HODGE-LEFSCHETZ THEORY, PERVERSE SHEAVES, AND SEMISMALL MAPS

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Abstract. This report concerns the topology of complex algebraic varieties and the maps between them, focusing on the role of Hodge-Lefschetz theory. As a highlight, we present de Cataldo and Migliorini's proof [3] of the decomposition theorem for semismall maps. This result states that the derived pushforward of the (shifted) constant sheaf along a semismall map is a direct sum of simple IC sheaves.

Contents

1. Introduction

What does a complex algebraic variety "look like"? More generally, what does a map of complex algebraic varieties "look like"? For smooth curves, some form of the answer was known to Riemann—they are orientable surfaces, and all nonconstant maps are branched coverings. For nonsingular projective varieties in higher dimensions, these questions were pioneered by Lefschetz, who correctly stated (but did not correctly prove) important results in this direction, including the *weak and hard Lefschetz theorems* (theorems 2.5 and 2.9). Later, the work of Hodge yielded correct proofs of some of these results, but by much different methods than Lefschetz.

While the work to this point mostly dealt in the absolute setting of a single variety, Deligne addressed the situation of a smooth map of quasiprojective varieties. By Ehresmann's theorem, such a map is a topological fiber bundle. By a surprising yet simple application of hard Lefschetz, Deligne showed [9] that such maps of varieties are the simplest kind of fiber bundle that one could hope for. In particular, he showed that the Leray spectral sequence for them always degenerates at the E_2 page (corollary 2.22). He actually showed a much stronger result (theorem 2.21(b)), that the derived pushforward of the constant sheaf (with field coefficients) along a smooth map, splits in the derived category as a sum of its shifted cohomology sheaves. Together with theorem 2.21(a)(c), this is one of the strongest known results on the topology of smooth maps.

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Dealing with singular varieties and non-smooth maps is a much harder issue. For starters, singular varieties are not manifolds, so basic tools like Poincare duality fail. The eventual realization was that usual (co)homology is not the right tool for the job when it comes to spaces with singularities. To this end, Goresky and MacPherson [13][14] developed *intersection homology*, a refined version of ordinary homology, which restores Poincare duality for a large class of singular spaces, including singular algebraic varieties. Amazingly, they also showed that the weak Lefschetz theorem holds for intersection homology. However, at this point there was still no replacement for Hodge theory or the hard Lefschetz theorem for singular varieties.

Following a suggestion of Deligne, Goresky and MacPherson realized their intersection homology theory as the hypercohomology of a certain complex of sheaves, called an *IC sheaf*[1]. One can show that IC sheaves are characterized by certain conditions on the supports of their cohomology sheaves. In a surprising turn of events, Beilinson, Bernstein, and Deligne [2] showed that by relaxing these conditions slightly, one obtained an abelian category of complexes of sheaves, which they called the category of *perverse sheaves*. In fact, the category of perverse sheaves corresponds to the heart of a nonstandard t-structure on the derived category of sheaves, called the *perverse t-structure*. The category of perverse sheaves is stable under Verdier duality, and is Artinian and Noetherian, properties not shared by the category of (constructible) sheaves. On a singular variety, the category of perverse sheaves is the "correct" abelian category of coefficients for cohomology.

In the book "Faisceaux Pervers" [2], Beilinson, Bernstein, and Deligne used the machinery of perverse sheaves to prove the *relative hard Lefschetz theorem* and the *decomposition theorem*. Together, these are some of the strongest known results on the topology of algebraic maps. In the case of a map to a point, the relative hard Lefschetz theorem specializes to give the hard Lefschetz theorem for intersection cohomology. The decomposition theorem is a vast generalisation of Deligne's theorem for smooth maps. In Deligne's theorem, the derived pushforward of the constant sheaf along a smooth map, is a sum of shifted simple local systems. Local systems belong to the usual category of sheaves, but in the presence of singularities one should really work with perverse sheaves. The decomposition theorem, states that for any proper map, the derived pushforward of the constant coefficient IC sheaf is a sum of shifted simple perverse sheaves. This recovers Deligne's theorem, as well as numerous other powerful results. For example, if Y is singular, the decomposition theorem implies that the intersection cohomology of Y is (non-canonically) a direct summand of the cohomology of a resolution of singularities of Y.

The proof of the decomposition theorem and the relative hard Lefschetz theorem in [2] proceeds by reduction to a statement over finite fields, and then uses Deligne's work on the Weil conjectures. Another proof was later given by Saito [22][23] using the theory of mixed Hodge modules. A more elementary third proof was given by de Cataldo and Migliorini [7], this time only using classical Hodge theory. Their proof fits quite naturally within the framework of classical Hodge-Lefschetz theory, and is much closer to the spirit of this report. A precursor to [7], was the paper [3], where de Cataldo and Migliorini prove the decomposition theorem for *semismall maps*, a certain simple yet important class of maps. The proof for semismall maps is considerably simpler than the general case, but the proof still contains much of de Cataldo and Migliorini's insight.

The goal of this report is to give a survey of some of the topics mentioned above. In section 2, we discuss Hodge-Lefschetz theory and Deligne's theorem (theorem 2.21). In section 3, we review some aspects of the theory of perverse sheaves. Finally, in section 4 we exposit the paper [3], proving the decomposition theorem for semismall maps (theorem 4.11).

This report is not entirely self-contained, and it is mostly a demonstration of the author's struggle to understand this material. A fantastic survey that includes everything in this report and more, is [4]. For concise accounts of the theory of intersection homology and perverse sheaves, see [21] or [12], and for a textbook account see [19] or [16]. For more on de Cataldo and Migliorini's work on the decomposition theorem, one should also see [8], [26], and [6]. The decomposition theorem has numerous applications to geometric representation theory and combinatorics, but unfortunately we do not discuss any of them. The survey [4] contains a wealth of information on these applications.

The main background we assume is algebraic geometry over the complex numbers, at the level of [1] or [15]. Familiarity with Hodge theory will also be helpful, for which a concise account is [5]. Starting in section 2.4, we assume familiarity with derived categories, and starting in section 3, we assume familiarity with the

[1]By sheaf here, we mean perverse sheaf, which really refers to a complex of sheaves.

"six functor formalism" on sheaves and Verdier duality. A nice reference for this material (mostly without proofs) is the first three chapters of [11].

Let us conclude the introduction by fixing some notation. We will always use X to denote a complex projective algebraic variety of pure dimension n. By dimension, we always mean complex dimension. All maps between varieties are assumed to be proper. Unless otherwise stated, we always work with $\mathbb{R}$ coefficients. Many of the results in this report work for more general types of spaces (complex analytic varieties or Kähler manifolds), and with more general (but usually field) coefficients. We let $D^b(X)$ denote the bounded derived category of sheaves (with $\mathbb{R}$ coefficients) on X, and let $\tau_{\leq k}, \tau_{\geq k}$ denote the usual truncation functors. For $\mathcal{F} \in D^b(X)$, $\mathcal{H}^k(\mathcal{F})$ denotes the k'th cohomology sheaf of $\mathcal{F}$. Finally, $\mathbb{H}^k(X;\mathcal{F})$ denotes the k'th hypercohomology of X with coefficients in $\mathcal{F}$.

2. Hodge-Lefschetz theory

2.1. **Hodge structures.** We start by reviewing the notions of *Hodge structure* and *polarization*, that will be used later. A reference here is [10, 2.1].

Let X be a smooth manifold. De Rham's theorem says that

$$H^k_{sing}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \simeq H^k_{dR}(X).$$

If X is a compact complex Kähler manifold, then Hodge theory tells us that there is a canonical decomposition

$$H^k_{dR}(X) \otimes_{\mathbb{R}} \mathbb{C} \simeq \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X)$ consists of complex de Rham classes u which can locally be written as

$$u(z) = \sum_{\substack{I,J\subset\{1,\dots,2n\}\\ |I|=p,|J|=q}} u_{IJ}(z) dz_I \wedge \overline{dz}_J$$

with respect to any local coordinate $(z_i)_{1\leq i\leq 2n}$. Moreover we have Hodge symmetry:

$$H^{p,q}(X) = \overline{H^{q,p}(X)}$$

for all p,q. The notion of a Hodge structure is meant to encapsulate this abundance of information.

Definition 2.1. Let k be $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. A *Hodge structure* on a k-module H, is a bi-grading on $H_{\mathbb{C}} := H \otimes \mathbb{C}$:

$$H_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}$$

such that $H^{p,q} = \overline{H^{q,p}}$.

We say that H is *pure of weight* l, if $H^{p,q} = 0$ unless $p+q = l$. Note that $H_{l,\mathbb{C}} := \bigoplus_{p+q=l} H^{p,q}$ is fixed under complex conjugation, and thus we have a *weight decomposition*:

$$H_{\mathbb{R}} := H \otimes \mathbb{R} = \bigoplus_l H_l$$

by pure Hodge structures.

Observe that giving a bi-grading on $H_{\mathbb{C}}$ is the same as specifying an action of $\mathbb{C}^\times \times \mathbb{C}^\times$ on $H_{\mathbb{C}}$. Indeed, (z_1, z_2) will act on $H^{p,q}$ by the factor $z_1^p z_2^q$. By Hodge symmetry, this action descends to an action of $\mathbb{C}^\times$ on $H_{\mathbb{R}}$, where $\mathbb{C}^\times$ embeds into $\mathbb{C}^\times \times \mathbb{C}^\times$ via $z \mapsto (z, \overline{z})$. More concisely, we can define the *Deligne torus*

$$\mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

where $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$. Then a Hodge structure on $H_{\mathbb{R}}$ is just an algebraic representation $h : \mathbb{S} \curvearrowright H_{\mathbb{R}}$. In particular, we have the *Weil operator*

$$C : H_{\mathbb{C}} \to H_{\mathbb{C}}, \qquad C(x) := h(i)\cdot x = i^{p-q}x \quad \text{for } x \in H^{p,q}.$$

Definition 2.2. A *polarization* of a real pure Hodge structure $H_{\mathbb{R}}$ of weight l is a bilinear form Ψ on $H_{\mathbb{R}}$ such that

$$\Psi(h(z)x, h(z)y) = (z\overline{z})^l \Psi(x,y) \qquad \text{for all } x,y \in H_{\mathbb{R}}, z \in \mathbb{C}^\times$$

and such that

$$\Psi(x, C(y))$$

is a symmetric and positive definite bilinear form. In particular, Ψ is invariant under the action of $\mathbb{C}^\times$ restricted to S^1.

We now collect some useful properties of polarizations.

Proposition 2.3. *Let Ψ be a polarization on a real pure Hodge structure $H_\mathbb{R}$. We let Ψ also denote its $\mathbb{C}$-linear extension to $H_\mathbb{C}$.*

(a) Ψ is symmetric if l is even and antisymmetric if l is odd.
(b) Ψ is nondegenerate.
(c) The form $(-1)^l i^{p-q}\Psi(x,\overline{y})$, restricted to $H^{p,q}$, is positive-definite Hermitian.
(d) If $H' \subset H$ is a Hodge sub-structure, than $\Psi|_{H'_\mathbb{R}}$ is a polarization.

Proof. (a): We have

$$\Psi(x,y) = \Psi(C(x),C(y)) = \Psi(y,C^2(x)) = \Psi(y,i^{2(p-q)}x) = (-1)^l\Psi(y,x).$$

(b): By assumption that Ψ is a polarization, $\Psi(x,C(x)) > 0$.
(c): That the form is Hermitian follows directly from (a). For positive-definiteness, let $0 \neq x = a+bi \in H^{p,q}$, where $a,b \in H_\mathbb{R}$. We have

$$\begin{aligned}(-1)^l i^{p-q}\Psi(x,\overline{x}) &= \Psi(x,\overline{(-1)^l i^{q-p}x}) \\ &= \Psi(x,\overline{C(x)}) \\ &= \Psi(a+bi, C(a)-C(b)i) \\ &= \Psi(a,C(a)) + \Psi(b,C(b)) + i[\Psi(b,C(a)) - \Psi(a,C(b))] \\ &= \Psi(a,Ca) + \Psi(b,C(b)) > 0.\end{aligned}$$

(d): A Hodge sub-structure is preserved under the action of $\mathbb{S}$, so the definition of polarization still makes sense. □

2.2. **The Lefschetz theorems.** One of Lefschetz's main ideas was to try and understand the topology of projective varieties by studying their hyperplane sections. Perhaps the most immediately striking result in this regard is the weak Lefschetz theorem (also called the Lefschetz hyperplane theorem). This can be proved using the purely topological methods of Lefschetz, but a stronger result was proven using Morse theory by Andreotti and Frankel:

Theorem 2.4 (Andreotti-Frankel)**.** *Let Y be a smooth complex affine variety of complex dimension n. Then Y has the homotopy type of a CW complex of real dimension at most n.*

For a proof, see [20, §7]. The weak Lefschetz theorem follows immediately from the above theorem.

Theorem 2.5 (Lefschetz hyperplane/weak Lefschetz theorem)**.** *Let X be a smooth projective variety of dimension n, and let $i : D \hookrightarrow X$ be the inclusion of a smooth hyperplane section. Then the restriction*

$$i^* : H^k(X) \to H^k(D)$$

is an isomorphism for $k < n-1$ and is injective for $k = n-1$. Also, the Gysin map

$$i_! : H^k(D) \to H^k(X)$$

is an isomorphism for $k > n+1$ and a surjection for $k = n+1$.

Proof. By Poincare duality, it suffices to only prove the former statement. From the long exact sequence of relative cohomology for the pair (X,D), we get the exact sequence

$$H^k(X,D) \to H^k(X) \xrightarrow{i^*} H^k(D) \to H^{k+1}(X,D).$$

Let $Y := X \setminus D$. Then Y is a smooth complex affine variety of dimension n. Now Lefschetz duality and theorem 2.4 give the vanishing

$$H^k(X,D) \simeq H_{2n-k}(Y) = 0 \quad \text{for } k \leq n-1.$$

□

Next, we turn to the hard Lefschetz theorem, which in some ways is a much deeper result than weak Lefschetz. As motivation, we first review the role of intersection forms in topology.

For any compact orientable (real) manifold M of (real) dimension n, the cup product on cohomology induces a bilinear form

$$Q(-,-) : H^k(M) \times H^{n-k}(M) \to H^n(M) \simeq \mathbb{R}, \qquad Q(-,-) := \int_M - \wedge -. \tag{2.6}$$

Poincare duality states that the form $Q(-,-)$ is nondegenerate. After passing to Poincare duals, $Q(-,-)$ corresponds to taking cycles of complementary dimension on M, moving them within their homology classes so they become transverse, and computing the (signed) number of intersections. Poincare duality is then equivalent to the fact that for every nonzero homology class on M, there is a cycle of complementary dimension which intersects the given cycle transversely, and which is nonzero in homology. Also, an immediate consequence is the equality of complementary Betti numbers, i.e. that $\dim H^k(M) = \dim H^{n-k}(X)$. These two facts are the first examples of how the nondegeneracy of an intersection form has strong topological implications. Below, we'll see another classical example of this phenomenon in the complex setting, and in section 4, we'll discuss a more recent application of this principle to the topology of maps.

As before, X is a complex projective variety of (complex) dimension n, and let L be an ample line bundle on X. Cup product with the first Chern class of L induces morphisms

$$L \wedge - : H^k(X) \to H^{k+2}(X). \tag{2.7}$$

By Poincare duality and the divisor-line bundle correspondence, (2.7) corresponds, up to a scalar multiple (since a positive power of L is very ample), with transversely intersecting a cycle on X with a hyperplane section of X. Now for every $r \geq 0$, we consider the bilinear form

$$Q(L^r -, -) : H^{n-r}(X) \times H^{n-r}(X) \to \mathbb{R}, \qquad Q(L^r -, -) \int_X L^r \wedge - \wedge -. \tag{2.8}$$

Under Poincare duality, we have the following geometric interpretation (again up to a scalar): Let E be the intersection of r generic hyperplane sections of X. Note that E has real dimension $2n-2r$, so the transverse intersection of an $(n+r)$-cycle with E has dimension

$$(n + r) + (2n - 2r) - 2n = n - r.$$

Thus if we intersect two $(n+r)$-cycles with E, we get two cycles of middle (real) dimension $n-r$ in E, and we can compute their usual intersection within E. The form (2.8) computes exactly this intersection. Just as Poincare duality asserts the nondegeneracy of the usual intersection form, the following theorem states the nondegeneracy of this new form. For a modern proof using Hodge theory, see [1], [15], or [24].

Theorem 2.9 (Hard Lefschetz)**.** *Let X be a smooth complex projective variety of dimension n, and let L be an ample line bundle on X. Then for every $r \geq 0$, cupping with the first Chern class of L induces isomorphisms*

$$L^r \wedge - : H^{n-r}(X) \xrightarrow{\simeq} H^{n+r}(X).$$

Equivalently, the bilinear form (2.8)

$$Q(L^r -, -) = \int_X L^r \wedge - \wedge -$$

is nondegenerate.

This theorem clearly recovers the equality of complementary Betti numbers. We already knew this from Poincare duality, but in fact we can deduce more numerology.

Corollary 2.10 (Unimodality of Betti numbers)**.** *Let $b_i := \dim H^i(X)$ denote the i'th Betti number of X. Then*

$$b_0 \leq b_2 \leq \cdots \leq b_{2\lfloor d/2 \rfloor} = b_{2\lceil d/2 \rceil} \geq \cdots \geq b_{2n-2} \geq b_{2n}$$

and similarly,

$$b_1 \leq b_3 \leq \cdots \leq b_{2\lfloor d/2 \rfloor +1} = b_{2\lceil d/2 \rceil -1} \geq \cdots \geq b_{2n-3} \geq b_{2n-1}.$$

Proof. Theorem 2.9 implies that the map
$$L \wedge - : H^k \to H^{k+2}$$
is injective if $k < n$, and surjective if $k \geq n$. □

Definition 2.11. For $0 \leq r \leq n$, define the space of *$(n-r)$-primitive cohomology* to be
$$P^{n-r}(X) := \ker L^{r+1} \subseteq H^{n-r}(X).$$

Geometrically, the $(n-r)$-primitive cohomology corresponds under Poincare duality to $(n+r)$-cycles which intersect trivially with the intersection of $r+1$ generic hyperplane sections. The following theorem is a mostly formal consequence of hard Lefschetz and the properties of Hodge structures.

Theorem 2.12 (Primitive Lefschetz decomposition)**.** *We have a direct sum decomposition of pure weight-$(n-r)$ Hodge structures*
$$H^{n-r}(X) = \bigoplus_{j \geq 0} L^j P^{n-r-2j}(X)$$
such that the summands are mutually orthogonal with respect to the bilinear form (2.8).

Theorem 2.9 says that the form (2.8) is nondegenerate, but in fact we can determine its precise sign when restricted to the primitive subspaces:

Theorem 2.13 (Hodge-Riemann bilinear relations)**.** *For every $0 \leq r \leq n$, the bilinear form*
$$\Psi(-,-) := (-1)^{(n-r)(n-r+1)/2} \int_X L^r \wedge - \wedge -$$
is a polarization of the pure weight $n-r$ Hodge structure $P^{n-r}(X)$. In particular, it follows from Proposition 2.3(c) that
$$(-1)^{(n-r)(n-r-1)/2} i^{p-q} \int_X L^r \wedge \alpha \wedge \overline{\alpha} > 0 \qquad \text{for all } 0 \neq \alpha \in (P^{n-r}(X) \otimes_{\mathbb{R}} \mathbb{C}) \cap H^{p,q}(X).$$

2.3. **Inductive "proof" of hard Lefschetz.** The weak and hard Lefschetz theorems imply that hyperplane sections carry a great deal of topological information about a variety. This gives the hope of understanding varieties inductively on dimension, by successively cutting with hyperplanes. This procedure often fails, however, since these theorems don't say anything about the middle degree cohomology. We now explain, following [5, 8.5], how to "almost" prove hard Lefschetz and the Hodge-Riemann relations by induction. While the argument fails to prove anything in this setting, it will be very important in section 4. We'll need the following lemma:

Lemma 2.14. *Let D denote a generic divisor corresponding to L, and let $i : D \hookrightarrow X$ denote the inclusion. Then the map $L \wedge - : H^k(X) \to H^{k+2}(X)$ factors as*
$$\begin{array}{ccc} H^k(X) & \xrightarrow{\quad L\wedge- \quad} & H^{k+2}(X) \\ & {\scriptstyle i^*}\searrow \quad \nearrow{\scriptstyle i_!} & \\ & H^k(D) & \end{array}.$$

Proof. See [1, Lemma 14.3.2(c)]. Let $\alpha \in H^k(X)$. Then for all $\beta \in H^{n-k-2}(X)$, we have
$$\int_X L \wedge \alpha \wedge \beta = \int_X i_! 1_D \wedge \alpha \wedge \beta = \int_D i^*\alpha \wedge i^*\beta = \int_D i_! i^* \alpha \wedge \beta.$$
□

Proposition 2.15. *Suppose we know hard Lefschetz and the Hodge-Riemann relations in dimension $n-1$. Then we can deduce hard Lefschetz in dimension n, and the Hodge-Riemann relations in all degrees except for $H^n(X)$.*

Proof. It follows from Lemma 2.14 that we can factor the map $L^r : H^{n-r}(X) \to H^{n+r}(X)$ as

$$\begin{array}{ccccc} H^{n-r}(X) & & \xrightarrow{\quad L^r \quad} & & H^{n+r}(X) \\ & \searrow^{i^*} & & \nearrow^{i_!} & \\ & H^{n-r}(D) & \xrightarrow{L^{r-1}} & H^{n+r-2}(D) & \end{array}.$$

For $2 \leq r \leq n$, the weak Lefschetz theorem implies that i^* and $i_!$ are isomorphisms, and the assumption that hard Lefschetz holds for D implies that $L^{r-1} : H^{n-r}(D) \to H^{n+r}(D)$ is an isomorphism. Thus the crucial case is showing hard Lefschetz for $r = 1$.

That is, we want to show that the composition $i_! \circ i^* : H^{n-1}(X) \to H^{n+1}(X)$ is an isomorphism. By weak Lefschetz, i^* is injective and $i_!$ is surjective, so this is equivalent to showing that $i^*H^{n-1}(X) \cap \ker i_! = 0$. Note that $i_!$ is the dual map to i^*, with respect to Poincare duality on $H^{n-1}(X)$ and $H^{n-1}(D)$. Thus $\ker i_! = (i^*H^{n-1}(X))^{\perp}$, where the orthogonal complement is with respect to the intersection pairing (Poincare pairing) on $H^{n-1}(D)$. Observe that $(i^*H^{n-1}(X))^{\perp} = \ker i_!$ is contained in the primitive subspace $P^{n-1}(D)$. So by the assumption that Hodge-Riemann holds on D, $(i^*H^{n-1}(X))^{\perp}$ is polarized by the intersection form, and in particular this form is nondegenerate on $(i^*H^{n-1}(X))^{\perp}$. Hence $i^*H^{n-1}(X) \cap (i^*H^{n-1}(X))^{\perp} = 0$, proving hard Lefschetz for $r = 1$.

For $r \geq 1$, Hodge-Riemann on $P^{n-r}(X)$ follows from the fact that $i^* : P^{n-r}(X) \to P^{n-r}(D)$ is an injection of Hodge structures, so the polarization on $P^{n-r}(D)$ induces a polarization on $P^{n-r}(X)$ by proposition 2.3(d). □

Somewhat tragically, there is no known way to establish the Hodge-Riemann relations on $P^n(X)$, so we can't close the induction. Note that the above argument crucially uses this middle dimensional case, and thus we have really proved nothing. This argument is still very interesting though, and provides insight into the mechanism of Hodge-Lefschetz theory. In section 4, we will be in a miraculous situation where we want to prove an analogue of hard Lefschetz, and the middle dimensional Hodge-Riemann relations can be deduced, so this argument goes through perfectly.

There is also another version of the inductive "proof," where hard Lefschetz is deduced from the semisimplicity of the monodromy in a Lefschetz pencil (see [5, 8.5]). This is an old idea, closer to the way Lefschetz would have thought about these theorems. Remarkably, Deligne was able to deduce this using his work on the Weil conjectures, and thus prove hard Lefschetz for the l-adic cohomology over any algebraically closed field.

2.4. **Smooth maps.** The goal of this section is to explain a theorem of Deligne [9], which shows that the hard Lefschetz theorem implies strong results on the topology of smooth maps. By Ehresmann's theorem, a smooth proper map of varieties is a topologically locally trivial fibration, i.e. a fiber bundle. What we'll show is that fiber bundles in the algebraic setting have much simpler topology than someone familiar with the C^∞ case might expect. Specifically, we'll show in theorem 2.21, that for a smooth proper map of varieties $f : X \to Y$, the derived direct image $Rf_*\mathbb{Q}_X$ splits "as semisimply as possible" in the derived category. In particular, this implies (corollary 2.22) that the Leray spectral sequence for f degenerates on the E_2 page.

First we prove a very general criterion for splitting in derived categories. Let $\mathcal{A}$ be an abelian category. Recall that a functor F from $D^b(\mathcal{A})$ into some abelian category is called *cohomological* if it takes exact triangles to exact sequences. As usual, we set $F^i(X) := F(X[i])$, for $X \in D^b(\mathcal{A})$. For every cohomological functor F, there is a spectral sequence associated to the composition $F \circ \mathcal{H}$:

$$E_2^{pq} = F^p(\mathcal{H}^q(X)) \Rightarrow F^{p+q}(X). \tag{2.16}$$

Lemma 2.17 ([9, Proposition 1.2])**.** *Let $X \in D^b(\mathcal{A})$. The spectral sequence (2.16) degenerates at the E_2 page for every cohomological functor F if and only if there is a decomposition in $D^b(\mathcal{A})$:*

$$X \simeq \bigoplus_i \mathcal{H}^i(X)[-i].$$

Proof. $\Leftarrow$ For a complex concentrated in only one degree, the spectral sequence (2.16) clearly degenerates at the E_2 page. By additivity of spectral sequences, a sum of such complexes also has a degenerate spectral

sequence.
$\Rightarrow$ For all $i \in \mathbb{Z}$, consider the cohomological functor

$$F_i(K) := \mathrm{Hom}_{D(\mathcal{A})}(\mathcal{H}^i(X), K).$$

The corresponding spectral sequence is

$$E_2^{pq} = \mathrm{Ext}^p(\mathcal{H}^i(X), \mathcal{H}^q(K)) \Rightarrow \mathrm{Hom}_{D(\mathcal{A})}(\mathcal{H}^i(X), K[p+q]).$$

By assumption, this degenerates at the E_2 page, so in particular the edge homomorphism

$$F_i^i(K) = \mathrm{Hom}_{D(\mathcal{A})}(\mathcal{H}^i(X)[-i], K) \to \mathrm{Hom}(\mathcal{H}^i(X), \mathcal{H}^i(K)) = E_2^{0i}$$

is surjective. Now setting $K = X$, we can lift the identity map in $\mathrm{Hom}(\mathcal{H}^i(X), \mathcal{H}^i(X))$ to a map

$$a_i : \mathcal{H}^i(X)[-i] \to X.$$

The map a_i induces an isomorphism on the i'th cohomology of X, and is 0 on all other degrees. Thus since X has bounded cohomology, we can define

$$\oplus_i a_i : \bigoplus_i \mathcal{H}^i(X)[-i] \to X,$$

and this is an isomorphism in the derived category. □

Next, we show how a hard Lefschetz type phenomenon lets us verify the above criterion.

Proposition 2.18. *Let $X \in D^b(\mathcal{A})$, and let L be a morphism $X \to X[2]$ such that the iterate*

$$L^i : \mathcal{H}^{n-i}(X) \to \mathcal{H}^{n+i}(X)$$

is an isomorphism for all $i \geq 0$. Then

$$X \simeq_{D^b(\mathcal{A})} \sum_i \mathcal{H}^i(X)[-i].$$

Proof. For more details see [9] or [25, 4.2]. We'll verify the degeneration criterion in lemma 2.17. Let F be a cohomological functor coming out of $D^b(\mathcal{A})$. For $i \geq 0$ define the *primitive cohomology* to be

$$P^{-i} := \ker(L^{i+1} : \mathcal{H}^{-i}(X) \to \mathcal{H}^{i+2}(X)).$$

For all $k \geq 0$ there is an injection

$$L^k P^{-i-2k} \hookrightarrow \mathcal{H}^{-i}(X).$$

Each $L^k P^{-i-2k}$ inside $\mathcal{H}^{-i}(X)$ is killed by a different power of L, so they are mutually disjoint subspaces. Hence

$$\mathcal{H}^{-i}(X) \simeq \bigoplus_{k \geq 0} L^k P^{-i-2k}, \tag{2.19}$$

and similar considerations show that

$$\mathcal{H}^{i}(X) \simeq \bigoplus_{k \geq 0} L^{k+i} P^{-i-2k}. \tag{2.20}$$

Functoriality of F induces a morphism

$$L : F^p(\mathcal{H}^q(X)) \to F^p(\mathcal{H}^{q+2}(X)),$$

i.e. a morphism $E_2^{pq} \to E_2^{p,q+2}$. Define PE_2^{pq} to be

$$\ker L^{i+2} : E_2^{p,-i} \to E_2^{p,i+2}.$$

Then the decompositions (2.19) and (2.20) induce decompositions

$$E_2^{p,-i} \simeq \bigoplus_{k \geq 0} L^k PE_2^{p,-i-2k}$$

$$E_2^{p,i} \simeq \bigoplus_{k \geq 0} L^{k+i} PE_2^{p,-i-2k}.$$

Thus, to show that the d_2 differentials vanish it suffices to show this on the primitive subspaces $PE_2^{p,-i}$. The spectral sequence (2.16) is functorial in X and compatible with L in such a way that L commutes with the differentials d_r. Thus we have a commutative diagram

$$\begin{array}{ccc} PE_2^{p,-i} & \xrightarrow{d_2} & E_2^{p+2,-i-1} \\ \downarrow L^{i+1} & & \downarrow L^{i+1} \\ E_2^{p,i+2} & \xrightarrow{d_2} & E_2^{p+2,i+1} \end{array}.$$

Observe that the left vertical arrow is zero by the definition of the primitive subspace. But the right vertical arrow is an isomorphism by assumption. Thus d_2 must vanish on the top row. To complete the proof, for $r > 2$ we inductively assume that $E_r^{pq} = E_2^{pq}$, and an identical argument shows that $d_r = 0$. □

Theorem 2.21. *Let $f : X \to Y$ be a smooth and projective map of complex quasi-projective varieties of dimensions n and m, respectively. Let L be a relatively ample line bundle on X.*
(a) Cup product with the first chern class of L induces isomorphisms for all $r \geq 0$

$$L^r : R^{n-r} f_* \mathbb{Q}_X \simeq R^{n+r} f_* \mathbb{Q}_X \qquad \textit{(Smooth Relative Hard Lefschetz)}$$

(b) There is an isomorphism in the derived category

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_k R^k f_* \mathbb{Q}_X[-k] \qquad \textit{(Deligne's Decomposition Theorem)}$$

(c) The local systems $R^k f_* \mathbb{Q}_X[-k]$ are semisimple. *(Deligne's Semismplicity Theorem)*

Proof. (a) See [25, 4.2.2] for more details. By assumption, the restriction of L to each fiber of f is ample, and the fiberwise action of cupping with L assembles to a a map of local systems

$$L : R^k f_* \mathbb{Q}_X \to R^{k+2} f_* \mathbb{Q}_X.$$

Then the result follows by hard Lefschetz on the fibers of f.
(b) Follows from part (a) and proposition 2.18.
(c) Follows from Deligne's theorem that a local system underlying a polarizable integral variation of Hodge structures is semisimple [10, Théorème 4.2.6]. □

By applying proposition 2.17 to the Leray spectral sequence, we get:

Corollary 2.22. *The Leray spectral sequence for f degenerates at the E_2 page. In particular,*

$$H^k(X, \mathbb{Q}) \simeq \bigoplus_{p+q=k} H^p(Y, R^q f_* \mathbb{Q}_X)$$

for all k. If Y is simply connected, we get

$$H^k(X, \mathbb{Q}) \simeq \bigoplus_{p+q=k} H^p(Y, \mathbb{Q}) \otimes H^q(F, \mathbb{Q}),$$

where F is the fiber of f.

The right hand sides of the equations in the above corollary are exactly what we would have expected from the Kunneth formula for $Y \times F$. Thus we have shown the incredible fact that X behaves cohomologically just as if it were a product space. This is in complete opposition to the theory of real fiber bundles, where a typical fiber bundle may have infinitely many nonzero pages in the Leray spectral sequence.

3. Perverse sheaves

3.1. **Constructible sheaves.** We begin by briefly reviewing the theory of stratifications, due to Thom, Whitney, and others. The idea is that a singular space should be broken up into nonsingular pieces, called strata, that are glued together in a reasonable way.

A *stratification* of an algebraic variety X is a decomposition of X into a disjoint union of connected, locally closed, and smooth *strata*

$$X = \bigsqcup_\alpha X_\alpha$$

such that the closure of each stratum is itself a union of strata. We would also like to impose a "local normal triviality condition," which essentially means that a small slice in the normal direction "looks the same" everywhere along a given stratum. We won't dwell on what this means precisely, but an important consequence is that for any stratum X_α and $x \in X_\alpha$, x has a small Euclidean neighborhood W, which is homeomorphic in a stratum-preserving way to $\mathbb{C}^{\dim X_\alpha} \times \mathrm{cone}^o(L)$, for some stratified space L ($\mathrm{cone}^o(L)$ denotes the open cone on L). Moreover, L is independent of x. The space L is called the *link* of X_α, and $\mathrm{cone}^o(L) \subset W$ is called a *normal slice* at x.

Whitney introduced differential-geometric conditions that guarantee local normal triviality (see [19, 7.1]), and we call a stratification satisfying these properties a *Whitney stratification.* Any complex algebraic variety admits a Whitney stratification, and in fact any stratification can be refined to a Whitney stratification. We like to think that X is partitioned as

$$X = X_{reg} \sqcup X_{sing}$$

where X_{reg} is a dense open stratum (the "regular locus"), and X_{sing} is a closed codimension 2 union of strata (the "singular locus"). Note X_{sing} may not actually consist of singular points of X—for example, we can always treat a single point of a nonsingular variety as its own 0-dimensional stratum.

Let $\mathrm{Sh}(X)$ refer the category of sheaves of $\mathbb{R}$-vector spaces on X. For ease of exposition, we work with $\mathbb{R}$ coefficients unless otherwise stated.

Definition 3.1. A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is called *constructible* with respect to a Whitney stratification $\bigsqcup_\alpha X_\alpha$, if $\mathcal{F}|_{X_\alpha}$ is a locally constant for all α, and all stalks $\mathcal{F}_x$ are finite dimensional. If no stratification is given, we say that $\mathcal{F}$ is constructible if it is constructible with respect to some Whitney stratification.

We let $D^b_c(X)$ denote the bounded derived category of constructible sheaves on X.

3.2. **Perverse sheaves.** In this section we (very briefly) review the basic definitions and facts concerning perverse sheaves. For a more details see [19], and for complete proofs see [2].

Let $i_x : \{x\} \hookrightarrow X$ denote the inclusion of any point, and let $\mathcal{F} \in D^b_c(X)$. The *stalk cohomology* of $\mathcal{F}$ at x is [19, 7.2]

$$\mathcal{H}^i(i_x^*\mathcal{F}) = \operatorname*{colim}_{U \ni x} \mathbb{H}^k(U; \mathcal{F}) = \mathbb{H}^k(B_\varepsilon(x); \mathcal{F})$$

where $B_\varepsilon(x)$ is a small Euclidean neighborhood of x and where $\mathbb{H}$ denotes hypercohomology. Similarly, the *costalk cohomology* of $\mathcal{F}$ at x is

$$\mathcal{H}^k(i_x^!\mathcal{F}) = \lim_{U \ni x} \mathbb{H}^k_c(U; \mathcal{F}) = \lim_{U \ni x} \mathbb{H}^k_c(B_\varepsilon(x); \mathcal{F}) = \mathbb{H}^k(B_\varepsilon(x), B_\varepsilon(x) - \{x\}; \mathcal{F})$$

where $\mathbb{H}_c$ denotes compactly supported hypercohomology. The *support* of $\mathcal{F}$ is

$$\mathrm{supp}\, \mathcal{F} := \overline{\{x \in X : \mathcal{H}^k(i_x^*\mathcal{F}) \neq 0\}}$$

and similarly, the *cosupport* is

$$\mathrm{cosupp}\, \mathcal{F} := \overline{\{x \in X : \mathcal{H}^k(i_x^!\mathcal{F}) \neq 0\}}.$$

Definition 3.2. A complex $\mathcal{P} \in D^b_c(X)$ is called a *(middle perversity) perverse sheaf* if

$$\dim \mathrm{supp}\, \mathcal{H}^{-k}(\mathcal{P}) \leq k \quad \text{for all } i \qquad \text{(support condition)}$$

and

$$\dim \mathrm{cosupp}\, \mathcal{H}^{k}(\mathcal{P}) \leq k \quad \text{for all } i \qquad \text{(cosupport condition).}$$

Equivalently, if $\mathcal{P}$ is constructible with respect to the stratification $\bigsqcup X_\alpha$, then $\mathcal{P}$ is perverse if and only if for all $x \in X_\alpha$,

$$\mathcal{H}^k(i_x^*\mathcal{P}) = 0 \quad \text{for } k > -\dim X_\alpha$$

and

$$\mathcal{H}^k(i_x^!\mathcal{P}) = 0 \quad \text{for } k < \dim X_\alpha.$$

We can rewrite these conditions yet again, in terms of the inclusions $i_\alpha : X_\alpha \hookrightarrow X$, so that $\mathcal{P}$ is perverse if and only if for all α,

$$\mathcal{H}^k(i_\alpha^*\mathcal{P}) = 0 \quad \text{for } k > -\dim X_\alpha$$

and

$$\mathcal{H}^k(i_\alpha^!\mathcal{P}) = 0 \quad \text{for } k < -\dim X_\alpha.$$

The equivalence of the last condition follows from the fact that [19, p. 101]

$$i_x^! \mathcal{P} \simeq i_x^*(i_\alpha^! \mathcal{P})[-2\dim X_\alpha].$$

We say that $\mathcal{P}$ satisfies the *support condition* (resp. *cosupport condition*) if it satisfies the first (resp. second) condition in one (equivalently any) of the pairs of conditions in the above definition. It is not at all obvious from the definition, but the category of perverse sheaves is in fact an abelian category. This can be proven using the machinery of t-structures: One can show that there is a nonstandard t-structure on $D_c^b(X)$ defined by setting ${}^pD^{\leq 0}(X)$ to be the complexes $\mathcal{P}$ satisfying the support condition, and ${}^pD^{\geq 0}$ the complexes satisfying the cosupport condition. One shows that this t-structure in nondegenerate, hence the abelian category of perverse sheaves is recovered as the heart of this t-structure, i.e. ${}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}$.

Example 3.3. Let X be nonsingular of dimension n, and let $\mathcal{L}$ be a local system on X. Then $\mathcal{L}[n]$ is perverse with respect to the trivial stratification on X. More generally, $\mathcal{L}[n]$ is perverse if X is a local complete intersection [19, Theorem 8.3.12]. If X is singular then this is false in general.

Remark 3.4. It follows from basis properties of Verdier duality, that $\mathcal{P}$ satisfies the support condition if and only if its Verdier dual satisfies the cosupport condition. Thus the category of Perverse sheaves is stable under Verdier duality, a property not shared by the category of constructible sheaves.

It is very common to study perverse sheaves inductively, by starting with a (shifted) local system on the regular locus, and extending it one stratum at a time. As we'll see in the next section, the (co)support conditions amount to certain constraints on these extensions. In this process, there is a certain way to extend a perverse sheaves, which in some sense involves the "least amount of gluing data." This is called the *intermediate extension*.

Theorem 3.5. *Let $j : U \hookrightarrow X$ be the inclusion of an open stratum, and let $\mathcal{P}$ be a perverse sheaf on U. Then there exists a unique perverse sheaf $j_{!*}\mathcal{P}$ on X, called the intermediate extension of $\mathcal{P}$, such that for every stratum $V \subseteq X - U$, with inclusion $i_V : V \hookrightarrow X$,*

(a) $j_{!*}\mathcal{P}_{|U} \simeq \mathcal{P}$
(b) $\mathcal{H}^k(i_V^* j_{!*}\mathcal{P}) = 0$ *for all* $k \geq -\dim V$
(c) $\mathcal{H}^k(i_V^! j_{!*}\mathcal{P}) = 0$ *for all* $k \leq -\dim V$.

Intermediate extensions can be explicitly constructed, inductively on the strata of X:

Proposition 3.6. *Let U_l be the union of all strata of X of dimension at least l, and let $v := v_l$ denote the inclusion $U_{l+1} \hookrightarrow U_l$. If $\mathcal{P}$ is a perverse sheaf on U_l, then*

$$v_{!*}\mathcal{P} = \tau_{\leq -l-1} v_*\mathcal{P}.$$

Using intermediate extensions, we can easily define IC sheaves, a very important class of perverse sheaves.

Definition 3.7. Let $j : X_{reg} \hookrightarrow X$ be the inclusion of the regular locus of X (i.e. an open dense stratum), and let $\mathcal{L}$ be a local system on X_{reg}. The *intersection cohomology complex*, or *IC sheaf*, with coefficients in $\mathcal{L}$, is the perverse sheaf

$$IC_X(\mathcal{L}) := j_{!*}(\mathcal{L}).$$

Remark 3.8. It follows from theorem 3.5 that the IC sheaves are characterized as being the perverse sheaves such that strict inequalities hold in the support and cosupport conditions. That is, $IC_X(\mathcal{L}) \in D_c^b(X)$ is the unique complex such that

(a) $IC_X(\mathcal{L})_{|X_{reg}} \simeq \mathcal{L}[n]$
(b) $\dim \operatorname{supp} \mathcal{H}^{-k}(IC_X(\mathcal{L})) < k$ for all k
(c) $\dim \operatorname{cosupp} \mathcal{H}^{k}(IC_X(\mathcal{L})) < k$ for all k.

Let $i : Z \hookrightarrow X$ be the inclusion of a closed stratum. If $\mathcal{L}$ is a simple local system on an open set of Z, then $i_! IC_Z(\mathcal{L})$ is a perverse sheaf on X. Such perverse sheaves are called *Deligne-Goresky-MacPherson (DGM)* complexes. In addition to being abelian and stable under Verdier duality, the most important property of the category of perverse sheaves is the following.

Theorem 3.9. *(a) The category of perverse sheaves on X is Artinian and Noetherian, i.e. every perverse sheaf is a finite iterated extension of simple perverse sheaves.*

(b) The simple perverse sheaves are the DGM complexes.

Historically, the IC sheaves were discovered before the invention of perverse sheaves, through Goresky and MacPherson's work on intersection homology. Going in reverse historical order, we can define:

Definition 3.10. Let $\mathcal{L}$ be a local system on the regular locus of X. The *(middle perversity) intersection cohomology* of X with coefficients in $\mathcal{L}$ is

$$IH^k(X;\mathcal{L}) := \mathbb{H}^{k-n}(X; IC_X(\mathcal{L}))$$

and similarly, the *intersection cohomology with compact support* is

$$IH^k_c(X;\mathcal{L}) := \mathbb{H}^{k-n}_c(X; IC_X(\mathcal{L})).$$

The cohomology theories IH and IH_c agree with singular cohomology (resp. with compact support) when X is nonsingular. When X is singular, IH can be defined as singular cohomology, but only using chains that intersect the singular locus with a certain controlled degree of transversality. This definition will not come up again in this report, but it is an important part of the theory.

3.3. **Extending perverse sheaves over contractible strata.** To get a feel for the definition of perverse sheaves, we'll ask the following question: Suppose that $Z \subset X$ is a closed connected stratum of dimension d. If $\mathcal{P}$ is a perverse sheaf on X, how much of $\mathcal{P}$ is determined by its restriction to $X - Z$? See [4, 5.7] for an overview of two different approaches to this question, one due to MacPherson and Vilonen, and the other to Beilinson and Verdier. In this section we'll describe the first approach [18], in the case where Z is contractible.

Let $\mathcal{P}$ be a perverse sheaf on X, and let Z be a d-dimensional closed contractible stratum of X. Since Z is contractible, in particular simply connected, constructibility implies that $\mathcal{P}$ has constant cohomology sheaves on Z. Let i and j denote the respective closed and open inclusions

$$Z \xrightarrow{i} X \xleftarrow{j} X - Z,$$

and consider the "attaching triangle"

$$i_! i^! \mathcal{P} \to \mathcal{P} \to j_* j^* \mathcal{P} \xrightarrow{[1]} .$$

The complex $i_! i^! \mathcal{P}$ is zero on $X - Z$, so we consider the restriction of the triangle to Z, which is

$$i^! \mathcal{P} \to i^* \mathcal{P} \to i^* j_* j^* \mathcal{P} \xrightarrow{[1]} . \tag{3.11}$$

Then the gluing data is determined by a class in

$$\mathrm{Ext}^1(i^* j_* j^* \mathcal{P}, i^! \mathcal{P}) = \mathrm{Hom}_{D^b_c(X)}(i^* j_* j^* \mathcal{P}, i^! \mathcal{P}[1]).$$

Since Z is contractible, [18, Lemma 2.4] implies that for any complex $\mathcal{F} \in D^b(Z)$, we have that $\mathcal{F} \simeq \oplus_k \mathcal{H}^k(\mathcal{F})[-k]$. Since we are working with field coefficients, everything is injective and projective, and so

$$\mathrm{Ext}^1(i^* j_* j^* \mathcal{P}, i^! \mathcal{P}) = \bigoplus_k \mathrm{Hom}(\mathcal{H}^k(i^* j_* j^* \mathcal{P}), \mathcal{H}^{k+1}(i^! \mathcal{P})).$$

Now the the support condition for $\mathcal{P}$ over Z gives that

$$\mathcal{H}^k(i^* \mathcal{P}) = 0 \quad \text{for } k > -d,$$

and so the connecting map

$$\mathcal{H}^k(i^* j_* j^* \mathcal{P}) \to \mathcal{H}^{k+1}(i^! \mathcal{P})$$

is an isomorphism for $k > -d + 1$. Similarly, the cosupport condition gives that

$$\mathcal{H}^k(i^! \mathcal{P}) = 0 \quad \text{for } k < -d,$$

and so

$$\mathcal{H}^k(i^* \mathcal{P}) \simeq \mathcal{H}^k(i^* j_* j^* \mathcal{P}) \quad \text{for } k < -d - 1.$$

Thus the only part of the gluing data which is not determined by the restriction of $\mathcal{P}$ to $X - Z$, is contained in the following fragment of the long exact sequence for (3.11):

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{H}^{-d-1}(i^*\mathcal{P}) & \longrightarrow & \mathcal{H}^{-d-1}(i^*j_*j^*\mathcal{P}) \\ \mathcal{H}^{-d}(i^!\mathcal{P}) & \longrightarrow & \mathcal{H}^{-d}(i^*\mathcal{P}) & \longrightarrow & \mathcal{H}^{-d}(i^*j_*j^*\mathcal{P}) \\ \mathcal{H}^{-d+1}(i^!\mathcal{P}) & \longrightarrow & 0 & & \end{array}.$$

Note that the terms $\mathcal{H}^{-d-1}(i^*\mathcal{P})$ and $\mathcal{H}^{-d+1}(i^!\mathcal{P})$ are determined once we know the rest of the exact sequence. We have thus proved:

Theorem 3.12. *Let Z be a closed, connected, contractible, d-dimensional stratum of X. There is a bijection between the sets*

$$\{\text{Perverse sheaves } \mathcal{P} \text{ on } X\}$$

and

$$\left\{\begin{array}{c}\text{Perverse sheaves } \mathcal{P}' \text{ on } X - Z \text{ together with an exact sequence of local systems} \\ \mathcal{H}^{-d-1}(i^*j_*\mathcal{P}') \to V_1 \to V_2 \to \mathcal{H}^{-d}(i^*j_*\mathcal{P}')\end{array}\right\}.$$

Under this bijection, $\mathcal{P}$ is sent to its restriction to $X - Z$ and the exact sequence

$$\mathcal{H}^{-d-1}(i^*j_*j^*\mathcal{P}) \to \mathcal{H}^{-d}(i^!\mathcal{P}) \to \mathcal{H}^{-d}(i^*\mathcal{P}) \to \mathcal{H}^{-d}(i^*j_*j^*\mathcal{P}).$$

One might expect that the bijection of theorem 3.12 could upgraded to an equivalence of categories, by defining morphisms between exact sequences in the obvious way. This is incorrect though—see [4, p. 98] for an example of when this functor is not faithful. However, in [18], MacPherson and Vilonen give a refinement of this construction, which indeed gives an equivalence of categories.

3.4. **Germs of perverse sheaves in dimension one.** As a nice application of theorem 3.12, let's describe the set of perverse sheaves on $\mathbb{C}$, constructible with respect to the stratification $\mathbb{C} = \{0\} \sqcup (\mathbb{C} - \{0\})$. The restriction of such a perverse sheaf to $\mathbb{C}^\times$ is just a shifted local system $\mathcal{L}[1]$. Let L be a stalk of $\mathcal{L}$ at some base point, so that $\mathcal{L}$ is determined by L together with its monodromy operator $T : L \to L$. A simple calculation in Čech cohomology computes the stalk cohomologies

$$\mathcal{H}^0(j_*\mathcal{L})_0 = \ker(T - \mathrm{Id}), \qquad \mathcal{H}^1(j_*\mathcal{L})_0 = \operatorname{coker}(T - \mathrm{Id}), \qquad \mathcal{H}^k(j_*\mathcal{L})_0 = 0 \text{ for all } k \neq 0, 1.$$

Hence

$$i^*j_*\mathcal{L}[1] = \ker(T - \mathrm{Id})[1] \oplus \operatorname{coker}(T - \mathrm{Id}).$$

Thus by theorem 3.12, such a perverse sheaf on $\mathbb{C}$ can be identified with a local system $\mathcal{L}$ on $\mathbb{C}^\times$ and an exact sequence of vector spaces

$$\ker(T - \mathrm{Id}) \to V_1 \to V_2 \to \operatorname{coker}(T - \mathrm{Id}).$$

For example, if V is a vector space (i.e. a sheaf on $\{0\}$), then i_*V restricts to the zero local system on $\mathbb{C}^\times$, so the corresponding exact sequence is

$$0 \to V \xrightarrow{\simeq} V \to 0.$$

Next, since j is an affine open immersion, $j_*\mathcal{L}[1]$ and $j_!\mathcal{L}[1]$ are perverse [19, Theorem 8.6.3]. For $j_*\mathcal{L}[1]$, we have that

$$\mathcal{H}^0(i^*j_*\mathcal{L}[1]) = \mathcal{H}^1(i^*j_*\mathcal{L}) = \operatorname{coker}(T - \mathrm{Id})$$

and

$$\mathcal{H}^0(i^!j_*\mathcal{L}[1]) = \mathcal{H}^0(i^*j_!\mathcal{L}^\vee[1]) = 0,$$

so the corresponding exact sequence is

$$\ker(T - \mathrm{Id}) \to 0 \to \operatorname{coker}(T - \mathrm{Id}) \xrightarrow{\simeq} \operatorname{coker}(T - \mathrm{Id}).$$

The dual calculation shows that the corresponding exact sequence for $j_!\mathcal{L}[1]$ is

$$\ker(T - \mathrm{Id}) \xrightarrow{\simeq} \ker(T - \mathrm{Id}) \to 0 \to \operatorname{coker}(T - \mathrm{Id}).$$

Finally, the intermediate extension $j_{!*}\mathcal{L}[1]$ corresponds to the exact sequence

$$\ker(T - \mathrm{Id}) \to 0 \to 0 \to \mathrm{coker}(T - \mathrm{Id}). \tag{3.13}$$

We can see this from theorem 3.5, since $j_{!*}\mathcal{L}[1]$ is characterized by that it restricts to $\mathcal{L}[1]$ on $\mathbb{C}^\times$, and that

$$\mathcal{H}^0(i^* j_{!*}\mathcal{L}[1]) = \mathcal{H}^0(i^! j_{!*}\mathcal{L}[1]) = 0.$$

The fact that the maps in (3.13) are all zero supports the idea that $j_{!*}\mathcal{L}[1]$ is the extension of $\mathcal{L}[1]$ which has the "least amount of gluing data."

3.5. **Artin vanishing.** See [19, 8.6] for details here. Artin vanishing is a cohomological vanishing theorem which generalizes the result of Andreotti and Frankel (theorem 2.4) to constructible coefficients:

Theorem 3.14 (Artin vanishing)**.** *Let Y be a complex affine variety of dimension n, and let $\mathcal{F}$ be a constructible sheaf on Y. Then $H^k(Y;\mathcal{F}) = 0$ for $k > n$.*

One can deduce from this a version of Artin vanishing for perverse sheaves:

Theorem 3.15 (Artin vanishing for perverse sheaves)**.** *Let Y be a complex affine variety of dimension n, and let $\mathcal{P}$ be a perverse sheaf on Y. Then*

$$\mathbb{H}^k(X;\mathcal{P}) = 0 \quad \text{for } k > 0$$

and

$$\mathbb{H}^k_c(X;\mathcal{P}) = 0 \quad \text{for } k < 0.$$

Recall that the classical weak Lefschetz theorem follows from the vanishing given by Andreotti and Frankel's theorem. Similarly, we can prove a version of weak Lefschetz with perverse coefficients, this time using Artin vanishing. We'll use this theorem later in section 4.3.

Theorem 3.16 (Weak Lefschetz for perverse sheaves)**.** *Let X be a complex projective variety, and let $i : D \hookrightarrow X$ be the inclusion of (any!) hyperplane section. Then for a perverse sheaf $\mathcal{P}$ on X, the restriction*

$$\mathbb{H}^k(X;\mathcal{P}) \to \mathbb{H}^k(D; i^*\mathcal{P})$$

is an isomorphism for $k < -1$ and is injective for $k = -1$. The pushforward

$$\mathbb{H}^k(D; i^!\mathcal{P}) \to \mathbb{H}^k(X;\mathcal{P})$$

is an isomorphism for $k > 1$ and is injective for $k = 1$.

Proof. Let $j : X - D \hookrightarrow X$ denote the inclusion of the open complement of D. Consider the distinguished triangle

$$j_! j^! \mathcal{P} \to \mathcal{P} \to i_* i^* \mathcal{P} \xrightarrow{[1]} .$$

From the long exact sequence in compactly supported hypercohomology we get the exact sequence

$$\mathbb{H}^k_c(X - D; j^*\mathcal{P}) \to \mathbb{H}^k(X;\mathcal{P}) \to \mathbb{H}^k(Y; i^*\mathcal{P}) \to \mathbb{H}^{k+1}_c(X - D; j^*\mathcal{P}).$$

Since X and Y are projective, hence compact, compactly supported hypercohomology for them is the same as regular hypercohomology. Also, $j^*\mathcal{P}$ is still perverse since it is the restriction to an open set. Then the required vanishing is given by theorem 3.15. Similarly, considering the long exact sequence in hypercohomology for the dual triangle

$$i_! i^! \mathcal{P} \to \mathcal{P} \to j_* j^* \mathcal{P} \xrightarrow{[1]}$$

together with theorem 3.15 yields the second statement. ∎

4. The topology of semismall maps

In section 2.4, we saw how Hodge-Lefschetz theory yielded strong results on the topology of smooth maps. In particular, the hard Lefschetz theorem played a key role, and we discussed how this theorem is equivalent to the nondegeneracy of a certain intersection form. In this section, we'll paint an analogous picture for a class of maps which are called "semismall." The goal will be to prove the decomposition theorem for semismall maps (theorem 4.11), which is the analogous result to theorem 2.21 for smooth maps. The original reference for this story is [3], and other references are [4] and [26].

From the perspective of the decomposition theorem, semismall maps are nice because they are characterized (proposition 4.2) by the fact that they preserve perverse sheaves under pushforward. In section 4.3, we'll explain how semismall maps are also characterized by a certain hard Lefschetz-type phenomenon, which will be the Hodge-Lefschetz-type input to the decomposition theorem.

In the following let $f : X \to Y$ be a proper surjective map of irreducible complex varieties, with X smooth of dimension n. The theory of stratifications carries over beautifully to maps between complex varieties [19, p. 152]. Any algebraic map $f : X \to Y$ between complex algebraic varieties can be *stratified*, meaning that there exist an algebraic Whitney stratifications of X and Y, such that for every stratum S of Y,

(a) The restriction $f : f^{-1}(S) \to S$ is a topologically locally trivial fibration.
(b) $f^{-1}(S)$ is a union of strata of X. In particular, each fiber $f^{-1}(y)$ has a Whitney stratification induced by the stratification of X.

From now on, we always assume that X and Y are stratified with respect to f. We may now define semismall maps. Let $d_S := \dim f^{-1}(y)$, where y is any point in S.

Definition 4.1. A map $f : X \to Y$ as above, is called *semismall* if $\dim S + 2d_S \leq \dim X$ for all strata S of Y.

Intuitively, semismall maps have fibers that are "not too large." In particular, a semismall map is generically finite, hence $\dim X = \dim Y$. Also, the dimension of a fiber is at most half the dimension of X, and equality can only occur at finitely many fibers. Surjective maps of surfaces are always semismall. While these maps may seem quite simple, there are many important examples, such as the Springer resolution of the nilpotent cone of a semisimple Lie algebra.

Proposition 4.2. *Let X be nonsingular. A map $f : X \to Y$ is semismall if and only if $Rf_*\mathbb{Q}_X[n]$ is perverse on Y*

Proof. Since f is proper, $Rf_* = Rf_!$, so $\mathcal{D}(Rf_*\mathbb{Q}_X[n]) = Rf_*(\mathcal{D}\mathbb{Q}_X[n]) = Rf_*\mathbb{Q}_X[n]$, where $\mathcal{D}$ denotes the Verdier dual functor. Thus by remark 3.4, it suffices to show the support condition for $Rf_*\mathbb{Q}_X[n]$, i.e. that

$$\mathcal{H}^k(Rf_*\mathbb{Q}_X[n])_y = 0 \quad \text{for } k > -\dim S \tag{4.3}$$

for every stratum S and $y \in S$. By proper base change, $\mathcal{H}^k(Rf_*\mathbb{Q}_X[n])_y \simeq H^{k+n}(f^{-1}(y))$, hence (4.3) holds if and only if

$$2\dim f^{-1}(y) \leq n - \dim S$$

for all S, i.e. if and only if f is semismall. □

For the remainder of the report, $f : X \to Y$ will denote a semismall map, with X projective and nonsingular of dimension n.

4.1. Intersection forms associated to strata.

Recall that our discussion of the topology of varieties in section 2.2 revolved around the nondegeneracy of certain intersection forms. In the following sections we'll extend this philosophy to study the topology of semismall maps. The first step is to define the analogous intersection forms. Specifically, we will define an intersection form I_S associated to every stratum S of Y.

We continue to let $f : X \to Y$ be semismall, but the following can be generalized to arbitrary proper maps (see [8][7]). Let $y \in Y$, and denote the closed inclusions by $\alpha : \{y\} \hookrightarrow Y$ and $i : f^{-1}(y) \hookrightarrow X$. For any k, there are natural maps

$$H_{n-k}(f^{-1}(y)) \xrightarrow{i_!} H^{BM}_{n-k}(X) \overset{PD}{\simeq} H^{n+k}(X) \xrightarrow{i^*} H^{n+k}(f^{-1}(y)).$$

We define the *refined intersection form* on $H_{n-k}(f^{-1}(y))$ to be the induced bilinear form

$$H_{n-k}(f^{-1}(y)) \times H_{n+k}(f^{-1}(y)) \to \mathbb{R}.$$

Geometrically, this form corresponds to taking cycles supported on $f^{-1}(y)$, and computing their intersection number on X. Note that if y lies on a positive-dimensional stratum S, then the refined intersection form at y is necessarily zero. Indeed, by the local triviality of $f^{-1}(S) \to S$, any two cycles on $f^{-1}(y)$ can be moved to cycles lying on disjoint fibers.

A more interesting intersection form is one obtained by letting cycles intersect not in all of X but just in $f^{-1}(N_y)$, where N_y is a normal slice to S at y. Let $d := \dim S$. Then the corresponding sequence of maps is

$$H_{n-d-k}(f^{-1}(y)) \xrightarrow{i_!} H^{BM}_{n-d-k}(f^{-1}(N_y)) \overset{PD}{\simeq} H^{n-d+k}(f^{-1}(N_y)) \xrightarrow{i^*} H^{n-d+k}(f^{-1}(y)).$$

We'll take $k = 0$, so we get a symmetric bilinear form

$$I_{S,y} : H_{n-d}(f^{-1}(y)) \times H_{n-d}(f^{-1}(y)) \to \mathbb{R}. \tag{4.4}$$

This is just the middle-dimension refined intersection form for the map $f^{-1}(N_y) \to N_y$, where now $\{y\}$ is its own zero dimensional stratum. Since f is topologically locally trivial along S, the form (4.4) induces a symmetric bilinear form on local systems (after taking duals)

$$I_S : (R^{n-d} f_* \mathbb{R}_X)_{|S} \times (R^{n-d} f_* \mathbb{R}_X)_{|S} \to \underline{\mathbb{R}}.$$

We call I_S the intersection form associated to S. The following proposition will be crucial in the next section.

Proposition 4.5. *Let $i : S \hookrightarrow Y$ denote the closed embedding. The intersection form $I_{S,y}$ is nondegenerate if and only if the natural map of stalks*

$$\mathcal{H}^{-d}(i_! i^! f_* \mathbb{R}_X[n])_y \to \mathcal{H}^{-d}(f_* \mathbb{R}_X[n])_y$$

is an isomorphism.

Proof. Let $B := B_\varepsilon(y)$ be a small Euclidean neighborhood of y, and let $S' := S \cap B$. The map in the proposition is identified with the natural map in relative cohomology

$$H^{n-d}(f^{-1}(B), f^{-1}(B - S')) \to H^{n-d}(f^{-1}(B)).$$

The proposition then follows from the discussion in [3, pp. 15-16]. □

4.2. **The decomposition theorem.** Deligne's decomposition theorem (theorem 2.21), shows that the derived pushforward of a constant sheaf along a smooth map splits completely, as a sum of its shifted cohomology sheaves. Recall that this had strong implications for the topology of smooth maps, in particular showing that their cohomology is "as simple as possible." The conclusion of theorem 2.21 fails for non-smooth maps, but there is a replacement called the *decomposition theorem*, which says that the derived pushforward of the constant sheaf along any proper map splits as a sum of shifted simple IC sheaves. This theorem beautifully illustrates the technology of perverse sheaves, and how intersection cohomology is the "correct" replacement for ordinary cohomology.

The goal of this section is to prove the decomposition theorem for semismall maps, modulo theorem 4.10, whose proof will occupy the next section.

We will need the following lemma of Cataldo and Migliorini [3, Proposition 3.1.2].

Proposition 4.6. *Let $C \xrightarrow{u} A \xrightarrow{v} B \xrightarrow{[1]}$ be a distinguished triangle in some derived category, and let k be such that $A \simeq \tau_{\leq k} A$ and $C \simeq \tau_{\geq k} C$. Then $\mathcal{H}^k(u) : \mathcal{H}^k(C) \to \mathcal{H}^k(A)$ is an isomorphism if and only if*

$$A \simeq \tau_{\leq k-1} B \oplus \mathcal{H}^k(A)[-k]$$

and the map v is the projection onto the first factor.

We'll prove the decomposition theorem inductively, by attaching one stratum at a time and checking the decomposition at each stage. The theory of stratifications says that after fixing a Whitney stratification of Y compatible with f, the variety Y admits a filtration

$$Y = Y_n \supseteq Y_{n-1} \supseteq \cdots \supseteq Y_1 \supseteq Y_0 \supseteq Y_{-1} = \emptyset$$

by closed subvarieties such that

$$S_l := Y_d - Y_{d-1}$$

is the union of the dimension-d strata of Y. Our induction step is the following:

Theorem 4.7. *Let* $U' := Y - Y_{d-1}$*. Let* $i : S := S_d \hookrightarrow U'$ *denote the closed inclusion of the union of the d-dimensional strata, and let* $j : U := Y - Y_d \hookrightarrow U'$ *denote the complementary open inclusion. Then we have a decomposition*

$$Rf_*\mathbb{Q}_X[n]_{|U'} \simeq j_{!*}(Rf_*\mathbb{Q}_X[n]_{|U}) \oplus (R^{n-d}f_*\mathbb{Q}_X)_{|S})[d]$$

if and only if the intersection form I_S *over each connected component of* S *is nondegenerate.*

Proof. Consider the attaching triangle

$$i_!i^!(Rf_*\mathbb{Q}_X[n]_{|U'}) \xrightarrow{u} Rf_*\mathbb{Q}_X[n]_{|U'} \xrightarrow{v} j_*(Rf_*\mathbb{Q}_X[n]_{|U}) \xrightarrow{[1]} .$$

The complex $Rf_*\mathbb{Q}_X[n]_{|U'}$ is perverse by proposition 4.2, so the support condition implies that

$$\tau_{\leq -d}Rf_*\mathbb{Q}_X[n]_{|U'} \simeq Rf_*\mathbb{Q}_X[n]_{|U'}.$$

Also, the cosupport condition over S implies that

$$\tau_{\geq -d}i^!Rf_*\mathbb{Q}_X[n]_{|U'} = i^!Rf_*\mathbb{Q}_X[n]_{|U'}.$$

Thus we are in the setup of proposition 4.6, which says that $\mathcal{H}^{-d}(u)$ is an isomorphism if and only if

$$\begin{aligned} Rf_*\mathbb{Q}_X[n]_{|U'} &\simeq \tau_{-d-1}j_*(Rf_*\mathbb{Q}_X[n]_{|U}) \oplus \mathcal{H}^{-d}(Rf_*\mathbb{Q}_X[n]_{|U'})[d] \\ &\simeq j_{!*}(Rf_*\mathbb{Q}_X[n]_{|U}) \oplus (R^{n-d}f_*\mathbb{Q}_X)_{|S}[d], \end{aligned}$$

where in the second isomorphism we use the support condition on the second direct summand. Now by proposition 4.5, the maps of stalks

$$\mathcal{H}^{-d}(u) : \mathcal{H}^{-d}(i_!i^!(Rf_*\mathbb{Q}_X[n]_{|U'}))_y \to \mathcal{H}^{-d}(Rf_*\mathbb{Q}_X[n]_{|U'})_y$$

is identified stalkwise with the intersection form $I_{S,y}$, for every $y \in S$, so the proposition follows. □

Definition 4.8. A stratum S of Y is called *relevant* if $\dim S + 2\dim f^{-1}(y) = n$ for any $y \in S$, i.e. equality holds in the definition of semismallness.

Note that relevant strata have even codimension, and that the open dense stratum is always relevant.

Remark 4.9. The local system $(R^{n-\dim S}f_*\mathbb{Q}_X)_{|S}$ occurring in theorem 4.7 is dual to the local system with stalk $H^{BM}_{n-\dim S}(f^{-1}(y))$, for $y \in S$. If S is not relevant, this is just the zero local system. If S is relevant, this is the local system generated by the monodromy action on the irreducible components of maximal dimension of $f^{-1}(y)$. Monodromy acts by homeomorphisms on $f^{-1}(y)$, so it permutes the irreducible components of maximal dimension. It follows that the associated monodromy representation factors through a finite symmetric group, and so $(R^{n-\dim S}f_*\mathbb{Q}_X)_{|S}$ is in fact a semisimple local system.

Theorem 4.7 has put us in the situation where we can inductively prove the decomposition theorem, granted we know the nondegeneracy of I_S at each stage. The goal of the next section is to prove the following result, which lets us do exactly this.

Theorem 4.10. *(Semismall index theorem) Let* $f : X \to Y$ *be a semismall map between complex projective varieties of even dimension* $n = 2m$*, with* X *nonsingular. Let* y *be a point of* Y *such that* $\dim f^{-1}(y) = m$*. Then the intersection form at* y

$$I_{\{y\},y} : H_n(f^{-1}(y)) \times H_n(f^{-1}(y)) \to \mathbb{Q}$$

is $(-1)^m$ *definite.*

Now the decomposition theorem follows:

Theorem 4.11 (Decomposition theorem for semismall maps)**.** *Let* $f : X \to Y$ *be a semismall map between complex projective varieties of dimension* n*, with* X *nonsingular. Let* Y_{rel} *denote the set of relevant strata of* Y*. Then there is a canonical decomposition*

$$Rf_*\mathbb{Q}_X[n] \simeq \bigoplus_{S \in Y_{rel}} IC_{\overline{S}}(\mathcal{L}_S),$$

where

$$\mathcal{L}_S = (R^{n-\dim S}f_*\mathbb{Q})_{|S}.$$

Moreover, the local systems $\mathcal{L}_S$ *are all semisimple.*

Proof. The statement on semisimplicity follows from remark 4.9. We proceed inductively on the strata of Y, starting with the largest stratum, and attaching strata of lower dimensions one at a time. By theorem 4.7, it is enough to show that the intersection form I_S is degenerate, for every stratum S of Y, for then we get the necessary decomposition at each stage. If S is not relevant, $\mathcal{L}_S$ is the zero local system by remark 4.9, so let's assume that S is relevant of dimension d.

We'll show that $I_{S,y}$ is nondegenerate for a given $y \in S$. Let H be the complete intersection of d generic hyperplane sections sections in Y, all passing through y. Let $g : f^{-1}(H) \to H$ be the restriction of f, which is still a semismall map. Note that $\{y\}$ is now a zero dimensional stratum of H with respect to g. Also, the intersection of H with a small euclidean neighborhood of y is a normal slice to y, so the intersection form for f at y is identified with the intersection form for g at y. Hence it suffices to show that the intersection form for g at y is nondegenerate.

Since S is relevant, $\dim H = \dim f^{-1}(H) = n - d$ is even, and $\dim g^{-1}(y) = \frac{n}{2}$. Moreover, [3, Proposition 2.17] ensures that $f^{-1}(H)$ can be chosen to be nonsingular. Thus g satisfies the hypotheses of theorem 4.10, and the required nondegeneracy follows. ☐

4.3. **The semismall index theorem via Hodge theory.** In this section we prove the semismall index theorem (theorem 4.10), completing the proof of the decomposition theorem for semismall maps. The tool we use is a version of Hodge-Lefschetz theory for semismall maps, developed by de Cataldo and Migliorini in [3].

We continue to let $f : X \to Y$ be a semismall map, with X nonsingular and projective of dimension n. Let L be an ample line bundle on Y, and let $L' := f^*L$.

If f has any positive dimensional fibers, say $f^{-1}(y)$, then L' is not ample, for then L' would be trivial on $f^{-1}(y)$. However, one of the insights of [3] is that L' behaves very much like an ample line bundle, with respect to Hodge-Lefschetz theory. In [3], such line bundles are called *lef* (*Lefschetz effettivamente funziona*). As we explain below, lef line bundles satisfy the conclusions of weak Lefschetz, hard Lefschetz, and the Hodge-Riemann relations.

Theorem 4.12. *Let $f : X \to Y$ be a semismall morphism with X smooth and projective of dimension n. Let L be an ample line bundle on Y, and let $L' := f^*L$. Let D be a generic hyperplane section of Y, corresponding to the zero locus of a generic section of L, and let be $D' := f^{-1}(D)$ be the zero locus of a generic section section of L'. Let $i : D \hookrightarrow Y$ denote the inclusion.*

(a) Weak Lefschetz holds on X with respect to D'.
(b) Hard Lefschetz holds for L' on X.
(c) The Hodge-Riemann relations hold for L' on X. Explicitly, for $0 \le r \le n$, let $P^{n-r}_{L'}(X) := \ker L'^{r+1} \subseteq H^{n-r}(X)$. Then $P^{n-r}_{L'}(X)$ is polarized by the bilinear form

$$\Psi_{L'}(-,-) := (-1)^{\frac{(n-r)(n-r+1)}{2}} \int_X L'^r \wedge - \wedge -.$$

Proof. (a) Since X is smooth, $\mathbb{R}_X[n]$ is perverse on X, and since f is semismall, proposition 4.2 implies that $Rf_*\mathbb{R}_X[n]$ is perverse on Y. Thus by theorem 3.16, weak Lefschetz holds for the restriction

$$H^{n+k}(X) = \mathbb{H}^k(Y; Rf_*\mathbb{R}_X[n]) \xrightarrow{i^*} \mathbb{H}^k(D; i^*Rf_*\mathbb{R}_X[n]) = H^{n+k}(D')$$

and the pushforward

$$H^{n+k}(D') = \mathbb{H}^k(D; i^!Rf_*\mathbb{R}_X[n]) \xrightarrow{i_!} \mathbb{H}^k(Y, Rf_*\mathbb{R}_X[n]) = H^{n+k}(X).$$

We prove (b) and (c) simultaneously, by induction on n:
(b in case $\dim X = 1$) X is a curve, L' is ample, and f is finite, so there is nothing to prove.
(b in case $\dim X = n$, assuming that (b) and (c) hold for $\dim X = n-1$) Just as in the proof of proposition 2.3, we can factor the map $L'^r : H^{n-r}(X) \to H^{n+r}(X)$ as

$$\begin{array}{ccc} H^{n-r}(X) & \xrightarrow{\quad L'^r \quad} & H^{n+r}(X) \\ {\scriptstyle i^*}\searrow & & \nearrow{\scriptstyle i_!} \\ H^{n-r}(D') & \xrightarrow{L'^{r-1}} & H^{n+r-2}(D') \end{array}.$$

Weak Lefschetz holds for i^* and $i_!$ by (a). Also, [3, Proposition 2.1.7] guarantees that a generic D' is smooth, so we can inductively assume that (b) and (c) hold for the semismall map $D' \to D$. That is, hard Lefschetz and Hodge-Riemann hold for L' on D'. At this point, we are exactly in the setup of the inductive "proof" of hard Lefschetz of proposition 2.3, and an identical argument follows.
(c in case $\dim X = n$, assuming that (b) holds for $\dim X = n$) For $r > 0$, recall that the fact that

$$i^* : P^{n-r}_{L'}(X) \to P^{n-r}_{L'}(D)$$

is an injection of Hodge structures, implies that $\Psi_{L'}$ is a polarization of $P^{n-r}_{L'}(X)$. Thus it remains to prove the case of $r = 0$. Recall that in proposition 2.3 we had no way of doing this. In this case however, there is an amazing approximation trick which allows us to close the induction:

Let η be an ample line bundle on X. For all $\varepsilon > 0$, $L' + \varepsilon\eta$ is an ample line bundle on X by [17, Corollary 1.4.10]. So by usual Hodge-Lefschetz theory, $\Psi_{L'+\varepsilon\eta}$ defines a polarization of the subspace

$$P^n_\varepsilon := \ker((L' + \varepsilon\eta) : H^n(X) \to H^{n+2}(X)).$$

Additionally, the primitive Lefschetz decomposition holds for $L' + \varepsilon\eta$. The analogous decomposition also holds for L', by assumption that hard Lefschetz holds for L'. Hence

$$\dim P^n_\varepsilon = P^n_{L'} = \dim H^n(X) - \dim H^{n-2}(X) \quad \text{for all } \varepsilon > 0.$$

It follows that in the Grassmannian of $\dim H^n(X) - \dim H^{n-2}(X)$ dimensional subspaces of $H^n(X)$, we have

$$\lim_{\varepsilon \to 0} P^n_\varepsilon = P^n_{L'}. \tag{4.13}$$

The fact that $\Psi_{L'+\varepsilon\eta}$ is a polarization of P^n_ε implies that $\Psi_{L'+\varepsilon\eta}(-, C(-))$ is positive-definite on P^n_ε, where C denotes the Weil operator (recall definition 2.2). Thus (4.13) implies that $\Psi_{L'}(-, C(-))$ is positive semi-definite on $P^n_{L'}$.

Hard Lefschetz for L' implies that $\Psi_{L'}$ is nondegenerate on $H^n(X)$. By orthogonality of the primitive Lefschetz decomposition for L', $\Psi_{L'}$ is still nondegenerate on $P^n_{L'}$. The form $\Psi_{L'}(-, C(-))$ is then nondegenerate on $P^n_{L'}$ and also symmetric, so it follows that $\Psi_{L'}(-, C(-))$ is positive definite. Hence $\Psi_{L'}$ is a polarization of $P^n_{L'}$. □

In fact, lef line bundles are characterized by the fact that they satisfy hard Lefschetz. The point is that if f has "large fibers," then L' will be trivial on them, and thus cupping with L' kills too much cohomology. The following proposition gives a converse to theorem 4.12(b).

Proposition 4.14. *Let $f : X \to Y$ be surjective and projective, let L be a line bundle on Y, and let $L' := f^*L$. If L' satisfies hard Lefschetz, then f is semismall.*

Proof. Suppose that f were not semismall. Let S be a stratum of Y that violates the condition of semismallness, i.e. $d_S > \frac{\dim X - \dim S}{2}$, where $d_S := \dim f^{-1}(y)$ for any $y \in S$. Let T be an irreducible subvariety of $f^{-1}(S_k)$ of maximal dimension, so

$$\dim T = \dim S + d_S > \frac{\dim S + \dim X}{2}.$$

In particular, the dimension of T is greater than half the dimension of X. Let $[S]$ be the fundamental class of S, and let $[T] := f^*([S])$ be the fundamental class of T. Then

$$c_1(L')^{\dim S} \cup [T] = f^*(c_1(L)^{\dim S} \cup [S]) = 0.$$

As $\dim S \leq 2\dim T - \dim X$, this violates hard Lefschetz for L'. □

We now turn to the setting of the semismall index theorem (theorem 4.10). Let $f : X \to Y$ be a semismall map, where now X has even dimension $n = 2m$. Suppose that $y \in Y$ is a point such that $F := f^{-1}(y)$ has dimension m. Let $F_1, \ldots, F_r$ denote the irreducible components of maximal dimension of F. The fundamental classes $[F_1], \ldots, [F_r]$ are a basis for $H_m(F)$.

Consider the cycle class map

$$cl : H_n(F) \to H^{BM}_n(X) \xrightarrow{PD} H^n(X).$$

Lemma 4.15. *The map cl is injective.*

Before proving lemma 4.15, we state a theorem from mixed Hodge theory, which de Cataldo and Migliorini aptly call the "weight miracle."

Theorem 4.16. *Let $Y \hookrightarrow U \hookrightarrow X$ be maps of complex varieties, with X projective and nonsingular, U Zariski-open in X, and Y a projective sub-variety. Then the natural restriction maps*

$$H^k(X) \to H^k(Y), \quad H^k(U) \to H^k(Y)$$

have identical images.

Proof. This follows from the strictness of morphisms of mixed Hodge structures. See [25, Proposition 4.23]. □

Proof of lemma 4.15. Since the Poincare duality map PD is an isomorphism, it is enough to show that the dual restriction map

$$H_n^{BM}(X) \to H_n(F)$$

is surjective. Let $U \subset Y$ be an open affine neighborhood of y, and let $U' := f^{-1}(U)$. Let $i : \{y\} \hookrightarrow U$ denote the inclusion, and $j : U - \{y\} \hookrightarrow U$ denote the inclusion of the open complement. Now consider the distinguished triangle

$$j_! j^! Rf_* \mathbb{R}_{U'}[n] \to Rf_* \mathbb{R}_{U'}[n] \to i_* i^* Rf_* \mathbb{R}_{U'}[n] \xrightarrow{[1]},$$

where by abuse of notation f denotes the restriction of f to U'. Artin vanishing for constructible complexes implies, in particular, that $\mathbb{H}^k(U, j_! j^! Rf_* \mathbb{R}_{U'}[n]) = 0$ for $k > 0$. Thus it follows from the long exact sequence of hypercohomology that the restriction map

$$H^n(U') = \mathbb{H}^0(U, Rf_* \mathbb{R}_{U'}[n]) \to \mathbb{H}^0(U, i_* i^* Rf_* \mathbb{R}_{U'}[n]) = H^n(F)$$

is surjective. By the "weight miracle" (theorem 4.16), the images of the restrictions $H^n(X) \to H^n(F)$ and $H^n(U) \to H^n(F)$ agree. Hence $H^n(X) \to H^n(F)$ is surjective and we're done. □

Now note that the local intersection form

$$I_{\{y\},y} : H_n(F) \times H_n(F) \to \mathbb{R}$$

is just the middle-dimension intersection form on $H^n(X)$, precomposed with cl. Intuitively, the local intersection $cl([F_i]) \cdot cl([F_j])$ corresponds to moving the cycles $[F_i]$ and $[F_j]$ on X until they intersect transversely, and then computing their intersection on X. Now we can prove the semismall index theorem:

Proof of theorem 4.10. Pick an ample line bundle L on Y, and let $L' := f^* L$. We can assume that a generic hyperplane section of Y does not intersect the point y, and thus the zero locus of a generic section of L' does not intersect F. Hence $cl(H_n(F))$ belongs to the subspace of primitive cohomology with respect to L'. Furthermore, F is a union of algebraic cycles, so the image of cl consists of classes of pure Hodge type (m, m). Finally, since cl is injective by lemma 4.15, we can identify $H_n(F)$ as a Hodge sub-structure

$$H_n(F) \subseteq P^n_{L'} \cap H^{m,m}(X).$$

By theorem 4.12, $P^n_{L'}$ is polarized by

$$\Psi_{L'}(-,-) = (-1)^{\frac{n(n+1)}{2}} \int_X - \wedge - = (-1)^m \int_X - \wedge -.$$

Thus $\Psi_{L'}(-, C(-))$ is positive definite on $P^0_{L'}$, but C acts as the identity on $H^{m,m}(X)$, so $\Psi_{L'}$ is positive definite. It follows that the intersection form is $(-1)^m$ definite on $P^n_{L'} \cap H^{m,m}(X)$, and the same holds on $H_n(F)$. □

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References

[1] Donu Arapura. *Algebraic geometry over the complex numbers*. Springer, 2012.
[2] A. A. Beilinson, J. Bernstein, and Pierre Deligne. Perverse sheaves. *Astérisque*, 100, 1982.
[3] Mark Andrea de Cataldo and Luca Migliorini. The hard lefschetz theorem and the topology of semismall maps, 2000.
[4] Mark Andrea de Cataldo and Luca Migliorini. The decomposition theorem and the topology of algebraic maps, 2009.
[5] Mark Andrea A De Cataldo. *The Hodge Theory of projective manifolds*. World Scientific, 2007.
[6] Mark Andrea A de Cataldo. Perverse sheaves and the topology of algebraic varieties (2015 pcmi). *arXiv preprint arXiv:1506.03642*, 2015.
[7] Mark Andrea A. de Cataldo and Luca Migliorini. The hodge theory of algebraic maps, 2004.
[8] Mark Andrea A. de Cataldo and Luca Migliorini. Intersection forms, topology of maps and motivic decomposition for maps of threefolds, 2005.
[9] Pierre Deligne. Théoreme de lefschetz et criteres de dégénérescence de suites spectrales. *Publications Mathématiques de l'IHÉS*, 35:107–126, 1968.
[10] Pierre Deligne. Théorie de Hodge : II. *Publications Mathématiques de l'IHÉS*, 40:5–57, 1971.
[11] Alexandru Dimca. *Sheaves in topology*. Springer Science & Business Media, 2004.
[12] Mark Goresky. Lecture notes on sheaves and perverse sheaves. *arXiv preprint arXiv:2105.12045*, 2021.
[13] Mark Goresky and Robert MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980.
[14] Mark Goresky and Robert MacPherson. Intersection homology ii. *Inc. Mat*, 71:77–129, 1983.
[15] Daniel Huybrechts. *Complex geometry: an introduction*, volume 78. Springer, 2005.
[16] Frances Kirwan and Jonathan Woolf. *An introduction to intersection homology theory*. CRC Press, 2006.
[17] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer, 2017.
[18] Robert MacPherson and Kari Vilonen. Elementary construction of perverse sheaves. *Inventiones mathematicae*, 84(2):403–435, 1986.
[19] L.G. Maxim. *Intersection Homology & Perverse Sheaves: with Applications to Singularities*. Graduate Texts in Mathematics. Springer International Publishing, 2020.
[20] John Willard Milnor. *Morse theory*. Princeton university press, 1963.
[21] Konstanze Rietsch. An introduction to perverse sheaves, 2003.
[22] Morihiko Saito. Modules de hodge polarisables. *Publications of the Research Institute for Mathematical Sciences*, 24(6):849–995, 1988.
[23] Morihiko Saito. Decomposition theorem for proper kahler morphisms. *Tohoku Mathematical Journal, Second Series*, 42(2):127–148, 1990.
[24] C. Voisin and L. Schneps. *Hodge Theory and Complex Algebraic Geometry I: Volume 1*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002.
[25] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II: Volume 2*, volume 77. Cambridge University Press, 2003.
[26] Geordie Williamson. The hodge theory of the decomposition theorem (after de cataldo and migliorini), 2016.