EXISTENCE AND UNIQUENESS OF STOCHASTIC OPTIMAL CONTROLS

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Abstract. Historically, there have been two broad approaches to optimal control problems: those that rely on the dynamic programming method pioneered by Bellman, and those that rely on the maximal approach of Pontryagin. This is true in stochastic control as well. Despite both methods being developed to deal with the same set of problems, it is often that those who apply one method are unfamiliar with the other. This paper seeks to outline and discuss both approaches and the connections between them, especially as relates to the generalized Hamiltonian and stochastic verification theorems.

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1. Introduction

Historically, there have been two broad approaches to optimal control problems: those that rely on the dynamic programming method pioneered by Bellman, and those that rely on the maximal approach of Pontryagin. This dichotomy holds continuously in stochastic control theory. Despite both methods being developed to deal with the same set of problems, it is often that those who apply one method are unfamiliar with the other. Oftentimes, in fact, a stochastic control problem is presented entirely within the context of dynamic programming or a maximal,

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Hamiltonian, framework. This paper seeks to outline and discuss both approaches and the connections between them.

The connections between the two methods can clarify the way formal solutions to the problem arise, especially in the case of dynamic programming, for which the literature on formal solutions is often secondary to computational methods. The connections should also be known better as to inspire a more unified view of the field, as the development of two entirely separate frameworks for the same problem ensures effort is wasted on rediscovery of facts known in one approach in the other.

2. Formulations of stochastic control problems

Some conditions must be established on the objects to be used in optimal control problems. As these definitions are not yet standardized, they are repeated here. First, as stochastic analysis revolves around continuous spaces with a useful notion of probability measure (that events with zero probability are all known), we create the *regular* space:

Definition 2.1 (Regular)**.** If a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[S,T]}, \mathbb{P})$ is complete and $\{\mathcal{F}_t\}_{t\in[S,T]}$ is right-continuous such that

$$\mathbb{P}(\omega) = 0 \implies \omega \in \mathcal{F}_S,$$

we say $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[S,T]}, \mathbb{P})$ is regular.

We then want a set of functions that works with the filtration. Thus we define *feasible* controls:

Definition 2.2 (Feasible control)**.** For a given probability space with filtration $\{\mathcal{F}_t\}_{t\in[S,T]}$, define

$$\mathcal{U}[S,T] \coloneqq \{u(\cdot)\colon [S,T]\times\Omega \to U \mid u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t\in[S,T]}\text{-adapted}\},$$

or equivalently, that any $u(\cdot) \in \mathcal{U}[S,T]$ is not dependent on any σ-algebra not contained in the filtration. A member of this set is a feasible control or is said to satisfy the feasibility condition.

The next definition outlines the type of stochastic differential equation which we work with in stochastic control. All *controlled stochastic differential equations* are stochastic differential equations which take some time-dependent input whose functional form is one the system controller ostensibly can modify. In this case, we only consider ones with finite, deterministic time horizons, but easy extensions to infinite time and random stopping do exist.

Definition 2.3 (Controlled stochastic differential equation)**.** Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if it is adapted with a filtration $\{\mathcal{F}_t\}_{t\in[S,T]}$ such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[S,T]}, \mathbb{P})$ satisfies Definition 2.1 and is equipped with a m-dimensional Brownian motion $W(t)$, consider the stochastic differential equation $\mathbf{X}$ in $\mathbb{R}^n$ of the form

$$\begin{cases} dX(t) = f(t, X(t), u(t))dt + \varrho(t, X(t), u(t))dW(t) \\ X(S) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $f\colon [S,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n$, $\varrho\colon [S,T]\times\mathbb{R}^n\times U \to \mathbb{R}^{n\times m}$, U is a separable metric space, $S, T \in [0,\infty)$ with $S < T$, and $u(\cdot)$ satisfies Definition 2.2. We call this equation a controlled stochastic differential equation, or specifically a controlled stochastic differential equation $\mathbf{X}$ over $[S,T]$ with initial condition $X(S) = x_0$, $\mathbf{X}^{x_0}_{[S,T]}$. A solution to this equation is deemed $\hat{X}_{[S,T],x_0}(t)$.

The other component to a stochastic control problem is an object to take an infimum or supremum (we choose to use infimum in this text, which is arbitrary as one can simply negate the function). This object represents the payoff or value that the controller is trying to maximize.

Definition 2.4 (Cost functional)**.** The cost functional for $\mathbf{X}$ with solution X is defined as

$$J^{\mathbf{X}}_{[S,T]}(\Omega, \mathcal{F}, \mathbb{P}, W(\cdot); u(\cdot)) = \mathbb{E}\left[\int_S^T g(t, X(t), u(t))dt + h(X(T))\right]$$

where $\mathbf{X}$ is defined for $(\Omega, \mathcal{F}, \mathbb{P}, W(\cdot))$

2.1. **Strong formulation.** Usually, we wish to solve the problem given a fixed probability space and filtration. This approach is generally considered to more closely reflect the real world. However, this approach is not analogous to solving a deterministic system due to the unique nature of Brownian motion. Thus, the techniques and theorems for deterministic systems do not always apply.

Definition 2.5 (Strong admissibility)**.** A control $u(\cdot)$ is called strongly admissible for a given controlled stochastic differential equation $\mathbf{X}_{[S,T],x_0}$ if

i) $u(\cdot)$ satisfies Definition 2.2.
ii) $\hat{X}^u_{[S,T],x_0}(t)$ is the weakly-unique solution to $\mathbf{X}_{[S,T],x_0}$ under $u(\cdot)$.
iii) $g(\cdot, X(\cdot), u(\cdot)) \in \mathcal{L}^1_{\mathcal{F}}([S,T]; \mathbb{R}^n)$ and $h(X(T)) \in \mathcal{L}^1_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$.

are all met. The set of all strongly admissible controls is denoted $\mathcal{U}^s_{a,d}[S,T]$.

Definition 2.6 (Strong optimal control)**.** A strong optimal stochastic control is defined as $u^*(\cdot) \in \mathcal{U}^s_{a,d}[S,T]$ such that

$$J_{[S,T]}(u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}^s_{a,d}[S,T]} J(u(\cdot)). \tag{2.7}$$

The corresponding optimal state process is referred to as $X^*_{[S,T],x_0}(t)$.

2.2. **Weak formulation.** Sometimes, varying the probability space and filtration allows a solution to be more easily obtained, which is useful in some approaches to the problem, like the dynamic programming approach. It also is relevant for problems where we wish to obtain the probability law of a solution, such as in robustness problems in economics. However, unlike in the previous approach where we study a set of functions, in this case we study a set of tuples.

Definition 2.8 (Weak admissibility)**.** A 5-tuple $\pi := (\Omega, \mathcal{F}, \mathbb{P}, W(\cdot); u(\cdot))$ is called a weakly admissible control for a given controlled stochastic differential equation $\mathbf{X}^{x_0}_{[S,T]}$ if

i) $\{W(t)\}_{t\in[S,T]}$ is a m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with $W(S) = 0$ almost surely.
ii) For $\hat{\mathcal{F}}_t = \sigma\{W(r) \colon S \leq r \leq t\}$, the σ-algebra generated by the Brownian motion up to time t, we have $(\Omega, \mathcal{F}, \{\hat{\mathcal{F}}_t\}_{t\in[S,T]}, \mathbb{P})$ regular.
iii) $u(\cdot)$ is feasible for $(\Omega, \mathcal{F}, \{\hat{\mathcal{F}}_t\}_{t\in[S,T]}, \mathbb{P})$.
iv) $\hat{X}^u_{[S,T],x_0}(t)$ is the weakly-unique solution to $\mathbf{X}_{[S,T],x_0}$ on $(\Omega, \mathcal{F}, \{\hat{\mathcal{F}}_t\}_{t\in[S,T]}, \mathbb{P})$ under $u(\cdot)$.
v) $g(\cdot, X(\cdot), u(\cdot)) \in \mathcal{L}^1_{\mathcal{F}}([S,T]; \mathbb{R}^n)$ and $h(X(T)) \in \mathcal{L}^1_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ such that $\mathcal{L}^1_{\mathcal{F}}([S,T]; \mathbb{R}^n)$ and $\mathcal{L}^1_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ are defined for $(\Omega, \mathcal{F}, \{\hat{\mathcal{F}}_t\}_{t\in[S,T]}, \mathbb{P})$.

The set of all weakly admissible controls is defined as $\mathcal{U}^w_{a,d}[S,T]$.

Definition 2.9 (Weak optimal control)**.** An optimal stochastic control is defined as $\bar{\pi} \in \mathcal{U}^w_{a,d}[S,T]$ such that

$$J_{[S,T]}(\bar{\pi}) = \inf_{\pi \in \mathcal{U}^w_{a,d}[S,T]} J(\pi). \tag{2.10}$$

The corresponding optimal state process is referred to as $\bar{X}_{[S,T],x_0}(t)$.

3. Dynamic Programming

The goal of the dynamic programming approach is to remove the stochastic element from the differential equation. On an abstract level, formal dynamic programming involves deriving a partial differential equation out of the problem. Practically, the approach enables computational methods to be used to find optimal controls.

In the deterministic case, the approach is built around using the principle of optimality to solve a first order differential equation for the path of the optimal cost function. In the stochastic case, we see that we have to pay attention to the existence of the stochastic term in the differential equation. Thus, the solution we obtain is not just that of the deterministic problem, but also involves the ϱ term as well.

3.1. **Stochastic dynamic programming.**

3.1.1. *Stochastic framework for dynamic programming.* We essentially wish to solve a weak problem for every single possible intermediate time horizon in order to learn about the path of a function that *a priori* is the path of the cost functional under the optimal control.

Definition 3.1 (Value function)**.** For a given strong stochastic control problem with controlled stochastic differential equation $\mathbf{X}_{[S,T],x_0}$, let $s \in [S,T]$ and $X(s) = y$. Then develop a weak control problem for $\mathbf{X}_{[s,T],y}$.

The value function V of a given strong control problem is defined

$$\begin{cases} V(s,y) = \inf_{\pi \in \mathcal{U}^w_{a,d}[S,T]} J^{\mathbf{x}}_{[s,T]}(\pi), \forall \mathbf{x} \in \{\mathbf{X}_{[s,T],s}\}_{(s,y)\in[S,T)\times\mathbb{R}^n}, \\ V(T,y) = h(y), \forall y \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

3.1.2. *Bellman's Principle of Optimality.* Bellman's Principle can be stated as essentially "global optimality is local optimality". It is key to many proofs in dynamic programming and thus we repeat it here for clarity.

Theorem 3.3 (Principle of Optimality)**.** *Let (U,d) be Polish and f, ϱ, g, h Lipschitz continuous. Then for any $(s,y) \in [S,T) \times \mathbb{R}^n$ and $u \in \pi$,*

$$\begin{aligned} V(s,y) = \inf_{\pi \in \mathcal{U}^w_{a,d}[S,T]} \mathbb{E}\Bigg[\int_s^{s+b} & g(t, X^u_{[s,T],y}(t), u(t))dt \\ & + V(s+b, X^u_{[s,T],y}(t))\Bigg], \forall 0 \leq b \leq T-s. \end{aligned} \tag{3.4}$$

It's clear, therefore, that the path of the value function depends only on the solution we obtain for the differential equation. We should therefor be able to state a relationship between the control and the value function.

3.1.3. *Hamilton-Jacobi-Bellman equation.* Solving a Hamilton-Jacobi-Bellman equation, along with the Verification theorem, provides the main method for formally obtaining a solution by dynamic programming.

Theorem 3.5 (Hamilton-Jacobi-Bellman equation)**.** *Let (U,d) be Polish, f, ϱ, g, h Lipschitz continuous, and $V \in C^{1,2}([S,T] \times \mathbb{R}^n)$. Then we have for $Y \coloneqq X^u_{[s,T],y}(\cdot)$ that*

$$\frac{\partial V}{\partial t} = \sup_{u \in U} \left\{ f(t, Y(t), u(t)) \frac{\partial V}{\partial Y} + \frac{1}{2} \operatorname{tr}[\varrho(t, Y(t), u(t)) \varrho^T(t, Y(t), u(t))] \frac{\partial^2 V}{\partial Y^2} \right\} \tag{3.6}$$

for all $(t,x) \in [S,T) \times \mathbb{R}^n$, and $\frac{\partial V}{\partial t} = h(x) \forall x \in \mathbb{R}^n$.

It is therefore easy to see that the optimal control is the argument $u \in U$ that maximizes the value function. As the above problem takes the form of a deterministic partial differential equation, the existence and uniqueness of solutions depends on the theorems for that class of problem. This is immensely useful, as we have brought the problem into an area already well-understood by analysts, and for the most part, removed the unfortunate effects of the rough-path nature of Brownian motion.

However, as the value function is not necessarily smooth, we consider the existence of a different class of solution in a later section.

4. Maximum Principle

4.1. **Stochastic Hamiltonians.** Another approach to the problem involves transforming the stochastic *partial* differential equation into a mere stochastic differential equation. This is done by breaking the system down into multiple interrelated stochastic differential equations (the original system and its *adjoint processes*).

Notations 4.1. $(*) \coloneqq (t, X^*_{[S,T],x_S}(t), u^*(t)), Y \coloneqq X^*_{[S,T],x_S}$.

Definition 4.2 (Adjoint process of the first order)**.**

$$\begin{cases} dp(t) &= -[\frac{\partial}{\partial Y} f(*) p(t) + \frac{\partial}{\partial Y} \varrho^T(*) q(t) - \frac{\partial^2}{\partial Y^2} g(*)] dt + q(t) dB(t) \\ p(T) &= -\frac{\partial}{\partial Y} h(X^*_{[S,T],x_S}(T)) \end{cases} \tag{4.3}$$

Definition 4.4 (Adjoint process of the second order)**.**

$$\begin{cases} dP(t) &= -[\frac{\partial}{\partial Y} f^T(*) P(t) + P(t) \frac{\partial}{\partial Y} f(*) + \frac{\partial}{\partial Y} \varrho^T(*) P(t) \frac{\partial}{\partial Y} \varrho(*) \\ & \quad + \frac{\partial}{\partial Y} \varrho^T(*) Q(t) + Q(t) \frac{\partial}{\partial Y} \varrho(*) + \Xi] dt + Q(t) dB(t) \\ P(T) &= -\frac{\partial^2}{\partial Y^2} h(X^*_{[S,T],x_S}(T) \end{cases} \tag{4.5}$$

where $\Xi = -\frac{\partial^2}{\partial Y^2} g(*) + p(t) \frac{\partial^2}{\partial Y^2} f(*) + q(t) \frac{\partial^2}{\partial Y^2} \varrho(*)$

These processes are termed *backwards* stochastic differential equations because of the terminal condition.

A useful transformation of the system is taking its Hamiltonian, which is a well-known approach to the deterministic problem.

Definition 4.6 (Hamiltonian)**.** $\mathcal{H}(t, x, u, p, q) = \langle p, f(t, x, u)\rangle + \operatorname{tr}[q^T \varrho(t, x, u)] - g(t, x, u), (t, x, u, p, q) \in [S, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times m}$.

However, in the stochastic case, we must account for the trace properties of the second-order adjoint process. Thus we generalize the Hamiltonian.

Definition 4.7 (Generalized Hamiltonian)**.** $\mathcal{G}(t, x, u, p, q) = \mathcal{H}(t, x, u, p, q) - \operatorname{tr}[q^T \varrho(t, x, u)] + \frac{1}{2} \operatorname{tr}[\varrho(t, x, u)^T P \varrho(t, x, u)], (t, x, u, p, q) \in [S, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n$.

Thus the system becomes much easier to solve because it only involves one independent variable.

4.2. **Stochastic Maximum Principle.** Assuming that we have a maximum, we have a very convenient theorem to use the generalized Hamiltonian and adjoint processes to derive the optimal control. This maximum principle allows us to just worry about a system of equations in one variable instead of in multiple.

Theorem 4.8 (Stochastic Maximum Principle)**.** *Assume that* $f, \varrho, g \in \mathcal{C}_c^2(\mathbb{R}^n)$*. Let*

$$\mathcal{K}(t, x, u) := \mathcal{G}(t, x, u, b, B) + \varrho^T(t, x, u)(m - B\varrho(t, x, u)),$$

where (b, m) *and* (B, M) *are adjoint processes of* x *of the first and second order, respectively. Then we have that*

$$\mathcal{K}(t, X^u_{[S,T],x_S}(t), u^*(t)) = \max_{u \in U} \left\{ \mathcal{K}(t, X^u_{[S,T],x_S}(t), u(t)) \right\} \tag{4.9}$$

almost surely.

5. Connections between the Maximum Principle and Dynamic Programming

5.1. **Viscosity solutions.** As alluded to in Section 3, we need a new framework for nonsmooth solutions to Hamilton-Jacobi-Bellman equations.

Definition 5.1 (Viscosity solution)**.** A viscosity solution is a solution $v \in \mathcal{C}([S, T] \times \mathbb{R}^n)$ to a given Hamilton-Jacobi-Bellman equation that fulfills

$$\begin{cases} v(T, x) \leq h(x), \forall x \in \mathbb{R}^n \\ v(T, x) \geq h(x), \forall x \in \mathbb{R}^n \end{cases} \tag{5.2}$$

and for any $\varphi \in \mathcal{C}^{1,2}([S, T) \times \mathbb{R}^n)$, whenever $d(v, \varphi)$ attains a maximizing (or minimizing) point respectively at $(t, x) \in [S, T] \times \mathbb{R}^n$, we have

$$\begin{cases} \sup_{u \in U} \left\{ \mathcal{G}(t, x(t), u(t), -\frac{\partial \varphi}{\partial x}(t, x), -\frac{\partial^2 \varphi}{\partial x^2}(t, x)) \right\} \leq \frac{\partial \varphi}{\partial t}(t, x) \\ \sup_{u \in U} \left\{ \mathcal{G}(t, x(t), u(t), -\frac{\partial \varphi}{\partial x}(t, x), -\frac{\partial^2 \varphi}{\partial x^2}(t, x)) \right\} \geq \frac{\partial \varphi}{\partial t}(t, x). \end{cases} \tag{5.3}$$

We should also define the sets of differentials that arise in this chapter.

Definition 5.4 (Differential sets)**.**

$$\begin{cases} \mathcal{D}_x^{2,+} v(\hat{t}, \widehat{x}) = \{(p, P) \in \mathbb{R}^n \times \mathcal{S}^n \mid \\ \overline{\lim}_{x \to \widehat{x}} \frac{v(\hat{t}, x) - v(\hat{t}, \widehat{x}) - \langle p, x - \widehat{x}\rangle - \frac{1}{2}(x - \widehat{x})^\top P (x - \widehat{x})}{|x - \widehat{x}|^2} \leq 0 \Big\}, \\ \mathcal{D}_x^{2,-} v(\widehat{t}, \widehat{x}) = \{(p, P) \in \mathbb{R}^n \times \mathcal{S}^n \mid \\ \underline{\lim}_{x \to \widehat{x}} \frac{v(\hat{t}, x) - v(\hat{t}, \widehat{x}) - \langle p, x - \widehat{x}\rangle - \frac{1}{2}(x - \widehat{x})^\top P (x - \widehat{x})}{|x - \widehat{x}|^2} \geq 0 \Big\}, \end{cases} \tag{5.5}$$

$$\left\{ \begin{array}{l} \mathcal{D}^{1,+}_{t+} v(\hat{t},\widehat{x}) = \left\{ q \in \mathbb{R} \mid \overline{\lim}_{t\downarrow\hat{t}} \frac{v(t,\widehat{x})-v(\hat{t},\widehat{x})-q(t-\hat{t})}{|t-\hat{t}|} \leq 0 \right\}, \\ \mathcal{D}^{1,-}_{t+} v(\hat{t},\widehat{x}) = \left\{ q \in \mathbb{R} \mid \underline{\lim}_{t\downarrow\hat{t}} \frac{v(t,\widehat{x})-v(\hat{t},\widehat{x})-q(t-\hat{t})}{|t-\hat{t}|} \geq 0 \right\}. \end{array} \right. \tag{5.6}$$

5.2. **Relationship for stochastic systems.** The simplest connection to exposit is the relationship between the generalized Hamiltonian and the Hamilton-Jacobi-Bellman equation, which is immediately observable.

Proposition 5.7. *Let (U,d) be Polish, f,ϱ,g,h Lipschitz continuous, and $V \in C^{1,2}([S,T]\times\mathbb{R}^n)$. Then we have for $Y := X^u_{[s,T],y}(\cdot)$ that*

$$\frac{\partial V}{\partial t} = \sup_{u\in U}\left\{\mathcal{G}(t,Y,u,-\frac{\partial V}{\partial Y},-\frac{\partial^2 V}{\partial Y^2})\right\} \tag{5.8}$$

for all $(t,x)\in[S,T)\times\mathbb{R}^n$, and $\frac{\partial V}{\partial t} = h(x)\forall x\in\mathbb{R}^n$.

This then begs the question– does there exist a relationship between the value function and the object we maximize in the stochastic Hamiltonian approach? Indeed, we have the following theorem, which works for all solutions to the value function, regardless of the smoothness with respect to the time variable. The proof is involved but intuitive, and relies simply on properties of measure and continuity to show the inclusion.

Theorem 5.9. *Let (U,d) be Polish, f,ϱ,g,h and their derivatives be Lipschitz continuous, and $V\in\mathcal{C}^2([S,T]\times\mathbb{R}^n)$. Then*

$$\mathcal{K}(t,X^*_{[S,T],x_S}(t),u^*(t)) \in \mathcal{D}^{1,+}_{t+}V(t,X^*_{[S,T],x_S}(t)) \tag{5.10}$$

almost surely for almost every $t\in[S,T]$.

Proof. For any $t\in(s,T)$, take $\tau\in(t,T]$. Denote by $X_\tau(\cdot)$ the classical solution to the following stochastic differential equation on $[\tau,T]$:

$$X_\tau(t) = X^*_{[S,T],x_S}(t) + \int_\tau^r f(\theta,X_\tau(\theta),u^*(\theta))d\theta + \int_\tau^r \varrho(\theta,X_\tau(\theta),u^*(\theta))dW(\theta). \tag{5.11}$$

Set $\xi_\tau(r) = X_\tau(r) - X^*_{[S,T],x_S}(r)$ for $r\in[\tau,T]$. Under the new probability measure $\mathbb{P}\left(\cdot\mid\hat{\mathcal{F}}_\tau\right)$, we get for any $k\geq 1$ that:

$$\mathbb{E}\left\{\sup_{\tau\leq r\leq T}|\xi_\tau(r)|^{2k}\mid\hat{\mathcal{F}}_\tau\right\} \leq K|X^*_{[S,T],x_S}(\tau)-X^*_{[S,T],x_S}(t)|^{2k}, \quad \mathbb{P}\text{-a.s.} \tag{5.12}$$

Taking $\mathbb{E}\left(\cdot\mid\hat{\mathcal{F}}_t\right)$ on both sides, $\hat{\mathcal{F}}_t\subseteq\hat{\mathcal{F}}_\tau$ we obtain by $\hat{\mathcal{F}}_t\subseteq\hat{\mathcal{F}}_\tau$:

$$\mathbb{E}\left\{\sup_{\tau\leq r\leq T}|\xi_\tau(r)|^{2k}\mid\hat{\mathcal{F}}_t\right\} \leq K|\tau-t|^k, \quad \mathbb{P}\text{-a.s,} \tag{5.13}$$

for constant K. We omit here the details of the variational equations that $\xi_\tau(r)$ satisfies.[1] Those equations satisfy weak admissibility such that

$$\left(\Omega,\mathcal{F},\mathbb{P}\left(\cdot\mid\hat{\mathcal{F}}_\tau\right)(\omega),W(\cdot)-W(\tau),\,u^*(\cdot)|_{[\tau,T]}\right)\in\mathcal{U}^w_{a,d}[\tau,T], \quad \mathbb{P}\text{-a.s.} \tag{5.14}$$

[1] These can be found in [1].

Thus, by the definition of the value function V,

(5.15)
$$V(\tau, X^*_{[S,T],x_S}(t)) \leq \mathbb{E}\left\{\int_\tau^T g\left(r, X_\tau(r), u^*(r)\right) dr + h\left(x_\tau(T)\right) \mid \hat{\mathcal{F}}_\tau\right\}, \quad \mathbb{P}\text{-a.s.}$$

Taking $\mathbb{E}\left(\cdot \mid \hat{\mathcal{F}}_t\right)$ and as $t < \tau$, we have

(5.16)
$$V(\tau, X^*_{[S,T],x_S}(t)) \leq \mathbb{E}\left\{\int_\tau^T g\left(r, X_\tau(r), u^*(r)\right) dr + h\left(X_\tau(T)\right) \mid \hat{\mathcal{F}}_t\right\}, \quad \mathbb{P}\text{-a.s.}$$

Choose a subset $\Omega_0 \subseteq \Omega$ with $\mathbb{P}\left(\Omega_0\right) = 1$ such that for any $\omega_0 \in \Omega_0$, we have the above conditions and

$$\sup_{s \leq r \leq T} \left(\left|p\left(r, \omega_0\right)\right| + \left|P\left(r, \omega_0\right)\right|\right) < +\infty \tag{5.17}$$

for any rational $\tau > t$. Let $\omega_0 \in \Omega_0$ be fixed, and set $\mathbb{E}^t := \mathbb{E}\left(\cdot \mid \hat{\mathcal{F}}_t\right)(\omega_0)$. Then for any rational $\tau > t$, we have

$$\begin{aligned}
&V\left(\tau, X^*_{[S,T],x_S}\left(t, \omega_0\right)\right) - V\left(t, X^*_{[S,T],x_S}\left(t, \omega_0\right)\right) \\
&\leq \mathbb{E}^t\left\{-\int_t^\tau g(r)dr + \int_\tau^T \left[g^*\left(r, X_\tau(r), u^*(r)\right) - g^*(r)\right] dr \right. \\
&\qquad \left. + h\left(X_\tau(T)\right) - h(X^*_{[S,T],x_S}(T))\right\} \\
&= \mathbb{E}^t\left\{-\int_t^\tau g^*(r)dr + \int_\tau^T \left\langle \frac{\partial g^*}{\partial X}(r), \xi_r(r)\right\rangle dr \right. \\
&\quad + \left\langle \frac{\partial h}{\partial X}(X^*_{[S,T],x_S}(T)), \xi_\tau(T)\right\rangle + \frac{1}{2}\int_r^T \operatorname{tr}\left(\frac{\partial^2 g^*}{\partial X^2}(r)\xi_\tau(r)\xi_\tau(r)^\top\right) dr \\
&\quad \left. + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 h}{\partial X^2}(X^*_{[S,T],x_S}(T))\xi_\tau(T)\xi_\tau(T)^\top\right)\right\} + o(|\tau - t|).
\end{aligned} \tag{5.18}$$

Then

$$\begin{aligned}
&V\left(\tau, X^*_{[S,T],x_S}\left(t, \omega_0\right)\right) - V\left(t, X^*_{[S,T],x_S}\left\langle t, \omega_0\right)\right) \\
&\leq -\mathbb{E}^t\int_t^\tau g^*(r)dr - \mathbb{E}^t\left\{\langle p(\tau), \xi_\tau(\tau)\rangle + \frac{1}{2}\xi_\tau(\tau)^\top P(\tau)\xi_\tau(\tau)\right\} \\
&\quad + o\langle|\tau - t|).
\end{aligned} \tag{5.19}$$

As for any $\varphi, \psi \in \mathcal{L}^2_{\mathcal{F}}([S,T];\mathbb{R}^n)$, by Cauchy-Schwarz

$$
\begin{aligned}
&\mathbb{E}^t \left\langle \int_t^\tau \varphi(r)dr, \int_t^\tau \psi(r)dr \right\rangle \\
&\le \left\{ \mathbb{E}^t \left| \int_t^\tau \varphi(r)dr \right|^2 \right\}^{\frac{1}{2}} \left\{ E^t \left| \int_t^T \psi(r)dr \right|^2 \right\}^{\frac{1}{2}} \\
&= (\tau - t) \left\{ \int_t^\tau \mathbb{E}^t |\varphi(r)|^2 dr \int_t^T \mathbb{E}^t |\psi(r)|^2 dr \right\}^{\frac{1}{2}} \\
&= o(|\tau - t|), \quad \text{as } \tau \downarrow t, \quad \forall t \in [s,T), \quad \mathbb{P}\text{-a.s.},
\end{aligned} \tag{5.20}
$$

and

$$
\begin{aligned}
&\mathbb{E}^t \left\langle \int_t^T \varphi(r)dr, \int_t^T \psi(r)dW(r) \right\rangle \\
&\le \left\{ \mathbb{E} \left| \int_t^\tau \varphi(r)dr \right|^2 \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left| \int_t^T \psi(r)dW(r) \right|^2 \right\}^{\frac{1}{2}} \\
&= (\tau - t)^{\frac{1}{2}} \left\{ \int_t^\tau \mathbb{E}^t |\varphi(r)|^2 dr \int_t^\tau \mathbb{E}^t |\psi(r)|^2 dr \right\}^{\frac{1}{2}} \\
&= o(|\tau - t|), \quad \text{as } \tau \downarrow t, \quad \text{a.e. } t \in [s,T), \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{5.21}
$$

The last equality comes from the property that sets of Lebesgue points have full Lebesgue measures for integrable functions and $t \mapsto \hat{\mathcal{F}}_t$ is continuous in t. Thus

$$
\begin{aligned}
\mathbb{E}^t \langle p(\tau), \xi_\tau(\tau) \rangle = \mathbb{E}^t \left\{ - \left\langle p(t), \int_t^\tau f^*(r)dr \right\rangle - \int_t^\tau \operatorname{tr}\left[q(r)^\top \varrho^*(r) \right] dr \right\} \\
+ o(|\tau - t|).
\end{aligned} \tag{5.22}
$$

Similarly,

$$
\mathbb{E}^t \xi_\tau(\tau)^\top P(\tau) \xi_\tau(\tau) = \mathbb{E}^t \int_t^\tau \operatorname{tr}\left(\varrho^*(r)^\top P(t) \varrho^*(r) \right) dr + o(|\tau - t|). \tag{5.23}
$$

It follows that for any rational $\tau > t$ and at $\omega = \omega_0$

$$
\begin{aligned}
&V(\tau, X^*_{[S,T],x_S}(t)) - V(t, X^*_{[S,T],x_S}(t)) \\
&\le \mathbb{E}^t \left\{ \left\langle p(t), \int_t^\tau f^*(r)dr \right\rangle + \sum_{j=1}^m \int_t^\tau \left\langle q_j(r), \varrho^{*,j}(r) \right\rangle dr \right. \\
&\left. -\frac{1}{2} \int_t^\tau \operatorname{tr}\left(\varrho^*(r)^\top P(t) \varrho^*(r) \right) dr - \int_t^\tau f^*(r)dr \right\} + o(|\tau - t|) \\
&= (\tau - t)(\mathcal{G}(t, X^u_{[S,T],x_S}(t), u^*(t), p(t), P(t)) \\
&+\varrho^T(t, X^u_{[S,T],x_S}(t), u^*(t))(q(t) - P(t)\varrho(t, X^u_{[S,T],x_S}(t), u^*(t)))) + o(|\tau - t|).
\end{aligned} \tag{5.24}
$$

As $o(|\tau - t|)$ is of aribtrary size, by the continuity of V, we have that

$$\begin{aligned}\mathcal{G}(t, X^u_{[S,T],x_S}(t), u^*(t), p(t), P(t)) \\ + \varrho^T(t, X^u_{[S,T],x_S}(t), u^*(t))(q(t) - P(t)\varrho(t, X^u_{[S,T],x_S}(t), u^*(t))) \\ \in \mathcal{D}^{1,2,+}V(t, X^*_{[S,T],x_S}(t)),\end{aligned}$$

and the proof is complete. □

5.2.1. *Smooth case.* The above theorem leads to an extraordinarily powerful result for smooth value functions. The proof is immediate from the above theorem.

Corollary 5.25. *Let (U, d) be Polish, f, ϱ, g, h and their derivatives be Lipschitz continuous, and $V \in \mathcal{C}^3([S,T] \times \mathbb{R}^n)$. Then*

$$\mathcal{K}(t, X^*_{[S,T],x_S}(t), u^*(t)) \in \mathcal{D}^{1,2,+}_{t+}V(t, X^*_{[S,T],x_S}(t)). \tag{5.26}$$

This result essentially says that $-\frac{\partial V}{\partial Y}$ is equal to $p(t)$ and $-\frac{\partial^2 V}{\partial Y^2}\varrho(t, Y(t), u(t))$ is equal to $q(t)$. As a corollary to Corollary 5.25, we obtain the following in [1].

Corollary 5.27.

$$\begin{aligned}V(t, Y(t)) = V(s, y) - \int_s^t g(r, Y(r), u^*(r))dr \\ + \int_s^t \frac{\partial V}{\partial Y}(r, Y(r))^T \varrho(r, Y(r), u^*(r))dW(r).\end{aligned} \tag{5.28}$$

This theorem expresses the value function as an Ito process, or that an optimal process (z, u) makes $t \to V(t, z) + \int_s^t g(t(r), z(r), u(r))dr$ a Martingale for all t.

5.2.2. *Nonsmooth case.* An equivalent theorem exists in [1] for value functions nonsmooth in the state variable.

Corollary 5.29.

$$\{-p(t)\} \times [-P(t), \infty) \subseteq \mathcal{D}^{2,+}_x V(t, X^u_{[s,T],y}(t)), \forall t \in [s, T], \tag{5.30}$$

$$\mathcal{D}^{2,-}_x V(t, X^u_{[s,T],y}(t)) \subseteq \{-p(t)\} \times [-P(t), \infty), \forall t \in [s, T] \tag{5.31}$$

almost surely.

This result again expresses the value function as related to the adjoint processes, but this time with less precision.

5.3. **Stochastic verification theorems.** There are many different modes for the verification theorems, which are normally presented as a core part of solving formally the Hamilton-Jacobi-Bellman equation. Instead, here I present a theorem that relates any given solution for the value function to the generalized Hamiltonian. This theorem does not require smoothness, as it allows for the expectation to be used instead. It is a convenient criterion for checking if a given solution we obtain to a Hamilton-Jacobi-Bellman equation is valid.

The theorem is typically stated as in [2]:

Proposition 5.32. $V(s, y) < J(s, y; u(\cdot)), \forall u(\cdot) \in \mathcal{U}^w_{a,d}[s, T], (s, y) \in [s, T] \in \mathbb{R}^n$.

However, the following formulation is much more useful, as it does not rely on the optimal cost functional.

Theorem 5.33. *A given admissible pair* $(X^u_{[S,T],x_S}(\cdot), u(\cdot))$ *is optimal if and only if for almost every* t *there exists separable* $(v(t), p(t), P(t)) \in \mathcal{D}^{1,2,+}_{t+,x} V(t, X^u_{[S,T],x_S}(t))$ *such that*

$$\mathbb{E}[v(t)] \leq \mathbb{E}[\mathcal{G}(t, X^u_{[S,T],x_S}(t), u(t), -p(t), -P(t))]. \tag{5.34}$$

Proof. First we have that for any $V(\cdot,\cdot) \in \mathcal{C}([S,T]\times\ \mathbb{R}^n)$, there exists a $\varphi(\cdot,\cdot) \equiv \varphi(\cdot,\cdot;t,x,v,p,P) \in \mathcal{C}^{1,2}\left([S,T] \times \mathbb{R}^n\right)$ with $(t,x) \in [S,T) \times \mathbb{R}^n$ and $(v,p,P) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ such that whenever $(t,x) \in [S,T) \times \mathbb{R}^n$ and $(v,p,P) \in \mathcal{D}^{1,2,+}_{t+,x} V(t,x)$, one has for $Y = X^u_{[S,T],x_S}$

$$\left(\frac{\partial \varphi}{\partial t}(t,Y), \frac{\partial \varphi}{\partial Y}(t,Y), \frac{\partial^2 \varphi}{\partial Y}(t,Y)\right) = (v,p,P),$$

and $v - \varphi$ attains a strict maximum over $[t,T] \times \mathbf{R}^n$ at (t,x). Now, for almost all $\omega \in \Omega$, let

$$\phi(r,z) \equiv \phi(r,z;\omega) := \varphi(r,z;t,Y(t;\omega),v(t;\omega),p(t;\omega),P(t;\omega)),$$

where $(v(\cdot),p(\cdot),P(\cdot))$ are the processes satisfying the given conditions. Then, on the probability space $\left(\Omega, \mathcal{F}, \mathbf{P}\left(\cdot \mid \hat{\mathcal{F}}_t\right)(\omega)\right)$, applying Itô's formula to $\phi(r,Y(r))$, we have

$$\begin{aligned}
&\mathbb{E}\left[V(t+b,Y(t+b)) - V(t,Y(t)) \mid \hat{\mathcal{F}}_t\right] \\
&\leq \mathbb{E}\left[\phi(t+b,Y(t+b)) - \phi(t,Y(t)) \mid \hat{\mathcal{F}}_t\right] \\
&= \mathbb{E}\left\{\int_t^{t+b} \left[\frac{\partial \phi}{\partial t}(r,Y(r)) + \left\langle \frac{\partial \phi}{\partial Y}(r,Y(r)), f(r)\right\rangle\right.\right. \\
&\left.\left. + \frac{1}{2}\operatorname{tr}\left(\varrho(r)^T \frac{\partial^2 \phi}{\partial Y^2}(r,Y(r))\varrho(r)\right)\right] dr \mid \hat{\mathcal{F}}_t\right\}, \mathbb{P}\text{-a.s.}
\end{aligned}$$

Taking the expectation in the above and letting t be the Lebesgue point of the integrand, we have

$$\begin{aligned}
&\mathbb{E}[V(t+b,Y(t+b)) - V(t,Y(t))] \\
&= b\mathbb{E}\left[\frac{\partial \phi}{\partial t}(t,Y(t)) + \left\langle \frac{\partial \phi}{\partial t}(t,Y(t)), f(t)\right\rangle\right. \\
&\quad \left. + \frac{1}{2}\operatorname{tr}\left[\varrho(t)^T \frac{\partial^2 \phi}{\partial Y^2}(t,Y(t))\varrho(t)\right]\right] + o(h) \\
&= b\mathbb{E}\left\{v(t) + \langle p(t), f(t)\rangle + \frac{1}{2}\operatorname{tr}\left[\varrho(t)^T P(t)\varrho(t)\right]\right\} + o(b).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\overline{\lim_{b\to 0^+}} \frac{\mathbb{E}[V(t+b,Y(t+b))] - \mathbb{E}[V(t,\bar{x}(t))]}{b} \\
&\quad \leq \mathbb{E}\left[v(t) + \langle p(t), f(t)\rangle + \frac{1}{2}\operatorname{tr}\left(\varrho(t)^T p(t)\varrho(t)\right)\right].
\end{aligned}$$

Now, applying a technical lemma from [1], we arrive at

$$\begin{aligned}&\mathbb{E}[V(T,Y(T))-V(s,y)]\\&\quad\le\mathbb{E}\int_s^T\left\{v(t)+\langle p(t),f(t)\rangle+\frac{1}{2}\operatorname{tr}\left(\varrho(t)^TP(t)\varrho(t)\right)\right\}dt\\&\quad\le-\mathbb{E}\int_s^T f(t)dt,\end{aligned}$$

where the last inequality is due to the theorem. This leads to

$$V(s,y)\ge J(s,y;u^*(\cdot)).$$

Thus, combining with Proposition 5.32, we obtain the optimality of the pair $(X^*_{[S,T],x_S},u^*(\cdot))$. □

As a convenient corollary, if we do have a smooth value function, we obtain a more exact result [1].

Corollary 5.35. *A given admissible pair $(X^u_{[S,T],x_S}(\cdot),u(\cdot))$ is optimal if and only if for almost every t there exists $(v(t),p(t),P(t))\in\mathcal{D}^{1,2,+}_{t+}V(t,X^u_{[S,T],x_S}(t))$ such that*

$$v(t)\le\mathcal{G}(t,X^u_{[S,T],x_S}(t),u(t),-p(t),-P(t)).\tag{5.36}$$

This means that when it comes to smooth functions, we can actually obtain functional forms to an arbitrary degree of precision via a guess-and-check method.

References

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