ON THE PATTERSON-SULLIVAN MEASURE FOR CONVEX COCOMPACT GROUPS

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Abstract. Given a discrete group of isometries, we go through the construction of the Patterson-Sullivan measure on the boundary of compactified hyperbolic space which describes the density of limit points. We then show that the critical exponent (that is exponent of divergence) of its Poincaré series is equal to the Hausdorff dimension of the limit set for convex cocompact groups such as Schottky and Fuchsian groups.

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1. Introduction

Patterson-Sullivan theory is a rich field within geometry that looks at how groups act on hyperbolic spaces. Particularly, it provides insight into how group orbits evolve as they undergo multiple iterations of the group action and can help predict how orbits converge and are distributed on the limit set. It does this by constructing a family of measures as a weighted series (dependent on which group is acting on the space) and taking their weak limit to get a measure that lives on the boundary. One interesting result from this beautiful subject is that the exponent of divergence of the Poincaré series coincides with the Hausdorff dimension of the limit set. While this is true more generally, here we look at the result primarily for convex cocompact groups as the proof for these groups only depends on the existence of the measure and some of its local properties.

2. Preliminaries

2.1. Fuschian Groups.

Recall that $\mathbb{Z} \oplus \mathbb{Z}$ acts on $\mathbb{C}$ by translation. We can find a *fundamental domain* for this action by passing to orbits. This leaves us with a

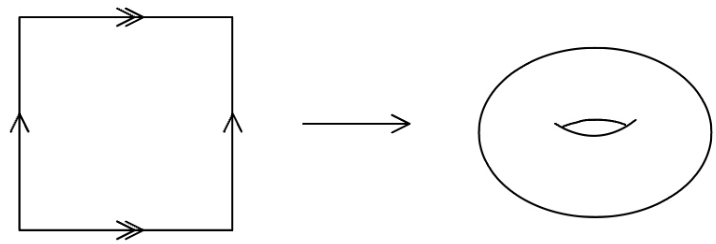

FIGURE 1. We can match up sides of a unit square to make a torus.

unit square where we must match opposite sides, or a torus, which can be tiled to generate the complex plane.

Definition 2.1. Given a topological space X and a group that acts on it G, a fundamental domain is a subset of our space containing exactly one point from each orbit in its interior. Formally, we can say that a closed $F \subset X$ is a fundamental domain if $\bigcup_{g\in G} g \cdot F = X$ and $F^\circ \cap (g^n \cdot F)^\circ = \varnothing$ for all n.

A Fuschian group Γ is a discrete, torsion-free subgroup of PSL(2, $\mathbb{R}$) a.k.a. a group of orientation-preserving isometries of the hyperbolic plane. The question is, what are its fundamental domains?

Definition 2.2. A Dirichlet polygon, centered at a point $z_0 \in \mathbb{H}^2$, is defined to be the following set:

$$F := \{z \in \mathbb{H}^2 : d_{hyp}(z, z_0) \leq d_{hyp}(z, \gamma z_0) \quad \forall \gamma \in \Gamma\},$$

where Γ is any Fuschian group. Equivalently, this can be characterized as the intersection of half-spaces. Namely, we can rewrite the definition as

$$F = \cap_{\gamma\in\Gamma}\{z \in \mathbb{H}^2 : d_{hyp}(z, z_0) \leq d_{hyp}(z, \gamma z_0)\}.$$

Convexity follows directly — we have that the half-spaces are convex and the intersection of convex spaces is convex. To see this, we consider that for two points in the intersection of half-spaces, both points must be in each half-space. Using their convexity, then the segment between x and y must also be in each half-space. But by definition, if the segment is in each half-space, it must be in the intersection, implying that the intersection of half-spaces, our Dirichlet polygon, is also convex. The associated group acting on the fundamental domain can generate a tiling of hyperbolic space.

Proposition 2.3. *The Dirichlet polygon is a fundamental domain for* Γ *acting on* $\mathbb{H}^2$.

Proof. We must check if the following properties of a fundamental domain hold:

(1) that the intersection between an arbitrary orbit and a Dirichlet polygon is non-empty and

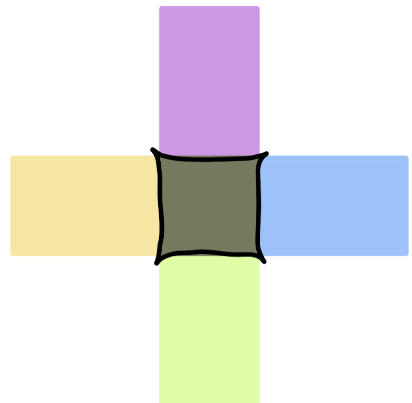

FIGURE 2. The central polygon is an intersection of the half spaces.

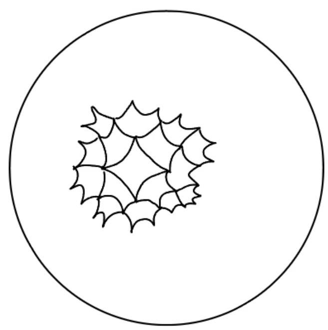

FIGURE 3. Tiling the Poincaré disk model of hyperbolic space with its fundamental domain.

(2) that the above intersection is no more than one point except on the boundary.

Since Γ is Fuschian, we know that it is a discrete group. Then Γz (for $z \in \mathbb{H}^2$) is a discrete set, meaning we can find a minimum distance from our central point z_0. Call a point at which this minimum is achieved $z^* \in \Gamma z$. Furthermore, $d_{hyp}(z^*, z_0) \leq d_{hyp}(\gamma z^*, z_0) = d_{hyp}(z^*, \gamma^{-1} z_0)$. Now we know that a group element must have an inverse element, meaning $d_{hyp}(z^*, \gamma^{-1} z_0) = d_{hyp}(z^*, \gamma z_0)$ for some other γ, which implies that $d_{hyp}(z^*, z_0) \leq d_{hyp} z^*, \gamma z_0)$. Since a Dirichlet polygon by is definition $F = \{z \in \mathbb{H}^2 : d_{hyp}(z, z_0) \leq d_{hyp}(z, \gamma z_0) \quad \forall \gamma \in \Gamma\}$, the above inequality implies that $z^* \in F$, showing that an arbitrary orbit always intersects the Dirichlet polygon.

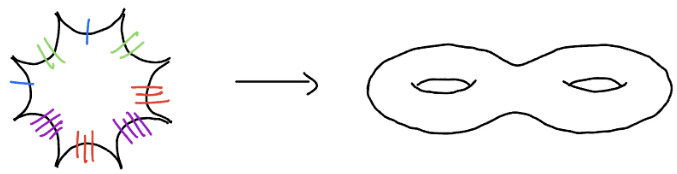

FIGURE 4. We can match up sides of this octogon to make a 2-torus.

Now, we need to show that an orbit intersects the interior no more than once. We can proceed by contradiction. To obtain a contradiction, suppose there are 2 points z_1, z_2 from the same orbit that lie in the interior of the same Dirichlet polygon. By definition of the Dirichlet polygon, $d_{hyp}(z_1, z_0) < d_{hyp}(z_1, \gamma z_0) = d_{hyp}(\gamma^{-1} z_1, z_0)$. Since z_1, z_2 are from the same orbit, we can choose a γ such that $d_{hyp}(\gamma^{-1} z_1, z_0) = d_{hyp}(z_2, z_0)$, giving the inequality $d_{hyp}(z_1, \gamma z_0) \leq d_{hyp}(z_2, \gamma z_0)$. If we repeat the calculation switching the places of z_1 and z_2, we can also get the inequality $d_{hyp}(z_2, \gamma z_0) \leq d_{hyp}(z_1, \gamma z_0)$, which would only be possible if the equality is achieved. But the equality is only achieved on the boundary which is a contradiction, hence proving the claim. □

Now that we know that a Dirichlet polygon is a fundamental domain for a Fuschian group, let's consider an octagon for example and see what genus surface it translates to as we did with the complex plane. Suppose our group acts by pairing every other side up. Then we can see, we are left with a genus 2 surface. Using similar constructions, we can in fact create other surfaces with genus greater than 2 as well.

2.2. **Schottky Groups.** *A Kleinian group is a discrete subgroup of PSL(2, $\mathbb{C}$) a.k.a. a group of orientation-preserving isometries of $\mathbb{H}^3$. We can think about it in a similar way to Fuschian groups, except instead of thinking about the Poincaré Disk model with a circle at infinity, we have the Poincaré ball with a sphere at infinity. A special case of this is a Schottky group. A Schottky group is a special kind of finitely-generated Kleinian or Fuchsian group.*

The way a Schottky group is generated is as follows: suppose you have two intervals A and B, and let g_{AB} be a hyperbolic isometry taking the complement of A to B and A to the complement of B. Let g_{CD} be a similar hyperbolic isometry but for intervals C and D. Composing $g_{AB}, g_{AB}^{-1}, g_{CD}, g_{CD}^{-1}$ in different orders will produce the self-similar Schottky group.

In some sense, the intuition for the generation of a Schottky group is almost opposite to the intuition from earlier for a Fuschian group. Particularly, in our discussion about Fuschian groups, upon finding the fundamental domain, we reverse engineered the surfaces they came from. But for Schottky groups we are starting with a surface, specifically a pair of pants, deconstructing it to form the fundamental domain, and generating the group from there.

Definition 2.4. A limit set, or set of limit points, is the set of points to which a sequence of limits converges. More specifically here, $\lim_{n\to\infty} S_n$ where S_n is the nth generation of the Schottky group. Here a generation is a way of describing the level of replication. Namely, after applying the action (or in our case the set of actions) n times, we are left with n "layers" of replication.

Lemma 2.5. *The limit set of a Schottky group is a Cantor set.*

Proof. There are 3 properties we need to prove to show that the limit set is a cantor set - compact, totally disconnected, and perfect. By the definition of being a limit set, we have that it is closed. This is because a limit set is the set of all the accumulation points and a closed set is just a set that contains all of its accumulation points. We know from the construction of the group that the closures of the generating circles (domains in the disk model) are disjoint and that intervals of the same generation are non-overlapping. Then by containment within the intervals, the limit set must be bounded as well, making it compact.

Now to show that the limit set is totally disconnected. We know from above that intervals within the same generation won't intersect (Here a generation refers to how many layers deep we have gone into the fractal. For example the image below is in its 3rd-generation). We then have sequences of nonempty closed subsets of our intervals nested within each other (each generation is a subset of the last). We also know that as n goes to infinity, the diameter of each of the disjoint intervals of the nth generation Schottky group goes to 0. Then by the nested intersection property of metric spaces, we have that the intersection for each of these sequences is a single point. Since we know that the intervals are disjoint from each other, we can use the fact that the topology on the limit set is generated by the intervals in each generation of the Schottky group, to see that the set is totally disconnected. Finally, we have that this set is perfect. To see this, we recognize that by being a limit set, each point in the set, being a limit point, we can find a sequence of points converging to that value. Particularly, for any epsilon neighborhood of the limit point, we can always go back a finite number of generations such that we contain it (just by the definition of a limit and convergence). By these three properties, we have a Cantor set. □

Definition 2.6. A group is considered convex cocompact if its action on the convex hull has a compact fundamental domain. From this definition, we recognize that Schottky groups are convex cocompact. Furthermore, finitely generated Fuchsian groups without cusps also have this property.

3. Poincare Series

Definition 3.1. A Poincare Series is a power series that intuitively looks at a growth rate of a discrete group acting on a space, particularly our Schottky group. Mathematically, it can be written as

$$g_s(x,y) = \sum_{\gamma\in\Gamma} e^{-s(x,\gamma y)},$$

where s is a positive real number and $(x, \gamma y)$ is the hyperbolic distance.
Our goal is to construct a measure by starting from a point x and looking at the orbit of a point y under our group. From here, we will have a basis to look at how the orbit would look at ∞.

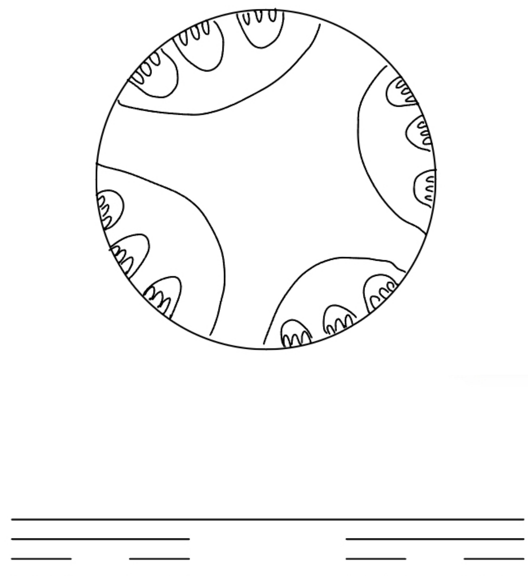

Figure 5. We see in that the 3rd generation Schottky closely resembles the 3rd generation Cantor set below it, except the intervals lie on the boundary of the disk.

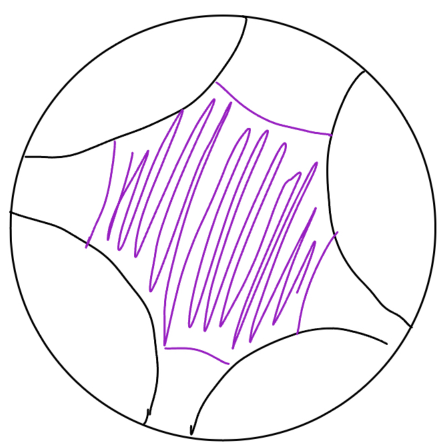

Figure 6. The convex hull of the deconstructed pair-of-pants is highlighted in purple.

Definition 3.2. Two functions are proportional if their ratio is bounded both above and below by finite positive constants.
If we fix x and y, the Poincare series is intuitively looking at the orbit points of y under Γ as viewed from x. Hence it would make sense that it is proportional to

$$\sum_{k=0}^{\infty} s_k e^{-ks}, \tag{3.3}$$

where s_k is the number of orbit points in a half-open shell centered about x with a radius within $\left(k - \frac{1}{2}, k + \frac{1}{2}\right]$. When does (3.3) converge?

Definition 3.4. The critical exponent can be defined as:

$$\delta = \limsup_{k\to\infty} \frac{1}{k} \ln(s_k). \tag{3.5}$$

If $s > \delta$, (3.3) converges and if $s < \delta$ it diverges.

Lemma 3.6. *The critical exponent is no bigger than the dimension of the ambient hyperbolic space and only depends on our discrete group.*

Proof. Specifically, now we can use some properties of the Schottky group. Particularly, since we have that it is a discrete group, there must be some minimum separation between points. Hence we can bound s_k above by ce^{dk}, for some constant c and for dimension d.

$$\begin{aligned}
\delta &= \limsup_{k\to\infty} \frac{1}{k} \ln(s_k) \\
&\leq \limsup_{k\to\infty} \frac{1}{k} \ln(ce^{dk}) \\
&= \limsup_{k\to\infty} \frac{1}{k} [\ln(c) + dk] \\
&= \limsup_{k\to\infty} \frac{dk}{k} \\
&= d,
\end{aligned}$$

which shows that $\delta \leq d$. Furthermore, this critical exponent doesn't depend on x or y, rather only our discrete group Γ. We can see this using the two triangle inequalities:

- $(x, \gamma y) \leq (x, y) + (y, \gamma y)$
- $(x, \gamma y) \geq (y, \gamma y) - (x, y)$

Then we have that:

$$\begin{aligned}
& e^{-s[(x,y)+(y,\gamma y)]} \leq e^{-s(x,\gamma y)} \leq e^{-s[(y,\gamma y)-(x,y)]} \qquad \forall \gamma \in \Gamma \\
\implies & \sum_{\gamma\in\Gamma} e^{-s[(x,y)+(y,\gamma y)]} \leq \sum_{\gamma\in\Gamma} e^{-s(x,\gamma y)} \leq \sum_{\gamma\in\Gamma} e^{-s[(y,\gamma y)-(x,y)]} \\
\implies & e^{-s(x,y)} \sum_{\gamma\in\Gamma} e^{-s(y,\gamma y)} \leq \sum_{\gamma\in\Gamma} e^{-s(x,\gamma y)} \leq e^{s(x,y)} \sum_{\gamma\in\Gamma} e^{-s(y,\gamma y)} \\
\implies & e^{-s(x,y)} g_s(y, y) \leq g_s(x, y) \leq e^{s(x,y)} g_s(y, y)
\end{aligned}$$

So far, there is a characterization of the poincare series when it is either greater than or less than the critical exponent, but what happens exactly at it? At the moment, suppose it diverges at this value.

4. Construction of the Measure

Consider the family of measures taking the form of

$$\mu_s = \frac{1}{g_s(y,y)} \sum_{\gamma \in \Gamma} e^{-s(x,\gamma y)} \delta(\gamma y), \tag{4.1}$$

These measures are densities, which are formed by weighting a Poincaré series with point masses at orbit points of y. Each of these measures lives inside hyperbolic space, on the boundary of a ball contained within the Poincaré ball. However, we want our final measure to live on the boundary of the Poincaré ball. To arrange this, we need to define a notion of *weak convergence*.

Definition 4.2. The weak convergence of probability measures can refer to multiple equivalent notions outlined by the Portmanteau theorem, two of which include:

- $|f|_{P_n} \to |f|_P$ for all f bounded, continuous
- $\lim P_n(A) \to P(A)$ for all continuity sets A (Borel sets with $\mu(\partial A) = 0$)

where $\delta(\gamma y)$ is the Dirac delta centered at γy. Using our inequality from above, we can bound the mass of these measures above and below for any s.

Definition 4.3. Let $\mu(x) = \lim_{s_i \to \delta} \mu_{s_i}(x)$ be the weak limit of our family of measures. We can call this limit the Patterson-Sullivan measure. Interestingly, our initial measure family lived on the interior points, but upon taking the limit, our new measure lives on the limit set.

Proposition 4.4. *$\mu(x)$ gives no mass to the interior*

Proof. We consider definition 4.3, i.e. what happens when we take s to ∞. We see that $g_s(y,y)$ goes to ∞ as s goes to δ since we assume it diverges at the critical exponent. Then given our definition above, since we are taking our limit toward the boundary, our measure is only non-zero at the limit points and 0 in the interior.

Definition 4.5. A horosphere is a hypersurface that can be described as a limit of hyperspheres that share a tangent hyperplane as their radii go to infinity. The shared point of tangency is called the base point. A two-dimensional horosphere, better known as a horocycle, also has the property that all of its normal geodesics converge in the same direction to its base point.

Claim 4.1. For another point x', $\lim_{s_i \to \delta} \mu_{s_i}(x') \to \mu(x')$, an equivalent measure s.t. the Radon-Nikodym Derivative $\frac{d\mu(x')}{d\mu(x)} = e^{s(x,x')_\xi}$, where $(x,x')_\xi$ is the signed distance between the horospheres based at ξ passing through x and x'.

Proof. For orbit points near the base point, ξ, we know that $(x,\gamma y) - (x',\gamma y) \approx (x,x')_\xi$. To see this, we notice that as we approach infinity, the difference goes to 0, as shown in figure 8. Since the purple segment in figure 8 goes to length 0 as y approaches infinity in the hyperbolic metric, we are allowed to approximately equate the two sides. Now we can think about the ratio of the coefficients of the

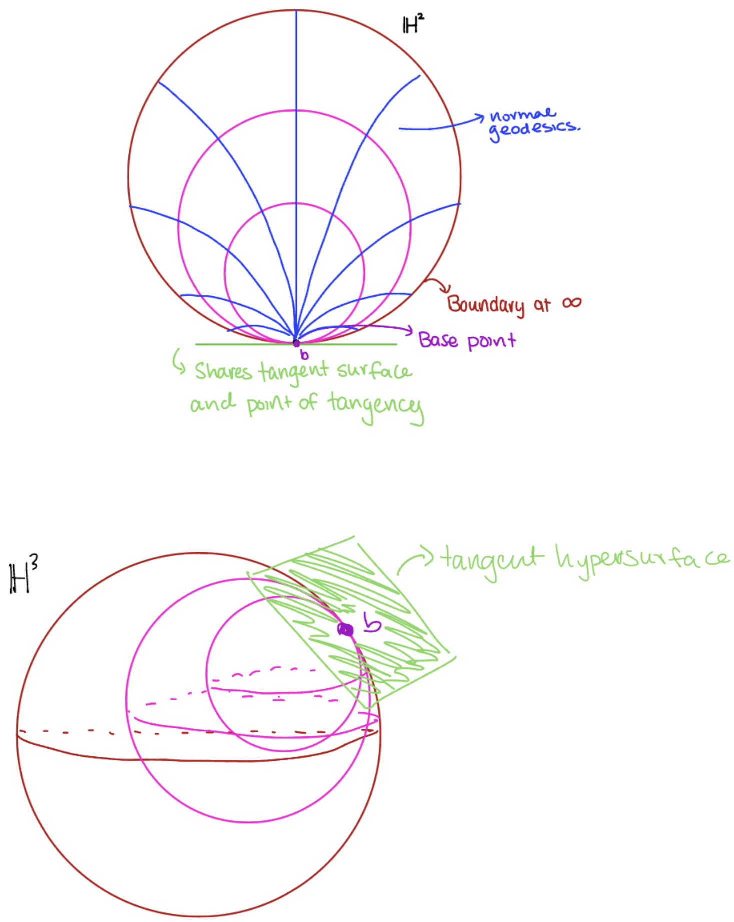

FIGURE 7. Pictured above in pink are horospheres in Hyperbolic 2 and 3 space.

delta mass in the measure. Particularly using the definition from above, we have the ratio of the coefficients to be:

$$\frac{e^{-s(x',\gamma y)}}{e^{-s(x,\gamma y)}}$$
$$= e^{-s[(x',\gamma y)-(x,\gamma y)]}$$
$$\approx e^{s(x,x')_\xi}.$$

Now, we recognize that as we approach the critical exponent, these points are the only ones that contribute (since we are computing the measure of a neighborhood of the limit point and base point of our horospheres, ξ), meaning we have the ratio to be $e^{s(x,x')_\xi}$ as desired. Finally, we realize that the equality (rather than an approximation) is achieved as the approximation error term goes to 0 upon taking the limit by the definition of the hyperbolic metric. This gives the existence of the measure specifically if the dimension is δ. Now we deal with the assumption from

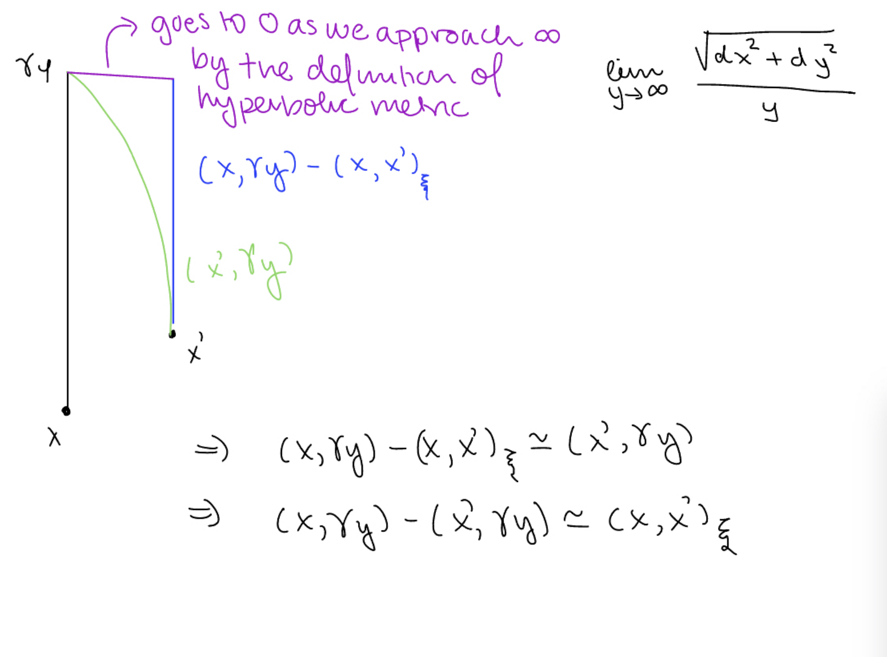

FIGURE 8. Pictured here are the geodesics in the upper-half plane model. We see that as we go to infinity, the distance between the two go to 0 by the hyperbolic metric, allowing us to equate the desired difference to a Busemann function/ horospherical distance between x and x'

earlier regarding the divergence at the critical exponent.
What do we do if the Poincaré series doesn't diverge at δ? We can observe that defining the measure as a weak limit of the weighted series as we did before doesn't make sense if it doesn't diverge at the critical exponent. So how does one deal with this? There needs to be a way to increase the weights of the delta masses such that the series would diverge. Choose a continuous and non-decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that the series, $\sum_k h(k) s_k e^{-ks}$ diverges at the critical exponent and that $\left|\frac{h(r+d)}{h(r)} - 1\right| < \varepsilon$ for a bounded distance and positive epsilon. This effectively removes dependence on the choice of the function h because as s_i approaches the critical exponent, we see that $\frac{h(r+d)}{h(r)} \to 1$.

5. Showing that our conformal density can be bounded

Definition 5.1. A conformal density is a family of finite positive Borel measures that live on the boundary of our space X taking the form of $\mu = (\mu_x)_{x \in X}$. Furthermore, for any two elements of X, the associated measures are absolutely continuous with respect to each other.
Consider an invariant conformal density μ of dimension α associated to our group Γ. From above, we have the existence of such a density if $\alpha = \delta$. Our goal is to estimate the measure of a ball of radius r scaled by $\frac{1}{r^\alpha}$ with respect to the metric

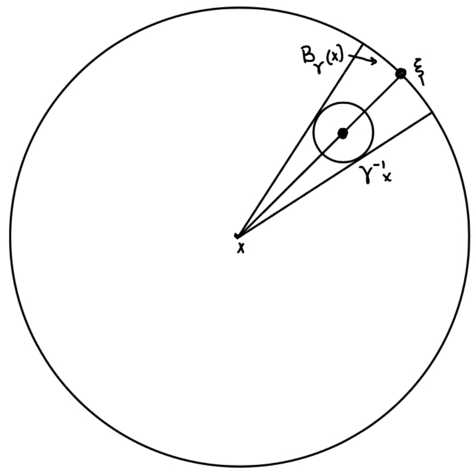

FIGURE 9. Uniform expansion to engulf all but a ball of radius $\leq \varepsilon$. We can use the boundedness from this construction to prove the critical exponent is equal to the Hausdorff dimension.

on the sphere associated to a point x in hyperbolic space (ie $\mu_x(B_\gamma)/r^\alpha$). We need to be careful to change our metric conformally to ensure that asymptotic properties of the radii when $r \to 0$ don't change.

Proposition 5.2. *Given x in $\mathbb{H}^{d+1}$, let $r_\gamma = e^{-(x,\gamma^{-1}x)}$. We can find balls B_γ of radius $k \cdot r_\gamma$ that are centered at the ends of rays from x to γ_x^{-1} for $\gamma \in \Gamma$ such that $\mu_x(B_\gamma)/r^\alpha$ can be bounded above and below.*

Proof. By applying a hyperbolic translation that preserves the axis of γ going through x, we can uniformly expand the ball with a radius of $k \cdot r_\gamma$. The idea is that if we choose some $\varepsilon > 0$, we can choose a corresponding $k = k(\varepsilon)$ large enough so that the image of our ball under this transformation will be expanded uniformly to include all but a small ball of radius $\leq \varepsilon$. Now, we can apply this process for all geodesics connecting x and $\gamma^{-1}x$, iterating through each group element γ in Γ. What results is that B_γ of radius $k(\varepsilon)e^{-(x,\gamma^{-1}x)}$ about ξ, the endpoint of our directed ray, is expanded uniformly to engulf all but a ball of radius $\leq \varepsilon$. Furthermore, we know, that if we are given μ isn't a single atom, we can pick some $\nu <$ mass μ such that for $\varepsilon > 0$, every ball on the sphere of radius $\leq \varepsilon$ has μ-measure $\leq \nu$. If μ has no atoms (eg Lebesgue measure), ν can be chosen arbitrarily small. Otherwise, we can choose ν to be the mass of the largest atom of μ + some small positive number.

Definition 5.3. A Lebesgue number, given a compact[ified] space and a [finite] open covering of that space $\{U_i\}$, is a positive number β such that for every $S \subset X$ with $diam(S) \leq \beta$, we can find $U_i \supset S$.

We can now choose ε to be less than the Lebesgue number of some finite subcover $\{U_\xi\}$, where U_ξ is a disk about ξ such that $\mu(U_\xi) \leq \nu$. Now by picking our ε this way and using the setup that we have created, the expansion rate is approximately $e^{(x,\gamma^{-1}x)}$ and we have essentially guaranteed ourselves bounds above and below for $\mu_x(B_\gamma)/r^\alpha$ where all γ dependence cancels out. Hence our bounds are independent of γ.

6. Showing the Equivalence of critical exponent and Hausdorff dimension.

Definition 6.1. The Hausdorff δ-measure looks at the size of a set by using shrinking set coverings. As δ approaches 0, the coverings become increasingly fine allowing for the approximation of the Hausdorff measure. Mathematically, the d-dimensional Hausdorff δ-measure of S, a subset of a metric space, is defined as:

$$H^d_\delta(S) := \inf\Big\{\sum_{i=1}^{\infty}(\mathrm{diam}(U_i))^d \Big| \cup_{i=1}^{\infty} U_i \supset S, \mathrm{diam}(U_i) < \delta\Big\}.$$

Definition 6.2. The d-dimensional Hausdorff measure H^d is defined as the limit as $\delta \to 0$ of the Hausdorff δ-measure for measurable sets (otherwise it is called an outer measure). Given the definition above, we can see that it is similar to the Lebesgue measure, but generalizes to allow for non-integer dimension, making it very useful for many things, including limit sets of self-similar groups.

Definition 6.3. The Hausdorff dimension is defined as:

$$\dim_H(X) := \inf\{d \geq 0 | H^d(X) = 0\}.$$

Theorem 6.4. *For a convex cocompact group, the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ is the critical exponent $\delta(\Gamma)$. It is also positive and finite.*

Proof. Choose a Γ-invariant conformal density μ of dimension δ, the critical exponent of our convex cocompact group. Using proposition 5.2, we can compute the ratios $\mu_x(B_\gamma)/r^\delta$ for all balls about ξ on the limit set. Consider a ray that originates at a point x in the convex hull and terminates at ξ. Since the action on the convex hull has a compact fundamental domain, we know that the orbit points of x will be a bounded distance from each point on the ray.

Now we want to consider our prior construction of B_γ. Given some small positive number ε, we can choose an appropriate constant such that upon scaling, for γ near the ray, B_γ will contain a ball of radius εr_γ centered at the endpoint of the ray, ξ. If we follow the ray, $B(\xi, \varepsilon r_\gamma) \subset B_\gamma$ will continue to shrink.

This is useful as we can take the boundedness above and below of $\mu_x(B_\gamma)/r^\gamma$ in proposition 5.2 and apply it to our ball centered at ξ. Namely, because of the subset relationship, the boundedness above and below of $\mu_x(B_\gamma)/r^\gamma$ automatically gives the boundedness above and below of $\mu_x(B(\xi, r))/r^\gamma$ for every r. Now, we recognize that we the choice of ξ was arbitrary, meaning we can use this same construction for any point in the limit set. Now, we are ready to evaluate the Hausdorff measure on the limit set.

Consider a Borel subset of the limit set $A \subset \Lambda$. Take a covering of this subset $\cup_i B_i$ where each B_i is a ball centered at a point at the limit set with a respective radius

r_i. Now, we can set up the following inequality which follows from boundedness, subadditivity, and definition of a cover respectively:

$$\sum_i r_i^\delta \geq \text{constant} \cdot \sum_i \mu_x(B_i) \geq \text{constant} \cdot \mu_x(\cup_i B_i) \geq \text{constant} \cdot \mu_x(A) \tag{6.5}$$

Then computing the Hausdorff measure and using this inequality, we have:

$$\lim_{\varepsilon \to 0} \quad \inf_{\vartheta = \cup o_i \supset A, r_i \leq \varepsilon} \sum_i r_i^d \quad \geq c_* \cdot \mu_x(A)$$

Since we ultimately want to prove equality, we now need to get the inequality in the reverse direction. Our strategy to approach this would be to construct a cover of the limit set in a systematic fashion for some $\varepsilon > 0$ and shrink the balls to ensure disjointness. Consider a sequence of balls that are centered at points on the limit set such that radius $B_i \geq$ radius B_{i+1} with radius $B_1 \leq \varepsilon$. At the ith step, we can choose the center of B_i such that center $B_i \notin (\cup_{j=1}^{i-1} B_j)$. While this construction creates some level of separation, it doesn't guarantee that the balls are entirely disjoint. To ensure disjointness between the balls, we can shrink the radii to half of the original radii. Denote this new disjoint union B. Now we are perfectly set up for the reverse inequality which gives that the Hausdorff measure of the subset is at most some constant times $\mu_x(A)$:

$$\sum_i r_i^d = 2^d \sum_i \left(\frac{1}{2} r_i\right)^d \leq c_{**} \cdot \mu_x(B) \leq c \cdot \mu_x(\Lambda)$$

As we have both directions, taking $\varepsilon \to 0$ proves the claim.

In addition to this theorem, the above proof gives information regarding the Hausdorff δ-measure of the limit set when confined to a ball. Given the inequalities above, we get that, the measure must be proportional (up to a proportionality constant determined by the constant terms in the inequality) to r^δ, giving a sense of how the limit set grows.
Now we just want to show uniqueness of the measure. From theorem 6.4, we have that for a Γ-invariant δ-conformal density on the limit set, δ must be the Hausdorff dimension and resultantly that all sets of the same measure class correspond to the measure classes under Hausdorff measure (just following from the equivalence).

Definition 6.6. Consider the following two conditions:

(1) $\gamma A = A \ \forall \gamma \in \Gamma$

(2) $\mu(A) = 1$ or $\mu(X \setminus A) = 1$

Ergodicity with respect to a probability measure μ and a group Γ means that condition (1) necessarily implies condition (2).

Claim 6.1. The Hausdorff measure class (hence equivalently that of μ) is ergodic under the action of Γ.

Proof. To prove this, consider a measurable set A whose measure is preserved under the group action. We observe that if A has positive Hausdorff dimension d, then it has a positive d- dimensional Hausdorff measure. Since the measure of A is preserved under the group action, there are two options: $\mu(X \setminus A) = 0$ or $\mu(A) = 0$, where the latter only occurs if the Hausdorff dimension was originally 0. But this is precisely the criterion for ergodicity, hence showing the desired result.

This is significant as it means that the Γ-invariant conformal density is unique. To see this, suppose for the sake of contradiction that it isn't unique. Then we can take two such densities and take their average, which would be 1 (by the above claim). Then the Radon-Nikodym derivatives with respect to the average measure for each starting measure would be constant. Taking limits, we get that the two densities are equivalent, contradicting our initial assumption that they were different.

Another thing we can note is that the Poincare series must diverge at this critical exponent in the convex cocompact case. We can check this using our construction of the balls within each other. Our covering is arbitrarily fine upon taking the limit, and since the measure inside each ball is proportional to the subsequent one (by our construction earlier), we would have to diverge as no matter how far out we go, we still have dependence on the next term. Recall at the end of section 3, we just took this to be the case to make sense of the construction that followed. Confirming this fact means that this assumption was reasonable.

That the critical exponent of the Poincaré series equals the Hausdorff dimension of the limit set is not unique to convex cocompact groups. For example, for cocompact groups, where the limit set is the full circle at infinity, we can relatively easily check manually that it holds (since the Hausdorff dimension of the limit set would just be 1). In this case, it would work out rather nicely as this would just make the Patterson-Sullivan measure equal the Lebesgue measure. However, in many other cases, the construction is far messier and while very interesting, requires far more machinery.

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