A MOTIVIC APPROACH TO THE HEIGHT ONE TELESCOPE CONJECTURE

RUSHIL MALLARAPU

Abstract. The telescope conjecture, which attempts to unify algebraic and topological perspectives on chromatic homotopy theory, has motivated a wide body of research in the subject. In particular, the resolution of the height one telescope conjecture led to novel techniques in using auxiliary spectral sequences to resolve Adams spectral sequence differentials, which have recently been better understood and generalized by motivic and synthetic homotopy theory. We first provide an introduction to the motivic and synthetic categories. Then, we will reprove the height one telescope conjecture at odd primes to illustrate these modern differential lifting techniques. No background is assumed aside from a good familiarity with stable homotopy theory, localizations of categories, and Adams spectral sequences.

Contents

1. Introduction

The telescope conjecture in chromatic homotopy theory has been a motivating source of both formal and computational advances in the subject since its formulation in [Rav84]. It is a family of conjectures, one for each prime p and natural number n, asserting that "geometric" and "algebraic" versions of localizations of the category of spectra coincide. While forthcoming work by Burklund-Hahn-Levy-Schlank shows this conjecture is in fact false for $n \geq 2$, it is a classical result of

Date: December 14, 2023.

Miller that the conjecture holds for $n = 1$ and $p > 2$ [Mil81]. Our goal is to reexamine the key computation underlying this result, and along the way recount some of the major advances motivic methods have brought to computational homotopy theory.

First, a lightning introduction to the telescope conjecture. More details can be found in [Rav92, Lur10, Pst20].

Theorem 1.1. *For each prime p, there exist spectra $K(n)$ for each $n \geq 0$, the **Morava K-theories**, such that*

(1) $K(0)$ is the rational Eilenberg-Mac Lane spectrum $H\mathbf{Q}$ and $K(1)$ is the Adams summand of mod p complex K-theory.
(2) For $n > 0$, $\pi_* K(n) \simeq \mathbf{F}_p[v_n^{\pm}]$, where $|v_n| = 2(p^n - 1)$.
(3) If X is a finite spectrum which is $K(n)$-acyclic (i.e. $K(n) \otimes X \simeq 0$), then it is $K(n-1)$-acyclic.
(4) If the p-localization of a finite spectrum X is not contractible, then $K(n) \otimes X \not\simeq 0$ for $n \gg 0$.

These Morava K-theories interpolate between rational homology, $H\mathbf{Q}$, and mod p homology, $H\mathbf{F}_p$, thought of as "$K(\infty)$" in a way that can be made precise. We think of rational homology, and more particularly rationalization, as being something easy to understand and compute, while $H\mathbf{F}_p$-localization is much more interesting. Thus, in light of properties (3) and (4) of 1.1, we can ask how complicated a finite p-local spectrum is:

Definition 1.2. The **type** of a nontrivial finite p-local spectrum X is the smallest integer n such that $K(n) \otimes X \not\simeq 0$. If X has type n, then a v_n**-map** on X is a map

$$\Sigma^d X \xrightarrow{f} X$$

with $K(n) \otimes f$ an isomorphism and $K(m) \otimes f$ nilpotent for all $m \neq n$.

As it turns out, there are v_n-self maps: this is the content of the Periodicity Theorem of Hopkins-Smith [HS98].

Theorem 1.3. *Let X be a finite p-local spectrum of type $\geq n$. Then there exists a v_n-self map f which is asymptotically unique: if g is another v_n-self map, then some iterates of f and g are homotopic.*

As a consequence of 1.3, we can uniquely define the telescope $X[f^{-1}]$ as

$$\operatorname{colim}\left(X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \xrightarrow{f} \dots\right),$$

In fact, we can do better: there is a localization functor $L_{T(n)}$ on the category of p-local spectra which sends finite p-local spectra of type $\geq n$ to their telescope as above, and sends those of type $< n$ to 0. Correspondingly, this is called the **telescopic localization**, or $T(n)$**-localization**. It is clear that if X is $T(n)$-acyclic, then it must be $K(n)$-acyclic, so $K(n)$-local spectra must be $T(n)$-local, implying a natural transformation of localization functors $L_{T(n)} \to L_{K(n)}$.

Conjecture 1.4. *For every prime p and $n > 0$, the natural transformation $L_{T(n)} \to L_{K(n)}$ is an equivalence.*

Here, telescopic localizations carry interesting homotopy-theoretic information; in particular, the existence of certain type n complexes and their corresponding

v_n-self maps lets us construct elements in the stable homotopy groups of spheres. On the other hand, $K(n)$-localizations are more algebraic in nature, and can be accessed directly through the Adams and Adams-Novikov spectral sequence.

Note that the telescope conjecture is something of tautology for $n = 0$, in which case multiplication by p is a v_0-self map; to rationalize a finite p-local spectrum, it suffices to simply invert p. For $n = 1$ and $p = 2$, 1.4 was proven by Mahowald [Mah82]. The remainder of this paper will discuss the case for height $n = 1$ and odd primes $p > 2$, where 1.4 is due to [Mil81].

By formal arguments, for a fixed height n and prime p, it suffices to prove the equivalence in telescope conjecture for one p-local type n finite spectrum. At height 1, we can use the mod p Moore spectrum S^0/p, which is the cofiber of the multiplication by p map

$$S^0 \xrightarrow{p} S^0 \to S^0/p \to S^1.$$

In [Ada66], Adams constructed a v_1-self map, denoted $\phi\colon \Sigma^{2(p-1)}S^0/p \to S^0/p$, on this Moore spectrum, and Bousfield shows that the localization map $S^0/p \to S^0/p[\phi^{-1}]$ is a $K(1)$-localization [Bou79, 4.2], under the assumption of the following computation by Miller:

Theorem 1.5. [Mil81, 4.11] *The homotopy groups of $S^0/p[\phi^{-1}]$ are*

$$\pi_*\left(S^0/p[\phi^{-1}]\right) = \begin{cases} \mathbf{F}_p & * \equiv 0, -1 \mod 2(p-1) \\ 0 & \textit{otherwise.} \end{cases}$$

The techniques in establishing 1.5 were novel when [Mil81] was released, but now, thanks to motivic and synthetic homotopy theory, we can intrinsically understand what once were handcrafted spectral sequence comparision results.

Let us briefly outline the remainder of the paper. In section 2, we will give an introductory overview of motivic and synthetic homotopy theory. In section 3, we will discuss Miller's original techniques in establishing 1.5, and explain a motivically-accesible generalization of this method. Finally, in section 4, we will walk through the desired computations, aiming to illustrate techniques that have revolutionized computational homotopy theory more broadly.

2. The Motivic and Synthetic Categories

In this section, we will briefly discuss the category of $\mathbf{C}$-motivic spectra and the category of synthetic spectra, presenting them as deformations of the classical category of spectra. Then, we will state an equivalence between the p-completions of nice versions of these categories, letting us translate intuitions from one into the other in the subsequent sections.

2.1. **Motivic Stable Homotopy Theory.** Let us start by overviewing the construction of the category of motivic spectra. Like the classical category of spectra, it can be thought of as a stabilization of a suitable category of motivic spaces, which we discuss first

Definition 2.1. Let $\mathbf{Sm}/\mathbf{C}$ denote the category of smooth finite type $\mathbf{C}$-schemes. A **distinguished Nisnevich square** is a pullback diagram in $\mathbf{Sm}/\mathbf{C}$

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] & p^{-1}(X-U) \arrow[l] \arrow[d, "\cong"] \\ U \arrow[r, hook, "i"'] & X & X-U \arrow[l] \end{tikzcd}$$

where p is étale, i is an open embedding, and p is an isomorphism away from U.[1] A presheaf of spaces $F \in \mathbf{Pre}(\mathbf{Sm}/\mathbf{C})$ is a **Nisnevich sheaf** if for every distinguished Nisnevich square,

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \times_X V) \end{tikzcd}$$

is a pullback diagram of spaces, and $F(\emptyset)$ is a final object. Let $\mathbf{Shv}_{Nis}(\mathbf{Sm}/\mathbf{C})$ denote the category of Nisnevich sheaves.

Now, to get to the unstable category of motivic spaces, we need to further invert $\mathbb{A}^1$, thought of as a Nisnevich sheaf under the Yoneda embedding. This is analogous to inverting the interval $[0,1]$ in the ordinary category of spaces to form the homotopy category.

Definition 2.2. Let $\mathcal{J}$ be the class of projections $\{\mathbb{A}^1 \times F \to F\}$ in $\mathbf{Shv}_{Nis}(\mathbf{Sm}/\mathbf{C})$. The localization of $\mathbf{Shv}_{Nis}(\mathbf{Sm}/\mathbf{C})$ at $\mathcal{J}$ is the **category of motivic spaces**, denoted by $\mathbf{Spc}_{\mathbf{C}}$.

Explicitly, replacing $\mathbf{Sm}/\mathbf{C}$ by Nisnevich sheaves formally adjoints limits and colimits, giving a category that behaves like the familiar category of spaces, while forming the $\mathbb{A}^1$-localization means that for any motivic space $X \in \mathbf{Spc}_{\mathbf{C}}$, $X(U) \to X(\mathbb{A}^1 \times U)$ is an equivalence of spaces for all $U \in \mathbf{Sm}/\mathbf{C}$.

Example 2.3. Importantly, the Nisnevich topology is subcanonical, meaning that representable presheaves are sheaves. Thus, for every smooth finite type $\mathbf{C}$-scheme X, we get a corresponding motivic space, which we will also call X. This gives motivic spaces such as $\mathbb{A}^1$ or $\mathbf{G}_m$. Note that by construction, the former is contractible in $\mathbf{Spc}_{\mathbf{C}}$. In a different direction, we can sheafify and $\mathbb{A}^1$-localize constant presheaves of spaces, giving, for example the motivic spaces S^n.

Remark 2.4. Before proceeding, a few disclaimers are in order. First, the above definition of the Nisnevich topology as that generated by distinguished squares is actually a property of it's "true" definition, in which covers are étale covers with an additional condition on points having preimages with isomorphic residue fields. See [MV99, 1.1,1.4] and [AE16] for more details. However, under extremely mild conditions on the base scheme (in our case Spec $\mathbf{C}$), these two definitions coincide, giving a much more hands-on perspective on this topology.

To motivate *why* we consider the Nisnevich topology, note that it sits between the Zariski and étale topologies, giving it the best properties of both. For instance, fields have no higher cohomology and algebraic K-theory has descent, as in the Zariski topology, but also smooth pairs are locally equivalent to linear ones, as in the étale

[1]The right square in this diagram is the actual distinguished Nisnevich square.

topology. This latter fact is crucial for the proof of the motivic purity theorem, which is the motivic analog of the tubular neighborhood theorem in topology.

Now that we have an unstable category of motivic spaces, we can try to stabilize it. As a preliminary step, we really want to consider $\mathbf{Spc}_{\mathbf{C},*}$, the category of pointed motivic spaces, i.e. spaces X with a distinguished map $\operatorname{Spec}\mathbf{C} \to X$ of spaces; this enables us to form suspensions and loop spaces as the appropriate limit/colimit diagrams.[2]

However, here we see the first major difference between classical and motivic homotopy theory; in the latter, there are two circles. We have the simplicial circle S^1, viewed as a constant (pre)sheaf of spaces, and the "Tate" circle $\mathbf{G}_m$, which comes from geometry. We want to invert both of these spheres, and while it is possible to do these localizations one after the other, we can be more clever. First, a lemma:

Lemma 2.5. *In $\mathbf{Spc}_{\mathbf{C},*}$, the suspension of the pointed space $(\mathbf{G}_m, 1)$ is $(\mathbf{P}^1, \infty)$.*

Proof. We have a distinguished Nisnevich square

$$\begin{tikzcd} \mathbf{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & (\mathbf{P}^1, 1) \end{tikzcd}$$

in $\mathbf{Sm}/\mathbf{C}$, so considering these as motivic spaces, this becomes a pushout diagram in $\mathbf{Spc}_{\mathbf{C},*}$ [AE16, 4.13]. Now, as $\mathbb{A}^1 \simeq *$, because we've $\mathbb{A}^1$-localized, this exhibits $(\mathbf{P}^1, 1)$ as the suspension $\Sigma\mathbf{G}_m = S^1 \wedge \mathbf{G}_m$. As $(\mathbf{P}^1, 1) \simeq (\mathbf{P}^1, \infty)$ are $\mathbb{A}^1$-homotopic, we are done. □

Thus, to invert both circles, we can simply invert $(\mathbf{P}^1, \infty)$.[3] Hopefully this process is familiar from the construction of the ∞-category of spectra:

Definition 2.6. Let $\Omega_{\mathbf{P}^1} \colon \mathbf{Spc}_{\mathbf{C},*} \to \mathbf{Spc}_{\mathbf{C},*}$ be the functor on pointed motivic spaces given by $\operatorname{Map}((\mathbf{P}^1, \infty), -)$.[4] The **stable motivic category** is the ∞-categorical limit

$$\mathbf{Sp}_{\mathbf{C}} = \lim \left(\dots \xrightarrow{\Omega_{\mathbf{P}^1}} \mathbf{Spc}_{\mathbf{C},*} \xrightarrow{\Omega_{\mathbf{P}^1}} \mathbf{Spc}_{\mathbf{C},*} \xrightarrow{\Omega_{\mathbf{P}^1}} \mathbf{Spc}_{\mathbf{C},*} \right)$$

Explicitly, an object of this category is a $\mathbf{P}^1$-spectrum; a sequence of motivic spaces $X_0, X_1, X_2, \dots$ with chosen equivalences $X_n \simeq \Omega_{\mathbf{P}^1} X_{n+1}$. The homotopy category of $\mathbf{Sp}_{\mathbf{C}}$ is $\mathbf{SH}(\mathbf{C})$, the **stable motivic homotopy category**.

The following list of properties follow formally from this construction.

Proposition 2.7. *$\mathbf{Sp}_C$ is a stable symmetric monoidal ∞-category and there is a monoidal functor $\Sigma^\infty_{\mathbf{P}^1} \colon \mathbf{Spc}_{C,\cdot} \to \mathbf{Sp}_C$ mapping $(\mathbf{P}^1, \infty)$ to an invertible object. Moreover, there is an adjunction $\Sigma^\infty_{\mathbf{P}^1} \dashv \Omega^\infty_{\mathbf{P}^1}$, where $\Omega^\infty_{\mathbf{P}^1}$ sends a $\mathbf{P}^1$-spectrum X_* to its 0-th space X_0.*

[2]We are considering $\mathbf{Spc}_{\mathbf{C}}$ and $\mathbf{Spc}_{\mathbf{C},*}$ as ∞-categories, so one should think of limits and colimits as what would classically be called "homotopy" limits and colimits.

[3]Compare to the situation in classical algebra, where if $f, g \in A$ are two elements of a ring, we have a canonical isomorphism $\left(A[f^{-1}]\right)[g^{-1}] \simeq A[(fg)^{-1}]$.

[4]Here Map denotes the internal pointed motivic mapping space.

Remark 2.8. Some sources to construct $\mathbf{Sp_C}$ by inverting S^1, then inverting $\mathbf{G}^m$, and finally taking the $\mathbb{A}^1$-localization, e.g. [Voe07]. However, the approach here is equivalent, and has the benefit of being far more compact.

As a result, spheres in $\mathbf{Sp_C}$ are now bigraded, as is the suspension functor and homotopy groups. The conventions below are standard, and are chosen to agree with grading conventions for motivic cohomology.

Definition 2.9. We have bigraded spheres $S^{p,q} \in \mathbf{Sp_C}$, with $S^1 \simeq S^{1,0}$ and $\mathbf{G}_m \simeq S^{1,1}$. Thus, for instance, $\mathbf{P}^1 \simeq S^{2,1}$. The **bigraded homotopy groups** $\pi_{p,q}(X)$ of a spectrum X are the abelian groups of maps $\pi_{p,q}X := [S^{p,q}, X]$ in $\mathbf{SH}(\mathbf{C})$, and the **bigraded mapping set** $\pi_{p,q}\operatorname{Map}(X,Y)$ is defined as $[\Sigma^{p,q}X, Y]_{\mathbf{SH}(\mathbf{C})}$.

Definition 2.10. Let E be a motivic spectrum and p, q, n integers with $2q-p \geq 0$. The E-cohomology of a motivic space X is

$$E^{p,q}(X) := \pi_{-p,-q}\operatorname{Map}(\Sigma^\infty_{P^1} X_+, E) \simeq \pi_{2p-q}\operatorname{Map}(\Sigma^\infty_{\mathbf{P}^1} X_+, \Sigma^q_{\mathbf{P}^1} E).$$

Example 2.11. Voevodsky constructed a motivic spectrum $\mathbf{F}_p^{\mathrm{mot}}$ which represents mod p motivic cohomology, in the sense of 2.10. Importantly for us, $\pi_{*,*}\mathbf{F}_p^{\mathrm{mot}} \simeq \mathbf{F}_p[\tau]$, where τ is in degree $(0,-1)$. This element τ lifts to a map

$$\widehat{S^{0,-1}} \xrightarrow{\tau} \widehat{S^{0,0}}$$

between the $\mathbf{F}_p^{\mathrm{mot}}$-completed sphere spectra, inducing a nonzero map on motivic homology. The cofiber of this map $C\tau$ in the $\mathbf{F}_p^{\mathrm{mot}}$-completed stable motivic category, which we'll denote $\mathbf{Sp}_\mathbf{C}^\wedge$, is a commutative algebra object for all primes [Ghe17]. We'll come back to understanding τ once we discuss synthetic homotopy theory.

Finally, for our purposes, we'll want to restrict to a smaller subcategory of $\mathbf{Sp_C}$.

Definition 2.12. The **p-complete cellular motivic category** is the smallest stable subcategory of $\mathbf{Sp}_\mathbf{C}^\wedge$ which contains the p-complete spheres $\widehat{S^{p,q}}$ and is closed under arbitrary colimits.

Intuitively, cellular motivic spectra are those which admit a description internal to the homotopy theory of $\mathbf{Sp_C}$. In fact, using synthetic spectra, we can make this intuition precise.

2.2. **Synthetic Homotopy Theory.** Fix a ring spectrum E, where we'll ask that E_*E is flat over E_* (more specifically, that E is Adams type.[5]) Classically, we can construct a stack

$$\mathscr{M}_E := \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} \operatorname{Spec}\left(\pi_*(E^{\otimes[n]}\right), \tag{2.13}$$

where for any spectrum X, the E-homology E_*X descends to a quasicoherent sheaf over $\mathscr{M}_E$. The Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathscr{M}_E}(E_*X, E_*Y) \Rightarrow [X, Y_E^\wedge]_{t-s} \tag{2.14}$$

lets us go back, from algebraic geometry over a stack to homotopy theory. The idea of synthetic spectra is to generalize this phenomena, and construct a "deformation" of the category of ordinary spectra into a purely algebraic category based on sheaves over $\mathscr{M}_E$, or as more commonly formulated in the literature, E_*E-comodules.

[5]This means E is a filtered colimit of finite E-projective spectra E_α which all satisfy a universal coefficient isomorphism; namely that $E^*E_\alpha \to \operatorname{Hom}_{E_*}(E_*E_\alpha, E_*)$ is an isomorphism. The sphere, any field, and Landweber-exact homology theories are all Adams-type.

Theorem 2.15. [Pst22, Mar22] *For E an Adams-type ring spectrum, there is a presentable symmetric monoidal stable ∞-category $\mathbf{Syn}_E$ of E-**synthetic spectra** and a diagram of functors*

$$\begin{tikzcd} \mathbf{Sp} \arrow[rr, equal] \arrow[dr, "\nu"'] & & \mathbf{Sp} \\ & \mathbf{Syn}_E \arrow[ur, "\tau^{-1}"'] & \end{tikzcd}$$

with (1) ν fully faithful, lax symmetric monoidal, and preserving filtered colimits and (2) τ^{-1} symmetric monoidal and preserving all colimits.

Remark 2.16. ν is symmetric monoidal on the subcategory of finite E-projectives. So when E is a field, e.g. $E = \mathbf{F}_p$ or $K(n)$, ν is symmetric monoidal on all finite spectra. In any case, ν carries commutative ring spectra to synthetic commutative ring spectra by lax monoidality.

In synthetic spectra, there are bigraded spheres, arising from taking the "derived" category of an already stable ∞-category:

Definition 2.17. The **bigraded synthetic sphere spectrum** $S^{t,w}$ is defined as $\Sigma^{t-w}\nu S^w$. The monoidal unit is $S^{0,0}$ and the bigraded suspension functor is $\Sigma^{t,w} = S^{t,w} \otimes -$. The bigraded homotopy groups are

$$\pi_{t,w} := [S^{0,0}, X]_{t,w} = \pi_0 \mathrm{map}(S^{t,w}, X).$$

The choice of the $t-w$ grading is motivated partly by comparision to the cellular motivic category, as described below. In fact, in light of the perspective of synthetic spectra as a deformation of spectra, we can abstractly describe this "deformation parameter."

Construction 2.18. The pushout comparision map

$$S^{0,-1} = \Sigma(\nu S^{-1}) \to \nu(\Sigma S^{-1}) = S^{0,0}$$

gives a canonical element $\tau \in \pi_{0,-1} S^{0,0}$. The cofiber $C\tau$ is a synthetic commutative ring spectrum.

Generically, if we invert τ, we just recover ordinary spectra:

Theorem 2.19. [Pst22, 4.33, 4.37] *Let $\mathbf{Syn}_E(\tau^{-1})$ be the subcategory of synthetic spectra on which τ acts as an equivalence. Then*

(1) $\mathbf{Syn}_E(\tau^{-1})$ is a localization of $\mathbf{Syn}_E$,

(2) there is a fully faithful functor $Y : \mathbf{Sp} \to \mathbf{Syn}_E$ giving an equivalence onto $\mathbf{Syn}_E(\tau^{-1})$,

*(3) defining the **underlying spectrum functor** $\tau^{-1} : \mathbf{Syn}_E \to \mathbf{Sp}$ via*

$$\tau^{-1}X = \mathrm{colim}\left(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \to \dots\right)$$

under this equivalence gives a symmetric monoidal left adjoint to Y,

(4) for any spectrum X, $\tau^{-1}(\nu X) \simeq X$.

One way to interpret this synthetic τ is that classical spectra are embedded into synthetic spectra as precisely those which are τ-invertible. This element arises from the different notions of suspension on a synthetic spectrum, and when this difference disappears, so do the non-topological phenomona in $\mathbf{Syn}_E$. Thus, any algebraic behavior in $\mathbf{Syn}_E$ is effectively τ-torsion.

While generically, $\mathbf{Syn}_E$ behaves like ordinary spectra, at the special fiber, i.e. taking modules over $C\tau$, we get a fully faithful monoidal embedding $\mathbf{Mod}_{C\tau}(\mathbf{Syn}_E) \to \mathbf{Stable}_{E_*E}$, where the right side is Hovey's stable ∞-category of E_*E-comodules, which is purely algebraic in nature [Pst22, 4.46]. This embedding is an equivalence if E is Landweber exact. As a practical consequence of this fact, we have the following theorem:

Theorem 2.20. *For any $X, Y \in \mathbf{Sp}$, there is a cofiber sequence*

$$\Sigma^{0,-1}\nu X \xrightarrow{\tau} \nu X \to \nu X \otimes C\tau \to \Sigma^{1,-1}\nu X \tag{2.21}$$

and an natural isomorphism $[\nu Y, \nu X \otimes C\tau]_{t,w} \simeq \mathrm{Ext}^{w-t,t}_{E_*E}(E_*Y, E_*X)$.

Notice that $\pi_{t,w}(\nu X \otimes C\tau) \simeq \mathrm{Ext}_{E_*E}(E_*, E_*X)$ is (up to regrading) the E_2-page for the E-Adams spectral sequence converging to $\pi_*(X^\wedge_E)$. This is just the surface of a surprising relation to the Adams spectral sequence: as a preliminary, note that taking homotopy groups of 2.21 gives an exact couple leading to a τ-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}_{E_*E}(E_*, E_*X) \Rightarrow \pi_{*,*}(\nu X^\wedge_\tau).$$

Alternatively, we could construct a νE-Adams spectral sequence internal to $\mathbf{Syn}_E$ computing $\pi_{*,*}(\nu X^\wedge_{\nu E})$. These spectral sequences are related as follows:

Theorem 2.22. [BHS22, A] *Let X be a spectrum.*

(1) There is a trigraded νE-Adams spectral sequence

$$E_2^{s,t,w} \simeq {}_{cl}E_2^{s,t} \otimes \mathbf{Z}[\tau] \Rightarrow \pi_{*,*}(\nu X^\wedge_{\nu E}) \tag{2.23}$$

with ${}_{cl}E_2^{s,t}$ the classical Adams E_2-page in tridegree (s,t,s) and $|\tau| = (0,0,-1)$ The synthetic differentials in this spectral sequence are $d_r(x) = \tau^{r-1}y$ for every classical differential $d_{r,cl}(x) = y$.

(2) X is E-nilpotent complete iff νX is νE-nilpotent complete iff it is τ-complete.

(3) Given X is E-nilpotent complete, the E-Adams spectral sequence, the νE-Adams spectral sequence, and the τ-Bockstein spectral sequence converge strongly, and the latter spectral sequences are isomorphic (up to regrading).

This leads to the motto that synthetic spectra "categorify" the E-Adams spectral sequence; the homotopy groups $\pi_{*,*}(\nu X^\wedge_{\nu E})$ and module structure over $\pi_{*,*}(S^{0,0})^\wedge_{\nu E}$ capture the differentials and extensions in the E-Adams spectral sequence via the presence of τ-torsion.

2.3. **Motivic and Synthetic Comparisions.** For our purposes, a key motivating property of the synthetic category is the following equivalence, proved in [Pst22, 7.34]:

Theorem 2.24. *For each prime p, there is an adjoint equivalence between the p-complete category of cellular $\mathbf{C}$-motivic spectra and the p-complete category of even MU-synthetic spectra.*

This equivalence qualifies our statement following 2.12: cellular motivic spectra are part of stable motivic homotopy theory that can be described internal to "ordinary" homotopy theory, and synthetic machinery gives us a handle on that description.[6] For simplicity, fix a prime p for the remainder of this section, and let everything in sight be p-complete.

Given the discussion in the previous subsection, one should expect that the element τ discussed in 2.11 has similar properties to the synthetic τ. Indeed, letting $\mathbf{Sp}_{\mathbf{C},cell}$ denote the (p-complete) cellular motivic category, we have functors

$$\tau^{-1}\mathbf{Sp}_{\mathbf{C},cell} \xleftarrow{\tau^{-1}} \mathbf{Sp}_{\mathbf{C},cell} \xrightarrow{-\otimes C\tau} \mathbf{Sp}_{\mathbf{C},cell}/\tau \tag{2.25}$$

with the left arrow being the generic fiber and the right being the special fiber. Importantly, we can identify the categories on the far sides of this span:

Definition 2.26. The **realization functor** $R\colon \mathbf{Sp}_{\mathbf{C}} \to \mathbf{Sp}$ is the one induced by the functor sending a smooth $\mathbf{C}$-scheme X to the topological space of its complex points $X(\mathbf{C})$. For instance, $R(S^{p,q}) = S^p$ and $R(\mathbf{F}_p^{\mathrm{mot}}) = \mathbf{F}_p$.

Proposition 2.27. *The category of τ-invertible cellular motivic spectra is equivalent to the ordinary p-complete category of spectra:*

$$\tau^{-1}\mathbf{Sp}_{\mathbf{C},cell} \simeq \mathbf{Sp}_p^{\wedge},$$

Moreover, under this equivalence, the τ-inversion functor corresponds to the realization functor in 2.26.

Proposition 2.28. *The category of cellular modules over the cofiber of τ, i.e. $\mathbf{Sp}_{\mathbf{C},cell}/\tau$, is equivalent to the* ***stable category of BP_*BP-comodules****, see* [Hov03].

Importantly, we have as before that the generic fiber of this deformation is the ordinary category of spectra, whereas the special fiber is a category which is purely describable by algebraic objects, and therefore more computationally-friendly. In the next section, we will see how this deformation can be leveraged to understanding relations between various Adams-type spectral sequences.

3. The Miller Square and Lifting Differentials

Recall the setup from the introduction: let $p > 2$ be an odd prime, and consider the mod p Moore spectrum $X = S^0/p$. Adams constructed a v_1-self map $\phi\colon \Sigma^{2(p-1)}X \to X$, which upon composing with the inclusion $S^0 \to X$ of the bottom cell, gives an element we will also denote ϕ in $\pi_{2(p-1)}X$. By compactness of the sphere spectrum, we find that $\phi^{-1}(\pi_* X) \simeq \pi_*(X[\phi^{-1}])$, and we can compute the former via a localized Adams spectral sequence.

Consider the mod p dual Steenrod algebra, $\mathcal{A}$, as in [Rav86, §3]. A standard computation shows that $H\mathbf{F}_{p_*}X$ is isomorphic to an $\mathbf{F}_p$-exterior algebra on the class τ_0 in degree 1, i.e. $E(\tau_0)$, as an $\mathcal{A}_*$-comodule. In [Mil81], Miller shows that the element ϕ is detected in the $\mathbf{F}_p$-Adams spectral sequence by a class

$$v_1 \in E_2^{1,2p-1} = \mathrm{Ext}_{\mathcal{A}}^{1,2p-1}(\mathbf{F}_p, E(\tau_0)).$$

Localizing the Adams spectral sequence at this element, he proves that the v_1-local E_2-page is isomorphic as an $\mathbf{F}_p$-algebra to

$$v_1^{-1}\,\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbf{F}_p, E(\tau_0)) \simeq \mathbf{F}_p[v_1^{\pm}] \otimes E(h_{n,0} | n \geq 1) \otimes \mathbf{F}_p[b_{n,0} | n \geq 1], \tag{3.1}$$

[6] In the context of writing this paper, "synthetic" ideas are used to this effect alone – providing intuition and understanding of ostensibly motivic computations. However, recent work has demonstrated the power of synthetic spectra, particularly $\mathbf{F}_2$-synthetic spectra, in greatly streamlining computations or in understanding filtrations of interest in higher algebra overall.

with $|h_{n,0}| = (1, 2(p^n - 1))$ and $|b_{n,0}| = (2, 2p(p^n - 1))$, and that this localized spectral sequence converges to $\phi^{-1}\pi_* X$.

Examining the spectral sequence in 3.1, it is immediate that the d_2 differentials are v_1 and $h_{1,0}$-linear. The more interesting result is the following

Proposition 3.2. *For $n > 1$, $d_2 h_{n,0} = v_1 b_{n-1,0}$.*

The following section will examine how this differential is computed. However, it affords us a drastic simplification. A spectral sequence exercise now shows that

$$v_1^{-1}E_3 \simeq \mathbf{F}_p[v_1^{\pm}] \otimes E[h_{1,0}]$$

and the spectral sequence degenerates on the E_3-page. As mentioned above, the class v_1 detects the v_1-self map ϕ, and let α denote the element of $\pi_{2p-3}X$ detected by $h_{1,0}$.[7] Thus, we have that

$$\pi_*(X[\phi^{-1}]) = \mathbf{F}_p[\phi^{\pm}] \otimes E(\alpha). \tag{3.3}$$

Counting degrees in 3.3, we get 1.5.

3.1. **The Classical Miller Square.** To motivate Miller's approach to 3.2, consider the following setup. Let X be the mod p Moore spectrum as above. Then, we know its mod p homology is $E(\tau_0)$, as an $\mathcal{A}$-comodule. The localized Adams spectral sequence uses this to compute the localized homotopy groups of X

$$v_1^{-1}\operatorname{Ext}_{\mathcal{A}}^{s,t}(\mathbf{F}_p, E(\tau_0)) \simeq E_2^{s,t} \Rightarrow \phi^{-1}(\pi_{t-s}X).$$

Likewise, let BP denote the Brown-Peterson spectrum at the prime p, with homotopy groups $\pi_* BP = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ and $|v_n| = 2(p^n - 1)$. The BP-homology of $X[\phi^{-1}]$ is now $v_1^{-1}BP_*/p$. Then, we can use the Adams-Novikov spectral sequences to compute the homotopy groups of $X[\phi^{-1}]$.

$$\operatorname{Ext}_{BP_*,BP}^{s,t}(BP_*, v_1^{-1}BP_*/p) \simeq E_2^{s,t} \Rightarrow \pi_{t-s}i(X[\phi^{-1}]).$$

These are useful tools, but even their E_2 page can be difficult to compute, leading to the following auxiliary spectral sequences.

Definition 3.4. Recall that the dual mod p Steenrod algebra is, as a $\mathbf{F}_p$-algebra, isomorphic to

$$\mathcal{A} \simeq \mathbf{F}_p[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots).$$

Letting $\mathcal{P}$ denote the polynomial part and $\mathcal{Q}$ the exterior part, we have an extension of Hopf algebras

$$\mathcal{P} \to \mathcal{A} \to \mathcal{Q}.$$

This gives rise to the **Cartan-Eilenberg spectral sequence** of signature

$$E_2^{s,k,t} = \operatorname{Ext}_{\mathcal{P}}^{s,t}(\mathbf{F}_p, \operatorname{Ext}_{\mathcal{Q}}^{k}(\mathbf{F}_p, E(\tau_0))) \Rightarrow \operatorname{Ext}_{\mathcal{A}}^{s+k,t}(\mathbf{F}_p, E(\tau_0)),$$

with differentials $d_r \colon E_r^{s,k,t} \to E_r^{s+r,k-r-1,t}$.

Remark 3.5. For odd primes, the Cartan-Eilenberg spectral sequence collapses, as differentials are required to preserve an additional grading on the generators of $\mathcal{A}$ [BX23, 2.2]. This does **not** occur at the even prime, and is one of many motivating sources of the lifting techniques described in the next section.

[7] It turns out that this element exists before localization, and is related to ϕ via a Bockstein map.

Definition 3.6. Let $I = (p, v_1, v_2, \ldots)$ denote the augmentation ideal of BP_*. Filtering BP_* by powers of I gives the **algebraic Novikov spectral sequence** of signature

$$\operatorname{Ext}^{s,t'}_{BP_*BP/I}(\mathbf{F}_p, I^k/I^{k+1}) \Rightarrow \operatorname{Ext}^{s,t'}_{BP_*BP}(BP_*, BP_*/p),$$

with differentials $d_r \colon E_r^{s,k,t'} \to E_r^{s+1,k+r-1,t'}$

Theorem 3.7. *The Cartan-Eilenberg spectral sequence 3.4 and algebraic Novikov spectral sequence 3.6 have isomorphic E_2 pages. More precisely, suppressing the grading, we have an isomorphism*

$$\operatorname{Ext}_{\mathcal{P}}(\mathbf{F}_p, \operatorname{Ext}_{\mathcal{Q}}(\mathbf{F}_p, E(\tau_0))) \simeq \operatorname{Ext}_{BP_*BP/I}(BP_*/I, I^*/I^{*+1})$$

induced by the isomorphism of Hopf algebroids $(BP_*/I, BP_*BP/I) \simeq (\mathbf{F}_p, \mathcal{P})$.

We can organize these spectral sequences into the Miller square below:

$$\begin{tikzcd}
& v_1^{-1}\operatorname{Ext}_{\mathcal{P}}(\mathbf{F}_p, \operatorname{Ext}_{\mathcal{Q}}(\mathbf{F}_p, E(\tau_0))) \arrow[dl, Rightarrow, "\text{Cartan-Eilenberg}"'] \arrow[dr, Rightarrow, "\text{algebraic Novikov}"] & \\
{v_1}^{-1}\operatorname{Ext}_{\mathcal{A}}(\mathbf{F}_p, E(\tau_0)) \arrow[dr, Rightarrow, "\text{Adams}"'] & & \operatorname{Ext}_{BP_*BP}(BP_*, v_1^{-1}BP_*/p) \arrow[dl, Rightarrow, "\text{Adams-Novikov}"] \\
& \pi_*X[\phi^{-1}] &
\end{tikzcd}$$

The key idea in this square is that Adams-Novikov d_2 differentials in the top right can induce Adams d_2 differentials in the bottom left. More precisely, Miller proves the following:

Theorem 3.8. *Suppose in the diagram above that the Cartan-Eilenberg spectral sequence collapses. Let $d_2(x) = y$ be a differential in the algebraic Novikov spectral sequence. Then, letting x and y denote the corresponding classes in the Adams spectral sequence, there is a d_2 differential $d_2(x) = \pm y$.*

This is how 3.2 was proven in [Mil81]: BP-theory gives concrete algebraic methods for computing the algebraic Novikov differentials on $h_{n,0}$, and 3.8 turns these into the key Adams differentials. However, the original proof of 3.8 relies on a complex diagram chase, and is quite limited in its scope and assumptions. As it turns out, there is a way to generalize this result to lift arbitrary algebraic Novikov differentials, relying crucially on the motivic Adams spectral sequence.

3.2. **Lifting through the Motivic Adams Spectral Sequence.** Now, we will introduce the motivic dual Steenrod algebra and motivic Adams spectral sequence, with the goal of better conceptualizing 3.8. Most of this material comes from [BX23, Sta21].

Definition 3.9. The **motivic dual Steenrod algebra** $\mathcal{A}^{\mathrm{mot}} := {\mathbf{F}_p^{\mathrm{mot}}}_{*,*}(\mathbf{F}_p^{\mathrm{mot}})$ is the Hopf algebra of motivic homology cooperations; the motivic homology of any spectrum is a comodule over $\mathcal{A}^{\mathrm{mot}}$.

In fact, we know the structure of the motivic dual Steenrod algebra, which is important due to its analogous role to the classical dual Steenrod algebra.

Theorem 3.10. *At odd primes $p > 2$, we have an $\mathbf{F}_p[\tau]$-algebra isomorphism.*

$$\mathcal{A}^{\mathrm{mot}}_{*,*} \simeq \mathbf{F}_p[\tau][\xi_1, \xi_2, \ldots] \otimes_{\mathbf{F}_p[\tau]} E_{\mathbf{F}_p[\tau]}[\tau_0, \tau_1, \ldots],$$

where $|\xi_n| = (2(p^n - 1), p^n - 1)$ and $|\tau_n| = (2p^n - 1, p^n - 1)$. The coproduct is

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i, \text{ and } \Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \tau_i.$$

We won't be focusing on the even prime in the rest of this paper, but the following is included for completeness.

Theorem 3.11. *For $p = 2$, we have an $\mathbf{F}_2[\tau]$-algebra isomorphism.*

$$\mathcal{A}^{\mathrm{mot}} \simeq \frac{\mathbf{F}_2[\tau][\tau_0, \tau_2, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 = \tau\xi_{i+1}}$$

Remark 3.12. Observe that the even motivic dual Steenrod algebra isn't simply the extension of the classical dual Steenrod algebra along $\mathbf{F}_p \to \mathbf{F}_p[\tau]$, as in the odd prime case. The extra relation $\tau_i^2 = \tau\xi_{i+1}$ means that when τ is inverted, we return to the classical dual Steenrod algebra tensored up to $\mathbf{F}_2[\tau^{\pm}]$, but when τ is killed, i.e. via forming $\mathcal{A}_{*,*} \otimes_{\mathbf{F}_2[\tau]} \mathbf{F}_2$, the resulting algebra appears to mimic the odd prime case. This is one potential explanation for the dramatic success of motivic methods at the even prime [IWX23].

Now we have the technology to set up the motivic Adams spectral sequence.

Construction 3.13. Let E be a motivic ring spectrum where $E_{*,*}$ is flat over $E_{*,*}E$ (such as for $E = \mathbf{F}_p^{\mathrm{mot}}$). Let $\overline{E}$ be the fiber of the unit $S^{0,0} \to E$, giving the family of cofiber sequences

$$\overline{E}^{\otimes s+1} \otimes X \to \overline{E}^{\otimes s} \otimes X \to E \otimes \overline{E}^{\otimes s} \otimes X.$$

for X a motivic spectrum. Splicing these sequences together gives the canonical $E_{*,*}$-Adams resolution

$$\begin{tikzcd}
X \arrow[d] & \overline{E} \otimes X \arrow[l] \arrow[d] & \bar{E}^{\otimes 2} \otimes X \arrow[l] \arrow[d] & \bar{E}^{\otimes 3} \otimes X \arrow[l] \arrow[d] & \ldots \arrow[l] \\
E \otimes X & E \otimes \overline{E} \otimes X & E \otimes \bar{E}^{\otimes 2} \otimes X & E \otimes \bar{E}^{\otimes 3} \otimes X & \ldots
\end{tikzcd}$$

Taking homotopy groups gives a trigraded exact couple, and by usual formal manipulations, a trigraded spectral sequence $E_r^{s,t,w}(X; E)$ with differentials

$$d_r \colon E_r^{s,t,w}(X; E) \to E_r^{s+r,t+r-1,w}(X; E).$$

This is the **motivic Adams spectral sequence**. Standard homological algebra identifies the E_2 page with trigraded Ext groups in $E_{*,*}E$-comodules

$$E_2^{s,t,w} \simeq \mathrm{Ext}^{s,t,w}_{E_{*,*}E}(E_{*,*}, E_{*,*}X).$$

Example 3.14. Applying 3.13 for the case when $E = \mathbf{F}_p^{\mathrm{mot}}$ and $X = S^{0,0}$, we get the motivic Adams spectral sequence

$$\mathrm{Ext}^{s,t,w}_{\mathcal{A}^{\mathrm{mot}}_{*,*}}(\mathbf{F}_p[\tau], \mathbf{F}_p[\tau]) \Rightarrow \pi_{s-t,w}\widehat{S^{0,0}}$$

converging to the homotopy groups of the p-completed sphere.

Example 3.15. Under the realization functor in 2.26, we get map $\mathcal{A}^{\mathrm{mot}} \to \mathcal{A}$. This sends $R(\tau) = 1$, $R(\tau_n) = \tau_n$, and $R(\xi_n) = \xi_n$. In addition, this gives a map from the motivic Adams spectral sequence to the classical Adams spectral sequence, corresponding to the τ-inversion arrow in 2.25.

While the previous example relates the motivic and classical Adams spectral sequences, we also want a spectral-sequence analog of the special fiber 2.28: this is the motivic Adams spectral sequence for the cofiber of τ

$$\mathrm{Ext}_{\mathcal{A}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], \mathbf{F}_p) \Rightarrow \pi_{*,*}(C\tau).$$

Given that modules over $C\tau$ have an algebraic description in terms of BP-theory, we should expect something similar for this spectral sequence, motivating the following critical theorem in motivic stable homotopy theory [GWX20, 1.17]:

Theorem 3.16. *There is an isomorphism of tri-graded spectral sequences between the motivic Adams spectral sequence for $C\tau$ and the algebraic Novikov spectral sequence*

$$\begin{tikzcd}
\mathrm{Ext}^{s+k,t'-k,t'/2}_{\mathcal{A}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], \mathbf{F}_p) \arrow[r, "\simeq"] \arrow[d, Rightarrow, "MASS\ for\ C\tau"'] & \mathrm{Ext}^{s,t'}_{BP_*BP/I}(\mathbf{F}_p, I^k/I^{k+1}) \arrow[d, Rightarrow, "algebraic\ Novikov"] \\
\pi_{t'-s,t'/2}(C\tau) \arrow[r, "\simeq"'] & \mathrm{Ext}^{s,t'}_{BP_*BP}(BP_*, BP_*)
\end{tikzcd}$$

Finally, we can understand how to lift differentials in the case of $X = S^0/p$. By naturality of Adams spectral sequences, we've translated the abstract motivic deformation of 2.25 into the following diagram of spectral sequences:[8]

$$\begin{tikzcd}
\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbf{F}_p, E(\tau_0)) \arrow[d, Rightarrow] & \mathrm{Ext}^{s,t,w}_{\mathcal{A}^{\mathrm{mot}}}\left(\mathbf{F}_p[\tau], E_{\mathbf{F}_p[\tau]}(\tau_0)\right) \arrow[l] \arrow[r] \arrow[d, Rightarrow] & \mathrm{Ext}^{s,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], E(\tau_0)) \arrow[d, Rightarrow] \\
\pi_{t-s}X & \pi_{t-s,w}X \arrow[l] \arrow[r] & \pi_{t-s,w}(C\tau \otimes X)
\end{tikzcd}$$

The leftward map is that of realization, setting $\tau = 1$, and the rightward map is induced by the inclusion of the bottom cell $S^{0,0} \to C\tau$, setting $\tau = 0$. We start by computing the E_2 pages of these various spectral sequences, using whatever strategies at hand. Next can use BP-theory combined with 3.16 to deduce differentials in the rightmost spectral sequence, and map differentials from the middle spectral sequence to classical ones on the left.

To see how to lift differentials across the right-hand map, consider a class x on the E_2 page of the middle spectral sequence such that its image x' in the $C\tau$ spectral sequence supports a nontrivial d_2-differential $d_2(x') = y' \neq 0$. Then, if we define $y \coloneqq d_2(x)$, y must map to y', implying $d_2(x)$ is nonzero. If in addition there was only one possible target for this differential, we would have computed $d_2(x)$. In general, considering d_r differentials for $r \geq 2$, there can be far more possibilities, but we can still force the existence of otherwise hard-to-compute differentials.

[8]We implicitly identify a classical spectrum with its "constant spectrum" motivic analog; that this embedding of $\mathbf{Sp}$ into $\mathbf{Sp}_{\mathbf{C}}$ is fully faithful is a theorem of Levine [Pst22].

4. Revisiting the Key Computation

To conclude, let us apply the program of the previous section to proving 1.5 vis-a-vis 3.2. For the sake of completeness, we record below our working definition of Ext of comodules over a Hopf algebroid:

Definition 4.1. Let (A, Γ) be a Hopf algebroid and H a Γ-comodule. Let $\overline{\Gamma}$ denote the unit coideal, i.e. the cokernel of the left unit where we mod out the image of 1. The **cobar complex** $C^*(H)$ is the cochain complex given by

$$C^s(H) = \overline{\Gamma}^{\otimes s} \otimes H,$$

where tensor products are taken over A. The differential $d_s \colon C^s(H) \to C^{s+1}(H)$ is given by

$$d_s[a_1|\dots|a_s]x = \sum_{i=1}^{s} (-1)^i [a_1|\dots|\Delta a_i|\dots|a_s]x + (-1)^{s+1}[a_1|\dots|a_s|x']x'',$$

where $\psi(x) = x' \otimes x'' \in \Gamma \otimes H$. Then, $\mathrm{Ext}^s_\Gamma(A, H)$ is defined as $H^s(C^*(H), d_s)$.

4.1. **Computing the Motivic E_2 Page.** We follow the strategy in [Mil78] to compute the MASS E_2 page for (the motivic analog of) $X = S^0/p$. As in the classical case, define

$$\mathcal{P}^{\mathrm{mot}} := \mathbf{F}_p[\tau][\xi_1, \xi_2, \dots]$$
$$\mathcal{Q}^{\mathrm{mot}} := E_{\mathbf{F}_p[\tau]}[\tau_0, \tau_1, \dots].$$

We have an extension of Hopf algebras

$$\mathcal{P}^{\mathrm{mot}} \to \mathcal{A}^{\mathrm{mot}} \to \mathcal{Q}^{\mathrm{mot}},$$

giving rise to a Cartan-Eilenberg spectral sequence 3.4

$$\mathrm{Ext}^{s,t,w}_{\mathcal{P}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], \mathrm{Ext}^k_{\mathcal{Q}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], E_{\mathbf{F}_p[\tau]}(\tau_0))) \Rightarrow \mathrm{Ext}^{s+k,t,w}_{\mathcal{A}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], E_{\mathbf{F}_p[\tau]}(\tau_0)).$$

with differentials $d_r \colon E_r^{s,k,t,w} \to E_r^{s+r,k-r+1,t,w}$. Moreover, as differentials must preserve the Cartan grading, with ξ_i in degree 0 and τ_i in degree 1, this spectral sequence collapses. Additionally, by a straightforward computation with 4.1, using that $\mathcal{Q}^{\mathrm{mot}}$ is a cocommutative, primitively generated exterior algebra, we have the following lemma:

Lemma 4.2. *We have an isomorphism*

$$\mathrm{Ext}_{\mathcal{Q}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], E_{\mathbf{F}_p[\tau]}(\tau_0)) \simeq \mathbf{F}_p[\tau][v_1, v_2, \dots]$$

with $v_n = [\tau_n] \in \mathrm{Ext}^{1,2p^n-1,p^n-1}$.

Proof. In general, taking Ext over the exterior Hopf algebra $(\mathbf{F}_p[\tau], \mathcal{Q}^{\mathrm{mot}})$ will produce a polynomial algebra [Rav86, 3.1.9]; the nuance is in why there is no "v_0" in the right-hand side. To see this, observe that the cobar differential on$[]\tau_0 \in C^0(E_{\mathbf{F}_p[\tau]}(\tau_0))$ is $[\tau_0]$. □

Now, inverting v_1 and using that localization is exact, the collapsing Cartan-Eilenberg spectral sequence gives an isomorphism (up to a regrading)

$$v_1^{-1}\,\mathrm{Ext}_{\mathcal{A}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], E_{\mathbf{F}_p[\tau]}(\tau_0)) \simeq \mathrm{Ext}_{\mathcal{P}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], \mathbf{F}_p[\tau][v_1^{\pm}, v_2, \dots]).$$

Here, we use the following change-of-rings isomorphism [Mil78, Rav91]

Lemma 4.3. *Let $\mathcal{B} = \mathcal{P}^{\mathrm{mot}}/(\xi_i^p)$; this is a primitively generated cocommutative Hopf algebra which acts trivially on $\mathcal{Q}^{\mathrm{mot}}$. We have isomorphisms*

$$\begin{aligned}\mathrm{Ext}_{\mathcal{P}^{\mathrm{mot}}}(\mathbf{F}_p[\tau], \mathbf{F}_p[\tau][v_1^{\pm}, v_2, \ldots]) &\simeq \mathrm{Ext}_{\mathcal{B}}(\mathbf{F}_p[\tau], \mathbf{F}_p[\tau][v_1^{\pm}]) \\ &\simeq \mathbf{F}_p[\tau][v_1^{\pm}] \otimes_{\mathbf{F}_p[\tau]} \mathrm{Ext}_{\mathcal{B}}(\mathbf{F}_p[\tau], \mathbf{F}_p[\tau]).\end{aligned}$$

Finally, computing $\mathrm{Ext}_{\mathcal{B}}(\mathbf{F}_p[\tau], \mathbf{F}_p[\tau])$ is a direct consequence of the following lemma, which is a good exercise in working with 4.1.

Lemma 4.4. *Let $\Gamma = A[x]/x^p$ be the truncated polynomial algebra on an even-degree generator x, considered as a primitively generated Hopf algebra over A. Then*

$$\mathrm{Ext}_\Gamma(A, A) = E(h) \otimes A(b),$$

where

$$h = [x] \in \mathrm{Ext}^1$$

and

$$b = \sum_{0<i<p} \frac{1}{p}\binom{p}{i}[x^i | x^{p-i}] \in \mathrm{Ext}^2.$$

Proof. See [Rav86, 3.2.4]. □

Combining 4.3 and 4.4, we have proved

Theorem 4.5. *The v_1-localized E_2 page of the motivic Adams spectral sequence for $X = S^0/p$ is, as an $\mathbf{F}_p[\tau]$-algebra,*

$$E_2^{s,t,w} \simeq \mathbf{F}_p[\tau][v_1^{\pm}] \otimes_{\mathbf{F}_p[\tau]} E_{\mathbf{F}_p[\tau]}(h_{1,0}, h_{2,0}, \ldots) \otimes_{\mathbf{F}_p[\tau]} \mathbf{F}_p[\tau][b_{1,0}, b_{2,0}, \ldots]$$

with

$$\begin{aligned} v_1 &\in \mathrm{Ext}^{1,2p-1,p-1}, \\ h_{n,0} &\in \mathrm{Ext}^{1,2(p^n-1),p^n-1}, \\ b_{n,0} &\in \mathrm{Ext}^{2,2p(p^n-1),p(p^n-1)}. \end{aligned}$$

Remark 4.6. In fact, replacing $\mathcal{A}^{\mathrm{mot}}$ with $\mathcal{A}$ and $\mathbf{F}_p[\tau]$ with $\mathbf{F}_p$, the process outlined above is how Miller computes the classical E_2 page in [Mil78]. Observe that the same change-of-ring isomorphism in 4.3 applies in the motivic world, as the constructions in [Mil78, §4] preserve motivic weights.

Corollary 4.7. *The v_1-localized E_2 page of the motivic Adams spectral sequence for $X \otimes C\tau$ is isomorphic to*

$$E_2^{s,t,w} \simeq \mathbf{F}_p[v_1^{\pm}] \otimes E(h_{1,0}, h_{2,0}, \ldots) \otimes \mathbf{F}_p[b_{1,0}, b_{2,0}, \ldots]$$

i.e. it is the localized E_2 page of 4.5 with $\tau = 0$.

4.2. **Computing the Key Differential.** Recall that $BP_*BP \simeq BP_*[t_1, t_2, \ldots]$, with $|t_n| = 2(p^n - 1)$. Moreover, we have

Lemma 4.8. [Mil81] *For $n \geq 2$, the coproduct of t_n in BP_*BP is*

$$\Delta t_n \equiv [t_n|1] + [1|t_n] - v_1 \sum_{0<i<p} \frac{1}{p}\binom{p}{i}[t_{n-1}^i | t_{n-1}^{p-i}] \mod I^2.$$

Additionally, tracing through the definitions, we have the following translations from $C\tau$ classes to algebraic Novikov classes:

Lemma 4.9. *In the algebraic Novikov spectral sequence*

$$E_2^{s,t',k} = \mathrm{Ext}^{s,t'}_{BP_*BP/I}(BP_*/I, v_1^{-1}I^k/I^{k+1}) \Rightarrow \mathrm{Ext}^{s,t'}_{BP_*BP}(BP_*, v_1^{-1}BP_*/p)$$

the classes $h_{n,0}$ and $b_{n,0}$ are represented under 3.16 by

$$h_{n,0} = [t_n] \text{ and } b_{n,0} = \sum_{0<i<p} \frac{1}{p}\binom{p}{i}[t_n^{p-i}|t_n^i].$$

Thus, computing the algebraic Novikov d_2 differential on $h_{n+1,0}$ becomes a problem of computing the cobar differential on t_n. Combining 4.8 and 4.9, we find

Proposition 4.10. *In the algebraic Novikov spectral sequence, we have*

$$d_2(h_{n+1,0}) = -v_1 \sum_{0<i<p} \frac{1}{p}\binom{p}{i}[t_n^i|t_n^{p-i}] = -v_1 b_{n,0},$$

for $n \geq 1$

With this, we can compute the motivic Adams d_2 differential on $h_{n+1,0}$. Observe that $y \coloneqq d_2(h_{n+1,0})$ has degree $(s,t,w) = (3, 2p^{n+1}-1, p^{n+1}-1)$ and maps under the special fiber to $v_1 b_{n,0}$. As there are no other classes in this tridegree in 4.5, we conclude that $d_2(h_{n+1}, 0) = v_1 b_{n,0}$. This differential then maps under realization to the desired Adams d_2 differential, completing the proof of 3.2, and with it, the height 1 telescope conjecture for odd primes.

Remark 4.11. While the lifting problem in this example was rather simple, recent work by Isaksen-Wang-Xu has used this general process, what they call the "$C\tau$-philosophy" at the even prime to compute the first 90 stable stems, greatly expanding on the range in which these groups were known from previous methods [IWX23].

Acknowledgments

I am delighted to thank my mentor, Mark Behrens, for his supportive mentorship and intuitive understanding of computational aspects of homotopy theory, which he graciously shared with me. I would also like to thank Peter May for his ongoing work in organizing the REU and for helping me develop a wider perspective on algebraic topology. A great deal of thanks goes to Annie Wei, Isaiah Dailey, and everyone else I lived with this summer for being incredibly helpful mathematicians and even better friends. Finally, I would like to thank my family for their unwavering love and support.

References

[Ada66] J.F. Adams, *On the groups j(x)—iv*, Topology **5** (1966), no. 1, 21–71 (en).
[Ada10] Michal Adamszek, *An elementary guide to the adams-novikov ext*, Oct 2010.
[AE16] Benjamin Antieau and Elden Elmanto, *A primer for unstable motivic homotopy theory*, arXiv:1605.00929 [math].
[AI23] Toni Annala and Ryomei Iwasa, *Motivic spectra and universality of k-theory*, arXiv:2204.03434 [math].
[BHS22] Robert Burklund, Jeremy Hahn, and Andrew Senger, *On the boundaries of highly connected, almost closed manifolds*, arXiv:1910.14116 [math].
[Bou79] A.K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281 (en).

[Bra18] Thomas Brazelton, *A1-homotopy theory from an $(\infty,1)$-categorical viewpoint*, Jun 2018.

[BX23] Robert Burklund and Zhouli Xu, *The adams differentials on the classes h_j^3*, arXiv:2302.11869 [math].

[Dé21] Frédéric Déglise, *An introductory course on motivic homotopy theory and rational mixed motives: Lecture 2*, Jul 2021.

[Ghe17] Bogdan Gheorghe, *The motivic cofiber of τ*, 2017.

[GWX20] Bogdan Gheorghe, Guozhen Wang, and Zhouli Xu, *The special fiber of the motivic deformation of the stable homotopy category is algebraic*, arXiv:1809.09290 [math].

[Hov03] Mark Hovey, *Homotopy theory of comodules over a hopf algebroid.*

[HS98] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory ii*, The Annals of Mathematics **148** (1998), no. 1, 1.

[Isa14] Daniel C. Isaksen, *Stable stems*, arXiv:1407.8418 [math].

[IWX23] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, *Stable homotopy groups of spheres: From dimension 0 to 90*, arXiv:2001.04511 [math].

[Lur10] Jacob Lurie, *Chromatic Homotopy Theory (Math 252x)*, Jan 2010.

[Mah82] Mark Mahowald, *The image of j in the ehp sequence*, The Annals of Mathematics **116** (1982), no. 1, 65.

[Mar22] Peter Marek, *$h\mathbb{F}_2$-synthetic homotopy groups of topological modular forms.*

[Mat12] Akhil Mathew, *Computing the first three stable stems i*, Jun 2012.

[Mil78] Haynes R. Miller, *A localization theorem in homological algebra*, Mathematical Proceedings of the Cambridge Philosophical Society **84** (1978), no. 1, 73–84 (en).

[Mil81] ———, *On relations between adams spectral sequences, with an application to the stable homotopy of a moore space*, Journal of Pure and Applied Algebra **20** (1981), no. 3, 287–312 (en).

[MV99] Fabien Morel and Vladimir Voevodsky, *A1-homotopy theory of schemes*, Publications mathématiques de l'IHÉS **90** (1999), no. 1, 45–143 (en).

[Pst20] Piotr Pstragowski, *Finite Height Chromatic Homotopy Theory (Math 252y)*, Jul 2020.

[Pst22] Piotr Pstrągowski, *Synthetic spectra and the cellular motivic category*, arXiv:1803.01804 [math].

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, American Journal of Mathematics **106** (1984), no. 2, 351.

[Rav86] ———, *Complex cobordism and stable homotopy groups of spheres*, Pure and applied mathematics, Academic Press, Orlando, 1986.

[Rav91] ———, *Progress report on the telescope conjecture*, 1991.

[Rav92] ———, *Nilpotence and periodicity in stable homotopy theory*, Annals of mathematics studies, Princeton University Press, Princeton, N.J, 1992.

[Rob15] Marco Robalo, *K-theory and the bridge from motives to noncommutative motives*, Advances in Mathematics **269** (2015), 399–550 (en).

[Sta21] Sven-Torben Stahn, *The motivic adams-novikov spectral sequence at odd primes over $\mathbb{C}$ and $\mathbb{R}$*, arXiv:1606.06085 [math].

[Voe03a] Vladimir Voevodsky, *Motivic cohomology with z/2-coefficients*, Publications mathématiques de l'IHÉS **98** (2003), no. 1, 59–104 (en).

[Voe03b] ———, *Reduced power operations in motivic cohomology*, Publications mathématiques de l'IHÉS **98** (2003), no. 1, 1–57 (en).

[Voe07] *Voevodsky's Nordfjordeid Lectures: Motivic Homotopy Theory*, Universitext, Springer e-books, Berlin Heidelberg, 2007 (eng).

[Voe10] Vladimir Voevodsky, *Motivic eilenberg-maclane spaces*, Publications mathématiques de l'IHÉS **112** (2010), no. 1, 1–99 (en).

[WZ23] Xiangjun Wang and Yu Zhang, *A correspondence between higher adams differentials and higher algebraic novikov differentials at odd primes*, arXiv:2305.12630 [math].