of Knapp's book [1]. $\square$

Lemma 9.24. *The only compact subgroup of $\mathbb{R}^+$ under multiplication is $\{1\}$.*

Proof. Let S be a nontrivial compact subgroup. There is an $x \in S$ such that $x \neq 1$. Without loss of generalization, let $x > 1$. If this is not the case, we consider x^{-1}. Then $\{x^k\}_{k=1}^{\infty}$ is a monotonically increasing sequence and is also a subset of S. However, since this sequence is unbounded, S cannot be compact. It must be that $\{1\}$ is the only compact subgroup of $\mathbb{R}^+$. $\square$

Lemma 9.25. *All 1-dimensional representations of a semisimple Lie algebra are trivial. Thus all 1-dimensional representations of a semisimple Lie group are trivial.*

Proof. Let $\mathfrak{g}$ be a Lie algebra. Let $\phi : \mathfrak{g} \to \mathbb{C}$ be a 1-dimensional representation of $\mathfrak{g}$ into the abelian real Lie algebra $\mathbb{C}$. By Corollary 4.23, $\phi(\mathfrak{g}) = \phi([\mathfrak{g}, \mathfrak{g}]) = [\phi(\mathfrak{g}), \phi(\mathfrak{g})] = 0$. For simplicity, we only give the second part of the lemma for closed linear group G. From Proposition 3.6, $\exp \mathfrak{g} = G$ since G is connected. Let $\psi : G \to \mathbb{C}$ be a 1-dimensional representation of G. Then $d\psi$ is a 1-dimensional Lie algebra representation on $\mathfrak{g}$. By Theorem 3.8,

$$\psi(G) = \psi \circ \exp(\mathfrak{g}) = \exp \circ d\psi(\mathfrak{g}) = \exp(0) = 1. \tag{9.26}$$

For general Lie group, we have a similar formula to Theorem 3.8 constructed by lifting of homomorphisms (Section I.10 of Knapp's book [1]). Using this formula, we can rederive the analogue of (9.26). $\square$

Corollary 9.27. *The modular function Δ for G has the following properties:*

- *(a) $\Delta : G \to \mathbb{R}^+$ is a smooth homomorphism,*
- *(b) $\Delta(t) = 1$ for all t in any compact subgroup of G or in any semisimple analytic subgroup of G,*
- *(c) $d_l(x^{-1})$ and $\Delta(x)d_l x$ are equal as right Haar measures,*
- *(d) $d_r(x^{-1})$ and $\Delta(x)^{-1}d_r x$ are equal as left Haar measures,*
- *(e) $d_r(t\cdot) = \Delta(t)d_t(\cdot)$ for any right Haar measure on G.*

Proof. Proposition 9.23 gives an explicit formula for Δ in terms of composition of smooth homomorphisms. From this, we get part (a). For (b), the image of a compact subgroup of G under Δ is a compact subgroup K of $\mathbb{R}^+$. From Lemma 9.24, $\Delta(K) = \{1\}$. The statement for semisimple Lie groups follows from Lemma 9.25. For (c), define $d\mu(x) = \Delta(x)d_l x$. Under this new measure, continuous functions are still measurable since Δ is continuous (from (a)). We have

$$\begin{aligned}
\int_G f(xt)d\mu(x) = \int_G f(xt)\Delta(x)d_l x &= \int_G f(x)\Delta(xt^{-1})d_l(xt^{-1}) \\
&= \int_G f(x)\Delta(x)\Delta(t^{-1})\Delta(t)d_l x \\
&= \int_G f(x)\Delta(x)d_l x = \int_G f(x)d\mu(x)
\end{aligned}$$

The second equality comes from the substitution $x \mapsto xt^{-1}$. The third equality is the result of Δ being a homomorphism. Hence $d\mu(x)$ is a right Haar measure. We also have that $d_l(x^{-1})$ is a right Haar measure since $d_l((xt)^{-1}) = d_l(t^{-1}x^{-1}) = d_l(x^{-1})$. By Theorem 9.14, $d_l(x^{-1}) = c\Delta(x)d_l x$ for some constant $c > 0$. Performing a change of variable $x \mapsto x^{-1}$, we get

$$d_l x = c\Delta(x^{-1})d_l(x^{-1}) = c^2\Delta(x^{-1})\Delta(x)d_l x = c^2 d_l x.$$

Thus we have $c = 1$, proving (c). The proof for (d) is similar so we omit the details. Finally, for (e), without loss of generalization, let $d_r x = d_l(x^{-1}) = \Delta(x)d_l x$. Then,

$$\begin{aligned}\int_G f(x)d_r(tx) &= \int_G f(t^{-1}x)d_r x = \int_G f(t^{-1}x)\Delta(x)d_l(x)\\ &= \int_G f(x)\Delta(tx)d_l x\\ &= \Delta(t)\int_G f(x)\Delta(x)d_l x = \Delta(t)\int_G f(x)d_r x,\end{aligned}$$

so we conclude that $d_r(t\cdot) = \Delta(t)d_t(\cdot)$. □

Definition 9.28. We call a Lie group G **unimodular** if every left Haar measure on G is also a right Haar measure (and vice versa). Then, we can speak of a single Haar measure on G. By (9.22), G is unimodular if and only if $\Delta(t) = 1$ for all $t \in G$.

Corollary 9.29. *The following types of Lie groups are unimodular:*

(a) abelian Lie groups,
(b) compact Lie groups,
(c) semisimple Lie groups,

Proof. Part (a) follows directly from (9.22), that is

$$d_l(xt) = d_l(tx) = d_l(x) = \Delta(x)^{-1}d_l(x).$$

So $\Delta(t) = 1$. Parts (b) and (c) follow from Corollary 9.27(b). □

10. Weyl Integration Formula

The Weyl Integration Formula outlines a method to integrate over compact connected Lie groups by first integrating over each conjugacy class, then integrating over the set of conjugacy classes. In this section, let G be a compact connected Lie group, let T be a maximal torus in G, and let $\mathfrak{g}_0$ and $\mathfrak{t}_0$ be the respective Lie algebras. Set $m = \dim G$ and $l = \dim T$.

Definition 10.1. An element g of G is called **regular** if the eigenspace of $\mathrm{Ad}(g)$ for eigenvalue 1 has dimension l.

Remark 10.2. For $g \in G$, we can consider the set of velocity vectors at g

$$\mathfrak{g}_0^g = \{d'(0) \mid d : \mathbb{R} \to G \text{ is a smooth curve with } d(0) = g\}$$

We can identify $\mathfrak{g}_0^g$ with the tangent space at g. There is a natural isomorphism between $\mathfrak{g}_0^g$ and $\mathfrak{g}_0$ given by the map $d \to g^{-1}d$. In fact, we can show that this is a Lie algebra homomorphism. Then, we can identity $\mathfrak{g}_0$ itself with the tangent space at g.

Remark 10.3. From Theorem 7.23, the map $G \times T \to G$ by $\psi(g,t) = gtg^{-1}$ is surjective. For fixed $g \in G$ and $t \in T$, we can identity the tangent spaces at g, t, and gtg^{-1} with $\mathfrak{g}_0$, $\mathfrak{t}_0$ and $\mathfrak{g}_0$ respectively by Remark 10.2. The formula (7.24) computes the differential of ψ at (g,t) as

$$d\psi(X,H) = \mathrm{Ad}(g)((\mathrm{Ad}(t^{-1}) - I)X + H), \tag{10.4}$$

where $X \in \mathfrak{g}_0$ and $H \in \mathfrak{t}_0$.

We can define the descended map of ψ restricted to $G/T \times T \to G$ by $\psi(gT,t) = gtg^{-1}$. This new map is well-defined since T is abelian. By abuse of notation, we also call this descended map ψ. We may identify the tangent space G/T with the orthogonal complement $\mathfrak{t}_0^{\perp}$ to $\mathfrak{t}_0$ in $\mathfrak{g}_0$ (relative to the invariant inner product in Proposition 7.1). We can modify (10.4) for the descended map

$$d\psi(X,H) = \mathrm{Ad}(g)((\mathrm{Ad}(t^{-1}) - I)X + H), \tag{10.5}$$

where $X \in \mathfrak{t}_0^{\perp}$ and $H \in \mathfrak{t}_0$. Now, we can rewrite the differential $d\psi$ at (g,t) as the matrix

$$(d\psi)_{(g,t)} = \mathrm{Ad}(g)\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Ad}(t^{-1}) - I \end{pmatrix}.$$

The first column of $(d\psi)_{(g,t)}$ describes the action on $\mathfrak{t}_0$ and the second column describes that on $\mathfrak{t}_0^{\perp}$.

Proposition 10.6. *We have* $\det Ad(g) = 1$ *for all* $g \in G$*.*

Proof. Since G is compact, by Corollary 9.27 (b), $\Delta(g) = |\det \mathrm{Ad}(g)| = 1$. This implies that $\det \mathrm{Ad}(g)$ is either 1 or -1. From Corollary 9.27(a), it follows that $\det \mathrm{Ad}$ is continuous. With $\det \mathrm{Ad}(I) = 1$ and G being connected, we use the intermediate value property to conclude that $\det \mathrm{Ad}(g) = 1$ for all $g \in G$. □

Corollary 10.7. *From Proposition 10.6, we have that*

$$\det(d\psi)_{(g,t)} = \det((Ad(t^{-1}) - I)|_{\mathfrak{t}_0^{\perp}}). \tag{10.8}$$

Remark 10.9. Following the process in Remark 9.5 and 9.7, we can build a left-invariant $(m-l)$ form on G/T from the duals of elements in $\mathfrak{t}_0^{\perp}$ and a left-invariant l form on T from the duals of the elements in $\mathfrak{t}_0$. A left-invariant m form on G is then the wedge product of these two forms. Each left-invariant differential m form on G defines a left Haar measure by Theorem 9.8. To integrate over the group G, it is reasonable to first integrate over T, then over G/T.

$$\begin{aligned}
\int_G f(g)dg &= \int_G f(x)d\mu_l^G(x) \\
&= \int_G f\omega \\
&= \int_{G/T\times T} (f\circ\psi)\psi^*\omega \\
&= \int_{G/T\times T} (f\circ\psi)(gT,t)|\det(d\psi)_{(g,t)}|d(gT)dt \\
&= \int_T \left[\int_{G/T} f(gtg^{-1})d(gT)\right]\left|\det((\mathrm{Ad}(t^{-1})-I)|_{\mathfrak{t}_0^\perp})\right|dt \qquad (10.10)
\end{aligned}$$

The third equality follows from Proposition 8.33. The fourth equality uses (8.15) with $F_a = 1$. And the fifth equality is a result of 10.8. However, the application of Proposition 8.33 fails in two ways:

(1) the Jacobian of $\psi : G/T \times T \to G$ has determinant 0 at some points in the domain,
(2) ψ is not one-one even if we exclude points of Jacobian determinant 0.

Remark 10.11. To fix the first issue, we consider the possibility of excluding the problematic points from the integral. Since dim(G) $= \dim(G/T \times T)$, the points of determinant 0 are exactly the critical points of ψ. By Sard's Theorem (Theorem 8.27) and Proposition 8.24, this set of critical values has Haar measure zero in G. By (10.8), we can exclude these points if we restrict ψ to a map $\psi : G/T \times T' \to G'$.

Remark 10.12. To demonstrate the extent of the second issue, let w be in the normalizer of T in G, denoted $N_G(T)$. Then

$$\psi(gwT, w^{-1}tw) = (gw)(w^{-1}tw)(w^{-1}g^{-1}) = gtg^{-1} = \psi(gT,t) \qquad (10.13)$$

When w is not in $Z_G(T) = T$ (Proposition 7.26), we have $gwT \neq gT$. Then each member of G' has at least $|N_G(T)/Z_G(T)|$ preimages. In fact, this bound is strict for all elements of G'.

Definition 10.14. We define the **analytic Weyl group** $W(G,T)$ as the quotient group of the normalizer and centralizer

$$W(G,T) = N_G(T)/Z_G(T)$$

In fact, $W(G,T)$ is a finite group.

Lemma 10.15. *Each member of G' has exactly $|W(G,T)|$ preimages under the map $\psi : G/T \times T' \to G'$.*

Proof. See Lemma 8.57 of Knapp's book [1]. □

Remark 10.16. We revisit Proposition 8.33. Instead of $\Phi : M \to N$ being an orientation-preserving diffeomorphism, we let Φ be an everywhere regular (nonzero Jacobian determinant) n-to-1 map of M onto N with equal dimension. To account for the double-counting, we have the modification

$$n \int_N f\omega = \int_M (f \circ \Phi)\Phi^* \omega. \tag{10.17}$$

Remark 10.18. For $t \in T'$, consider $\mathrm{Ad}(t^{-1}) - I$ as an endomorphism of $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$. From the definition of regular elements, $\mathrm{Ad}(t^{-1}) - I$ is diagonalizable with eigenvalues 0 with multiplicity l. If $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ is the set of roots, then $\xi_\alpha(t^{-1}) - 1$ are eigenvalues each with multiplicity 1. These are the multiplicative characters described in (7.19). Then we have $\left|\det(\mathrm{Ad}(t^{-1}) - I)|_{\mathfrak{t}_0^\perp}\right| = \left|\Pi_{\alpha\in\Delta}(\xi_\alpha(t^{-1}) - 1)\right|$. Fixing a positive root system Δ^+ and recognizing that $\xi_\alpha(t^{-1}) = \xi_{-\alpha}(t^{-1})$, we see that

$$\left|\det(\mathrm{Ad}(t^{-1}) - I)|_{\mathfrak{t}_0^\perp}\right| = \left|\Pi_{\alpha\in\Delta^+}(\xi_\alpha(t^{-1}) - 1)\right|^2 \tag{10.19}$$

Theorem 10.20 (Weyl Integration Formula)**.** *Let G be a compact connected Lie group with a maximal torus T. For every Haar measurable function $F \geq 0$ on G, we have*

(10.21)
$$\int_G F(x)dx = \frac{1}{|W(G,T)|}\int_T \left[\int_{G/T} F(gtg^{-1})d(gT)\right]\left|\Pi_{\alpha\in\Delta^+}(\xi_\alpha(t^{-1}) - 1)\right|^2 dt.$$

Proof. We fix the two problems of the naive formula in (10.10) with using the modified formula in (10.17). In addition, we express the Jacobian determinant as products of multiplicative characters by (10.19). Then, we get (10.21). □

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