CONSTRUCTION OF SMOOTH MANIFOLDS AND TANGENT SPACES

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Abstract. Manifolds are the tool and language with which much of differential topology and geometry is expressed. By relaxing the condition on a space from being globally Euclidean to locally Euclidean, we are able to retain the tools of real analysis and apply them to a much more general range of topological spaces. This paper provides an overview of the construction of smooth manifolds.

Contents

1. Topology, Continuity, and Homeomorphisms

We begin by defining the most important notions in general topology.

Definition 1.1. Let X be any set. A topology on X is a set T of subsets of X such that the following hold:

1. The empty set and X are both members of T
2. Any union of (possibly infinitely many) elements in T is itself an element in T.
3. Any intersection of finitely many elements in T is itself an element in T

Definition 1.2. A topological space is a pair (X,T), where X is a set and T is a topology on X. Subsets of X which are elements of T are referred to as "open" in X.

Definition 1.3. Let (X, T) be a topological space, and let $x \in X$ be any point in X. A neighborhood U of x is an open subset $U \subset X$ such that $x \in U$.

Most often, when referring to topological spaces, the topology is understood from context or unimportant. In this case, it is common to refer to the set alone. We convey the fact that a set X has a topology by saying X is a topological space.

Though it is not intuitive from the above definition, open sets are used to define a notion of closeness among points in a set, and topologies allow us to extend this notion to sets where a distance between points is not defined.

A fundamental example of this is the way open sets can be used to extend the notion of continuity to arbitrary topological spaces. Intuitively, we can think of a continuous function as a function in which points that are close together in the domain get mapped to points that are close together in the codomain. Drawing a graph on a sheet of paper without lifting your pencil can convince you that this is a sensical way to describe continuous functions. For a function between metric spaces (spaces for which distance between points is a well-defined notion), we can formalize the definition of a continuous function by the familiar delta-epsilon definition: a function $f : X \to Y$ between metric spaces is continuous at a point $a \in X$ if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X : d(x, a) < \delta$, we have $d(f(x), f(a)) < \epsilon$, where $d : X \times X \to \mathbb{R}$ returns the distance between two points in X and $d : Y \times Y \to \mathbb{R}$ returns the distance between two points in Y.

Using open sets as our notion of closeness allows us to express our idea that "close points get mapped to close points" using topologies instead of metrics, thus allowing for a more general definition of continuity that can apply to arbitrary topological spaces, particularly spaces where no metric is defined.

Definition 1.4. Let X, Y be topological spaces. A function $f : X \to Y$ is continuous if for every open set $U \subset Y$, its preimage $f^{-1}(U)$ is open in X.

A reader who is familiar with the topology of metric spaces can verify that the above definition reduces to the familiar delta-epsilon definition when the spaces are metric spaces and their topologies are the ones induced by their respective metrics.

A common theme throughout algebra and topology is our attempt to find ways to identify when certain sets or spaces share a property in common that makes them "similar enough" to consider to be essentially the same set. Of course, our standard for two sets to be "similar enough" to be considered the same depends on what we are using the sets for. Therefore, many different notions of "similar enough" have been defined, such as the isomorphism which may be familiar from algebra. The common way to classify sets as "similar enough" is to come up with a type of function between the two sets that preserves the desired properties.

The most fundamental example in topology is when two spaces are "topologically equivalent", roughly meaning that they share the same topology. Since the topology on a set is defined by its open subsets, and continuous functions preserve open sets (albeit only in their inverse direction), it is natural to exploit continuity in forming our definition.

Definition 1.5. Let X,Y be topological spaces. A function $f : X \to Y$ is a homeomorphism if it is continuous and bijective and its inverse $f^{-1} : Y \to X$ is continuous.

Definition 1.6. Let X,Y be topological spaces. X is homeomorphic to Y if there exists a homeomorphism $f : X \to Y$.

Note that the definition of a homeomorphism implies that X is homeomorphic to Y if and only if Y is homeomorphic to X. Thus, homeomorphism of spaces is a symmetric relation, and it therefore makes sense to say X and Y are homeomorphic without specifying the direction of the homeomorphism.

To conclude our discussion on the basic notions of topology, we will describe the intuition behind one last powerful use of open sets, which we will demonstrate rigorously when we introduce manifolds. Because open sets can be used to describe "closeness", they can be used to describe the local properties of a topological space. We can say that a topological space X exhibits a property locally if for all points $p \in X$, p has a neighborhood U such that the property holds in U. Essentially, we are saying that each point has a small space around it for which the property holds. This powerful concept allows us to generalize certain topological properties where locality is enough to spaces that may not globally exhibit the property in question.

2. Further Topics in Topology

Before defining manifolds, there are two more notions in topology we must define. Since the space $\mathbb{R}^n$ will provide an important example with which to understand these notions, we will first define a topology for $\mathbb{R}^n$, which first requires defining open balls. Note that there exists a more general definition of an open ball that can apply to arbitrary metric spaces. For our purposes, the following definitions will suffice.

Definition 2.1. Let $x \in \mathbb{R}^n$ such that $x = (x_1, ..., x_n)$ for $x_1, ..., x_n \in \mathbb{R}$. The norm of x, denoted $|x|$, is the real number $|x| = \sqrt{(x_1)^2 + ... + (x_n)^2}$.

It is helpful to visualize a point's norm as the length of an arrow connecting the origin to the point, computed using the n-dimensional generalization of the Pythagreon theorem. In fact, this visualization is exactly the motivation for the above definition.

Definition 2.2. Let $x \in \mathbb{R}^n$, and let $\epsilon > 0$. The open ball of radius ϵ centered at x is the set $B_\epsilon(x) = \{y \in \mathbb{R}^n : |y - x| < \epsilon\}$

We consider $|y - x|$ to be the distance between x and y. It is helpful to draw out open balls in $\mathbb{R}$ and $\mathbb{R}^2$ in order to visualize them, and one can then easily understand why they are called open balls.

Definition 2.3. The Euclidean topology of $\mathbb{R}^n$ is the topology on $\mathbb{R}^n$ defined as follows: $U \subset \mathbb{R}^n$ is open if either it is equal to a union of open balls in $\mathbb{R}^n$, or it is the empty set.

Whenever we refer to $\mathbb{R}^n$, we will assume it is given the Euclidean topology unless otherwise specificed. It is a worthwhile exercise to confirm that the above definition is in fact a topology on $\mathbb{R}^n$, meaning it is consistent with Definition 1.1.

Definition 2.4. A topological space X is a Hausdorff space if for all distinct points $p, q \in X$, there exist disjoint open sets $U, V \subset X$ such that $p \in U$ and $q \in V$. If X is a Hausdorff space, we also say X is Hausdorff.

The Hausdorff property allows us to separate the spaces around distinct points, and generally provides us with a way to seek out nicely behaving spaces. For example, Hausdorff spaces have the useful property that the limits of all sequences are unique, which is not true for topological spaces in general.

Theorem 2.5. $\mathbb{R}^n$ *is Hausdorff.*

Proof. Let x, y be distinct points in $\mathbb{R}^n$, and write $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ where $x_1, ...x_n, y_1, ..., y_n \in \mathbb{R}$. There exists some i with $1 \leq i \leq n$ such that $x_i \neq y_i$. (Otherwise, we would have $x = y$.) Let $\epsilon = \frac{|x_i - y_i|}{2}$. Then we have the disjoint open sets $B_\epsilon(x)$ and $B_\epsilon(y)$. □

The remaining topic in this section concerns the basis of a topology, which can be considered to be a smaller set of "building blocks" which can generate a topology.

Definition 2.6. Let X be a set. A basis $\mathcal{B}$ of X is a collection of subsets of X such that the following properties hold:

(1) For all $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in B$
(2) For all pairs $B_1, B_2 \in \mathcal{B}$, for all $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Note that if the pair B_1, B_2 is disjoint, this condition is fulfilled.

Definition 2.7. Let X be a set and let $\mathcal{B}$ be a basis of X. The topology on X induced by $\mathcal{B}$ is the topology on X determined in the following way: A subset $U \subset X$ is open if for all $x \in U$, there exists some $B \in \mathcal{B}$ such that $x \in B \subset U$. Put differently, the open sets in X are precisely the unions of elements of $\mathcal{B}$.

Definition 2.8. A topological space X is second countable if there exists a basis $\mathcal{B}$ for its topology such that $\mathcal{B}$ is countable.

The above property is called "second countability" to distinguish it from a weaker property which we will not define, called "first countability". Second countability is, esssentially, a statement about the minimum size of a set of "building blocks" with which one can form the topology of a space.

Lemma 2.9. *Let $B_\epsilon(x)$ be an open ball in $\mathbb{R}^n$. Then, for all $y \in B_\epsilon(x)$, there exists a rational $\epsilon' > 0$ such that $B_{\epsilon'}(y) \subset B_\epsilon(x)$*

Proof. Write $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ where $x_1, ..., x_n, y_1, ..., y_n \in \mathbb{R}$. For $1 \leq i \leq n$, let $d_i = \epsilon - |y_i - x_i|$. Let $d = \min\{d_i | 1 \leq i \leq n\}$. Clearly, $0 < d \leq \epsilon$, with $d = \epsilon$ if and only if $y = x$. (If we had $d \leq 0$, then y would not be in $B_\epsilon(x)$). Because the rational numbers are dense in the real numbers, there exists a rational ϵ' such that $0 < \epsilon' < d$, and for any such ϵ' we have $B_{\epsilon'}(y) \subset B_\epsilon(x)$. □

Lemma 2.10. *Let $B_\epsilon(x)$ be an open ball in $\mathbb{R}^n$. Then, for all $y \in B_\epsilon(x)$, there exists a rational z and a rational $\epsilon' > 0$ such that $y \in B_{\epsilon'}(z) \subset B_\epsilon(x)$, where by a rational point in $\mathbb{R}^n$, we mean a point in $\mathbb{Q}^n$ with $\mathbb{Q}^n$ considered a subset of $\mathbb{R}^n$ in the natural way.*

Proof. Let $y \in B_\epsilon(x)$ be given. By Lemma 2.9, there exists a rational $\epsilon' > 0$ such that $B_{\epsilon'}(y) \subset B_\epsilon(x)$. Because the rational numbers are dense within the real numbers, there exists a $z \in \mathbb{Q}^n$ such that $|y - z| < \frac{\epsilon'}{2}$. Note that $\frac{\epsilon'}{2}$ is rational. We then have $y \in B_{\frac{\epsilon'}{2}}(z) \subset B_{\epsilon'}(y) \subset B_\epsilon(x)$.

□

Corollary 2.11. *Any open ball in $\mathbb{R}^n$ is the union of a set of open balls whose centers and radii are rational.*

The proof follows immediately from Lemma 2.10.

Lemma 2.12. *The set of open balls in $\mathbb{R}^n$ whose centers and radii are rational is a basis for the topology of $\mathbb{R}^n$.*

Proof. First, we must show that this set is a basis. For any $x \in \mathbb{R}^n$ and any rational $\epsilon > 0$, clearly $x \in B_\epsilon(x)$. If $x \in B_\epsilon(y) \cap B_{\epsilon'}(z)$ for rational $y, z, \epsilon, \epsilon'$, there exsists a rational $\delta > 0$ such that $B_\delta(x) \subset B_\epsilon(y) \cap B_{\epsilon'}(z)$. Next, we must show that the topology of $\mathbb{R}^n$ is that induced by this basis. This follows directly from Corollary 2.11 and the fact that open sets in $\mathbb{R}^n$ are exactly the unions of open balls. □

Lemma 2.13. *The finite union of countable sets is countable.*

For a proof, see Theorem 2.12 of "Principles of Mathematical Analysis" by Walter Rudin

Lemma 2.14. *The set of all rational numbers is countable.*

For a proof, see the corollary to Theorem 2.13 of "Principles of Mathematical Analysis" by Walter Rudin.

Lemma 2.15. $\mathbb{Q}^n \times \mathbb{Q}^+$ *is a countable set, where* $\mathbb{Q}^+ = \{x \in \mathbb{Q} | x > 0\}$

The proof follows trivially from Lemma 2.12 and Lemma 2.13.

Lemma 2.16. *There exists a bijection between the set of all open balls in* $\mathbb{R}^n$ *with rational center and rational radii and the set* $\mathbb{Q}^n \times \mathbb{Q}^+$.

Proof. To each such open ball, assign the point in $\mathbb{Q}^n$ that is its center followed by the point in $\mathbb{Q}^+$ that is its radius. □

Corollary 2.17. *The set of all open balls in* $\mathbb{R}^n$ *with rational center and rational radii is countable.*

The proof follows immediately from Lemma 2.15 and Lemma 2.16.

Theorem 2.18. $\mathbb{R}^n$ *is second-countable.*

Proof. By Lemma 2.12, the set of open balls in $\mathbb{R}^n$ whose centers and radii are rational forms a basis for the topology of $\mathbb{R}^n$. By Corollary 2.17, this set is countable. Therefore, there exists a countable basis for the topology. □

It can now be seen that the Euclidean topology of $\mathbb{R}^n$ is the topology that is induced by the set of all open balls with rational centers and rational radii. A more general treatment of the topology of metric spaces will demonstrate how a basis for a metric space can be defined by the metric. In this case, we say that the topology is "induced by the metric". This is the reason for the name "Euclidean topology", as it is the topology that is induced by the Euclidean metric.

3. Manifolds

Having defined the prerequisite notions in topology, we may proceed with defining manifolds and constructing their smooth structures.

Definition 3.1. A manifold is a topological space M such that the following properties hold:

(1) M is Hausdorff
(2) M is second countable
(3) There exists a positive integer n such that for all $p \in M$, p has a neighborhood $U \subset M$ that is homeomorphic to an open set $\hat{U} \subset \mathbb{R}^n$.

We say that M has dimension n.

Following the discussion at the end of section 1, the third condition above can be summarized as "M is locally homeomorphic to $\mathbb{R}^n$". When we wish to clarify the dimension of M, we might call M an "n-dimensional manifold." or a "manifold of dimension n".

Definition 3.2. Let M be a manifold of dimension n, and let $p \in M$. A coordinate chart on p is a pair (U, φ), where $U \subset M$ is a neighborhood of p and φ is a homeomorphism $\varphi : U \to \hat{U}$ for an open subset $\hat{U} \subset \mathbb{R}^n$. The domain of a chart (U, φ) is U.

Alternatively, we could have defined the domain of a chart (U, φ) as the domain of φ, which by definition is U.

The name "coordinate chart" suggests a useful way to visualize this object. Any point in $\mathbb{R}^n$ can be represented by an ordered n-tuple of real numbers. Thus, the function φ assigns to each point in its domain an n-tuple of real numbers to represent it, which we consider to be the coordinates of that point.

Definition 3.3. An atlas for M is a collection $\mathcal{A}$ of charts whose domains cover M.

By the definition of a manifold, every point on a manifold is contained within some chart. Thus, every manifold has an atlas. As we will soon see, simply having an atlas is often not enough to derive the properties that will allow us to apply the methods of calculus to manifolds. We will need a stronger condition, which we will find in smooth manifolds.

4. Smooth Structures

We have seen that a chart on a point is a function that assigns to the point coordinates of real numbers, and that this allows us to represent the point and points near it as their coordinate representations in some open subset of Euclidean space, which is a much "nicer" place to do math. A problem presents itself when we have multiple coordinate charts that "disagree" with each other, roughly meaning that the method with which one chart assigns coordinates to the same set of points is not compatible with the method the other chart uses. We will solve this problem by appealing to our notion of two distinct items being "similar enough", and then selecting charts that are "similar enough" to be compatible.

Because homeomorphism will not be enough for our purposes, we must borrow an important concept from analysis.

The following definition allows us to assign partial derivatives to a vector valued function by rewriting it as multiple real valued functions.

Definition 4.1. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$. Let F be a function $F : U \to V$ such that $F(y) = (x^1, x^2, ...x^m)$, where $y \in U$ and $x^1, x^2, ...x^m \in \mathbb{R}$. The kth component function of F is the function $F^k : U \to \mathbb{R}$ such that $F(y) = (F^1(y), F^2(y), ..., F^m(y))$.

Definition 4.2. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets. A function $F : U \to V$ is smooth if each of its component functions has continuous partial derivatives of all orders.

Having introduced smooth functions, we can now define our notion of "similar enough" that is stronger than homeomorphism.

Definition 4.3. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets. A function $F : U \to V$ is a diffeomorphism if F is smooth and bijective and its inverse F^{-1} is smooth. If there exists a diffeomorphism between U and V, we say that U and V are diffeomorphic.

Note that any diffeomorphism is a homeomorphism, because smooth functions are continuous. This is why we are justified in referring to diffeomorphisms as a stronger condition than homeomorphisms. Having introduced the necessary concepts from analysis, we can return to solving our problem of determining whether charts are compatible. The key is to ensure that charts whose domains overlap are the same up to diffeomorphism, but we need a way to apply the concept of diffeomorphism to manifolds.

Definition 4.4. Let (U, φ) and (V, ψ) be charts on a manifold such that $U \cap V \neq \emptyset$. The transition map from φ to ψ is the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$.

Note that a transition map is a composition of homeomorphisms and is therefore itself a homeomorphism. Also note that the domain and codomain of a transition map are both open subsets of Euclidean spaces, so usual analysis terminology, such as smoothness, is well-defined.

Definition 4.5. Let (U, φ) and (V, ψ) be charts on a manifold. (U, φ) and (V, ψ) are smoothly compatible if either $U \cap V = \emptyset$ or the transition map from φ to ψ is a diffeomorphism.

Definition 4.6. Let M be a manifold. A smooth atlas on M is an atlas $\mathcal{A}$ on M such that any two charts in $\mathcal{A}$ are smoothly compatible.

A technical issue that arises in our attempt to come up with a well-defined smooth structure for a manifold is that there may be more than one smooth atlas whose charts are compatible. We therefore consider only maximal smooth atlases, which we will presently define.

Definition 4.7. Two smooth atlases on a manifold M are compatible if their union is also a smooth atlas on M

Notice that two smooth atlases are compatible if and only if none of their charts are pairwise incompatible.

Definition 4.8. Let $\mathcal{A}$ be a smooth atlas on a manifold M. The maximal smooth atlas on M determined by $\mathcal{A}$ is the union of $\mathcal{A}$ with all smooth atlases on M that are compatible with $\mathcal{A}$.

Definition 4.9. Let M be a manifold. A smooth structure on M is a maximal smooth atlas $\mathcal{A}$ on M.

Definition 4.10. A smooth manifold is a pair $(M, \mathcal{A})$, where M is a manifold and $\mathcal{A}$ is a smooth structure on M.

We normally refer to a smooth manifold simply as M, with its smooth structure being understood or unimportant.

Now that we have defined smooth manifolds, we are properly equipped to define smooth functions on manifolds. We will first examine functions from smooth manifolds to Euclidean spaces, and then we will examine functions between smooth manifolds.

Definition 4.11. Let M be a manifold of dimension n. A function $f : M \to \mathbb{R}^k$ is smooth if for all $p \in M$, there exists a smooth chart (U, φ) whose domain contains p such that $f \circ \varphi^{-1}$ is smooth on $\varphi(U)$.

Note $\varphi(U)$ is an open subset of $\mathbb{R}^n$, and $f \circ \varphi^{-1}$ is a function from $\varphi(U)$ to $\mathbb{R}^k$, so smoothness is well-defined.

It is often helpful to use coordinates on M to view f as a function between Euclidean spaces. To do this, for each point $p \in M$, we can consider the function that takes the associated coordinates of p in $\mathbb{R}^n$ to the image of p in $\mathbb{R}^k$. We formalize this idea with the following definition:

Definition 4.12. Let M be a manifold of dimension n and let $f : M \to \mathbb{R}^k$ be a function defined on M. Let (U, φ) be a chart on M. The coordinate representation of f for φ is the function $\hat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\hat{f}(x) = f \circ \varphi^{-1}(x)$.

We now define smooth maps between manifolds.

Definition 4.13. Let M, N be smooth manifolds. A function $f : M \to N$ is smooth if for all $p \in M$ there exist smooth charts (U, φ) and (V, ψ) with $p \in U$ and $f(U) \subset V$ such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.

Note that M and N do not need to be of the same dimension. As the last step of our construction of smooth structures on manifolds, we can define diffeomorphisms between smooth manifolds.

Definition 4.14. Let M, N be smooth manifolds. A function $f : M \to N$ is a diffeomorphism is it is smooth and bijective and its inverse $f^{-1} : N \to M$ is smooth.

Definition 4.15. Let M, N be smooth manifolds. M and N are diffeomorphic if there exists a diffeomorphism $f : M \to N$.

5. Tangent Spaces

Our discussion of tangent spaces will follow that found in "Differential Topology" by Victor Guillemin and Alan Pollack. Following Guillemin and Pollack's approach, we will focus only on manifolds that are defined as subsets of $\mathbb{R}^n$. This greatly simplifies our constructions as we can more directly use concepts such as the directional derivative as in ordinary real analysis. We also do not lose much generality because by the Whitney Embedding Theorem, any smooth manifold is diffeomorphic to a smooth manifold which is a subset of Euclidean space. For a more general and thorough treatment of these topics which can apply directly to arbitrary smooth manifolds, the reader is directed to Chapter Three of "Introduction to Smooth Manifolds" by John Lee. For the statement and proof of the Whitney Embedding Theorem, see Chapter Six of "Introduction to Smooth Manifolds".

We begin by reminding the reader of the following definition from analysis:

Definition 5.1. Let $U \subset \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}^m$ be smooth, and let x be any point in U and h be any point in $\mathbb{R}^n$. Then, the directional derivative of f in the direction of h at x is defined as follows:

$$df_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}$$

The directional derivative tells us how much f changes at x for a small change in the input in the h direction. In single-variable calculus, after defining the derivative of a function at a point, the next step is to allow x to become a variable and define the derivative to be a new function of x. This time, however, we will keep x fixed and rather allow h to be a variable. Thus, for every x in the domain we can define a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ as follows:

Definition 5.2. Let $U \subset \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}^m$ be a smooth function, and let $x \in U$. Then, the derivative of f at x is the map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ which maps $h \in \mathbb{R}^n$ to $df_x(h) \in \mathbb{R}^m$ as defined by Definition 5.1.

We see from the above definitions that for a multivariable function f, the derivative of f at x is a linear transformation which represents the best linear approximation of f at x. Recall that a coordinate chart is a function which provides a local representation of a manifold near a point. We can therefore use the derivative of a coordinate chart to derive a local linear model of a manifold near each point, and this model will be the tangent space.

Definition 5.3. Let $M \subset \mathbb{R}^N$ be a smooth manifold of dimension k. Let $x \in M$ and let (U, φ) be a chart on x centered at x (meaning $\varphi(x) = 0$), and let $V \subset \mathbb{R}^k$ be the codomain of φ. Let $\phi : V \to M$ be defined by $\phi(y) = \varphi^{-1}(y)$. The tangent space of M at x, denoted TM_x, is the image $d\phi_0(\mathbb{R}^k)$.

To unpack the above definition, we demonstrate how we generalized the spaces on which functions are defined:

$$\varphi^{-1} : V \subset \mathbb{R}^k \to U \subset M \subset \mathbb{R}^N$$

$$\phi : V \subset \mathbb{R}^k \to M \subset \mathbb{R}^N (\phi = \varphi^{-1})$$

$$d\phi_0 : \mathbb{R}^k \to \mathbb{R}^N (\text{Linear})$$

Just as the derivative at a point of a smooth function between open subsets of Euclidean spaces is defined as a linear transformation of the ambient Euclidean spaces, it is possible to define the differential (at a point) of a smooth function between manifolds as a linear transformation of the tangent spaces at that point and its image. For such a construction, the reader is directed to Chapter 1, Section 2 of “Differential Topology” by Victor Guillemin and Alan Pollack.

6. Morse Theory

Morse Theory is a topic in differential topology which provides a beautiful unification of algebraic and differential topology by using tools from analysis to classify topological spaces algebraically. In particular, Morse Theory involves analyzing the critical points of smooth, real-valued functions defined on manifolds.

For example, consider a torus M standing upright over a flat plane, and let $f : M \to \mathbb{R}$ be a smooth function which maps each point of M to its height above the plane. For each $a > 0$, let $M^a = \{p \in M | f(p) \leq a\}$. We find that for all $a, b \in \mathbb{R}$ with $a < b$, M^a is homeomorphic to M^b exactly when there are no critical values of f between a and b.

For details on Morse Theory, the reader is directed to “Morse Theory” by John Milnor.

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