TOPOLOGICAL FIELD THEORY AND FINITE GAUGE THEORY

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Abstract. This paper provides an introduction to the ideas of topological field theory including non-extended and fully extended theories. The classification of topological field theories in 1 and 2 dimensions is discussed. Additionally, finite gauge theory is constructed and used as the primary example of the ideas of topological field theory developed within.

Contents

1. Introduction

The idea of a topological field theory first appeared in a paper by Witten [22]. The mathematical formulation of topological field theories using bordism categories was then laid out in a paper of Atiyah's [1]. A topological field theory can be seen to have motivations from two different perspectives. The concept of a bordism group and bordisms arose in the work of Thom [20], where he computed the unoriented bordism ring, in effect computing all of the unoriented cobordism groups. An interesting concept to study is then bordism invariants. These are integers that are attached to a manifold that are the same whenever two manifolds are bordant. If these invariants have other nice properties such as additivity under disjoint union, then these then give a morphism from the bordism group to $\mathbb{Z}$. In general, any such morphism would give a bordism invariant, but they become relevant primarily when there is some geometric interpretation of the number attached to a manifold, i.e. when there is a method for computing the invariant given an arbitrary manifold. This concept can then be categorified. Categorification is a concept that has arosen recently where algebraic objects are lifted a categorical level in an attempt to learn more about structures involved. In this case, categorifiying the bordism group then leads to the bordism category. The proper categorical notion of an abelian group here is a symmetric monoidal category. Bordism invariants then become symmetric monoidal functors from this bordism category to another nice category which is typically complex vector spaces.

The other perspective from which these ideas arise is in an atempt to make sense of a simplified model of physics. Atiyah's definition of topological field theories was inspired by Segal's work on conformal field theories, defining them as a functor out of what is roughly a geometric bordism category [18]. These ideas were then expanded on in the general setting of a wick rotated quantum field theory in [12]. The limit in which a field theory only depends on the topology of space and not the geometry then reduces to a functor out of a topological bordism category. With inspiration coming from physics, it then becomes natural to ask that such a theory should be completely local, which leads to the idea of a fully extended topological field theory and the cobordism hypothesis originally introduced by Baez and Dolan [2]. A sketch of a prooof of this theorem was then laid out by Lurie in 2009.

This paper provides an overview of some of the basic ideas of topological field theories. Throughout, the primary example of a topological field theory is finite gauge theory. Section 2 contains an introduction to the idea of bordisms and the bordism group. This is first done in the unoriented case, but in the second half of this section the idea is expanded to general G-tangential structures, with a particular emphasis on oriented and framed theories. Nevertheless, there is a brief discussion of general G-tangential structures. Section 3 then proceeds to define the notion of a topological field theory. This section also discusses the classification of 1D and 2D oriented topological field theories, but is only explicit in this demonstration in one direction. The section ends with a brief discussion of how actions of the mapping class group are seen to arise from a topological field theory. Section 4 then introduces a specific example of a topological field theory called finite gauge theory. Machinery is built up that allows for the computation of the theory in any dimension, and the details of the theory are given in dimensions 1 and 2.

Section 5 then introduces the concept of an extended topological field theory and the notion of higher dualizability in n-categories. Higher categories are necessary in this discussion, but they are dealt with naively as they are not used that heavily or intricately. This section is primarily intended to introduce the ideas necessary to understand the cobordism hypothesis. Section 6 then applies these ideas to to construct a 2D fully extended finite gauge theory. A certain 2-category is introduced to achieve this, and dualizability is studied in this category.

Throughout the paper, manifold is used to mean smooth manifold. Some basic familiarity with smooth manifolds is assumed as can be found in the course notes [7]. Besides this, some familiarity with category theory is assumed as can be found in [14] and the first chapter of [21]. Knowledge of symmetric monoidal categories and dualizability is also necessary, but the basic definitions and results needed are provided in the appendix 7. The theory of higher categories will enter into the picture in the second half of the paper when dealing with extended field theories. Minimal knowledge of these concepts is necessary, and they will be dealt with informally at this point. Only an understanding of 2-categories is necessary to understand end of the paper, and as such any mention of higher categories can be restricted to that of 2-categories at little expense.

2. Bordisms

In subsection 2.1 the basic ideas of a bordism of manifolds is introduced. It is then shown that this gives an equivalence relation on all closed manifolds of a specified dimension, and furthermore that these equivalence classes form a group. In subsection 2.2, the concept of G-tangential structure is introduced. This allows me to define more general bordisms, specifically those between framed manifolds and oriented manifolds. The primary examples presented throughout the paper will all be oriented theories, so this notion is particularly useful. A reference for basic information on bordisms and bordism groups can be found [5]. More information on ideas similar to G-tangential structures can be found in section 2.4 of [13] and section 2.5 of [10].

2.1. **Unoriented Bordisms.**

The basis of all that is to follow is the concept of a bordism between smooth manifolds. This notion provides a notion of a manifold smoothly changing topology over time. This notion is captured by the following definition.

Definition 2.1. Given two closed $n-1$ manifolds M and N, a *bordism* from M to N is the data (X, p, i_0, i_1), where X is an n-manifold with boundary ∂X, $p : \partial X \to \{0, 1\}$ is a continuous map, and diffeomorphisms

$$i_0 : M \times [0, 1) \to X$$

$$i_1 : N \times (0, 1] \to X,$$

such that $i_0(0) = p^{-1}(0)$ and $i_1(1) = p^{-1}(1)$. Two manifolds M and N are called bordant if there exists a bordism from M to N.

Remark 2.1. Moralistically, a bordism is really a manifold such that its boundary can be identified with the two manifolds M and N. However, the data includes a diffeomorphism onto collar neighborhoods of the boundary for technical reasons in order to make gluing easier and later to make identifying tangential structures easier. It is a theorem that there exists such a collar neighborhood of the boundary of any smooth manifold, see theorem 1.14 and exercise 1.15 in [5]. As such, when the information of a bordism is considered up to equivalence in this case, the choice of collar neighborhood will be irrelevant.

In view of a bordism being a change in topology of some manifold over time, gives an interpretation of the manifold M being the incoming manifold, and N being the outgoing manifold. Bordisms will often be drawn in the following way, where time is moving from left to right in the picture.

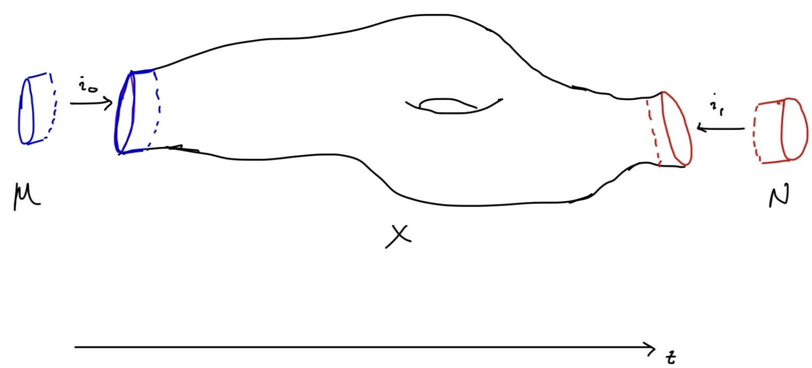


FIGURE 1. A depiction of a bordism X from M and N. The flow of "time" is shown at the bottom. The incoming manifold will always be depicted in blue and the outgoing one in red.

In order for this to be a useful notion in what follows, it is necessary that being bordant defines an equivalence relation on manifolds. This allows the construction of a set and eventually a category out of this data.

Theorem 2.2. *The condition of being bordant forms an equivalence relation on smooth n-dimensional manifolds.*

Proof. First, note that M is bordant to itself via the $n + 1$ dimensional manifold $M \times I$. Additionally, the condition is symmetric as well. To see this, note that if M is bordant to N, then there exists the data (X, p, i_0, i_1) of a bordism from M to N. This equivalently defines the notion of a bordism from N to M by interchanging the roles of i_0 and i_1 and composing p with the automorphism of $\{0, 1\}$ switching 0 and 1.

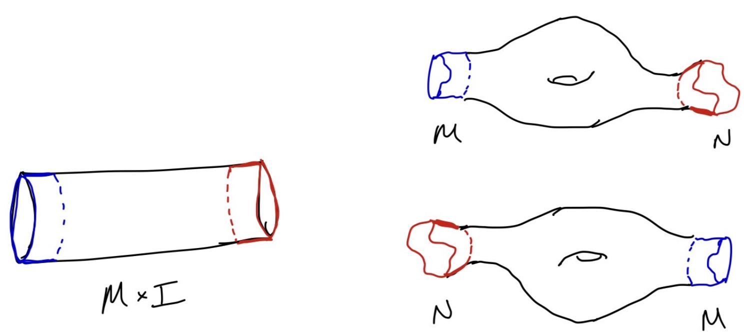


FIGURE 2. The picture on the left depicts the bordism giving reflexivity, and the one on the right depicts the symmetric property of the bordant relation.

The final condition of associativity is slightly more technical to verify. Suppose that M is bordant N via a bordism $(X, p_x, i_{0,x}, i_{1,x})$ and N is bordant to K via a bordism $(Y, p_y, i_{0,y}, i_{1,y})$. The goal is then to construct a bordism from M to K. Roughly, the idea is to glue the two bordisms, X and Y, along the component of the boundaries of X and Y diffeomorphic to N.

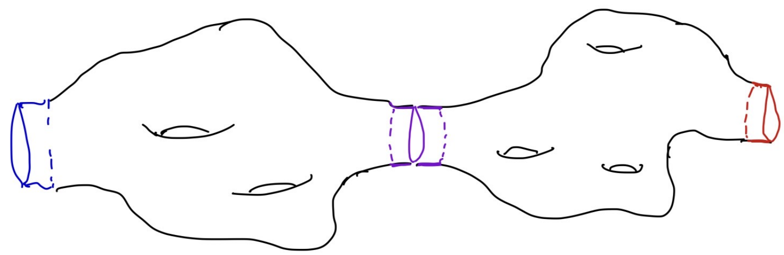


FIGURE 3. A depiction of the gluing giving transitivity. The intermediate manifolds will always be depicted in purple.

First take $N \times (-1, 1)$ and glue this to X via the diffeomorphism data from the bordism, and likewise glue this to Y. The conditions for this to be locally euclidean can then be checked locally on either X, Y, or $N \times (-1, 1)$. The conditions of Hausdorff, second countable, and compact also easily follow. □

Thus, it follows naturally that the set of equivalence classes of n-dimensional closed manifolds under bordisms from a set, called Ω_n. In a way these equivalence classes can be viewed as containing all the possible topologies reachable from the evolution of a specific space. As such, each equivalence class represents different sectors of a theory inaccessible to each other. It will turn out that this set naturally carries the structure of a group. Note that disjoint union of manifolds gives a natural way to get a new closed n-dimensional manifold from two old ones. Additionally, this operation is well-defined on bordism classes, by taking the disjoint union of the manifolds defining the bordisms. This operation can be seen as taking two different systems or spaces and allowing them to interact.

Theorem 2.3. *The operation of disjoint union $\sqcup$ endows the set Ω_n with the structure of a group where the inverse of a manifold is itself, and unit $\emptyset$ the empty set viewed as an n-dimensional manifold.*

Proof. First note that given any closed n-dimensional manifold M, it follows that $M \sqcup \emptyset = M$, and so $\emptyset$ is a unit on this set. Now to see that $M \sqcup M$ is null-bordant, i.e. bordant to the empty set, take the cylinder $M \times I$. The function p in the definition of the bordism then just maps the entire boundary to 0, and the boundary is then isomorphism to $M \sqcup M$. □

These groups can be easily computed in low dimensions. The zero dimensional case gives $\Omega_0 = \mathbb{Z}/2$ with the only non-trivial bordism class being a point. Two points are then null-bordant, or bordant to the empty set, via an arc as shown in figure 2.1. This is reminiscent of physics where particles must be created or annihilated in pairs.

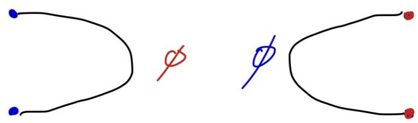


FIGURE 4. A depiction of the arcs that annihilate and create pairs of points.

The one dimensional case gives $\Omega_1 = 0$ as the circle is the only connected closed 1-manifold and it is null-bordant via a disk. The two dimensional case gives $\Omega_2 = \mathbb{Z}/2$ with generator $\mathbb{RP}^2$ as it does not bound a compact 3-manifold. For a proof of this see proposition 1.32 in [5].

2.2. **G-Tangential Structures and Bordisms.** The above discussion is more specifically called unoriented bordism and the unoriented bordism group. By modifying this dimension to include different tangential structure allows for a much richer theory. In order to define a tangential structure, recall that the classifying space $\mathrm{BO}(n)$ classifies n-dimensional vector spaces. There is a universal vector bundle $\mathcal{V} := \mathrm{EO}(n) \times_{O_n} \mathbb{R}^n$ where $\mathrm{EO}(n)$ is the contractible space such that $\mathrm{BO}(n)$ is the quotient of this space by $\mathrm{O}(n)$. Given any n-dimensional vector bundle V over M, there is a map $f : M \to \mathrm{BO(n)}$ such that $V \simeq f^*\mathcal{V}$. Additionally, given any map of groups $G \to O(n)$, this induces a map $\mathrm{BG} \to \mathrm{BO}(n)$. This then gives the technology to define a G-tangential structure on a manifold M.

Definition 2.4. Let M be an n-dimensional manifold, and let $t : M \to \mathrm{BO}(n)$ be a map such that $t^*\mathcal{V} \simeq \mathrm{T}M$. Additionally, let $G \to O(n)$ be a group homorphism, and $\mathrm{BG} \to \mathrm{BO}(n)$ be the associated map of classifying spaces. A *G-tangential structure* on M is then a map $f : M \to \mathrm{BG}$ such that the following diagram commutes.

$$\begin{array}{ccc} & & \mathrm{BG} \\ & \overset{f}{\nearrow} & \downarrow \\ M & \xrightarrow{t} & \mathrm{BO}(n) \end{array}$$

Example 2.5. (1) Take G to be $\mathrm{SO}(n)$, and the map $\mathrm{SO}(n) \to \mathrm{O}(n)$ to be the natural inclusion. The notion of G-tangential structure in this case reduces to an orientation of the manifold. This follows as $\mathrm{BSO}(n)$ classifies oriented n-dimensional vector bundles.
(2) Take G to be the trivial group $\{*\}$, and the map $G \to \mathrm{O}(n)$ is the unique one. In this case the tangential structure is called a framing of the manifold, and it is an isomorphism of $\mathrm{T}M$ with the trivial vector bundle $\mathbb{R}^n \times M$. This follows as pulling back the universal bundle $\mathcal{V}$ to the space $B\{*\} = \{*\}$, gives $\mathbb{R}^n$, which then pulls back to the constant bundle along a map $M \to \{*\}$. This pullback must be isomorphic to $t^*\mathcal{V} = \mathrm{T}M$ if the diagram commutes.
(3) For a third example, take $G = \mathrm{Spin}(n)$ to be the spin group, and the map $\mathrm{Spin}(n) \to \mathrm{O}(n)$ the natural map. In this case a G-tangential structure is a spin structure on the manifold.

The first two of the above examples will be the most relevant throughout the rest of this paper. The definition of a bordism can now be modified to the case where all the manifolds involved have some G-tangential structure, but first it is necessary to know how to restrict and compare G-tangential structures on open submanifolds.

Definition 2.6. Let M be an n-dimensional manifold with G-tangential structure $f : M \to \mathrm{B}G$, and $\iota : U \hookrightarrow M$ an open submanifold of M. Then the restriction of the G-tangential structure to U is the composition $f \circ \iota : U \to BG$.

Remark 2.2. Note that in this case $\iota^*(TM) \simeq TU$, so that if $t : M \to \mathrm{BO}(n)$ classifies the tangent bundle of M, then $t \circ \iota$ classifies that of U. This means that the map in the definition above does indeed give a G-tangential structure.

Definition 2.7. A G-bordism between two closed $n-1$ dimensional manifolds M and N with G-tangential structures on their product with the intervals is the same data (X, p, i_0, i_1) as for a regular bordism given in definition 2.1. The only difference now is that the n-dimensional manifold is now required to carry a G-tangential structure such that it restricts to the induced G-tangential structures on the images of i_0 and i_1.

This definition will again lead to an equivalence relation by a similar argument as given for the unoriented case. It is also a fact that the equivalence classes will form a group under disjoint union, but the inverse of a manifold with G-tangential structure (M, f), might correspond to a M with a different G-tangential structure on $M \times I$. For example, in the oriented case, the inverse will be the manifold with opposite orientation. The corresponding group is denoted by Ω_n^G.

Remark 2.3. There exists a more general notion of a bordism that can be found in [13]. This notion is useful in sketching the proof of a theorem in the paper, but the most important examples of bordisms are those arising from tangential structures.

There is another way of approaching extra structures on bordism groups, especially relevant to the ideas of physics, as this approach can also be used in the construction of geometric ideas of bordism. This approach defines fields, $\mathcal{F}$, on n-dimensional manifolds to be sheafs

$$\mathcal{F} : \mathrm{Man}_n \to \mathrm{sSet}$$

from the category of n-dimensional manifolds with morphisms local diffeomorphisms to that of simplicial sets. The theory of bordisms can be defined in relation to any such sheaf. A topological theory of bordisms arises when these fields are actually locally constant sheaves. For more details see [11] and [8].

The rest of this section will be focused on working out some details in the oriented and framed cases.

To do so, first note that a G-tangential structure on $M \times I$ can be simplified to come from something intrinsic to M in both of these cases. In the framed case, this is an isomorphism $\mathrm{T}M \oplus \underline{\mathbb{R}} \simeq \underline{\mathbb{R}}^n$, which is also called an n-framing of M. This follows by tracing through the definitions and using the fact that fiber-wise the tangent bundle of a product is the direct sum of the tangent spaces of the factors along with the fact that I has no non-trivial vector bundles. The oriented case corresponds to choosing an orientation on the manifold M. This is a lot a less transparent than in the framed case and so deserves more details as to how this works.

Recall that given a submanifold N of another manifold M, there exists a short exact sequence of vector bundles over N.

$$0 \to TN \to TM|_N \to \nu \to 0$$

Here ν denotes the normal bundle to the submanifold N. From this it follows that choosing an orientation on any two of these bundles will determine an orientation on the third. There is already an orientation on $TM|_N$ coming from the orientation on M, so that giving an orientation of the normal bundle ν is then enough to specific an orientation on TN and hence on the manifold N.

Here the relevant case arises from considering $M \times [0, 1)$ or $M \times (0, 1]$ for some $n-1$ dimensional manifold M. In this case, M has codimension 1 and so orienting the normal bundle amounts to choosing a direction in the interval. A natural choice is to choose the direction from 0 to 1, or the direction of "time". Note that the typical orientation on the incoming boundary as the boundary of an oriented manifold would be opposite this orientation. Some sources will use this convention and then take the opposite orientation of the incoming manifold when constructing the bordism. See lecture 24 of [7] and lecture 2 of [5] for more details.

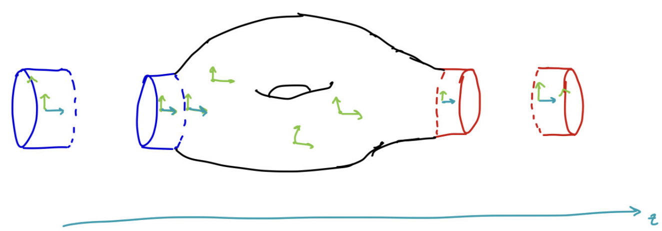


Figure 5. A depiction of how the orientation on the bordism and that coming from the "time" induce an orientation on the boundary manifolds. The induced orientation on the circles is depicted as a direction on them.

As an example of how this changes the theory, the oriented bordism groups can be defined by taking equivalence classes of oriented manifolds under oriented bordism. These groups are then different from the unoriented bordism groups. In dimension 0, $\Omega_0^{SO} = \mathbb{Z}$ where each class is determined by the number of positive points minus the number of negative points. The arcs that killed off pairs of points in the unoriented case can only annihilate or create two points if one is positive and one is negative. In the one-dimensional case $\Omega_1^{SO} = 0$ is unchanged as the circle and the disk are both oriented, and in two dimensions, $\Omega_2^{SO} = 0$ now since $\mathbb{RP}^2$ is not orientable.

3. Topological Field Theories

In subsection 3.1, the notion of a topological field theory is introduced. First, the bordism group must be enhanced to a symmetric monoidal category of bordisms. See appendix A for a review of the theory of symmetric monoidal categories. In this way, not just the possible sectors of the theory are being remembered, but the different ways spaces within a sector can deform into one another. It is then possible to define a topological field theory as a symmetric monoidal functor out of this bordism category. This subsection then ends with a brief explanation of the data contained in an oriented 1-D topological field theory. The theory in this dimension is sometimes referred to as topological quantum mechanics. The next subsection 3.2 contains the case of an oriented 2-D topological field theory, where it is shown how the theory then gives a vector space attached to the circle the structure of a commutative Frobenius algebra. Finally, in subsection 3.2 the action of the mapping class group on the vector spaces in the theory is introduced.

3.1. The Definition of a Topological Field Theory.

There is a categorified version of the bordism group Ω_n. This follows a general pattern in mathematics where instead of just viewing equivalence classes of objects, one keeps all of the objects and remembers how they are equivalent. This leads to the following definition of the bordism category.

Definition 3.1. The category $\mathrm{Bord}^G_{\langle n-1,n\rangle}$ is defined to have objects closed $n-1$ manifolds M with a G-tangential structure on $M \times (-1, 1)$, and morphisms diffeomorphism classes of bordisms with G-tangential structure between them. The operation of disjoint union of manifolds gives this category the structure of symmetric monoidal category.

This category roughly remembers the different configurations of space and how they can evolve. In a field theory, attached to every space should be some vector space of states on the space. This leads to the concept of a topological field theory, which is a functor out of this category.

Definition 3.2. A *(non-extended) topological field theory* is a symmetric monoidal functor $F : \mathrm{Bord}^G_{\langle n-1,n\rangle} \to \mathrm{Vect}_{\mathbb{C}}$. More generally $\mathrm{Vect}_{\mathbb{C}}$ can be replaced by any symmetric monoidal category.

Remark 3.1. Note that a closed n-manifold in such a theory is assigned a linear map from $\mathbb{C}$ to $\mathbb{C}$, as $\mathbb{C}$ is the unit of the monoidal category Vect. Such a map is determined by a complex number. Thus, it can be said that a topological field theory is a tool that assigns numbers to closed n-manifolds and vector spaces to closed $n-1$ manifolds in some way that respects gluing and locality. This idea can be extended further down in dimension with the notion of an extended theory. The rough idea is that each level down, the categorical level of the invariants increases.

Remark 3.2. Other common choices for the symmetric monoidal category could include Mod_R, the category of modules over a commutative ring R or $\mathrm{Ch}(R)$, the (possibly derived) category of chain complexes of R-modules. A common choice in relation to physics would be the category $\mathrm{sVect}_{\mathbb{C}}$ of super, i.e. $\mathbb{Z}/2$-graded vector spaces.

The bordism category has very nice properties as a symmetric monoidal category. Every single one of its object satisfies some sort of finiteness condition called dualizability. See section 7.2 for the definitions and some basic results.

Theorem 3.3. *Every object of* $\mathrm{Bord}^{\mathrm{SO}}_{\langle n-1,n\rangle}$ *is dualizable, with the dual of a manifold M being M, or $\bar{M}$ in the oriented bordism category.*

Proof. The pictures in the following proof will be done completely in the 1D case, where the object is the point. The general case can be seen by taking the cartesian product of all the pictures to follows with an arbitrary closed $n-1$ dimensional manifold.

Working in the oriented case, the dual to a manifold M is the manifold with reversed orientation $\bar{M}$. The duality data, $e : M \sqcup \bar{M} \to \emptyset$ and $c : \emptyset \to M \sqcup \bar{M}$ are the bordisms pictured below.

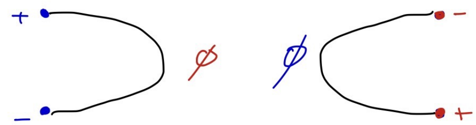


Figure 6. A depiction of the two bordisms giving duality for the positively oriented point.

The underlying manifold in both of these cases is $M \times I$. Now note that the identity bordism $M \times I$ can be decomposed in the following way.

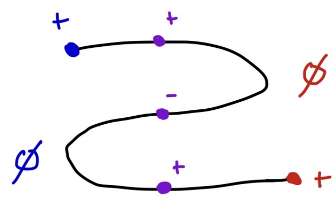


Figure 7. A depiction demonstrating the condition (7.12) for the positively oriented point

This says that $(e \sqcup \mathrm{id}_M) \circ (\mathrm{id}_M \sqcup c) = \mathrm{id}_M$. Taking the same picture with orientations reversed then gives the other condition for the above maps to be dualization data. Thus, M is dualizable with dual M. □

Remark 3.3. In general, every object M of $\mathrm{Bord}^G_{\langle n-1,n\rangle}$ will also be dualizable with dual the inverse of M in the bordism group Ω^G_{n-1}.

Notation 3.4. From here on out, $\mathrm{Bord}_{\langle n-1,n\rangle}$ will denote the oriented theory.

This is important due to the fact that it forces the same constraints on what objects in the target category can be attached to closed n-manifolds.

Corollary 3.5. *Any topological field theory $F : \mathrm{Bord}_{\langle n-1,n\rangle} \to C$, where C is a symmetric monoidal category, factors through the full subcategory of C of dualizable objects denoted by C^d. In particular, when $C = \mathrm{Vect}_{\mathbb{C}}$, then it factors through the full subcategory of finite dimensional vector spaces.*

Proof. This follows directly from theorem 3.3 and theorem 7.17 in the appendix. □

Example 3.6. To illustrate some of these ideas, examine the simplest case of an oriented 1-D topological field theory $F : \mathrm{Bord}^{SO}_1 \to \mathrm{Vect}_{\mathbb{C}}$. Tracing through the definitions, it follows that it is equivalent to the data of a assigning a vector space V to the positively oriented point pt_+, and its dual V^* to the negatively oriented point pt_-. There is then a linear map assigned to the following 2-manifolds with boundary.

Note that by definition it must assign the first one to the identity map on V. The next two are assigned to the coevaluation and evaluation maps of the duality data by theorem 3.3 and corollary 3.5. Finally, the circle is assigned the dimension of V. This is a demonstration of why the objects have to be dualizable in general. The dimension is only a well-defined number if V is finite dimensional. In general, the dimension of an object in a symmetric monoidal category can only be defined if the object is dualizable.

Remark 3.4. In general, in any n-dimensional theory with values in Vect_k, there is a manifold assigned to any closed $n-1$ dimensional manifold M a vector space V, and assigns the number $\dim V$ to $S^1 \times M$.

Even more generally, if the field theory has targets in a symmetric monoidal category, C, and some closed $n-1$ manifold M maps to an object C, then $M \times S_1$ maps to $\dim(C)$. See 7.15 for the definition of dimension in a dualizable category.

From the above example, it follows that the only data that was put into the 1-D field theory was that of a vector space, and from here all of the data was uniquely determined. In fact, given any finite dimensional vector space such a theory can be constructed. This leads to a classifcation of 1-D oriented field theories.

Theorem 3.7. *There is an equivalence of categories between the category of* 1*-D topological field theories and* $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{f.d.}}$*, the category of finite dimensional vector spaces.*

Actually proving this statement fully rigorously uses Morse-theoretic arguments. The idea is to decompose every bordism into a collection of simple bordisms using Morse theory, and then enforcing any conditions that arise from the ambiguity in this decomposition. This is roughly the pattern that other classifications of topological field theories are proved as well.

In the unoriented case, there is no distinction between the positively and negatively oriented points. Thus, it follows that V is self-dual since the point is now self-dual. This data amounts to an identification $\varphi : V \xrightarrow{\simeq} V^*$ or equivalently a non-degenerate bilinear form on V. This leads to the following classification.

Theorem 3.8. *There is an equivalence between 1-D unoriented topological field theories and finite dimensional vector spaces V equipped with a non-degenerate bilinear form.*

Note that in this case there is no separate theory of framed bordisms to consider as oriented and framed manifolds are the same in 0 and 1 dimensions.

3.2. **2-Dimensional Field Theories.** In this subsection, the structure of a 2-D topological field theory will be studied in order to see what type of data it forces onto the vector space assigned to S^1. Note that S^1 is the only connected, closed 1-dimensional manifold, and so it suffices to specify the vector space that the theory assigns to S^1. It will be seen that the different 2-manifolds with boundary manage to give some algebraic structure to this vector space. To start, first examine the pair of pants as shown below, or its equivalent formulation as a bordism embedded in the plane.

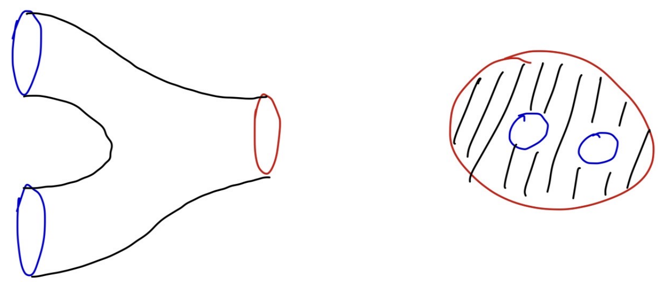


FIGURE 8. A depiction of the pair of pants giving the multiplication and the bordism viewed within the plane.

This is a bordism from $S^1 \sqcup S^1$ to S^1 and so it becomes a linear map of the form $m : A \otimes A \to A$. This is the exact type of map that would be used to define a multiplication on a vector space in order to turn it into an algebra. Indeed, the map has even been suggestively labelled m for multiplication. There are a number of nice properties that one would expect such a map to satisfy, such as associativity. This property can be visually seen to hold.

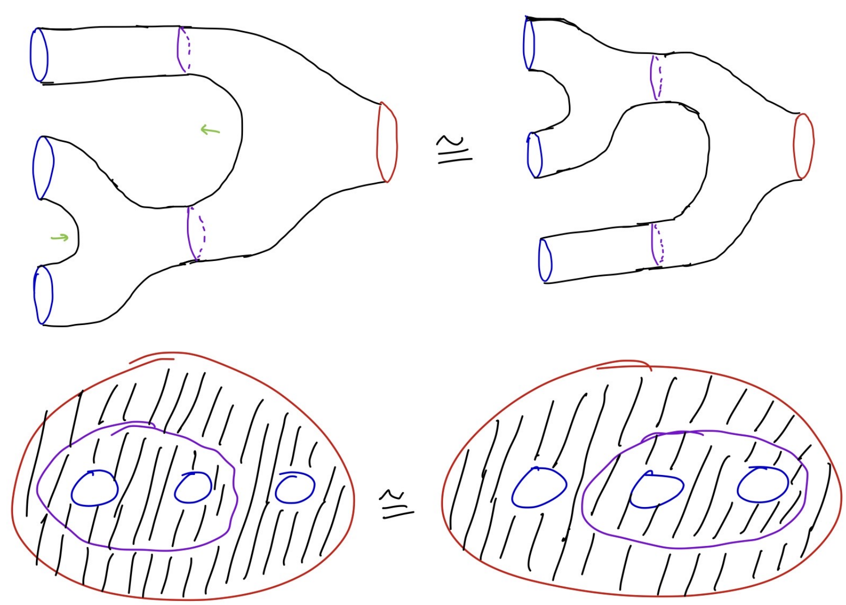


FIGURE 9. A depiction of the diffeomorphism of bordisms that demonstrates associativity of the multiplication.

In this picture, the two bordisms corresponding to $m \circ (m \otimes \mathrm{id})$ and $m \circ (\mathrm{id} \otimes m)$ are seen to be diffeomorphic. This means that they are both in fact the same morphism in the bordism category, and so it follows that they are mapped to the same map of vector spaces, which gives the desired result.

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$$

Therefore, it follows that m is an associative multiplication. An important thing to note throughout this section is that orientable compact 2 manifolds with boundary are classified by the number of circles making up the boundary and the genus of the surface. Thus, to verify that two bordisms are equivalent it suffices to verify that they have the same number of incoming and outgoing boundary components, and the same genus.

The next question is the existence of a unit for this multiplication. Note that a unit of a multiplication is an element of the vector space A. Such an element can be represented by a morphism $\eta : \mathbb{C} \to A$. Note that such a morphism makes sense as a unit if and only if

$$m \circ (\eta \otimes \mathrm{id}) = m \circ (\eta \otimes \mathrm{id}) = m(\mathrm{id} \otimes \eta).$$

The claim is that the unit of the multiplication is the disk, or spherical cap, which is a bordism from $\emptyset$ to S^1.

Figure 10. A depiction of the bordism giving the unit of the multiplication.

This bordism then gives the map $\eta : \mathbb{C} \to A$. Indeed the equations from above follow from the below pictures.

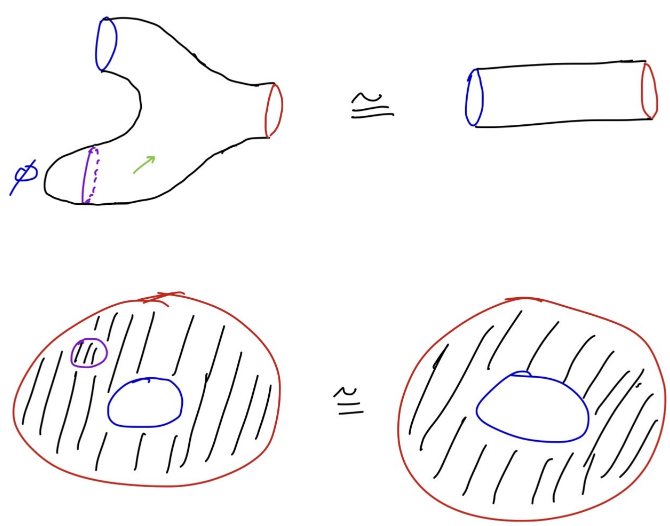


Figure 11. A depiction of the diffeomorphism of bordisms showing that figure 3.2 is indeed a unit of the multiplication.

The final condition on such a multiplication that one might want to see is a commutativity property. To see this, note that there is a second multiplication that can be defined.

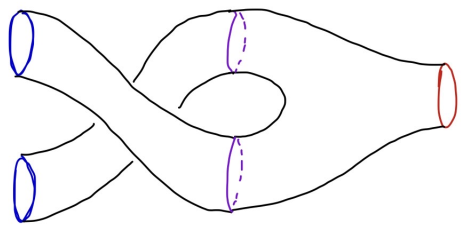


FIGURE 12. A depiction of the bordism giving the second multiplication operation.

In this picture the twist map is not identical to the identity morphism on $S^1 \sqcup S^1$ since the diffeomorphisms of bordisms must be trivial on the boundary, and so the circles are effectively being swapped. This is the map that gives the monoidal category Bord_2 a symmetric structure. Note that this new multiplication, $A \otimes A \to A$ is a map of algebras under m and $m : A \otimes A \to A$ is a map of algebras under this map. Therefore, it follows by a classical argument that these two multiplications are equal and commutative.

Remark 3.5. More exactly, the above says that $\mathcal{F}(S^1)$ would now have the structure of an E_2 algebra, i.e. a structure with two compatible multiplications. In this case, this turns out to be equivalent to a commutative algebra structure via the Eckmann-Hilton argument. In generalizations of the theory, this no longer holds. Viewing the bordisms in the plane as disks inside of disks, this gives a picture reminiscent of the little disks operad. In fact, such picture makes it plausible to see that in higher dimensions S^k will inherit the structure of a E_{k+1}-algebra.

There is one further piece of information on V given by flipping every picture above.

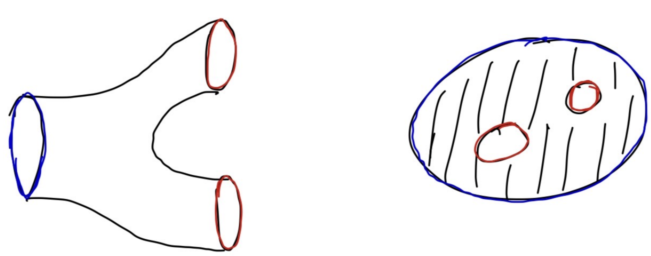


FIGURE 13. A depiction of the bordism giving the comultiplication.

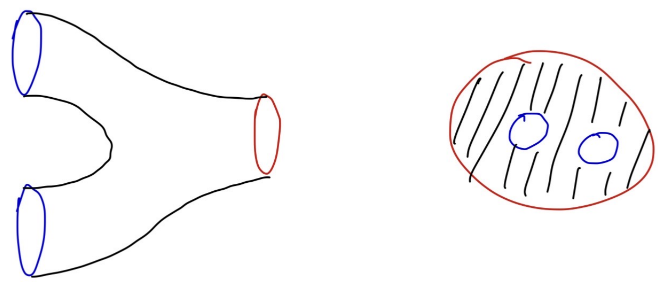


FIGURE 14. A depiction of the bordism giving the counit of the comultiplication.

The first of these pictures gives a so called comultiplication, $\mu : A \to A \otimes A$. By the same argument for the associativity of the multiplication, but flipping everything around, it follows that μ is coassociative.

$$(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$$

The second of these pictures gives a map $\varepsilon : A \to \mathbb{C}$, which is a counit of comultiplication μ. This means that $(\epsilon \circ \mathrm{id}) \circ \mu = \mathrm{id}$. This can be seen by flipping around the picture that was used to find the unit of the multiplication m. Similarly, it follows that μ is cocommutative, i.e. that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\mu} & A \otimes A \\ & {\scriptstyle \mu} \searrow \quad \swarrow {\scriptstyle \tau_V} & \\ & A \otimes A & \end{array}$$

This makes V into what is called a cocommutative counital coalgebra as well. Now the vector space V has two different structures on it, and is natural to wonder if there is any sort of compatibility between these two different structures. This compatibility arises by considering the following picture.

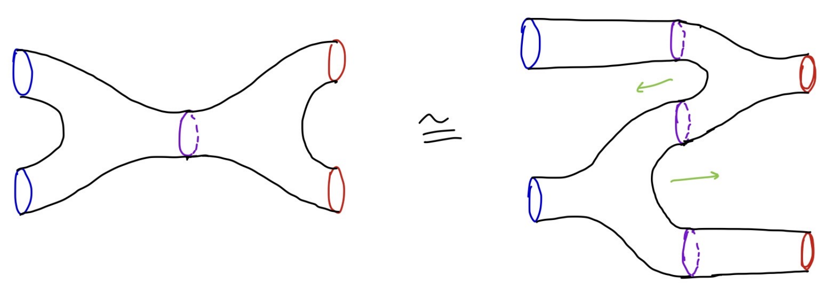


FIGURE 15. A depiction of the diffeomorphism of bordisms giving rise to the compatability condition between the algebra and coalgebra structures.

This picture translates to the following diagram commuting on the algebra side.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu\otimes \mathrm{id}} & A \otimes A \otimes A \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \mathrm{id}\times m} \\ A & \xrightarrow{\mu} & A \otimes A \end{array}$$

This gives rise to what is called the Frobenius condition

$$m \circ \mu = (\mu \otimes \mathrm{id}) \circ (m \otimes \mathrm{id}) \tag{3.9}$$

There is also a second Frobenius condition obtained by flipping things around to arrive at

$$m \circ \mu = (\mathrm{id} \otimes \mu) \circ (\mathrm{id} \otimes m). \tag{3.10}$$

However, since everything in sight is commutative or cocommutative, these two conditions are actually equivalent. This leads to the following definition.

Definition 3.11. An algebra A is a *Frobenius algebra* if it has the structure of a unital algebra (m, η) and a cocommutative counital algebra (μ, ε) satisfying (3.9) and (3.10). It is called a *commutative Frobenius algebra* if the multiplication m is commutative.

Remark 3.6. There is a useful way to interpret this definition and remember the condition. The Frobenius condition is equivalent to the requirement that the comultiplication $\mu : A \otimes A \to A$ is a map of A-modules. Therefore, it follows that a Frobenius algebra can be interpreted as an object having an algebra and coalgebra structure such that the comultiplication is a map of A-modules.

Note then that the entire analysis above effectively says that a oriented 2D topological field theory attached a commutative Frobenius algebra to the circle. It turns out that this is all the data that needs to be provided to determine such a theory.

Theorem 3.12. *Any commutative Frobenius algebra A determines an oriented 2D topological field theory F.*

This theorem again uses Morse theoretic arguments to decompose any bordism into fundamental bordisms, which in this case would be the bordisms determining the (co)multiplication and the (co)unit maps. The different conditions on these operations then arising by the ambiguity that occurs in the decomposition into fundamental bordisms.

Before moving on, there are a couple more pictures that should be noted.

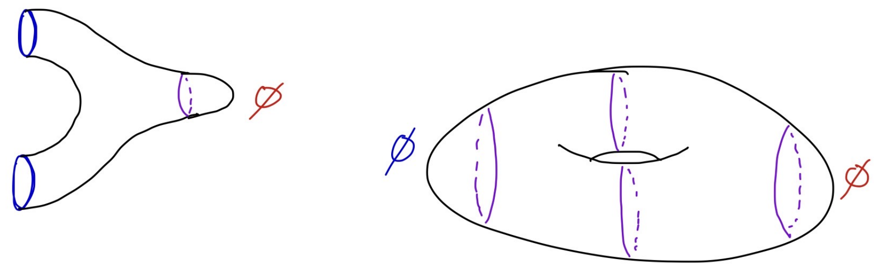


FIGURE 16. A depiction of two bordisms giving rise to a non-degenerate bilinear pairing on A and the dimension of the underlying vector space of A.

The first of these pictures gives a non-degenerate pairing $A \otimes A \to A$, which is arising by the standard duality data in the bordism category. This pairing is $\varepsilon \circ m$. The second of these is the torus, which is $S^1 \times S^1$, and so the number attached it is the $\dim_{\mathbb{C}} A$. Decomposing the torus as shown in the picture then tells us that $\dim_{\mathbb{C}} A = \varepsilon(m(\mu(\eta(1))))$. Similar formulas can be obtained for the higher genus g surfaces by taking the decomposition shown below.

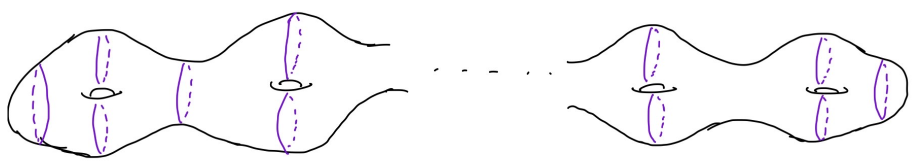


FIGURE 17. A depiction of how any closed orientable surface can be decomposed into the pieces described above.

3.3. **The Mapping Class Group.** Note that in the above subsection working through the data present in an oriented 2-D topological field theory, there was never a distinction given between the two different orientations on the circle. This follows from the fact that there are bordisms from a manifold M to itself for every diffeomorphism of the manifold. In fact, it will turn out that M possesses an action of its mapping class group

$$\mathrm{MCG}(M) := \pi_0 \mathrm{Diff}(M)$$

where Diff is the space of diffeomorphisms of M to itself.

Throughout this argument, for ease of notation, the definition of bordism will be altered by only requiring the maps i_0 and i_1 in definition 2.1 to be diffeomorphisms of M with $p^{-1}(0)$ and N with $p^{-1}(1)$. This gives a slightly different definition than the original one. However, the definitions give rise to the same bordism categories, as it does not change the diffeomorphism classes. This is basically a result of the fact that their always exist a collar neighborhood of the boundary, and that they are all diffeomorphic. Nevertheless, the arguments given in this section can be modified to use definition 2.1.

To see the bordism associated to a diffeomorphism $\varphi : M \to M$, take the cylinder $M \times I$. The data of a bordism effectively contained the information of identifications of M with the boundary of the cylinder. In particular, take i_0 that glues by the identity, and i_1 that glues by φ. Denote this bordism by C_φ The fact that this action factors through the mapping class group then follows from the following theorem.

Theorem 3.13. *The bordism C_φ is diffeomorphic as a bordism to $C_{\varphi'}$ if φ is homotopic to φ'.*

Proof. Note that if φ is smoothly homotopic to φ', then it follows that there is a map $F : M \times I \to M$ such that $F(m, 0) = \varphi(m)$ and $F(m, 1) = \varphi'(m)$. In particular, this gives rise to a homotopy $G : M \times I \to M$ from id_M to $\varphi' \circ \varphi^{-1}$ by $G(m, t) = F(\varphi^{-1}(m), t)$. From this it is now easy to construct a diffeomorphism of the cylinder $G' : M \times I \to M \times I$ defined by $G'(m, t) = (G(m), t)$. It only remains to check that this morphism commutes with the diffeomorphisms on the boundaries. This easily follows on the left boundary as everything is the identity. It also follows by construction on the right boundary as the commutative diagram is now

$$\begin{array}{ccccc} & & M & & \\ & \swarrow^{\varphi} & & \searrow^{\varphi'} & \\ M & & \xrightarrow[\varphi'\circ\varphi^{-1}]{} & & M \end{array} .$$

Thus, it follows that any homotopic diffeomorphisms give rise to diffeomorphic bordisms.

Now suppose that there is a diffeomorphism of bordisms $G' : C_\varphi \to C'_\varphi$. □

The big idea in relation to the 2D oriented case is that there is an orientation reversing diffeomorphism of the circle. This results in a bordism from one orientation of a circle to the other, and hence an identification of the circle with its dual. Therefore, there is no reason to keep track of the orientation of the circle. This identification of the Frobenius algebra with its dual can be seen to come from the perfect pairing $\epsilon \circ m : V \otimes V \to \mathbb{C}$. It actually turns out that this pairing, and consquently the map ϵ often called the trace pairing is enough to give the structure of the frobenius algebra. See section 3.8 of [16] for a proof of this fact.

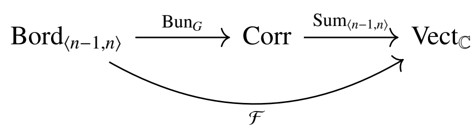
Remark 3.7. The above theory is still an oriented theory as its values on unoriented manifolds, such as the Möbius band and real projective space, were not specified.

In a general oriented theory of arbitrary dimension, the two possible orientations of a manifold only need to be distinguished if that manifold does not possess an orientation reversing diffeomorphism. The circle possesses such a diffeomorphism given by reflection. Additionally, an oriented surfaces Σ_g also possesses such a diffeomorphism given by a reflection of the fundamental polygon of Σ_g. Thus, orientations can be safely ignored in a $2D$ oriented theory. However, in dimension 1, the only diffeomorphism of a point is the identity, and thus has no orientation reversing diffeomorphism. This resulted in the fact that the positively and negatively oriented points had different vector spaces attached to them. Examples of manifolds without orientation reversing diffeomorphisms also exist in higher dimensions, and so orientations are important in higher dimensional oriented theories as well.

Another consequence of theorem 3.13 is that is that it means topological field theories give representations of the mapping class group of manifolds. In the oriented case, it actually gives a representation of only the orientation preserving diffeomorphisms. In the case of the circle, $MCG(S^1) \simeq \mathbb{Z}/2$, and so this data is exactly the identification with the dual, and there is no non-trivial action on the vector space assigned to S^1.

4. 2D Finite Gauge Theory

The above describes all the possible 2D and 1D oriented topological field theories. There are links between topological field theories and representation theory that lie within a special class of theories. Given any finite group G, there is a procedure that attaches to this group a topological field theory of any dimension called finite gauge theory. This procedure makes use of an important concept in the construction of field theories called the category of correspondences as will be described in section 4.1. In this section, a functor $\mathrm{Sum}_{\langle n-1,n\rangle}$ from the category of correspondences to that of vector spaces will be described. Then, in section 4.2 a functor from Bord_n to this category of correspondences will be constructed by passing through the groupoid of principal G-bundles on a space X $\mathrm{Bun}_G(X)$. Together, these two functors give the construction of finite gauge theory. Diagrammatically, this process looks like



In $2D$, this reproduces the topological field theory attached to a Frobenius algebra structure on the center of the group algebra of G. Some of these details will be discussed in section 4.3. It turns out that in some ways the information contained in this field theory is that contained in the finite dimensional representation theory of G. This is reflected more clearly when the extended $2D$ finite gauge theory is considered in section 6, where the theory assigns the group algebra of G to a point. The ideas of this section and generalizations of them can be found in [9].

4.1. **The Category of Correspondences.**

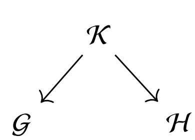
In this section, the notion of correspondences of groupoids will be introduced. This category will serve as an intermediary in the construction of finite gauge theory. The relevant functor $\mathrm{Sum}_{\langle n-1,n\rangle}$ out of the category of correspondences will also be introduced. In order to develop these notions, some preliminaries on groupoids are needed first.

Definition 4.1. Let $\mathcal{G}$ be a groupoid. Let $\pi_0(\mathcal{G})$ by the set of isomorphism classes of objects of $\mathcal{G}$. Given an object $g \in \mathcal{G}$, let $\pi_1(\mathcal{G}, g)$ denote the set of isomorphisms of g.

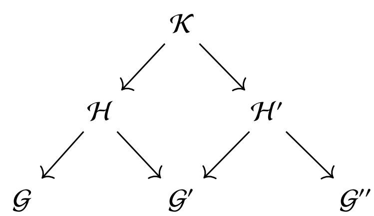
Definition 4.2. A groupoid $\mathcal{G}$ is *finite* if $\pi_0(\mathcal{G})$ and $\pi_1(\mathcal{G}, g)$ for every $g \in \mathcal{G}$ are a finite sets.

It is now time to introduce the category of correspondences that will play a key role for the remained of this section.

Definition 4.3. The category Corr of correspondences has as objects finite groupoids. Morphisms from a groupoid $\mathcal{G}$ to $\mathcal{H}$ are another groupoid $\mathcal{K}$ along with maps from $\mathcal{K}$ to $\mathcal{G}$ and $\mathcal{H}$. This data is called a correspondence between $\mathcal{G}$ and $\mathcal{H}$, and is often depicted by a diagram of the form



The correspondences compose via pullback, i.e via the diagram

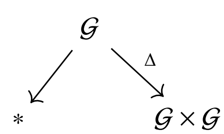


where the top square is a pullback square.

This category is endowed the structure of a symmetric monoidal category via the normal product of groupoids. Therefore, it is possible to define a topological field theory that has target Corr according to defintion 3.2. A natural question to ask at this point is what the dualizable objects in this symmetric monoidal category are as any topological field theory factors through the full subcategory of dualizable objects. It turns out that this actually provides no condition.

Theorem 4.4. *Every object of* Corr *is dualizable.*

Proof. Given a groupoid, $\mathcal{G}$, then it has dual itself, with coevaluation map



where $*$ is the groupoid with one element and just the identity morphism. The evaluation map is the same diagram but mirrored. The two conditions then boil down to one, which amounts to checking that

$$(\mathcal{G} \times \mathcal{G}) \times_{\mathcal{G}\times\mathcal{G}\times\mathcal{G}} (\mathcal{G} \times \mathcal{G}) \simeq \mathcal{G}.$$

This condition would be very straightforward to check if these were set valued correspondences, or in general valued in any category with finite products. It is a little trickier to check in this case. The first is the groupoid of quadruples of elements of $\mathcal{G}$ with compatible pairwise isomorphisms between them. It can be checked that the functor that sends every quadruple to the first element is an equivalence of categories. □

The idea here though is not to end up with a field theory valued in this category, but instead use it as an intermediate step in order to define the desired field theory valued in $\mathrm{Vect}_{\mathbb{C}}$. The rest of this subsection will be devoted to explaining the construction of the functor $\mathrm{Sum}_{\langle n-1,n\rangle}$ out of this category into $\mathrm{Vect}_{\mathbb{C}}$ to this end.

The map on objects sends a groupoid $\mathcal{G}$ to the vector space $\mathrm{Fun}(\mathcal{G}, \mathbb{C})$, where $\mathbb{C}$ is viewed as a discrete category. This has a much more concrete description which makes the vector space structure easier to see. Note that functors from $\mathcal{G}$ into any discrete category will factor through $\pi_0(\mathcal{G})$ viewed as a discrete category. Therefore, this functor category is actually just the set of maps $\mathrm{Map}(\pi_0(\mathcal{G}), \mathbb{C})$. This set carries a natural vector space structure induced pointwise by that on $\mathbb{C}$.

Now it is necessary to give a map of vector spaces from a correspondence of groupoids.

$$\begin{array}{ccc} & \mathcal{K} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathcal{G} & & \mathcal{H} \end{array} \tag{4.5}$$

This will be done in two distinct steps, in a pull-push procedure. The first step, pulling back along the map f is rather simple. The basic idea is that given a map $\varphi : \pi_0(\mathcal{G}) \to \mathbb{C}$, then the functor $f : \mathcal{K} \to \mathcal{G}$ induces a map $f_0 : \pi_0(\mathcal{K}) \to \pi_0(\mathcal{G})$. The pullback along f is then defined by precomposition with f_0.

$$f^*\varphi = \varphi \circ f_0$$

The second map, the pushforward along g is not as transparent to define. Given a map $\psi : \pi_0(\mathcal{K}) \to \mathbb{C}$, the idea of the pushforward is to sum along the fibers of the map g. To formalize this idea, the fiber of a functor between groupoids has to be defined.

Definition 4.6. Take a map of groupoids $g : \mathcal{K} \to \mathcal{H}$, and an element $h \in \mathcal{H}$. This determines a map $h : * \to \mathcal{H}$ that sends the unique object of $*$ to h. The fiber groupoid of the map g over h is defined to be the fiber product of $*$ and $\mathcal{K}$ over $\mathcal{H}$ with respect to these maps. In other words, the following diagram is cartesian.

$$\begin{array}{ccc} g^{-1}(h) & \longrightarrow & \mathcal{K} \\ \downarrow & & \downarrow{\scriptstyle g} \\ * & \xrightarrow{\;h\;} & \mathcal{H} \end{array}$$

Unwinding this definition, $g^{-1}(h)$ has objects (k, φ) where k is an object of $\mathcal{K}$ and φ is an isomorphism $g(k) \simeq h$. A morphism between two objects (k, φ) and (k', φ') is a map $\psi : k \to k'$ such that $\varphi' \circ g(\psi) = \varphi$. There is then a functor $F : g^{-1}(h) \to \mathcal{K}$ given by forgetting the isomorphism.

It is now possible to give the formula for the pushforward of ψ along g. Let h be an element of $\mathcal{K}$. In the following formula, the isomorphism has been suppressed for ease of notation.

$$g_*\psi(h) = \sum_{k \in \pi_0(g^{-1}(h))} \frac{1}{\left|\pi_1(g^{-1}(h), k)\right|} \psi(k)$$

This is almost just summing over the values of ψ along the fiber, but there is this strange factor which is division by the automorphisms of the objects in the fiber. With these maps now in hand, it is possible to give the definition of the desired functor from correspondences to vector spaces.

Definition 4.7. The functor $\mathrm{Sum}_{\langle n-1,n\rangle} : \mathrm{Corr} \to \mathrm{Vect}_{\mathbb{C}}$ is defined by sending a finite groupoid $\mathcal{G}$ to $\mathrm{Map}(\pi_0(\mathcal{G}, \mathbb{C})$. It sends a correspondence (4.5) to the map $g_* \circ f^*$ on vector spaces.

4.2. **Principal G-Bundles.** In this section, the map Bun_G from the bordism category to the category of correspondences described in the introduction is constructed. The key object in this construction is the groupoid $\mathrm{Bun}_\mathrm{G}(X)$ of principal G-bundles over X, where G is a finite group.

Definition 4.8. A principal G-bundle over a manifold X is a covering space $\pi : P \to X$ with an action of G on P by deck transformations such that the action restricted to any fiber is simply transitive.

Definition 4.9. Given a manifold X, there is a groupoid $\mathrm{Bun}_G(X)$, whose objects are the principal G-bundles over X and morphisms are bundle isomorphisms.

This object can be worked with geometrically by thinking of normal covering spaces of a manifold X. However, there are more algebraic ways of thinking of this groupoid as well. The algebraic description is arises from the idea of monodromy, i.e. by considering the action of the group on a fiber induced by going around a non-trivial loop.

Theorem 4.10. *Let X be a connected manifold. Then there is an equivalence between isomorphisms classes of G-principal bundles on X and group homomorphisms $\pi_1(X, x_0) \to G$ up to conjugation.*

Proof. The fundamental group, $\pi_1(X, x_0)$ acts by deck transformations on X and so in particular has an induced action on the fiber over x_0 which is a G-torsor. Choosing a point in this fiber $\tilde{x}_0$, there is a unique element $g \in G$ such that any other point in the fiber is $g \cdot \tilde{x}_0$. Thus, this gives a group homomorphism $\pi_1(X, x_0) \to G$, which sends an element $\gamma \in \pi_1(X, x_0)$ to the unique element of G such that $g \cdot \tilde{x}_0 = \gamma \cdot \tilde{x}_0$. □

This map can be extended to an equivalence of groupoids and the case when X is not connected by associating a groupoid to X enhancing the fundamental group. This is a classical object, called the fundamental groupoid, as can be found in [15].

Definition 4.11. Let X be a topological space. The *fundamental groupoid* of X $\Pi_{\leq 1} X$ has objects the points of X, and morphisms between $x, y \in X$ are homotopy classes of maps from x to y.

Remark 4.1. By choosing basepoints, x_i in each path component of X, this groupoid is equivalent to the groupoid $\sqcup B\pi_1(X, x_i)$ where BG in this case refers to the groupoid with one element and morphisms given by elements of G. This is called the classifying groupoid of G.

This description makes it easier to algebraically compute the groupoid associated to a manifold.

Example 4.12.
1. By noting that a point $*$ has trivial fundamental group so that any map $\pi_1(*, *) = \{1\} \to G$ is trivial. As conjugating by an arbitrary element of G gives an automorphism of this map, it follows that $\mathrm{Bun}_G(*) \simeq */G = \mathrm{B}G$. This can also be seen by noting that principal G bundles on $*$ are G-torsors, and this category is known to be $\mathrm{B}G$. See for example [15].
2. Similarly, the closed interval I and any contractible space, X, for that matter will have $\mathrm{Bun}_G(X) = */G$. This follows as this object is invariant under homotopy since $\Pi_{\leq 1} X$ is a homotopy invariant of the manifold.
3. For a circle, $\pi_1(S^1, *) = \mathbb{Z}$, and maps $\mathbb{Z} \to G$ in bijection with elements of G by looking at the image of $1 \in \mathbb{Z}$ under such a map. Under this bijection, the G action on the maps corresponds to the normal conjugation action of G on itself.

Thus $\text{Bun}_G \simeq G/G$, where this is the groupoid quotient of G acting on itself by conjugation. The objects are the elements of G, and the morphisms between $g, h \in G$ are the set of $k \in G$ such that $kgk^{-1} = h$.

(4) In the case of the pair of pants P, $\text{Bun}_G(P) = G/G \times G/G$. This follows as $\pi_1(P, *) = \mathbb{Z} * \mathbb{Z}$.

This construction provides a symmetric monoidal functor $\text{Bun}_G : \text{Bord}_n \to \text{Corr}$. Combining this functor with the functor $\text{Sum}_{\langle n-1,n \rangle}$ from definition 4.7, then gives the construction of a topological field theory valued in $\text{Vect}_{\mathbb{C}}$. This theory is n-dimensional (non-extended) finite gauge theory.

4.3. **Construction of Finite Gauge Theory in Low Dimensions.** In this section, the 1D and 2D finite gauge theories will be constructed. The examples given at the end of the last section provide the computations of all the relevant groupoids of principal G-structures for this purpose.

The 1D case is rather trivial. Note that this theory is determined by its value on the point, which is $\text{Map}(*/G \to \mathbb{C}) \simeq \mathbb{C}$. The value assigned to the circle would then be 1. This can be confirmed by direction computation through the functors constructed above.

The 2D case is more interesting, being effectively equivalent to the finite dimensional representation theory of the group G. Note that the vector space assigned to the circle S^1 is $\text{Map}(G/G, \mathbb{C}) = \text{Map}(G, \mathbb{C})^G$, which is the functions on G invariant under conjugation. This is the class functions, which can be viewed as the center of the group algebra $Z(\mathbb{C}[G])$. This space carries a natural algebra structure and a trace map that sends a function to its value on the identity of G divided by $|G|$. This is exactly the data necessary to construct a 2D topological field theory. It turns out that in going through the construction given above produces the same field theory as that given by this commutative Frobenius algebra. The most convenient basis to work with in describing the algebra and coalgebra maps explicitly is the characters χ_ρ of irreducible representations ρ. The algebra and coalgebra structure in this basis become

$$\eta(1) = \frac{1}{|G|} \sum_{\rho} \dim \rho \chi_\rho \tag{4.13}$$

$$m(\chi_\rho, \chi_\psi) = \delta_{\rho,\psi} \frac{|G|}{d} \chi_\rho \tag{4.14}$$

$$\varepsilon(\chi_\rho) = \frac{1}{|G|} \chi_\rho(e) = \frac{\dim \rho}{|G|} \tag{4.15}$$

$$\mu(\chi_\rho) = \frac{|G|}{\dim \rho} (\chi_\rho, \chi_\rho) \tag{4.16}$$

As an example of how the machinery developed in this section works, the counit (4.15) will be worked out. The counit arises from the disk viewed as a bordism from S^1 to the empty set. The correspondence of groupoids then becomes

$$\begin{array}{ccc} & \mathrm{Bun}_G(D^1) \simeq */G & \\ {}^{u}\swarrow & & \searrow^{v} \\ \mathrm{Bun}_G(S^1) \simeq G/G & & \mathrm{Bun}_G(\emptyset) \simeq * \end{array} \tag{4.17}$$

The map u is the map restricting a principal G-bundle G on the disk to the boundary. This is the inclusion which sends $*/G$ to the copy of $*/G$ in G/G with single object the identity $e \in G$. The other map v sends the unique object to the unique object and every morphism to the identity. Taking the space of maps attached to each of the objects leads to the following situation.

$$\mathrm{Map}(G, \mathbb{C})^G \xrightarrow{u*} \mathbb{C} \xrightarrow{v_*} \mathbb{C}$$

In computing the map v_*, note that the fiber of the map v over $*$ is just $*/G$ since there are no non-identity morphisms in the category $*$. The map v_* is then very easily computed to be multiplication by $1/|G|$. For the map u^* note that u just maps $*$ in $*/G$ to the object $e \in G/G$. Thus, it follows that $u^*\chi = \chi(e)$ for $\chi \in \mathrm{Map}(G, \mathbb{C})^G$. Putting these two maps together then gives (4.15).

This data can then be used to compute the number attached to any genus g surface Σ_g by making use of a nice decomposition of the space. This is the same decomposition as shown in figure 3.2.

This gives a composition of the maps given above, and it turns out that

$$\mathcal{F}(\Sigma_g) = \sum_{\rho} \left(\frac{|G|}{\dim \rho} \right)^{2g-2} \tag{4.18}$$

Using the machinery given above gives another way to arrive at the value $\mathcal{F}(\Sigma_g)$. To see this note that given a closed surface viewed as a bordism, the correspondence associated to it is

$$\begin{array}{ccc} & \mathrm{Bun}_G(\Sigma_g) & \\ \swarrow & & \searrow \\ * & & * \end{array}$$

Running through the summation functor on this gives

$$\sum_{\mathcal{P} \in \pi_0(\mathrm{Bun}_G(\Sigma_g))} \frac{1}{|\pi_1(\mathrm{Bun}_G, \mathcal{P}|}$$

Note that this value can be computed by using the equivalence
$\mathrm{Bun}_G(\Sigma_g) \simeq \mathrm{Fun}(B\pi_1(\Sigma_g), BG)$. Therefore, it follows that this value is

$$\frac{|\mathrm{Hom}(\pi_1(\Sigma_g), G)|}{|G|}$$

However, note that $\pi_1(\Sigma_g) = \langle x_1, ..., x_g, y_1, ..., y_g \mid [x_1, y_1]...[x_g, y_g] = 1 \rangle$. Using the universal properties of generators and relations, it then follows that

$$\mathrm{Hom}(\pi_1(\Sigma_g, G)) = \{(a_1, ..., a_g, b_1, ..., b_g) \in G^{2g} \mid [a_1, b_1]...[a_g, b_g] = 1\}.$$

Combining this with the other computation of $\mathcal{F}$ in (4.18) then gives the following formula about elements of the group

$$\{(a_1, ..., a_g, b_1, ..., b_g) \in G^{2g} \mid [a_1, b_1]...[a_g, b_g] = 1\} = \sum_{\rho} \frac{|G|^{2g-1}}{(\dim \rho)^{2g-2}} \tag{4.19}$$

This formula was known classically by Frobenius by algebraic means, but the machinery of topological field theories and finite gauge theory developed here are able to reproduce this result as well using only the computation of the Frobenius algebra structure using the summation functor.

5. Extended Field Theories and the Cobordism Hypothesis

In this section, a generalization of the the idea of a topological field theory is introduced, which possesses a powerful classification result called the cobordism hypothesis. The more general higher bordism category Bord_n is introduced in 5.1. Some technical details needed concerning dualizability in higher categories is then discussed briefly in 5.2. Finally, the cobordism hypothesis is stated in 5.3. A discussion of these concepts and a sketch of the proof can be found in [13]. Another leisurely discussion of these ideas can be found in [6], and it also discussed in the later lectures in [19].

5.1. **Definition of a Fully Extended Topological Field Theory.** Recall from section 3 that the entire data of a 1D topological field theory was determined by what it assigned to a point. In the two dimensional case, there was data beyond just the object that was attached to the circle to determine a 2D topological field theory, but it was still simple to decompose an arbitrary surface into pieces to compute the value of the theory on an arbitrary manifold. In higher dimensions, it is difficult to decompose manifolds into simpler pieces using codimension one manifolds. Additionally, there are more closed $n-1$ dimensional manifolds in the picture whose values need to be specified in the theory, frustrating classification attempts. If instead, manifolds were allowed to be decomposed along codimension submanifolds, the classification would become simpler. Effectively, since all manifolds are locally a disk, the hope would be that the entire theory could be computed from its value on a point. Such a theory must then allow bordisms between bordisms and manifolds with corners into the theory. The cobordism hypothesis says that it indeed happens that the theory is completely determined by its value on the point like the 1D case.

In order to accomplish this, we will need to construct a replacement for the category $\text{Bord}_{\langle n-1,n\rangle}$. As stated, we now need to include bordisms between bordisms, and so the morphisms in our category will now have morphisms and these morphisms will have morphisms and so on. This means that the replacement for $\text{Bord}_{\langle n-1,n\rangle}$ will be an n-category.

Definition 5.1. The n-category Bord_n is defined to have as objects closed 0 manifolds. Its 1-morphisms are bordisms between closed 0-manifolds. Its 2-morphisms are bordisms between these bordisms, which are now bordisms between the 1-manifolds giving 1-morphisms. These are now manifolds with corners. This goes all the way up to n-morphisms which are now diffeomorphisms classes of bordisms between $n-1$ morphisms, which are now manifolds with higher dimensional corners.

Remark 5.1. The theory of n-categories is a difficult subject to develop and there are no well studied formulations of this theory. Throughout this paper, to the extent that the subject will be used, it suffices to understand the basic idea. They will treated naively throughout the paper. The only higher category that will show up explicitly is a 2-category in 6.1. The case of 2-categories has been worked out a lot more in detail. For example, details on 2-categories and symmetric monoidal 2-categories can be found in [16]. A more thorough discussion of the ideas of n-categories can be found in [2]. The primary reference for the cobordism hypothesis [13] deals with an infinity categorical generalization of n-categories very briefly.

Now with this definition of bordism category in hand, it is possible to define a fully extended topological field theory.

Definition 5.2. A fully extended n-dimensional topological field theory is a symmetric monoidal functor of n-categories, $\mathcal{F} : \mathrm{Bord}_n \to C$ where C is a symmetric monoidal n-category.

This idea an extension of the original definition of a topological field theory, as it contains the information of a normal non-extended n-dimensional topological field theory by only considering closed $n-1$ dimensional manifolds and bordisms between them. In order to be more explicit, more notation must be developed first that will be helpful throughout what follows.

Note that the empty set $\emptyset$ is mapped to the tensor unit 1_C of C when it is viewed as a closed 0-manifold. This implies that any closed 1 dimensional manifold is mapped to $\mathrm{Hom}(1_C, 1_C)$ as this is a bordism from $\emptyset$ to $\emptyset$ viewed as a 0-manifold. There is a variation of the above definition that starts from closed 1-manifolds and gives a $n-1$ category. The above is saying that a fully extended n-dimensional field theory with target C contains such a modified theory with target $\mathrm{Hom}(1_C, 1_C)$. The empty set viewed as a closed 1 manifold then maps to the tensor unit of this $n-1$ symmetric monoidal category. This process can be carried out inductively, leading to the following definition.

Definition 5.3. Let C be a symmetric monoidal n-category. Then $\Omega C := \mathrm{Hom}(1_C, 1_C)$ where 1_C is the unit of the tensor product on C. This is symmetric monoidal $n-1$ category. We can then inductively define $\Omega^k C = \Omega(\Omega^{k-1})C$. This is called the k^{th} *loop space* of C.

In particular, $\Omega^k \mathrm{Bord}_n$ gives a variation of the category from definition 5.2 starting from closed k-manifolds that is now a $(n-k)$-category. In particular, $\Omega^{n-1}\mathrm{Bord}_n$ is the old bordism category from definition 2.1, and as stated above contains the closed $(n-1)$-manifolds and bordisms between them. This non-extended topological field theory now has target $\Omega^{n-1}C$. In the case of a non-extended topological field theory, the target category was often assumed to be $\mathrm{Vect}_{\mathbb{C}}$. Ideally, we would then require that the target category C in a fully extended topological field theory satisfies $\Omega^{n-1}\mathbb{C} \simeq \mathrm{Vect}_{\mathbb{C}}$. An example of such a category for $n = 2$ will given in the next section. Some discussion of categories for higher n appears in [9].

Remark 5.2. The rough idea that a non-extended topological field theory is a tool for attaching numbers to closed n-manifolds and vector spaces to closed $n-1$ manifolds in a way that behaves nicely under gluing can now be extended. Roughly, an extended topological field theory is a tool that attaches n-categorical data to a point, $n-1$ categorical data to a closed 1-manifold, and so on with numbers being attached to closed n-manifolds.

5.2. **Higher Duality Data.** Before giving the statement of the cobordism hypothesis, one more detail needs to be discussed. Recall that for a normal non-extended topological field theory, the theory factored through the full subcategory of dualizable objects of the target category. In the same vein, any fully extended topological field theory will factor through a category of fully dualizable objects. See [13] section 2.3 for more information. The correct idea for dualizability at the level of morphisms is having left and right adjoints.

Definition 5.4. Let C be a 2-category. Given a pair of 1-morphisms $f : A \to B$ and $g : B \to A$ where $A, B \in \mathrm{Ob}(C)$, then an *adjunction* between them is a pair of 2-morphisms $u : \mathrm{id}_A \to g \circ f$ and $v : f \circ g \to \mathrm{id}_B$ such that the following diagrams commute

$$\begin{array}{ccccc} f & \xrightarrow{\mathrm{id}\times u} & f \circ g \circ f & \xrightarrow{v\times \mathrm{id}} & f \\ & & \underbrace{\qquad\qquad\qquad}_{\mathrm{id}} & & \end{array}$$

and

$$\begin{array}{ccccc} g & \xrightarrow{\mathrm{id}\times u} & g \circ f \circ g & \xrightarrow{\mathrm{id}\times v} & g \\ & & \underbrace{\qquad\qquad\qquad}_{\mathrm{id}} & & \end{array}.$$

In this case, f is called a *left adjoint* to g and g is called a *right adjoint* f.

There are two things to note about this definition. First, as the name suggests, this definition is similar to that of adjunct functors. If C is taken to be the 2-category of small categories then this definition is that of adjoint functors. Second, this data looks similar to the duality data in a symmetric monoidal category. It turns out that duality data can be viewed as a special case of this definition as well.

Definition 5.5. Let C be a symmetric monoidal category. Then there exists a 2-category BC. It has one object, $*$, that has morphisms the category C. Composition of 1-morphisms, which are objects of C, is the tensor product from C. This is called the *classifying category of* C.

Duality data in C is then equivalent to the data of adjoint 1-morphisms in $\mathrm{B}C$. This makes the notion of adjoint 1-morphisms appear to be a natural extension of the notion of duality data. We now need to introduce a notion that generalizes that of a symmetric monoidal category having duals for all of its objects. First, the notion of a category having adjoints is needed, starting with the case of 2-categories.

Definition 5.6. A 2-category C is said to admit adjoints if every morphism has both a left and right adjoint.

With this it is now possible to define what it means for an n-category to have adjoints after one more definition is made.

Definition 5.7. The homotopy 2-category of an n-category C is the 2-category h_2C with objects and 1-morphisms the same as those of C, and 2-morphisms are isomorphism classes of 2-morphisms in C.

Definition 5.8. An n-category C has adjoints for 1-morphisms if the homotopy 2-category h_2C admits adjoints. Proceeding inductively, the category has adjoints for k morphisms if for every pair of objects X, Y in C, the $n-1$ category $\mathrm{Hom}(X, Y)$ has adjoints for $k-1$ morphisms. The category C is then said to *admit adjoints* if it admits adjoints for k-morphisms for all $1 \leq k \leq n$.

It is now possible to define what it means for a category to have duals, which will be the generalization of a symmetric monoidal category having all of its objects be dualizable.

Definition 5.9. A symmetric monoidal n-category C has duals for objects if every object in the homotopy category hC is dualizable, or equivalently the $n+1$ category $\mathrm{B}C$ has adjoints for 1-morphisms. The category C is said to have duals if it has duals for objects and adjoints.

Similar to how the dualizable objects of a symmetric monoidal category formed a full subcategory, given a symmetric monoidal n-category $\mathcal{C}$, there is a universal symmetric monoidal n-category with duals $\mathcal{C}^{f.d.}$ with a functor $\mathcal{C}^{f.d.} \to \mathcal{C}$. It is universal in the sense that given a symmetric monoidal n-category with duals $\mathcal{B}$ and a functor $\mathcal{B} \to \mathcal{C}$, there is an essentially unique functor $\mathcal{B} \to \mathcal{C}^d$ that fits into a commutative diagram.

$$\begin{array}{ccc} & & \mathcal{C}^{f.d.} \\ & \nearrow & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{C} \end{array}$$

The basic idea of the category $\mathcal{C}^{f.d.}$ is that you first throw out the objects without duals. Then throw away the objects such that the morphisms in duality data don't have adjoints, and then the objects such the adjoint data for the morphisms in the duality data don't have adjoints and so on. Then throw away all the maps between dualizable objects that don't have adjoints, and so on. The follow definition generalizes that of an object being dualizable in a symmetric monoidal category.

Definition 5.10. An object X of a symmetric monoidal n-category $\mathcal{C}$ is called *fully dualizable* if it is in the essential image of the functor $\mathcal{C}^{f.d.} \to \mathcal{C}$.

Effectively, an object X is fully dualizable if it admits a dual such the morphisms in its duality data admit adjoints that then admit adjoints and so on.

5.3. **The Cobordism Hypothesis.** In this subsection, the cobordism hypothesis and some of its variants will be stated and discussed.

Theorem 5.11 (The Cobordism Hypothesis)**.** *Let $\mathcal{C}$ be a symmetric monoidal n-category, and* $\mathrm{Fun}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ *denote the category of fully extended topological field theories with values in $\mathcal{C}$. There is an equivalence of categories*

$$\mathrm{Fun}(\mathrm{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\simeq} \tilde{\mathcal{C}}^{f.d.}$$

where $\tilde{\mathcal{C}}^{f.d.}$ is the n-groupoid obtained from $\mathcal{C}^{f.d}$ by throwing away the non-invertible morphisms. The functor sends a field theory $\mathcal{F}$ to $\mathcal{F}(\mathrm{pt})$.

Remark 5.3. For the rest of this section $\mathcal{C}$ will be assumed to be a symmetric monoidal n-category with duals for ease of notation.

This theorem will not be proved here, but a sketch of the proof can be found [13]. The main takeaway from this theorem is that a topological field theory is determined by the value it takes on point. Furthermore, the possible values that can be assigned to the point are the fully dualizable objects of a category. The theorem is moralistically saying that Bord_n^{fr} is the free symmetric monoidal n-category generated by a single fully dualizable object, the point with positive framing. Note that this encapsulates the case of oriented 1D topological field theories given in section 2. In the 1D case the framed and oriented theories coincide since $\mathrm{SO}(1) = *$.

It is convenient to be able to describe topological field theories coming from bordism categories with different tangential structures besides a framing. There is a generalized version of the cobordism hypothesis that takes these into account. The theory will still be determined by its value on the point, but the values that it can take will need to be modified. In order to describe this modification, it becomes necessary to first note that the category

$\mathrm{Bord}_n^{\mathrm{fr}}$ has an action of $\mathrm{O}(n)$ on it. An element of $\mathrm{O}(n)$ acts by changing the framing on the point, and then accordingly on every other manifold as well. Theorem 5.11 says that the fully dualizable objects of a category inherit an $\mathrm{O}(n)$ action. Given a group homomorphism $G \to \mathrm{O}(n)$, it is then possible to examine the fixed points of $\tilde{C}$ of the induced G-action, denoted by $\tilde{C}^G$. Here, taking fixed points is taken in some higher categorical sense. See section 2.4 of [13] for more details.

Notice that the setup here is the same as that given in definition 2.4 of a G-tangential structure on a manifold. Indeed, it turns out that taking the G fixed points of $\mathrm{Bord}_n^{\mathrm{fr}}$ gives the bordism category of manifolds with G-tangential structure. This gives rise to the desired generalization of the cobordism hypothesis.

Theorem 5.12 (Cobordism Hypothesis for G-Tangential Structures)**.** *Given a symmetric monoidal n-category with duals C and a group homomorphism $G \to \mathrm{O}(n)$, then there is an equivalence of categories*

$$\mathrm{Fun}(\mathrm{Bord}_n^G, C) \xrightarrow{\simeq} \tilde{C}^G.$$

The functor sends a topological field theory $\mathcal{F}$ to its value on a point.

Remark 5.4. The theory of G-tangential structures does not port over to the category Bord_n^G since it was developed with respect to a specific dimension. This can be fixed by considering m-manifolds M with a G-tangential structure on $M \times I^{n-m}$ and the data of these extra intervals are remembered in the gluing. Another way would be to modify the definition of G-tangential structure to that which is found in [13].

The next section will work with the example of finite gauge theory in order to demonstrate some of these ideas.

6. Fully Extended Finite Gauge Theory

In this section, finite gauge theory from section 3 will be enhanced to a fully extended theory. In the case of dimension 2, the idea is that one would want to attach some categorical data to the point. The most natural and what is effectively the correct idea is to assign the category of finite dimensional G-representations Rep_G to the point. However, working with some 2-category of $\mathbb{C}$-linear categories is tricky. Instead, there is a simpler category to work with, where all of the objects are algebraic objects as opposed to categories. The way to conceptually bridge the gap between these two perspectives is to note that given enough structure on the category of representations, it is possible to recover the underlying algebra. Nice enough morphisms between categories of modules can be given in formulas by tensoring with a bimodule, and then natural transformations are given maps of bimodules. This leads to a definition of a 2-category that is much easier to work with. This category and its fully dualizable objects will be characterized in 6.1. Topological field theories from these fully dualizable objects including fully extended finite gauge theory are then discussed in 6.2.

6.1. **The Morita 2-Category.**

In this subsection, the 2-category mentioned above will be described.

Definition 6.1. The *morita 2-category of algebras $\mathcal{A}lg_2$* has as objects $\mathbb{C}$-algebras. A 1-morphism from an algebra A to an algebra B is an (A, B)-bimodule M. The composition of an (A, B)-bimodule M and a (B, C)-bimodule N is the (A, C)-bimodule $M \otimes_B N$. The 2 morphisms between 1-morphisms are bimodule morphisms. The symmetric monoidal structure on this category is the normal tensor product of algebras.

Note that this satisfies the requirement that $\Omega\mathcal{A}lg_2 = \mathrm{Vect}_{\mathbb{C}}$. This can be seen as the tensor unit of $\mathbb{C}$-algebras is $\mathbb{C}$, and $\mathrm{Hom}(\mathbb{C},\mathbb{C})$ is the category of $(\mathbb{C},\mathbb{C})$-bimodules, i.e. $\mathbb{C}$ vector spaces. The morphisms are then also morphisms of $\mathbb{C}$-vector spaces.

It is now necessary to understand what fully dualizable objects in $\mathcal{A}lg_2$ are. The condition that an object be dualizable in the normal sense of symmetric monoidal categories turns out to be trivial.

Theorem 6.2. *Every object A in $\mathcal{A}lg_2$ is dualizable, with dual the opposite algebra A^{op}. The dualizing morphisms are A viewed as a $(\mathbb{C}, A\otimes A^{\mathrm{op}})$-bimodule and as a $(A\otimes A^{\mathrm{op}},\mathbb{C})$-bimodule.*

Proof. It suffices to verify that the following diagram commutes

$$A \xrightarrow{A\otimes A} A\otimes A^{\mathrm{op}}\otimes A \xrightarrow{A\otimes A} A, \qquad A \xrightarrow{\;A\;} A$$

as $(A^{\mathrm{op}})^{\mathrm{op}} = A$. This is then equivalent to showing that the tensor product

$$T = (A\otimes A)\otimes_{A\otimes A^{\mathrm{op}}\otimes A}(A\otimes A) \simeq A$$

as an (A, A)-bimodule. Here the first copy of A is acted on by $A\otimes A^{\mathrm{op}}$ on the right. The second copy of A is acted on the right by the third copy of A in $A\otimes A^{\mathrm{op}}\otimes A$. The third copy of A is acted on the left by the first A in $A\otimes A^{\mathrm{op}}\otimes A$, and the last copy of A is acted on the left by $A^{\mathrm{op}}\otimes A$. Note that there is a map of (A, A)-bimodules $A \to T$ given by $a \mapsto a\otimes 1\otimes 1\otimes 1$. The claim is that this map is an isomorphism. To see this note that any element $a\otimes b\otimes c\otimes d$ of T can be expressed as $\alpha\otimes 1\otimes 1\otimes 1$. This comes from the following chain of equalities.

$$a\otimes b\otimes c\otimes d = a\otimes 1\otimes c\otimes bd = ca\otimes 1\otimes 1\otimes 1\otimes bd = bdca\otimes 1\otimes 1\otimes 1$$

This gives the surjectivity. The injectivity then roughly follows from the fact that the actions above are roughly the unique possible way to get an element of the form $\alpha\otimes 1\otimes 1\otimes 1$. No matter what is done α will always be $bdca$. □

It still remains to see when the dualizing morphisms form adjoints. This is equivalent to seeing when A is dualizable as an object of $\mathrm{Vect}_{\mathbb{C}}$ and as an object of $\mathrm{Mod}_{A\otimes A^{\mathrm{op}}}$ (the category of bi-modules). The first is equivalent to A being finite dimensional as a $\mathbb{C}$ vector space, as has already been seen. Thus, it only remains to see when A is dualizable as an object of $\mathrm{Mod}_{A\otimes A^{\mathrm{op}}}$.

Theorem 6.3. *A finite-dimensional algebra A is dualizable in the symmetric monoidal category $\mathrm{Mod}_{A\otimes A^{\mathrm{op}}}$ if and only if A is semisimple.*

Proof. By theorem 7.21, it follows that it suffices to show that A is a finite dimensional projective $A\otimes A^{\mathrm{op}}$ module if and only if it is semisimple. By assumption, A is finite dimensional as a $\mathbb{C}$-algebra, and so the only condition to check is that it is a projective $A\otimes A^{\mathrm{op}}$ module if and only if A is semisimple.

Note that there is a natural surjective map of $A\otimes A^{op}$ modules

$$A\otimes A^{op} \xrightarrow{m} A$$

given by $a\otimes b \mapsto ab$. Showing that A is projective as an $A\otimes A^{op}$ module is then equivalent to showing that this map splits. Such a splitting is a map of $A\otimes A^{\mathrm{op}}$ modules

$$A \xrightarrow{\mu} A\otimes A^{op}.$$

This map μ is determined by its value on $1 \in A$, i.e. by an element of $A \otimes A^{op}$

$$\mu(1) = \sum_i a_i \otimes b_i \tag{6.4}$$

This element has to satisfy two conditions

$$\sum_i a_i b_i = 1 \tag{6.5}$$

$$\sum_i \alpha a_i \otimes b_i = \sum_i a_i \otimes b_i \alpha, \tag{6.6}$$

where α is any element of A. The first of these equations 6.5 is equivalent to the map μ being a splitting of m, i.e. $m \circ \mu = \mathrm{id}_A$. The second equation 6.6 is equivalent to the map μ being a map of $A \otimes A^{op}$ algebras. The theorem then follows from the following result. □

Definition 6.7. An algebra A over $\mathbb{C}$ is *separable* if there exists an element $\sum_i a_i \otimes b_i$ of $A \otimes A^{op}$ that satifies equations 6.5 and 6.6.

Theorem 6.8. *Separable algebras over $\mathbb{C}$ are the same as semisimple algebras.*

Proof. This is proposition 3.69 of [16]. The proof that semisimple algebras over $\mathbb{C}$ are separable will be given here. This direction suffices for the purposes of this paper to construct finite gauge theory.

Note that a semisimple algebra A over $\mathbb{C}$ is a finite product of matrix algebras over $\mathbb{C}$ by the Artin-Wedderburn theorem. Note that a finite product of separable algebras A_i is separable. This follows as the element of $A \otimes A^{op}$ can be taken to be a direct sum of the element from each $A_i \otimes A_i^{op}$ tensored with the identity from the other A_j.

It then suffices to show the result for an arbitrary matrix algebra $M_n(\mathbb{C})$. In this case the desired element is $\sum_i e_{1i} \otimes e_{i1}$, where e_{ij} is the matrix with a 1 in the i^{th} row and j^{th} column and 0's everywhere else. □

Remark 6.1. This theorem holds for algebras over any perfect field k. The direction not given above holds even if k isn't perfect. This says that separable is a stronger condition than semisimple in general. The proof fails for non-perfect k since non-separable field extensions are simple algebras over k that are not separable.

Putting all of the above together, then gives the desired characterization of fully dualizable objects of $\mathcal{A}lg_2$.

Corollary 6.9. *The fully dualizable objects of $\mathcal{A}lg_2$ are finite dimensional semisimple algebras A.*

6.2. **Topological Field Theories from the Morita 2-Category.** It follows from theorem 5.11 that given any such finite dimensional semisimple algebra A, one can construct a 2D fully extended topological field theory on framed manifolds assigning A to a point. Given such a theory, it is now natural to ask what the theory assigns to a the circle S^1. This data recovers the non-extended 2D theory associated to the fully extended one.

Theorem 6.10. *Take a fully extended 2D topological field theory on framed manifolds $\mathcal{F}$ that assigns a finite dimensional semisimple algebra A to the positively framed/oriented point. Then it assigns $A/[A, A]$ to the circle S^1.*

Proof. Note that S^1 can be decomposed into two half arcs. These half arcs then map to the coevaluation and evaluation maps associated to the duality data of A. Thus, the circle is assigned to the composition of maps $\mathbb{C} \to A \otimes A^{\mathrm{op}} \to \mathbb{C}$. The first morphism is A viewed as a $(\mathbb{C}, A \otimes A^{\mathrm{op}})$-bimodule, and the second by A viewed as a $(A \otimes A^{\mathrm{op}})$-bimodule. The composition is then $\mathbb{C}$-bimodule $A \otimes_{A \otimes A^{\mathrm{op}}} A$. This tensor product can be seen to be $A/[A, A]$. There is a map $A \to A \otimes_{A \otimes A^{\mathrm{op}}} A$ given by $a \mapsto a \otimes 1$. This map can be verified to be surjective with kernel $[A, A]$. □

Remark 6.2. Note that even though this is only a priori a vector space, it also carries the structure of a commutative algebra. This follows algebraically as $[A, A]$ is an ideal of A and then verifying that this quotient has a commutative multiplication. It can also be seen by noting that the object assigned to the circle carries an E_2-algebra structure from the pairs of pants and switch map, all of which are 2-framed. An E_2-algebra structure in the category of vector spaces is a commutative algebra by the classical Eckmann-Hilton argument.

Remark 6.3. Note that this construction does not require A to be finite dimensional or semisimple. In fact, it is possible to define what is effectively a once-categorified 1D theory from a 1-dualizable object in the terminology of [10]. This theory is defined in the same way, but it is no longer possible to construct the theory on 2-manifolds. Such a theory provides no numerical invariants. However, they are easier to construct due to the weaker dualizability constraints on the object assigned to a point.

Recall that finite gauge theory was constructed on oriented and not framed manifolds, and so we would like to examine fully extended theories on oriented manifolds. In order to construct an oriented theory from such an algebra A, the algebra must be a homotopy fixed point of the action of SO(2) on $\mathcal{A}lg_2$. In order to examine this action it is useful to introduce a higher categorical generalization of the fundamental groupoid. The case of $n = 2$ in the following definition will be of primary interest in what follows.

Definition 6.11. Let X be a topological space. The *fundamental n-groupoid* of X $\Pi_{\leq n}$ has objects the points of X and its 1-morphisms by paths between points. It's 2-morphisms are then homotopies between paths, and its 3 morphisms by homotopies of homotopies, and so on. It's n-morphisms are homotopy classes of homotopies between $n-1$ morphisms. In particular, $\Pi_{\leq 2}$ has objects the points of X, 1-morphisms paths in X, and 2-morphisms homotopy classes of homotopies of paths.

Roughly, an action of SO(2) can then be thought of in this 2-categorical world as a functor from $\Pi_{\leq 2}\mathrm{SO}(2)$ to $\mathcal{A}lg_2$ for every dualizable object A that sends a chosen base point in $\Pi_{\leq 2}\mathrm{SO}(2)$ to A. This then sends the generator of $\pi_1(\mathrm{SO}(2)) \simeq \mathbb{Z}$ to an automorphism of A. It turns out that this automorphism is the vector space dual A^* of A regarded as a (A, A)-bimodule. In order for A to be a fixed point, this automorphism must be trivialized, i.e., there must be an identification of A with A^* as (A, A)-bimodules. This is a nondegenerate bilinear form $b : A \otimes A \to \mathbb{C}$ that satisfies certain properties with respect to scalar multiplication by A.

$$b(a\alpha, \beta) = b(\alpha, \beta a) \tag{6.12}$$

$$b(\alpha a, \beta) = b(\alpha, a\beta) \tag{6.13}$$

Notice that since any element of A can be written as $a \cdot 1$, that this means that this form is determined by a non-degenerate trace map $\mathrm{tr} : A \to \mathbb{C}$, which factors through $A/[A, A]$ (see

[9] for more details). The data of an algebra A with this trace map is equivalent to the earlier definition of a Frobenius algebra. The trace map will then descend to $A/[A, A]$, making it into a commutative Frobenius algebra. An oriented 2D non-extended topological field theory is then recovered from this data being attached to S^1 as expected. The identification of A with A^*, allows for $A/[A, A]$ to be identified with the center of the algebra $Z(A)$.

Remark 6.4. In general this SO(2) action can be understood in terms of the serre automorphism. This automorphism is defined and discussed briefly in [13]. The details are discussed more in [16].

In particular, take $\mathbb{C}[G]$ with trace map $\sum_{g \in G} a_g g \mapsto \frac{1}{|G|} a_e$. This gives a fully extended topological field theory on oriented manifolds by the above discussion. It assigns $Z(\mathbb{C}[G])$ to the circle S^1. As this was the commutative Frobenius algebra assigned to circle in 2D finite gauge theory, it follows that this fully extended theory contains the information of the non-extended finite gauge theory. Thus, this fully extended theory is the fully extended 2D finite gauge theory. There is also a way to extend the construction we used to construct finite gauge theory in section 4, see [9] for more details on this.

7. Appendix: Monoidal Categories

This section contains a brief introduction to the theory of symmetric monoidal categories and dualizability conditions. A good source for information on these concepts is [4]. This book contains a lot of information towards a theory of categorified versions of algebras and representations, which are closely related to higher dimensional topological field theories. A briefer reference on some of these ideas is given in [3]. There is also some information in the first section of [13].

7.1. Definitions.

Definition 7.1. A *monoidal category* is a quintuple of data
$(\mathcal{M}, 1_{\mathcal{M}}, \otimes, \alpha, \iota_L, \iota_R)$. Here $\mathcal{M}$ is category, $1_{\mathcal{M}}$ is an object of $\mathcal{M}$, $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a functor, and $\alpha : \otimes \circ (\mathrm{id} \times \otimes) \to \otimes \circ (\otimes \times \mathrm{id})$, $\iota_L : 1_{\mathcal{M}} \otimes_{\to} \mathrm{id}_{\mathcal{M}}$, and $\iota_R :_{\otimes} 1_{\mathcal{M}} \to \mathrm{id}_{\mathcal{M}}$ are natural isomorphisms of functors. This data is required to satisfy the following coherence relations.

(7.2)

$$
\begin{array}{ccccc}
 & & (A \otimes B) \otimes (C \otimes D) & & \\
 & \nearrow^{\alpha^{-1}_{A,B,C\otimes D}} & & \searrow^{\alpha^{-1}_{A\otimes B,C,D}} & \\
A \otimes (B \otimes (C \otimes D)) & & & & ((A \otimes B) \otimes C) \otimes D \\
\uparrow {\scriptstyle \mathrm{id}_A \otimes \alpha^{-1}_{B,C,D}} & & & & \downarrow {\scriptstyle \alpha_{A,B,C} \otimes \mathrm{id}_D} \\
A \otimes ((B \otimes C) \otimes D) & & \xleftarrow{\quad \alpha^{-1}_{A,B\otimes C,D} \quad} & & (A \otimes (B \otimes C)) \otimes D
\end{array}
$$

$$
\begin{array}{ccc}
 & (A \otimes 1_{\mathcal{M}}) \otimes B & \\
\swarrow^{(\iota_R)_A \otimes \mathrm{id}_B} & & \searrow^{\alpha_{A,1_{\mathcal{M}},B}} \\
A \otimes B & \xleftarrow{\quad \mathrm{id}_A \otimes (\iota_L)_B \quad} & A \otimes (1_{\mathcal{M}} \otimes B)
\end{array}
\tag{7.3}
$$

The first of these is conditions is referred to as the pentagon identity and the second the triangle identity in reference to their shapes.

Remark 7.1. In principle, the idea of the coherence relations is that parentheses should not matter in a given expression. No matter how the parentheses are arranged, the resulting natural isomorphisms should be the same. The coherence conditions express this condition at the lowest possible level. It is then a theorem of MacLane that these conditions are sufficient to imply this with any number of terms.

Definition 7.4. A *symmetric monoidal category* is a monoidal category $\mathcal{M}$ along with a natural isomorphism $\sigma : \otimes \circ \tau \to \otimes$, where τ is the swap functor. This functor has to satisfy two conditions:

(1) $\sigma^2 = \mathrm{id}_{\mathcal{M}}$, i.e. $\sigma_{A,B}\sigma B, A = \mathrm{id}_{A\otimes B}$

(2)

$$
\begin{array}{ccccc}
 & & (A\otimes B)\otimes C & & \\
 & \swarrow^{\alpha_{A,B,C}} & & \searrow^{\sigma_{A,B}\otimes \mathrm{id}_C} & \\
A\otimes(B\otimes C) & & & & (B\otimes A)\otimes C \\
\downarrow{\scriptstyle \sigma_{A,B\otimes C}} & & & & \downarrow{\scriptstyle \alpha_{B,A,C}} \\
(B\otimes C)\otimes A & & & & B\otimes(A\otimes C) \\
 & \searrow_{\alpha_{B,C,A}} & & \swarrow_{\mathrm{id}_B\otimes\sigma_{C,A}} & \\
 & & B\otimes(C\otimes A) & &
\end{array}
$$

The last condition is called the hexagon identity.

Example 7.5. (1) Any category with finite products is a symmetric monoidal category whose tensor product is the categorical product. Similarly any category with finite coproducts is a symmetric monoidal category whose tensor product is the coproduct.

(2) The category Vect_k of vector spaces over a field k is a symmetric monoidal category with tensor being the usual tensor product of vector spaces. The swap map is the natural isomorphism $V \otimes W \simeq W \otimes V$.

(3) Assume that the field k does not have characteristic 2. The category of super vector spaces, i.e. the category of $\mathbb{Z}/2$ graded vector spaces, sVect_k is a symmetric monoidal category. It has the usual tensor product of vector spaces. The swap map is now modified so that given homogeneous elements $v \in V$ and $w \in W$, then $v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v$.

(4) Let R be a ring. The category of bi-modules over R, Mod_R, is a symmetric monoidal tensor category under the usual tensor product and isomorphism $M\otimes N \simeq N\otimes M$. This can be generalized to the case of quasicoherent sheaves over a scheme X.

(5) Let G be a finite group. Then the category of G representations Rep_G under the usual tensor product of representations and isomorphism $V \otimes W \simeq W \otimes V$. Note that by considering the group ring $\mathbb{C}[G]$, this example is a special case of the last one.

Definition 7.6. Let $(\mathcal{M}, 1_{\mathcal{M}}, \otimes_{\mathcal{M}}, \alpha_{\mathcal{M}}, \iota_{L,\mathcal{M}}, \iota_{R,\mathcal{M}})$ and $(\mathcal{N}, 1_{\mathcal{N}}, \otimes_{\mathcal{N}}, \alpha_{\mathcal{N}}, \iota_{L,\mathcal{N}}, \iota_{R,\mathcal{N}})$ be two monoidal categories. The data of a monoidal functor is a functor $F : \mathcal{M} \to \mathcal{N}$ and a natural transformation $J_{X,Y} : F(X) \otimes_{\mathcal{N}} F(Y) \to F(X \otimes_{\mathcal{M}} Y)$. This data is then required to satisfy $F(1_{\mathcal{M}}) \simeq 1_{\mathcal{N}}$ and the following commutative diagram.

(7.7)

$$\begin{array}{ccc}
 & (F(A) \otimes_{\mathcal{N}} F(B)) \otimes_{\mathcal{N}} F(C) & \\
\swarrow^{J_{A,B} \otimes \mathrm{id}_{F(C)}} & & \searrow^{(\alpha_{\mathcal{N}})_{F(A),F(B),F(C)}} \\
F(A \otimes_{\mathcal{M}} B) \otimes_{\mathcal{N}} F(C) & & F(A) \otimes_{\mathcal{N}} (F(B) \otimes_{\mathcal{N}} F(C)) \\
\downarrow J_{A \otimes_{\mathcal{M}} B, C} & & \downarrow \mathrm{id}_{F(A)} \otimes_{\mathcal{N}} J_{B,C} \\
F((A \otimes_{\mathcal{M}} B) \otimes_{\mathcal{M}} C) & & F(A) \otimes_{\mathcal{N}} F(B \otimes_{\mathcal{M}} C) \\
\searrow_{F(\alpha_{\mathcal{M}})_{A,B,C}} & & \swarrow_{J_{A, B \otimes_{\mathcal{M}} C}} \\
 & F(A \otimes_{\mathcal{N}} (B \otimes_{\mathcal{M}} C)) &
\end{array}$$

Definition 7.8. Let $\mathcal{M}$ be a symmetric monoidal category with symmetric structure $\sigma_{\mathcal{M}}$ and $\mathcal{N}$ be a symmetric monoidal category with symmetric structure $\sigma_{\mathcal{N}}$. Then a *symmetric monoidal functor* is a monoidal functor (F, J) subject to the further constraint that it satisfy the following commutative diagram.

$$\begin{array}{ccc}
F(A) \otimes_{\mathcal{N}} F(B) & \xrightarrow{J_{A,B}} & F(A \otimes_{\mathcal{M}} B) \\
\downarrow (\sigma_{\mathcal{N}})_{F(A),F(B)} & & \downarrow (\sigma_{\mathcal{M}})_{A,B} \\
F(B) \otimes_{\mathcal{N}} F(A) & \xrightarrow[J_{B,A}]{} & F(B \otimes_{\mathcal{M}} A)
\end{array} \tag{7.9}$$

Example 7.10.
(1) The forgetful functor $\mathrm{Rep}_G \to \mathrm{Vect}_k$ can be given the structure of a symmetric monoidal functor. The extra data J is easily given since the tensor product in Rep_G is the tensor product of the underlying vector spaces.
(2) Given a (R, S)-bimodule M, then the functor $- \otimes_R M$ gives a functor from Mod_R to Mod_S that can be given the structure of a symmetric monoidal tensor functor.

7.2. **Duality Data.** Take a symmetric monoidal category C. Then there is a notion of dualizability which is in a way a finiteness condition, which plays an essential role throughout the paper.

Definition 7.11. Take an object X in C. *Duality data* for X is a triple (X^*, e, c) where X^* is an object of C, while $e : X^* \otimes X \to 1_C$ and $c : 1_C \to X \otimes X^*$ are morphisms. This data must satisfy the following conditions

$$X \xrightarrow{c \otimes \mathrm{id}} X \otimes X^* \otimes X \xrightarrow{\mathrm{id} \otimes e} X, \qquad (\mathrm{id} \otimes e) \circ (c \otimes \mathrm{id}) = \mathrm{id} \tag{7.12}$$

$$X^* \xrightarrow{\mathrm{id} \otimes c} X^* \otimes X \otimes X^* \xrightarrow{e \otimes \mathrm{id}} X^*, \qquad (e \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c) = \mathrm{id} \tag{7.13}$$

Theorem 7.14. *The duality data of an object X is unique up to unique isomorphism. In other words, given different duality data $(X^\vee, e', c')$ then there is a unique isomorphism $X^* \to X^\vee$ that commutes with the evaluation and coevaluation maps.*

Proof. See proposition 2.10.5 of [4]. □

Remark 7.2. If there exists duality data associated to X, then the object X^* is referred to as the dual of X, and X is called a dualizable object.

Definition 7.15. Let X be a dualizable object of a symmetric monoidal category $\mathcal{M}$ with duality data (X^*, e, c). The *dimension* of X is defined to be the morphism

$$1_{\mathcal{M}} \xrightarrow{c} X \otimes X^* \xrightarrow{\sigma} X^* \otimes X \xrightarrow{e} 1_{\mathcal{M}}$$

When $\mathrm{Hom}(1_{\mathcal{M}}, 1_{\mathcal{M}}) \simeq \mathbb{C}$ the dimension is a complex number.

Example 7.16.
- Let C be $\mathrm{Vect}_{\mathbb{C}}$ the category of vector spaces and V a vector space. Then duality data for V is (V^*, e, c). Here $V^* := \mathrm{Hom}(V, \mathbb{C})$ is the normal dual vector space and $e : V \otimes V^* \to \mathbb{C}$ is the normal evaluation giving the perfect pairing. In order to express the coevaluation map, take a basis e_i of V and the dual basis e^i of V^*. The coevaluation map is defined by $1 \mapsto \sum_i e_i \otimes e^i$. This duality data only exists if the vector space is finite dimensional, so the dualizable objects are finite dimensional vector spaces. In this case, the dimension of V from definition 7.15 is the normal dimension of the vector space.
- Let Set be category of sets with tensor product given by direct product. The unit of this tensor product is the singleton set $*$. Note that the coevaluation map is then given by a point of $X \times X^*$, and the evaluation map is the unique map $X \times X^* \to *$. The first map in condition (7.12) is then $X \to X \otimes X^* \otimes X$ which has image a point times the last factor of X. The evaluation map collapses the last factor to a point though in the second map of condition (7.12). Therefore the only way for this composition to be the identity is for X to be a singleton set. Thus, there are no interesting dualizable objects in this symmetric monoidal category.
- Let Set be the category of sets with tensor product given by direct sum. The unit of this tensor product is the empty set. The evaluation map cannot exist unless X and X^* are the empty set as only the empty set can map to the empty set. Thus the empty set is the only dualizable object of this symmetric monoidal category.

The following theorems give useful properties of dualizable objects, building towards theorem 7.21, which will be important to the results of section 6.

Theorem 7.17. *Let A be a dualizable object of a symmetric monoidal category $\mathcal{M}$ with dual A^*, $\mathcal{N}$ another symmetric monoidal category, and a symmetric monoidal functor $F : \mathcal{M} \to \mathcal{N}$. Then, $F(A)$ is a dualizable object of $\mathcal{N}$ with dual $F(A^*)$.*

Proof. See exercise 2.10.6 of [4]. □

Theorem 7.18. *If X is dualizable with dual X^*, then $\_ \otimes X$ is both a left and right adjoint to $\_ \otimes X^*$.*

Proof. This follows from proposition 2.10.8 in [4] since in a symmetric monoidal category, right and left duals always coincide. □

Theorem 7.19. *Let C be a symmetric monoidal category with finite coproducts. Then coproducts of dualizable objects are dualizable.*

Proof. Note that $\otimes$ commutes with coproducts since it is a left adjoint by theorem 7.18. Therefore, it can easily be seen that the dual of $(X \otimes Y)$ is $Y^* \otimes X^*$ as the conditions break down into a coproduct of the conditions for X and Y respectively. □

Theorem 7.20. *If a symmetric monoidal category C has an internal hom functor $[-,-]$, i.e. a right adjoint to the tensor functor, then if an object X is dualizable, its dual is $[X, 1_C]$*

Proof. For all objects A of C, it follows from 7.18 that $\mathrm{Hom}(A \otimes X, 1_C) \simeq \mathrm{Hom}(A, X^*)$, and by the adjunction between internal hom and tensor that $\mathrm{Hom}(A \otimes X, 1_C) \simeq \mathrm{Hom}(A, [X, 1])$. Therefore, it follows that $X^* \simeq [X, 1]$ by the Yoneda lemma. □

As an application of the above results it is possible to classify dualizable objects in the symmetric monoidal category of modules over a ring.

Theorem 7.21. *Let R be a ring, and Mod_R be the category of bi-modules over R. It is a symmetric monoidal category with the tensor operation the usual tensor product of bi-modules. The dualizable objects of Mod_R are the finite dimensional projective modules.*

Proof. First note that R is a dualizable object of Mod_R as it is the tensor unit. Therefore, it follows that R^n is dualizable by theorem 7.19. It's duality data is the same as that of a vector space in example 7.16 with dual $\mathrm{Hom}(R^n, R)$ since Mod_R has an internal hom. Now note that a direct summand M of R^n is also dualizable. This follows as if $R^n = M \oplus N$, then it follows that $\mathrm{Hom}(R^n, R) \simeq \mathrm{Hom}(M, R) \oplus \mathrm{Hom}(N, R)$. Thus, it follows that

$$R^n \otimes \mathrm{Hom}(R^n, R) \simeq M \otimes \mathrm{Hom}(M, R) \oplus N \otimes \mathrm{Hom}(N, R) \oplus M \otimes \mathrm{Hom}(N, R) \oplus N \otimes \mathrm{Hom}(M, R).$$

The coevaluation map $R \to M \otimes \mathrm{Hom}(M, R)$ is then the composition of the coevaluation map for R^n followed by the projection of $R^n \otimes \mathrm{Hom}(R^n, R)$ to the $M \otimes \mathrm{Hom}(M, R)$ factor. Evaluation still has the same form. The conditions for this to be duality data follow from two observations. The evaluation map vanishes on the components $M \otimes \mathrm{Hom}(N, R)$ and $N \otimes \mathrm{Hom}(M, R)$. Additionally, the $N \otimes \mathrm{Hom}(N, R)$ factor does not contribute as the evaluation map is only applied to elements of M or its dual $\mathrm{Hom}(M, R)$.

To see the other direction, note that Mod_R has an internal hom given by the natural structure of an R-module on $\mathrm{Hom}(M, N)$. Therefore, it follows that if M is dualizable $M^* = \mathrm{Hom}(M, R)$ by theorem 7.20. The evaluation map is the image of the identity under the isomorphism $\mathrm{Hom}(M^*, M^*) \simeq \mathrm{Hom}(M^* \otimes M, R)$. This can be seen to be the classical evaluation map that sends (f, m) to $f(m)$. Now note that the coevaluation map $c : R \to M \otimes M^*$ if it exists is then determined by an element of $M \otimes M^*$. This element is a finite sum $\sum_{i=1}^{n} m_i \otimes f_i$. The condition (7.12) then gives that for all $m \in M$, $\sum_i f_i(m) m_i = m$. Note that this in particular implies that every $m \in M$ can be written as a sum of the m_i, so the m_i form a generating set of M. This then gives rise to a morphism $R^n \to M$. This morphism is then split by the map $M \to R^n$ sending m to $(f_1(m), ..., f_n(m))$. It follows that M is a direct summand of R^n and therefore projective and finite dimensional. □

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