A VARIANT OF LANDWEBER EXACT FUNCTOR THEOREM

ZHENPENG LI

Abstract. The Landweber exact functor theorem constitutes a fundamental result within chromatic homotopy theory, offering a wealth of useful even periodic spectra. In this paper, we shall establish the classical version of this theorem and subsequently delve into its broader form, as presented in [10]. Additionally, we will succinctly explore several applications, including chromatic convergence and elliptic cohomology.

Contents

1. Introduction

1.1. **History.** There is an interesting phenomenon in algebraic topology first observed by Quillen in the late 1960s. We know that in ordinary cohomology, $\mathrm{H}^*(\mathbb{CP}^\infty;\mathbb{Z})$ is isomorphic to $\mathbb{Z}[t]$ as a graded ring, where the degree of t is equal to 2. In Quillen's observation, a certain type of general multiplicative cohomology theory called complex-oriented cohomology theory satisfies the similar equation, $E^*(\mathbb{CP}^\infty) \simeq$

$E^*(*)[[t]]$, where we denote by E an arbitrary complex-oriented cohomology theory. The generator t in this (noncanonical) isomorphism can be regarded as the first E-Chern class of the tautological line bundle of $\mathbb{CP}^\infty$. Quillen's observation was that the first Chern class of the tensor product of two line bundles defined over $\mathbb{CP}^\infty$ can be calculated via a formal power series in $E^*(*)[[x, y]]$, and that the power series reflects some properties of complex-oriented cohomology theories. At least, we believe that spectra with nonisomorphic power series can not be equivalent to each other. For the deeper research, Quillen then used the algebraic concept called formal group to extract the essential properties of these power series given by complex-oriented cohomology. This observation is very meaningful because many important cohomology theories are complex-oriented, like complex K-theory and complex cobordism theory MU, and formal group is a simpler concept in terms of algebra.

After Quillen's work, it was Landweber who stepped much further. He contributed to constructing nicer spectra based on nice enough formal groups and gave us a simple criterion to determine whether a formal group is *nice*. His theorems are now called Landweber exact functor theorem. Afterwards, Morava, Ravenel, and many other homotopy theorists, introduced lots of beautiful and powerful spectra, based on diverse formal groups appearing in specific studies of mathematical branches. For example, Lubin-Tate spectra were constructed by the formal groups related to deformation theory. To our surprise, a majority of spectra can be recovered through the Bousfield localization with respect to Lubin-Tate spectra. A more precise statement is Theorem 2.52. Now, these theorems are classified into the field called chromatic homotopy theory. Many people in this field are devoted to analyzing the property of formal groups and constructing meaningful spectra so as to study stable homotopy theory.

With the relationship between $\mathbb{E}_\infty$-spaces and $\mathbb{E}_\infty$-ring spectra pointed out in [12], people found that $\mathbb{E}_\infty$-ring spectra play a prominent role in stable homotopy theory. First and foremost, many ring spectra hold the $\mathbb{E}_\infty$-property, like complex K-theory and the Eilenberg-MacLane spectra associated with ordinary commutative rings. Roughly speaking, an $\mathbb{E}_\infty$-ring spectrum can be thought of a space which is equipped with an addition and a multiplication in a weak sense. The addition and multiplication morphisms should satisfy several axioms not only up to homotopy, but also up to *coherent homotopy*. From this perspective, $\mathbb{E}_\infty$-ring spectra are the reasonable generalization of ordinary rings in a more homotopic sense. The basic difference between algebra and higher algebra is that the former is set-based and the latter is space-based. Properties in the former become structure in the latter. e.g. equality becomes a path/homotopy between two things. As we need ordinary commutative rings to define affine scheme in algebraic geometry, $\mathbb{E}_\infty$-ring spectra are the fundamental building blocks of derived algebraic geometry. There are many nice models about $\mathbb{E}_\infty$-ring spectra such as commutative monoids of S-modules in [12] and commutative algebra objects in the symmetric monoidal ∞-category of spectra Sp in [9]. We recommend readers unfamiliar with them to skim over these versions and have a rough grasp of what they are.

Consequently, people are curious about the $\mathbb{E}_\infty$-property of spectra constructed in chromatic homotopy theory. This problem is quite challenging because not only can $\mathbb{E}_\infty$-rings be very abstract, but also many homotopy theorists kept studying the ordinary objects instead of spectral objects. But Goerss-Hopkins-Miller in [5] took a different way. They developed an obstruction theory about a moduli problem in spectra that is related to the $\mathbb{E}_\infty$-problem. They finally verified the $\mathbb{E}_\infty$-structures of Lubin-Tate spectra, as presented in Theorem 4.15.

Their theorem was then generalized by Lurie in [10]. Lubin-Tate spectra are defined by the deformation of formal groups over a perfect field of character p. But Lurie chose to study the general deformation of a p-divisible group and found that there exists a nice formal group which induces an $\mathbb{E}_\infty$-ring spectrum. Actually, Lurie's theorem heavily relied on the enhancement of Goerss-Hopkins-Miller because in his proof, Lurie reduced the general cases to Lubin-Tate spectra by dévissage.

We have the next plain diagram to conclude this history.

$$\begin{tikzcd}
\text{Complex-oriented spectra} \arrow[d, shift right, "\text{Quillen}"'] & \mathbb{E}_\infty \text{ complex-oriented ring spectra} \\
\text{Formal groups} \arrow[u, shift right, "\text{Landweber}"'] \arrow[ur, dashed, "\text{Goerss-Hopkins-Miller, and then Lurie}"'] &
\end{tikzcd}$$

1.2. **Statements of several main theorems.** I would like to talk about our main theorems and the structure of the article before we step into following sections. Since We will devote ourselves to proving these theorems, I hope that this subsection can help readers understand our following theory better.

One version of Landweber exact functor theorem is the following, and for the more precise version, see Theorem 2.43.

Theorem 1.1 (Landweber). *(1) Let M be an ordinary*[1] *graded module over the Lazard ring L (See Definition 2.5). If for every prime p, the Hasse invariants (See Definition 2.12) $v_0, v_1, \cdots \in L$ form a regular sequence in M, then $X \mapsto MU(X) \otimes_L M$ is a homology theory. Here, the term regular sequence means that each v_i is a non-zero divisor in $M/(v_0, \cdots, v_{i-1})$.*

(2) When an ordinary commutative ring R is flat over the moduli stack of formal groups $\mathcal{M}_{\mathrm{FG}}$ (See Definition 2.7), we can construct a multiplicative complex-oriented spectrum whose homology theory is the one constructed in (1) (by virtue of a certain L-algebra R'). In addition, the construction is functorial and fully faithful.

Then Lurie's partial improvement can be summarized into one sentence.

Theorem 1.2 (Lurie). *Given an ordinary Noetherian $\mathbb{F}_p$-algebra R_0 such that the Frobenius map is finite and an ordinary nonstationary 1-dimensional p-divisible group $\mathbf{G}_0$, we can construct a classical universal deformation ring R flat over $\mathcal{M}_{\mathrm{FG}}$ whose associated ring spectrum in Theorem 1.1 can be lifted as an $\mathbb{E}_\infty$-ring.*

Don't worry if you don't understand all the notions in this statement. We'll define them carefully through our article.

This theorem is quite useful. For instance, the $\mathbb{E}_\infty$-property of Lubin-Tate spectra is just a special case when R_0 is a perfect field. Apart from it, there is a kind of elliptic cohomology theories which can be enhanced as $\mathbb{E}_\infty$-ring spectra in analogous ways. Last but not the least, in [2], the authors studied the *spherical Witt*

[1]We will frequently use the word *ordinary* or *classical* when talking about commutative rings and so on in our common sense. The reason for it is presented at the end of this section.

vectors whose $\mathbb{E}_\infty$-property is deduced from this theorem to analyze the number theory.

Let me explain the relationship between these two theorems and why we need these conditions in Theorem 1.2. Compared with the traditional way to construct ring spectra, we pay more attention to p-divisible groups instead of formal groups because the rigidity of power series limits the abundance of deformation and thus the ring spectra we can construct. The finite Frobenius morphism is an analogy of the regular sequence in Theorem 1.1. It suggests that the ordinary ring R has the nice regularity, providing convenience for our construction. The nonstationary property of $\mathbf{G}_0$ plays the same role. Last but not the least, the reason for the improvement brought by this theorem is that the author established his theory based on ring spectra instead of just on the ordinary rings.

In this article, I will show readers the basic knowledge on chromatic homotopy theory and pay more attention to Lurie's improvement. The diagram above also means the structure of this article. In section 2, we will introduce the basics of chromatic homotopy theory and its main outcomes, including Quillen's observation, Landweber's construction, and Lubin-Tate spectra in classical sense. Then in section 3, we begin to prepare for Lurie's theorem, Theorem 1.2, by defining some generalized notions about formal groups and p-divisible groups, and then construct formal groups associated to p-divisible groups. The proof of the main theorem is presented in section 4 on the basis of the deformation theory and the orientation.

Some technical results about the étale-connected decomposition and the existence of generalized universal deformation will be presented without proof because they can be easily understood as counterparts of similar theorems in classical algebraic geometry. For better reading experiences, readers are supposed to be familiar with the stable homotopy theory, basic algebraic geometry, and higher algebra theory. I know that many people cannot accept frequent change of topics due to diverse reasons, but I hope that everyone can understand our proof and the main constructions as presented above. Since it is really hard to convert hundreds of pages into a short article, I have to state some lemmas without proof. But I will give the related references in detail. The main references are [8] and [10].

Due to the abuse of notations in [10], we announce some usage of notations and terminologies in Convention 3.1. It doesn't influence reading before we step into the world of derived algebraic geometry in section 3.

2. Basic Chromatic Homotopy Theory

2.1. **Formal groups and the moduli stack.** Let's begin with one of most important concept, formal group laws. The following three axioms directly correspond to the unity, commutativity, and associativity of the tensor product of line bundles.

Definition 2.1. Consider an ordinary commutative and unital ring R, a formal group law is an element in $R[[x, y]]$ such that

$$f(x, 0) = f(0, x) = x,$$

$$f(x, y) = f(y, x),$$

$$f(f(x, y), z) = f(x, f(y, z)).$$

For two formal group laws f and f', a power series $g(t)$ is called a morphism from f to f' if $g(f(x, y)) = f'(g(x), g(y))$.

Example 2.2. For every ring R, $f(x+y) = x+y$ and $f(x,y) = x+y+xy$ are formal group laws over R.

Example 2.3. For every ring R and a formal group law f over R, if there is a formal power series $g(t) = b_0 t + b_1 t^2 + \cdots$, where b_0 is invertible, then there is a new formal group law $f'(x,y) = g(f(g^{-1}(x), g^{-1}(y)))$. According to the definition above, g is an isomorphism from f to f', and it can be regarded as the change of variables.

Remark 2.4. A formal group law $f(x,y)$ over R can be regarded as an abelian structure over $\mathrm{Spf}(R[[t]])$, the 1-dimensional affine formal scheme, because the data of f is the same as the data of $R[[t]] \to R[[x,y]]$ and then the data of $\mathrm{Spf}(R[[t]]) \times_R \mathrm{Spf}(R[[t]]) \to \mathrm{Spf}(R[[t]])$. The axioms of formal group laws deduce the commutative diagrams of abelian objects. Since there's no need to totally use the language of algebraic geometry in this section, we will talk about the story of formal schemes later.

Definition-Theorem 2.5 (Lazard ring)**.** In the polynomial ring, $\mathbb{Z}[c_{ij}]$, where i and j are natural numbers, we consider the ideal Q generated by the equations of coefficients in the three axioms of Definition 2.1 where $f(x,y) = \sum c_{ij} x^i y^j$. Then the Lazard ring L is defined to be $\mathbb{Z}[c_{ij}]/Q$, which corepresents the *set* of all the formal group laws of every commutative and unital ring.

Proof. It suffices to observe that every formal group law is uniquely determined by its coefficients. So the universal property of polynomial rings and quotients implies our claim. □

Remark 2.6. According to Lazard's theorem, L is isomorphic to a polynomial ring with infinitely many generators. As a result, there are lots of formal group laws over a fixed ring. For detailed proof, see [8].

Next, we can introduce the moduli stack of formal groups up to the isomorphisms. As for the theory of stacks, readers can refer to [6].

Definition 2.7. Let G^+ be the group scheme given by

$$G^+(R) = \left\{ g(t) = b_0 t + b_1 t^2 + \cdots \,\middle|\, b_0 \in R^\times \right\} \subseteq R[[t]]$$

with the group structure of composition. It acts on $\operatorname{Spec} L$ due to the corepresentability of L. We call the quotient stack the moduli stack of formal groups and denote it by $\mathcal{M}_{\mathrm{FG}}$.

In order to understand the moduli stack better, we need to generalize the definition of formal group laws.

Construction 2.8. Given a formal group law f over a ring R, there is a functor F_f from the category of R-algebras to the category of abelian groups sending an ordinary R-algebra A to its nilpotence ideal with the additive structure given by $a +_f b = f(a,b)$ for every nilpotent elements a and b. Although f includes infinitely many summands, the term $f(a,b)$ is well defined because every term of high enough degree vanishes. We will call every functor isomorphic to F_f for some f coordinated formal group.

Definition 2.9. Given an ordinary commutative ring R, a formal group over R is a functor G from the category of R-algebras to the category of abelian groups such that:

- G is a sheaf with respect to Zariski topology.
- G is locally given by some formal group law f as in Construction 2.8.

Based on the definition, one can show that $\mathcal{M}_{\mathrm{FG}}$ associate every ordinary commutative ring R with the groupoid encoding all formal groups and their isomorphisms induced by changing variables.

There is a notion called *heights* of formal groups and it leads to a stratification of $\mathcal{M}_{\mathrm{FG}}$. Let's begin with this lemma.

Lemma 2.10. *Let f and f' be two formal group laws over R where $p = 0$ for some prime number p, and there exists $h \in tR[[t]]$ such that h is a morphism from f to f'. Then one of the following claims holds:*

- $h = 0$.
- $h(t) = h'(t^{p^n})$ *for some n and $h'(t) = \lambda t + O(t^2)$ with $\lambda \neq 0$.*

Proof. Let's consider a special kind of Krull's differentials in $R[[t]]dt$. Here, we say that $g(t)dt$ is a *translation invariant* differential with respect to f if and only if $f^*(g(t)dt) = g(x)dx + g(y)dy$. By solving a series of linear equations, we know that there is a unique translation invariant differential ω_f of the form $dt + c_1t^2dt + \cdots$. Besides, h induces h^* from $R[[t]]dt$ to itself, with h^* carrying invariant differentials with respect to f' to invariant differentials with respect to f. In particular, note that $R[[t]]dt$ is a free $R[[t]]$-mod of rank 1, so $h^*\omega_{f'} = \lambda\omega_f + O(t)dt$ for some $\lambda \in R$. Unwinding the definitions, we see that $h(t) = \lambda t + O(t^2)$. If $h \neq 0$ but $\lambda = 0$, then $h^* = 0$ implies that $h(t) = h_0(t^p)$ for some h_0. We can replace f by $\tilde{f}$ where $\tilde{f}(x^p, y^p) = f(x, y)^p$. So we get

$$f'(h_0(x^p), h_0(y^p)) = f'(h(x), h(y)) = h(f(x, y)) = h_0(f(x, y)^p) = h_0(\tilde{f}(x^p.y^p)).$$

We can do the same operation for f', $\tilde{f}$, and h_0 again and again until the corresponding $\lambda \neq 0$. $\square$

Definition 2.11. Let $f(x, y) \in R[[x, y]]$ be a formal group law over an ordinary commutative ring R. For every nonnegative integer n, we define the n-series $[n](t) \in R[[t]$ as follows:

- If $n = 0$, we set $[n](t) = 0$.
- If $n > 0$, we set $[n](t) = f([n-1](t), t)$.

Easy calculation shows that $[n](t)$ satisfies the condition in Lemma 2.10. So the following definition makes sense.

Definition 2.12. Let f be a formal group law over an ordinary commutative ring R, and fix a prime number p. We let v_n denote the coefficient of t^{p^n} in the p-series $[p]$, which are called the Hasse invariants of f. We will say that f has height $\geq n$ if $v_i = 0$ for $i < n$. We will say that f has height exactly n if it has height $\geq n$ and $v_n \in R$ is invertible.

Remark 2.13. One can relate the heights above to some more algebro-geometric objects. Assume that F is a formal group of height exactly n. Let $F[p] = \mathrm{Ker}(F \xrightarrow{\times p} F)$. Then it is represented by a group scheme locally of the form $\operatorname{Spec} R[[t]]/[p](t)$. In this case, the height is determined by its rank. It is linked with p-divisible groups because locally constant height means globally *p-divisiblity of formal groups* over a ring spectra, which is discussed in [10, Theorem 4.4.14].

Although Hasse invariants are not G^+-invariant, the definition of height is not influenced when we change the variables. Consequently, we have an obvious stratification in $\mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}\,\mathbb{Z}_{(p)}$. We let $\mathcal{M}_{\mathrm{FG}}^{\geq n}$ be the closed substack $\left(\mathrm{Spec}\,L_{(p)}/(v_0, \cdots, v_n)\right)/G^+$. In other words, it classifies all formal groups of height no less than n. Then we let $\mathcal{M}_{\mathrm{FG}}^n$ be $\mathcal{M}_{\mathrm{FG}}^{\geq n} \backslash \mathcal{M}_{\mathrm{FG}}^{\geq n+1}$, which form up to the open strata for a stratification of the moduli stack.

2.2. **Complex-oriented spectra.** We will introduce Quillen's observation in this subsection and then provide readers with some examples.

Definition 2.14. A spectrum is called multiplicative if it is a commutative algebraic object in the stable homotopy category. A multiplicative spectrum A is called complex-oriented if the structure map $\mathbb{S} \simeq \Sigma^{\infty-2}\mathbb{CP}^1 \to A$ can be lifted to $\Sigma^{\infty-2}\mathbb{CP}^\infty \to A$. Some people like to call the structure map preorientation and the lifting map orientation.

Example 2.15. The Eilenberg-MacLand spectra of commutative rings are complex-oriented since $H^2(\mathbb{CP}^\infty; R) \simeq H^2(S^2; R)$.

Example 2.16. If all the homotopy groups of A in odd degree vanishes, the obstruction theory implies that A is complex-oriented. For example, according to Bott periodicity, the complex K-theory is oriented.

Example 2.17. The complex cobordism spectrum MU is complex-oriented. There is one proof in [8]. We will see a non-trivial enhancement in this subsection.

Construction 2.18. For a complex-oriented spectrum, one can construct a formal group law as follows. Note that the lifting element in $E^*(\mathbb{CP}^\infty)$ forces the Atiyah-Hirzebruch spectral sequence for E to degenerate, one can show that $E^*(\mathbb{CP}^\infty)$ is isomorphic to the formal power series ring $\pi_*(E)[[t]]$ where t is the element $\Sigma^{\infty-2}\mathbb{CP}^\infty \to A$ and its degree is 2, and that $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ is isomorphic to the formal power series ring $\pi_*(E)[[x, y]]$. We know that there is a tautological line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^\infty$. Then we can obtain another line bundle $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$ defined over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, where p_i means the projection morphism. As a consequence, this line bundle is classified by a map $u : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. In terms of cohomology, the orientation t is then mapped to a formal power series $f(x, y) \in \pi_*(E)[[x, y]]$. The unital and commutative property of the tensor product exhibits that $f(x, y) = f(y, x)$ and $f(x, 0) = x$. Similarly, if we consider the line bundle $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1) \otimes p_3^*\mathcal{O}(1)$ defined over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \times \mathbb{CP}^\infty$, we can prove that f is a formal group law over $\pi_*(E)$. Sometimes we will call f the (classical) Quillen formal group.

Remark 2.19. Sometimes we call the orientation t the universal first Chern class. In order to construct the higher Chern classes, it suffices to consider the universal map from $\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$ to $\mathrm{Gr}_n(\mathbb{C}^\infty)$, the classifying space of n-dimensional complex vector bundle, which classifies the bundle $p_1^*\mathcal{O}(1) \oplus \cdots \oplus p_n^*\mathcal{O}(1)$. We can deduce that c_n is nth elementary symmetric function on the orientations $t_1, \cdots, t_n$.

Example 2.20. The associated formal group law of ordinary cohomology is $f(x, y) = x + y$. As to complex K-theory, the formal group law is $x + y + xy$.

Example 2.21. Consider MU, then the formal group law over MU is realized by a homomorphism $L \to \pi_*(\mathrm{MU})$. In 1969, Quillen proved that it is an isomorphism on the basis of Lazard's theorem that $L \simeq \mathbb{Z}[t_1, t_2, \cdots]$. This isomorphism explains the graded structure of L. Readers interested in the proof of this isomorphism can refer to [14].

2.3. **Flatness and Landweber exact functor theorem.** We are ready to prove the first part of Theorem 1.1. Note that the direct construction $E_*(X) = \mathrm{MU}_*(X) \otimes_L M$ is not always a homology theory since we lose the long exact sequence, we need the flatness condition. We don't need M to be flat over L because flatness over L, a massive polynomial ring, is too strong. Instead, Landweber realized that the exact sequence of form $\cdots \to \mathrm{MU}_i(A) \to \mathrm{MU}_i(X) \to \mathrm{MU}_i(X/A) \to \cdots$ has a hidden structure of an exact sequence consisting of sheaves over $\mathcal{M}_{\mathrm{FG}}$ and proposed the following proposition.

Proposition 2.22. *Let $q : \operatorname{Spec} R \to \mathcal{M}_{\mathrm{FG}}$ be a map (classifying a formal group $\eta \in \mathcal{M}_{\mathrm{FG}}(R)$) and let N be an R-module which is flat over $\mathcal{M}_{\mathrm{FG}}$. Then the functor $M \mapsto M(\eta) \otimes_R N = q^* M \otimes_R N$ is an exact functor from $\mathrm{QCoh}(\mathcal{M}_{\mathrm{FG}})$ to the abelian category of ordinary R-modules.*

Proof. Due to the local property of exactness, we can assume that η is induced by a formal group law f. Then we have a pullback diagram.

$$\begin{tikzcd}
\operatorname{Spec} R[b_0^{\pm}, b_1, \cdots] \arrow[r] \arrow[d, "p'"] & \operatorname{Spec} L \arrow[d, "p"] \\
\operatorname{Spec} R \arrow[r, "\eta"] & \mathcal{M}_{\mathrm{FG}}
\end{tikzcd}$$

Here, p' is faithfully flat unwinding the definition. So the exactness can be detected by $M \to p'^*(q^* M \otimes_R N)$. Then the statement directly follows from the definition of flatness. □

Corollary 2.23. *Let M be an ordinary graded module over the Lazard ring L. If M is flat over $\mathcal{M}_{\mathrm{FG}}$, then the functor $X \mapsto \mathrm{MU}_*(X) \otimes_L M$ is a homology theory.*

Let's return to the first part of Theorem 1.1. In fact, a stronger version is:

Theorem 2.24. *Let M be an ordinary module over the Lazard ring L. Then M as a sheaf is flat over $\mathcal{M}_{\mathrm{FG}}$ if and only if for every prime p, the Hasse invariants $v_0, v_1, \cdots \in L$ form a regular sequence in M.*

Our approach involves employing the following lemma, which allows us to focus on simpler cases. There is a concise proof in the 16th lecture of [8].

Lemma 2.25. *Let R be an ordinary commutative ring containing a non-zero divisor x, and let M be an ordinary R-module. Then M is flat over R if and only if the following conditions are satisfied:*

- *The element x is a non zero-divisor on M.*
- *The quotient M/xM is a flat $R/(x)$-module.*
- *The module $M[x^{-1}]$ is flat over $R[x^{-1}]$.*

Since flatness is just a local property, we fix a prime number p and then reduce to the case of $M_{(p)}$. In this case, the Hasse invariants give us part of canonical generators in the isomorphism $L \simeq \mathbb{Z}[t_1, \cdots]$, which is proved in the second and third lecture in [8]. The hidden reason for colorful examples of formal groups is partially due to the calculation in this proof. Simultaneously, this tedious calculation implies the following fact.

Lemma 2.26. *Fix a prime number p, and let $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ be the L-module obtained by taking the quotient of $L_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, t_2, \ldots]$ by the ideal generated by $\{t_i\}_{i+1 \neq p^k}$. We claim that the map Spec $\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \to$ Spec $L \to \mathcal{M}_{\mathrm{FG}}$ is flat.*

Then we can reduce our problem to the flatness over $\operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$.

Lemma 2.27. *Let $q : \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \to \mathcal{M}_{\mathrm{FG}}$ be the flat map considered above. Let M be a quasi-coherent sheaf on $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. Then M is flat over $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$ if and only if $q'^{*}M$ is a flat $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$-module, where we assume that q factors through $\mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)} \to \mathcal{M}_{\mathrm{FG}}$ and generates $q' : \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$.*

Proof. The only nontrivial part is the "if" direction. According to commutative algebra, q' is faithfully flat. So given the following test pullback diagram, $R \to B$ is faithfully flat. Then it suffices that $f^{*}M \otimes_R B$ is flat over B. But $f^{*}M \otimes_R B = q'^{*}M \otimes_{\mathbb{Z}_{(p)}[v_1, v_2, \ldots.]} B$, which is flat over B since $q^{*}M$ is flat over $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$.

$$\begin{array}{ccc}
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \cdots] \\
\downarrow & & \downarrow {\scriptstyle q'} \\
\operatorname{Spec} R & \xrightarrow{f} & \mathcal{M}_{\mathrm{FG}} \times \mathbb{Z}_{(p)}
\end{array}$$

□

As for our theorem 1.1, let M be an ordinary module over the localized Lazard ring $L_{(p)}$ such that $v_0 = p, v_1, v_2, \ldots$ is a regular sequence on M. We wish to prove that M is flat along the map $\operatorname{Spec} L_{(p)} \to \mathcal{M}_{\mathrm{FG}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. Form a pullback square

$$\begin{array}{ccc}
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, \cdots] \\
\downarrow & & \downarrow \\
\operatorname{Spec} L_{(p)} & \xrightarrow{f} & \mathcal{M}_{\mathrm{FG}} \times \mathbb{Z}_{(p)}
\end{array}$$

We need to prove that $M_B = M \otimes_{L_{(p)}} B$ is flat as a module over the ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. Then by the technique of filtered colimits and the derived functors Tor, it is reduced to the flatness over every $\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n]$.

Lemma 2.28. *Consider the ideal I_m contained in B generated by the image of v_i, $0 \leq i \leq m-1$. For $m \leq n+1$, the quotient $M_B/I_m M_B$ is a flat module over the ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n] / (p, v_1, \ldots, v_{m-1})$.*

The case of $m = 0$ implies our theorem, and when $m = n+1$, there is nothing necessary to prove. Combined with lemma 2.25, it suffices to prove that for every integer $m \geq 0$, the module $(M_B/I_m M_B)\left[v_m^{-1}\right]$ is flat over $(\mathbb{Z}_p[v_1, v_2, \ldots] / (p, v_1, \ldots, v_{m-1}))\left[v_m^{-1}\right]$. It is related to the fact that every quasi-coherent sheaf on the stack $\mathcal{M}_{\mathrm{FG}}^{m}$ is flat, and this fact is proved in the 16th lecture of [8] in a very algebraic way.

2.4. **Phantom maps and even periodic cohomology theories.** We will prove a stronger version of the second part of Theorem 1.1 in the end of this subsection.

We have already known that homology theories are related to spectra. The more precise statement is Adams' variant of the Brown representability theorem in [1].

Theorem 2.29. *Let E be a spectrum and let h_* be a homology theory. Suppose we are given a map of homology theories $\alpha : E_* \to h_*$ (that is, a collection of maps $E_*(X, Y) \to h_*(X, Y)$, depending functorially on a pair of spaces $(Y \subseteq X)$ and compatible with boundary maps). Then there is a map of spectra $\beta : E \to E'$ and an isomorphism of homology theories $E'_* \simeq h'_*$ such that α is given by the composition $E_* \to E'_* \simeq h_*$.*

Corollary 2.30. *Every homology theory is represented by a unique spectrum up to homotopy equivalence, although there may be many choices of isomorphisms.*

Remark 2.31. The obstructions of unique isomorphism are the so-called phantom maps. We call a map $f : E \to E'$ between spectra a phantom if and only if it induces the zero map in the associated homology theories.

We will study the phantom maps between spectra given by Theorem 1.1, 2.23, and 2.30. First, we need several features of phantom maps.

Lemma 2.32. *Let $f : E \to E'$ be a map of spectra. The following conditions are equivalent:*

(1) The map f is a phantom.
(2) For every spectrum X, the map $E_*(X) \to E'_*(X)$ is zero.
(3) For every finite spectrum X, the map *$E_*(X) \to E'_*(X)$ is zero.*
(4) For every finite spectrum X, the map $E^*(X) \to E'^*(X)$ is zero.
(5) For every finite spectrum X and every map $g : X \to E$, the composition $f \circ g : X \to E'$ is nullhomotopic.

Proof. (1) is equivalent to (2) because we know that every spectra is approximated as the colimit of a family of spaces. The equivalence between (3) and (4) directly follows from Spanier-Whitehead duality, and it is trivial that (4) is equivalent to (5). As for (2) and (3), we only need to note that every spectrum is a filtered colimit of finite spectra. □

From then on, we will restrict our attention to evenly graded ordinary L-modules that are flat over $\mathcal{M}_{\mathrm{FG}}$ and the spectra constructed from them. For convenience, we will call these spectra *Landweber-exact* spectra.

Theorem 2.33. *Let E be a Landweber-exact spectrum, and let E' be a spectrum such that $\pi_k E' \simeq 0$ for k odd. Then every phantom map $f : E \to E'$ is nullhomotopic.*

Because Landweber-exact spectra satisfy the condition that $\pi_k \simeq M_k \simeq 0$ for each k odd, we know that:

Corollary 2.34. *Let E and E' be Landweber-exact spectra. Then every phantom map $f : E \to E'$ is nullhomotopic. In particular, every nontrivial endomorphism of E acts nontrivially on the homology theory E_*.*

More precisely speaking, since there's no phantom map between Landweber-exact spectra, Theorem 2.29 implies that the construction from a L-module M flat over $\mathcal{M}_{\mathrm{FG}}$ to a homology theory $\mathrm{MU}(-) \otimes_L M$, then to a spectrum realizing this homology theory is functorial.

Theorem 2.33 is the consequence of the following two propositions.

Proposition 2.35. *Every Landweber-exact spectrum E is evenly generated, i.e. for every finite spectrum X and every morphism $X \to E$, there exists a factorization $X \to X' \to E$ where X' is a finite spectrum admitting a decomposition such that there are no odd-dimensional cells in it.*

Proposition 2.36. *Let E be an evenly generated spectrum and let E' be a spectrum whose homotopy groups in odd degrees vanish. Then every phantom map $f : E \to E'$ is nullhomotopic.*

The reason for Proposition 2.35 is actually the nice structure of MU, Bruhat decomposition of complex Grassmannians, that is discussed in [8].

As for the Proposition 2.36, let E be an evenly generated spectrum. Here we take advantage of a common technique. Let A be a set of representatives for all homotopy equivalence classes of maps $X_\alpha \to E$, where X_α is an even finite spectrum, and form a fiber sequence

$$K \to \bigoplus_\alpha X_\alpha \xrightarrow{u} E.$$

Take the triangle, and we denote $E \to \Sigma(K)$ by u'. Since E is evenly generated, every map from a finite spectrum X into E factors through u, so the composite map $X \to E \to \Sigma(K)$ is null. On the basis of Lemma 2.32, u' is a phantom map. Furthermore, if $f : E \to E'$ is any phantom map, then $f \circ u$ is nullhomotopic, so that f factors as a composition $E \to \Sigma(K) \to E'$.

As a result, to prove Proposition 2.36, we need to prove that E'^{-1} vanishes. Since the homotopy groups of E' are concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence shows that $E'^{-1}(X) \simeq 0$ whenever X is a finite even spectrum. It will therefore suffice to prove the following:

Claim 2.37. The spectrum K is a retract of a direct sum of even finite spectra.

Proof. Consider the collection B of all the triples β, (α, α', f), such that $\alpha, \alpha' \in A$, and $f : X_\alpha \to X_{\alpha'}$ induces this commutative diagram.

$$\begin{tikzcd} X_\alpha \arrow[rr, "f"] \arrow[dr] & & X_{\alpha'} \arrow[dl] \\ & E & \end{tikzcd}$$

For each triple $\beta \in B$, we let $Y_\beta = X_\alpha$ and take the natural map from $\bigoplus_\beta Y_\beta$ to $\bigoplus_{\alpha'} X_{\alpha'}$ induced by all f's. Unwinding the definition, there are maps $K \to \bigoplus_\beta Y_\beta \to K$ giving us the retract diagram. More details is presented in [8]. $\square$

These constructions are still quite different from complex-oriented cohomology theories. More precisely, they should exhibit both multiplicative and Landweber-exact properties. But for a general flat graded module, the resultant spectrum might not inherently possess multiplicativity. In an attempt to step further, people studied *twisted* formal group laws. They can be regarded as various *coordinations* of generalized formal groups. In [8], the author proves that every formal group can be recovered by a twisted formal group law in the following way.

Definition 2.38 (Twisted formal group law)**.** For R a commutative ring and $\mathcal{L}$ an invertible R-module, we consider the twisted formal group law as a formal power series

$$f(x,y)=\sum a_{ij}x^iy^j,$$

where $a_{ij}\in\mathcal{L}^{\otimes(i+j-1)}$, satisfying the similar three axioms in Definition 2.1. Likewise, it induces a formal group by $A\mapsto \mathrm{Hom}_R(\mathcal{L},\sqrt{A})$. Here, A is an R-algebra and $\sqrt{A}$ is its nilpotent ideal. It is indeed a formal group because of the invertible property.

Remark 2.39. The data of $\mathcal{L}$-twisted formal group laws is the same as the morphism $L\to\bigoplus_{n\in\mathbb{Z}}\mathcal{L}^{\otimes n}$ with even grade structure in the latter ring. Besides, the following two statement is equivalent:

- The formal group in Definition 2.38 is classified by a flat morphism $\operatorname{Spec}R\to\mathcal{M}_{\mathrm{FG}}$.
- The graded L-module $\bigoplus_{n\in\mathbb{Z}}\mathcal{L}^{\otimes n}$ is Landweber-exact.

This is true because $\bigoplus_{n\in\mathbb{Z}}\mathcal{L}^{\otimes n}$ is flat over R. So for every flat $\operatorname{Spec}R\to\mathcal{M}_{\mathrm{FG}}$, we functorially obtain a spectrum determined uniquely up to canonical isomorphism in stable homotopy category due to Corollary 2.34, and denote it by E_R.

The following proposition endows E_R with a canonical commutative and associative ring structure up to homotopy.

Proposition 2.40. *In this pullback diagram, $E_B\simeq E_R\otimes E_{R'}$. Hence, the evident diagonal map* $\operatorname{Spec}R\to\operatorname{Spec}B$ *in the case of $R=R'$ verifies our claim above.*

$$\begin{tikzcd}
\operatorname{Spec}B \arrow[r] \arrow[d] & \operatorname{Spec}R' \arrow[d, "q'"] \\
\operatorname{Spec}R \arrow[r, "q"] & \mathcal{M}_{FG}
\end{tikzcd}$$

Proof. Consider the universal case $\operatorname{Spec}L\to\mathcal{M}_{\mathrm{FG}}$, and then we can get a universal spectrum MP, called *periodic complex bordism spectrum.* From the local perspective, we can assume that q and q' are formal groups induced by certain formal group laws. In this case, the definition of MP implies $B\simeq R\otimes(\mathrm{MP}_0\mathrm{MP})\otimes R'$. As a result, $(E_R\otimes E_{R'})_0(X)\simeq R\otimes_L R'\otimes_L(\mathrm{MP}\otimes\mathrm{MP})_0(X)$ is identical with $(E_B)_0(X)$. □

Obviously, these spectra constructed in this way are even periodic as follows.

Definition 2.41. Let E be a ring spectrum. We will say that E is even periodic if the following conditions are satisfied:

- The homotopy groups π_iE vanish when i is odd.
- The map $\pi_2E\otimes_{\pi_0E}\pi_{-2}E\to\pi_0E$ is an isomorphism.

Remark 2.42. For the even periodic spectrum, which is automatically complex-oriented due to Example 2.16, the formal group law on π_* can be seen as a twisted formal group law on π_0. It gives us a morphism $\pi_0\to\mathcal{M}_{\mathrm{FG}}$. Because for each even periodic spectrum, $\pi_0\to\pi_*$ is flat, the property that π_0 is flat over $\mathcal{M}_{\mathrm{FG}}$ is equivlent to that π_* is flat over $\mathcal{M}_{\mathrm{FG}}$.

Actually, even periodic spectra form up the essential image of our functorial construction above. To conclude, we have:

Theorem 2.43. *Let $\mathcal{C}$ be the full subcategory of $Sch^{aff}_{/\mathcal{M}_{FG}}$ spanned by all affine scheme flat over $\mathcal{M}_{FG}$. Then the construction $R \mapsto E_R$ is a fully faithful embedding into the category of commutative algebras in the homotopy category of spectra whose essential image is all of even periodic spectra such that the intrinsic map $\pi_0 \to \mathcal{M}_{FG}$ is flat.*

Proof. The reason for the property of fully faithful functor is that there are no nontrivial phantom maps. Then we can construct a right inverse $E \mapsto (\pi_*(E) \to \mathcal{M}_{\mathrm{FG}})$. After Landweber's construction, they are isomorphic since it is clear that they have the same homology theory but no nontrivial phantom maps exists. ☐

For future discussion, we define a weaker notion called complex periodic spectra.

Definition 2.44. We say that a spectrum A is complex periodic if and only if it is complex-oriented and weakly 2-periodic in the following sense.

- $\pi_2(A)$ is a projective module of rank 1 over $\pi_0(A)$.
- For every n, $\pi_2(A) \otimes_{\pi_0(A)} \pi_n(A) \simeq \pi_{n+2}(A)$.

2.5. **Example: Elliptic cohomology.** Elliptic curves have inherent formal groups. Hence, according to Landweber exact functor theorem, one can construct the spectrum associated to certain elliptic curves. Only if the conditions of Theorem 1.1 hold could we obtain an even periodic spectrum. More precisely, we have:

Construction 2.45. Let R be an ordinary commutative algebra and E be an elliptic curve over R, then we have a formal group $\widehat{E}$ which attaches every ordinary R-algebra A an abelian group consisting of the collection of the following diagrams and the natural additive structure induced by E. The affine scheme $\operatorname{Spec} A^{\mathrm{red}}$ can be seen as the point, while $\operatorname{Spec} A$ is its local thickening. Finally, when we accumulate enough data about the infinitesimal neighborhood of $\operatorname{Spec} A^{\mathrm{red}}$, we finish the procedure of formal completion near the zero point of E. For example, when R is a field k and we fix A^{red} to be k as well, then the thickening sequence $\operatorname{Spec}(k[\epsilon]/\epsilon^n)$ coincides with our thoughts.

$$\begin{tikzcd}
\operatorname{Spec} A^{\mathrm{red}} \arrow[r, hook] \arrow[d] & \operatorname{Spec} A \arrow[d] \\
\operatorname{Spec} R \arrow[r, "0"] & E
\end{tikzcd}$$

When this formal group satisfies Landweber exact functor theorem, the corresponding spectrum will be called elliptic cohomology.

Definition 2.46. For every commutative ring R, let $\mathrm{Ell}(R)$ denote the category of elliptic curves over R and let $\mathrm{Ell}(R)^{\simeq}$ denote its underlying groupoid. The construction $R \mapsto \mathrm{Ell}(R)^{\simeq}$ is (representable by) a Deligne-Mumford stack $\mathcal{M}_{\mathrm{Ell}}$, which we will refer to as the moduli stack of elliptic curves. The construction above implies the existence of the map $\mathcal{M}_{\mathrm{Ell}} \to \mathcal{M}_{\mathrm{FG}}$ which is flat according to [8].

Remark 2.47. The étale topos of $\mathcal{M}_{\mathrm{Ell}}$ can be identified with the topos of this Grothendieck category $\mathcal{U}$ with étale covering as topology, where $\mathcal{U}$ is defined as follows:

- The objects of $\mathcal{U}$ are pairs (R, E), where R is an ordinary commutative ring and E is an elliptic curve over R which is classified by an étale map $\operatorname{Spec}(R) \to \mathcal{M}_{\mathrm{Ell}}$.

- A morphism from (R,E) to (R',E') in the category $\mathcal{U}$ is given by a pullback diagram of schemes σ :
$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & E' \arrow[d] \\ \operatorname{Spec} R \arrow[r, "f_0"] & \operatorname{Spec} R' \end{tikzcd}$$
having the property that f carries the zero section of E into the zero section of E'.

Then the uniquely determined elliptic cohomology with respect to a fixed elliptic curve induces a presheaf $\mathcal{O}^h_{\mathcal{M}_{\mathrm{Ell}}}$ defined over $\mathcal{U}$ with even periodic spectra as values, which is an actual refinement of the structure sheaf. As before, this presheaf doesn't hint at the possibility of $\mathbb{E}_\infty$-structures. Goerss-Hopkins-Miller enhanced this construction in the following sense.

Theorem 2.48 (Goerss-Hopkins-Miller)**.** *The presheaf $\mathcal{O}^h_{\mathcal{M}_{\mathrm{Ell}}}$ can be promoted to a sheaf $\mathcal{O}^{top}_{\mathcal{M}_{\mathrm{Ell}}}$ with $\mathbb{E}_\infty$-ring spectra as values (with respect to the étale topology on the category $\mathcal{U}$), whose global sections are called topological modular forms* TMF*.*

In other words, our construction above can be lifted to $\mathbb{E}_\infty$-rings and satisfy the sheaf condition in this category. The reason for this lifting has something to do with our main target Theorem 1.2. Based on the similar definition of orientations in [10], the author proved this theorem briefly. Here, I just use this example to help our readers understand the value of our constructions in the next sections.

As to TMF, one of the most important features is that its ring of homotopy groups is rationally isomorphic to the ring of weakly holomorphic modular forms of integral weights. It is not so surprising since these rings are linked with the moduli stack of elliptic curves. Besides, both elliptic cohomology and topological modular forms are regarded as the higher data in our chromatic tower (See Theorem 2.52). For detailed theories on them, see [3].

2.6. **Example: Lubin-Tate theory, classical version.** Beyond the basic examples in chromatic homotopy theory, certain spectra fashioned through formal groups exhibit remarkable richness and potency. Among these, the Lubin-Tate spectrum stands out as its particular significance. Originating from the deformation of formal groups with a fixed height, this spectrum vividly portrays the behavior of an open substack of $\mathcal{M}_{\mathrm{FG}}$.

Fix a perfect field k of character p and a formal group law f of height n over k, then we will define the deformation of f.

Definition 2.49. An infinitesimal thickening of k is an ordinary commutative ring A with a surjective map $\phi : A \to k$ whose kernel $\mathfrak{m}_A = \ker(\phi)$ has the following properties:

(1) The ideal $\mathfrak{m}_A^a = 0$ for $a \gg 0$.
(2) Each quotient $\mathfrak{m}_A^a/\mathfrak{m}_A^{a+1}$ is a finite-dimensional vector space over k.

It means that A is a local Artin ring with residue field k.

Definition 2.50. Let A be an infinitesimal thickening of k. A deformation of f over A is a formal group law f_A over A, whose image under the map $\mathrm{FGL}(A) \to \mathrm{FGL}(k)$ is f. We say that two deformations of f are isomorphic if they differ by an invertible power series $g(t) \in A[[t]]$ such that $g(t) \equiv t \bmod \mathfrak{m}_A$. We will denote the collection of isomorphism classes of deformations of f over A by $\mathrm{Def}(A)$.

Lubin and Tate found a universal deformation under the settings here. The following theorem enables our idea at the beginning of the subsection. For the sketch of proof, one can search [8].

Theorem 2.51 (Lubin-Tate)**.** *There is a formal group law $\bar{f}$ over $R = W(k)\,[[v_1, \ldots, v_{n-1}]]$ that is a universal deformation of f in the following sense: for every infinitesimal thickening A of k, $\bar{f}$ gives a bijection*

$$\mathrm{Hom}_k(R, A) \to \mathrm{Def}(A).$$

Here, $W(k)$ means Witt vectors, and $R \to k$ is the quotient of the maximal ideal $(p, v_1, \cdots, v_{n-1})$. Then the deformation can be given by every lifting of $L_{(p)} \simeq \mathbb{Z}_{(p)}[t_1, \cdots] \to k$ such that the image of t_{p^i-1} is just v_i.

It is easy that this universal deformation makes R Landweber-exact because for $0 \le i < n$, v_i isn't zero divisors and v_n is invertible in $R/(p, v_1, \cdots, v_{n-1}) \simeq k$ due to our assumption of the height. As a result, we can obtain an even periodic spectrum E called Lubin-Tate theory or Morava E-theory. Despite the reliance on the prime, the height, the field, and the formal group law, many people tend to ignore them in notations. Let's end this section with a nice theorem in chromatic homotopy theory whose proof can be found in [13].

Theorem 2.52 (Chromatic convergence theorem)**.** *Fix the prime number p. Let $E(n)$ be the n-th Lubin-Tate theory. Then $E(n)$-acyclicity implies $E(n-1)$-acyclicity. Besides, given a finite p-local spectrum X, the following Bousfield localization chromatic tower converges to X.*

$$\cdots \to L_{E(2)}X \to L_{E(1)}X \to L_{E(0)}X$$

Remark 2.53. The fiber of each morphism in this sequence gives us the *monochromatic layers* that are slightly easier and important for calculations. Combined with nilpotence theorem, smash product theorem, and periodicity theorem, calculations within chromatic homotopy theory are intricately interwoven in a vibrant and interconnected manner.

Remark 2.54. Since the Lubin-Tate theory depicts the behavior of an open substack, there is a spectrum called the Morava K-theory that depicts the behavior of the locally closed substack $\mathcal{M}^n_{\mathrm{FG}}$. Many algebro-topological facts can be interpreted by this idea. For example, $E(n)$ is Bousfield equivalent to $E(n-1) \times K(n)$. Moreover, our theorem above can be regarded as a tool to recover all data of $\mathcal{M}_{\mathrm{FG}}$ from its stratification.

The Lubin-Tate theory is a nice example to illustrate the limitation of Landweber's theorem. Although we can obtain new spectra reflecting some behaviors of stable homotopy theory, we know nothing about their further structure. Some people, like Robinson in [15], wanted to develop an obstruction theory on a certain kind of cohomology to overcome this problem. Finally in [5], Goerss-Hopkins-Miller proved that Lubin-Tate theory admits an essentially unique $\mathbb{E}_\infty$-structure. Their approach is slightly different from Robinson's because they translated the additional structure into an algebro-geometric problem or, more precisely, a moduli problem first. We will adopt the latter approach to explore questions from an algebraic-geometric perspective.

3. p-Divisible Groups

In [10], the author studied the universal spectral deformation ring instead of the discrete ring $W(k)[[v_1, \cdots, v_{n-1}]]$. This is a generalization compared with classical theories. As a result, it is necessary to extend our theory of formal groups to the sense of ring spectra. We will first develop the theory of smooth coalgebras so as to define formal hyperplanes and generalized (higher-dimensional) formal groups. Then as the relationship between formal groups and p-divisible groups in classical theory, there is a construction called identity component with respect to a p-divisible group, sometimes providing us with a fully faithful bridge like Remark 3.25. At last, we will study the universal deformation in the context of p-divisible groups. Because there are many notions existing in the higher algebra, we'll introduce them at first.

3.1. **Preliminaries of higher algebra.** Many concepts in commutative algebra are introduced in higher algebra, and people tend to mix some notations in [10] with their common meanings. As a result, before we start our journey, I am obligatory to define several concepts and claim some usage of our notations. We still follow the convention in [10].

We will frequently use the notions of algebras in the sense of spectra, so I recommend reader unfamiliar with them to read [4], [9] and [11].

Convention 3.1. Here are our announcements.

- We will use the word *generalized* to emphasize our constructions over ring spectra, and the words *ordinary* or *classical* to emphasize our constructions over ordinary commutative algebras. I hope this action can distinguish the world of ordinary algebras and the world of ring spectra.
- As for $\mathbb{E}_\infty$-ring spectra, we use Lurie's version: that is a commutative algebra object in the symmetric monoidal ∞-category of spectra Sp, whose homotopy category is stable homotopy category.
- CAlg and CAlg_R are used to denote the (∞-)category of $\mathbb{E}_\infty$-ring spectra (defined over R). This does make sense in Lurie's theory because the nerve of the category of finite sets with base points assumes the role to detect $\mathbb{E}_\infty$-property.
- We don't distinguish the difference between an abelian group and its corresponding Eilenberg-MacLane spectrum. That is, we view the ordinary category of abelian groups as a full subcategory of Sp. Similarly, we view he ordinary category of commutative rings as a full subcategory of CAlg.
- When R is an $\mathbb{E}_\infty$-ring spectrum, we use *R-modules* to denote R-module spectra. Then Mod_R will denote the (∞-)category of R-modules.
- For an $\mathbb{E}_\infty$-ring spectrum R, we denote by $\mathrm{Spec}(R)$ the nonconnective spectral Deligne-Mumford stack or, equivalently, the functor $\mathrm{Map}_{\mathrm{CAlg}}(A, -)$.
- Due to the equivalences $\mathrm{cCAlg}_R \simeq \mathrm{cCAlg}_{\tau_{\geqslant 0} R}$ and so on in [10, Proposition 1.2.8], we can assume that our base ring spectrum is always connective from then on.

Then we need some concepts in higher algebra. They are tightly related to their counterparts in ordinary algebra. For example, flat module over an $\mathbb{E}_\infty$-ring deduces that on each stalk, this module behaves like a free module, so that it is a good candidate to define formal hyperplane. Adic ring spectra will appear when we study functions over a formal hyperplane and spectral deformation ring. As for completeness, it reflects the regularity of the adic structure.

Definition 3.2 (Flat modules)**.** Let M be a module over an $\mathbb{E}_\infty$-ring R. We say that M is flat if the following two conditions are satisfied:

(1) $\pi_0(M)$ is a flat module over $\pi_0(R)$ in the usual sense.
(2) The natural map $\pi_n(R) \otimes_{\pi_0(R)} \pi_0(M) \to \pi_n(M)$ is an isomorphism for every integer n.

Definition 3.3 (Adic $\mathbb{E}_\infty$-ring spectra)**.** An adic $\mathbb{E}_\infty$-ring spectrum consists of an $\mathbb{E}_\infty$-ring spectrum A and an adic topology of $\pi_0(A)$ in the classical sense. For two adic ring spectra A and A', we denote by $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(A, A')$ the summand of $\mathrm{Map}_{\mathrm{CAlg}}(A, A')$ spanned by morphisms that are continuous in π_0.

Definition 3.4 (Completeness with respect to an ideal)**.** We say that an $\mathbb{E}_\infty$-ring spectrum A is I-complete with respect to an ideal $I \subset \pi_0(A)$ if and only if for every $x \in I$, the limit of this tower vanishes.

$$\cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A$$

This definition is slightly different from classical completeness, but it won't cause any confusion because we will add the word *classical* whenever we discuss the classical situation.

3.2. **Formal hyperplanes.** Formal schemes is the nicer refinement than affine schemes of power series ring because in the completion process of ordinary rings, there are extra topological structures remembered by formal schemes. So we are willing to study ordinary formal groups over formal schemes. In order to study generalized formal groups in derived algebraic geometry, we also need the notion of formal hyperplane. In addition, there are two reasons for studying this more complexed definition.

(1) Even when dealing with oridinary rings, we don't want to just study coordinated formal groups. Besides, we cannot Zariski locally verify whether a formal scheme is isomorphic to the formal affine space $\hat{\mathbb{A}}^n$.
(2) The definition of $R[[x_1, \cdots, x_n]]$ is subtly ambiguous in the sense of ring spectra. There exist a lot of ring spectra whose homotopy groups are isomorphic to $\pi_*(R)[[x_1, \cdots, x_n]]$. We would allow all of them to become formal power series ring. It implies that we cannot obtain a good notion of formal group laws.

One way to solve it is that we enlarge the definition of so-called *hyperplanes* and replace the concrete formal group laws by the abelian objects in the category of formal hyperplanes. Since our definition tolerates the existence of globally *twisted* modules, we don't need to glue them together as we do in the Definition 2.9.

Lurie then used the notion of coalgebra to define formal hyperplanes. In ordinary algebraic geometry, the coalgebra of formal hyperplanes over a field κ is the coalgebra consisting of distributions: that is, the collection of all κ-linear maps $\mathcal{O}_X \to \kappa$ which vanish for some power of the maximal ideal $\mathfrak{m}$. Sometimes this coalgebra is easy to handle with because for two formal hyperplanes X and Y, the coalgebra of $X \times Y$ is the tensor product of those coalgebras of X and Y. Coalgebras provide more convenience than algebras $\mathcal{O}_X$ because algebras are endowed with topology.

Definition 3.5 (Flat coalgebras)**.** Given an $\mathbb{E}_\infty$-ring R, a flat commutative coalgebra over R is a flat R-module C with structure R-module homomorphisms comultiplication $\Delta : C \to C \otimes_R C$ and counit $\epsilon : C \to R$ which make these diagrams commute.

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes_R C \arrow[rr, "\mathrm{Swap}", "\sim"'] & & C \otimes_R C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\Delta"] \arrow[dr, "\sim"'] & & C \otimes_R C \arrow[dl, "\epsilon \otimes_R \mathrm{id}"] \\ & R \otimes_R C & \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_R C \arrow[d, "\Delta \otimes_R \mathrm{id}"] \\ C \otimes_R C \arrow[r, "\mathrm{id} \otimes_R \Delta"] & C \otimes_R C \otimes_R C \end{tikzcd}$$

Besides, we denote by $\mathrm{cCAlg}_R^\flat$ the full subcategory of flat coalgebras.

Construction 3.6. For R an ordinary commutative ring and M an ordinary flat R-module, we let $\Gamma_R^n(M)$ be the submodule of S_n-action invariant elements in $M^{\otimes n}$ and $\mathrm{Sym}_R^n(M)$ be the quotient module given by coinvariants for the action of S_n. Here, S_n is the symmetric group acting naturally on $M^{\otimes n}$. There are natural (co)algebra structures on $\Gamma_R^*(M) := \bigoplus_n \Gamma_R^n(M)$ and $\mathrm{Sym}_R^*(M) := \bigoplus_n \mathrm{Sym}_R^n(M)$. For example, since $\Gamma_R^*(M \oplus M') \simeq \Gamma_R^*(M) \oplus \Gamma_R^*(M')$, the diagonal map $M \to M \oplus M$ induces a coalgebra structure. We say that a flat coalgebra over R is smooth if and only if it is isomorphic to $\Gamma_R^*(M)$ for some projective R-module M whose rank is finite and called the dimension of smooth coalgebra.

Definition 3.7 (Smooth coalgebras)**.** Generally speaking, we define a smooth coalgebra C over an $\mathbb{E}_\infty$-ring R to be a flat coalgebra such that $\pi_0(C)$ is smooth over $\pi_0(R)$. We let $\mathrm{cCAlg}_R^{\mathrm{sm}}$ be the full subcategory of smooth coalgebras.

Next, let's define the cospectrum of a coalgebra as what we do in the case of algebras. Our choice here is *grouplike* elements.

Definition 3.8. Let R be an $\mathbb{E}_\infty$-ring and C be a flat commutative R-coalgebra. Then we define the grouplike elements of C as all morphisms of commutative coalgebras $R \to C$. We let $\mathrm{GLike}(C)$ denote the space of all grouplike elements. Since the extension of scalars preserves grouplike elements, we can get a functor from the category of connective R-algebras to spaces,

$$\mathrm{cSpec}(C) : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \quad A \mapsto \mathrm{GLike}(A \otimes_R C),$$

which is called the cospectrum of C.

Remark 3.9. The internal Hom in a symmetric monoidal category induces a contravariant functor $\mathrm{Hom}(-, 1)$, turning a coalgebra object into an algebra object. In terms of an $\mathbb{E}_\infty$-ring R and a flat coalgebra C, we denote the associated algebra by $C^\vee$. For a smooth coalgebra, [10, Proposition 1.3.10] implies that the dual is an *adic* $\mathbb{E}_\infty$-ring. The main idea is that we can turn Γ_R^* into Sym_R^* by duality. Besides, this construction is fully faithful when we consider the *continuous* morphisms between R-algebras according to [10, Theorem 1.3.15]. Then the following comparison map induces an equivalence $\mathrm{cSpec}(C) \simeq \mathrm{Spf}(C^\vee) \subseteq \mathrm{Spec}(C^\vee)$.

$$\begin{aligned}
\mathrm{cSpec}(C)(A) = & \ \mathrm{Map}_{\mathrm{cCAlg}_A}(A, A \otimes_R C) \\
\to & \ \mathrm{Map}_{\mathrm{CAlg}_A}\left((A \otimes_R C)^\vee, A^\vee\right) \\
\to & \ \mathrm{Map}_{\mathrm{CAlg}_A}(A \otimes_R C^\vee, A) \\
\simeq & \ \mathrm{Map}_{\mathrm{CAlg}_R}(C^\vee, A) \\
= & \ \mathrm{Spec}(C^\vee)(A)
\end{aligned}$$

Here we need the equivalence $\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}_R}(C^\vee, D^\vee) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})}(\mathrm{Spf}(D^\vee), \mathrm{Spf}(C^\vee))$ in [11, Theorem 8.1.5.1] to determine the image of this comparison map. Due to the same reason, we have:

Theorem 3.10. *Let R be an $\mathbb{E}_\infty$-ring. Then the construction $C \mapsto \mathrm{cSpec}(C)$ induces a fully faithful embedding of ∞-categories*

$$\mathrm{cCAlg}^{\mathrm{sm}}_R \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S}).$$

This gives us a nice realization of formal hyperplanes. Roughly speaking, for a smooth coalgebra C, $\mathrm{cSpec}(C) \simeq \mathrm{Spf}(C^\vee)$ means that cospectrum is a formal geometric object, and the coalgebra structure determines that the cospectrum behaves like a hyperplane.

Definition 3.11 (Generalized formal hyperplanes)**.** Given an $\mathbb{E}_\infty$-ring R, a functor $X : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}$ is called a (generalized) formal hyperplane if and only if it lies in the essential image of the fully faithful embedding in Theorem 3.10. The full subcategory of formal hyperplanes is denoted by $\mathrm{Hyp}(R)$. Actually, the construction of $\mathrm{Hyp}(R)$ is functorial with respect to R. Besides, we can take some smooth coalgebra C such that $X \simeq \mathrm{cSpec}(C)$ and let $\mathcal{O}_X$ be $C^\vee$ called the $\mathbb{E}_\infty$-ring of functions of X. For some reason, we also need to consider the pointed formal hyperplanes, which are defined as functors $\mathrm{CAlg}^{\mathrm{cn}}_R \to \mathcal{S}_*$ such that we get a formal hyperplane after we forget these base points of $\mathcal{S}_*$.

Convention 3.12. From then on, there are many notions that admits the extension of scalars, like formal hyperplanes, formal groups, p-divisible groups and so on. We always use the notation $(-)_{R'}$ for the morphism $R \to R'$ to denote this extension.

Example 3.13. With enough assumptions on regularity, the formal completion along a section is a formal hyperplane, whose reason is discussed in [10, Proposition 1.5.15]. It means that many formal hyperplanes are not as *clear* as we image.

Definition 3.14 (Generalized formal groups)**.** Given an $\mathbb{E}_\infty$-ring R, a formal group over R is a functor $\widehat{\mathbf{G}} : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$ such that the composition with Ω^∞ is a formal hyperplane. Sometimes we call this formal hyperplane the underlying hyperplane of $\widehat{\mathbf{G}}$. Similarly, there is the notion of $\mathbb{E}_\infty$-ring of functions of $\widehat{\mathbf{G}}$, denoted by $\mathcal{O}_{\widehat{\mathbf{G}}}$. Moreover, the full subcategory of formal groups is denoted by $\mathrm{FGroup}(R)$, which is also a functorial construction with respect to R.

Remark 3.15. Since $\mathrm{Ab}(\mathcal{S})$, the category of abelian objects in $\mathcal{S}$, is equivalent to $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$, we know that

$$\mathrm{Ab}(\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathcal{S})) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathrm{Ab}(\mathcal{S})) \simeq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}).$$

As a consequence, formal groups in the sense of Definition 3.14 can be seen as the abelian objects in the category of formal hyperplanes. This observation is the same as that in Remark 2.4.

3.3. **p-divisible groups and their identity components.** Barsotti and Tate introduced the notion of the p-divisible group of height h as an inductive system of finite groups schemes G_n over a base S such that the rank of G_n is p^{hn} and G_n is identified with subgroup scheme of G_{n+1} whose elements are those of order divisible by p^n. It was proposed because the p-torsion points over an abelian variety in characteristic p have the this formalism. First, let me define the generalization of p-divisible groups in derived algebraic geometry.

Definition 3.16. Let R be an $\mathbb{E}_\infty$-ring. A (generalized) p-divisible group over R is a functor $\mathbf{G} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$ with the following properties:

(1) For every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the $\mathbb{Z}$-module spectrum $\mathbf{G}(A)$ is p-nilpotent: that is, we have $\mathbf{G}(A)[1/p] \simeq 0$.
(2) For every finite abelian p-group M, the functor
$$(A \in \mathrm{CAlg}_R^{\mathrm{cn}}) \mapsto \left(\mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}}}(M, \mathbf{G}(A)) \in \mathcal{S}\right)$$
is corepresentable by a finite flat R-algebra.
(3) The map $p : \mathbf{G} \to \mathbf{G}$ is locally surjective with respect to the finite flat topology. In other words, for every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$ and every element $x \in \pi_0(\mathbf{G}(A))$, there exists a finite flat map $A \to B$ for which $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective and the image of x in $\pi_0(\mathbf{G}(B))$ is divisible by p.

For every finite abelian p-group M, we denote the finite flat R-algebra by $\mathbf{G}[M]$. Particularly, when $M \simeq \mathbb{Z}/p^k\mathbb{Z}$, we will denote it by $\mathbf{G}[p^k]$. The full subcategory of p-divisible groups over R is denoted by $\mathrm{BT}^p(R)$ to memorize Barsotti and Tate.

$\mathbf{G}[p^k]$ in this definition is the generalization of G_k in traditional definition. As for the point (3) in this definition, it is related to Grothendieck's refinement in 1971. He used the term p-divisibility with respect to the finite flat topology to denote the condition (3). Based on the definition, $\mathbf{G}$ is approximated by the sequence of $\mathbf{G}[p^k]$. In addition, we can define new p-divisible groups by extending scalars. The next theorem can be regarded as the construction of the Lie algebra of a Lie group.

Theorem 3.17 (Identity components)**.** *Let R be a (p)-complete $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. Then there exists an essentially unique formal group, called the identity component of $\mathbf{G}$, $\mathbf{G}^\circ \in \mathrm{FGroup}(R)$ with the following property: Let $\mathcal{E} \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ denote the full subcategory spanned by those connective R-algebras which are truncated and (p)-nilpotent. Then the functor $\mathbf{G}^\circ|_{\mathcal{E}}$ is given by the explicit construction $A \mapsto \mathrm{fib}\left(\mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}})\right)$.*

Similarly, A^{red} means the point, A means the thickening, and the whole data reflect the formal completion.

Sketch of Proof. Because [10, Theorem 2.1.1] detects this subcategory $\mathcal{E}$ and adjusts the object we need to consider, we can assume that R is a commutative $\mathbb{F}_p$-algebra. In this case, we can use the Frobenius morphisms to construct our formal group. More precisely, let X be a functor from the category $\mathrm{CAlg}_R^{\heartsuit}$ of commutative R algebras to some other category $\mathcal{C}$ (in practice, $\mathcal{C}$ will be either the category of sets or the category of abelian groups). For each $n \geqslant 0$, we let $X^{(p^n)} : \mathrm{CAlg}_R^{\infty} \to \mathcal{C}$

denote the functor given by the formula $X^{(p^n)}(A) = X(A^{1/p^n})$, where for every commutative R-algebra A, the nth power of the Frobenius map φ_A is denoted by $A \to A^{1/p^n}$. Simultaneously, it induces a map $X(A) \to X(A^{1/p^n}) = X^{(p^n)}(A)$. These maps are natural in A, and therefore define a natural transformation of functors $\varphi^n_{X/R} : X \to X^{(p^n)}$ which we will refer to as the relative Frobenius map.

After the pure but tedious calculation on affine spaces and then gluing them together given in [10], we know that if we let $X^{(p^\infty)}$ be the direct limit of the sequence of relative Frobenius maps

$$X \to X^{(p)} \to X^{(p^2)} \to \cdots ,$$

we can obtain an homotopy equivalence

$$X^{(p^\infty)}(A) \simeq X^{(p^\infty)}(A^{\mathrm{red}}) \tag{3.18}$$

every truncated $\mathbb{E}_\infty$-ring A.

Given a p-divisible group $\mathbf{G}$, we can do the same construction $G^{(p^n)}$, a functor from $\mathrm{CAlg}^{\heartsuit}_R$ to Ab. Then we can take the kernel of the relative Frobenius morphism, and we denote it by $\mathbf{G}[F^n]$. We can prove that $\mathbf{G}[F^n]$ is a finite flat group scheme over R. Our challenge is just the finiteness property because other properties follows from the fact that $\mathbf{G}[F^n]$ is a kernel. So the key point here is that there is also a so-called Verschiebung map $\mathbf{G}^{(p^n)} \to \mathbf{G}$. It can help us obtain a series of commutative diagrams which is related to finiteness. Besides, as a property in classical algebraic geometry tells, $G[F^n]$ is compatibly and Zariski locally isomorphic to $\mathrm{Spec}(R[x_1, \cdots, x_d]/(x_1^{p^n}, \cdots, x_d^{p^n}))$ for a fixed d dependent on $\mathbf{G}$.

After all of these preliminaries, we define $\mathbf{G}_0 : \mathrm{CAlg}^{\mathrm{cn}}_R \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$ by $\mathbf{G}_0(A) = \mathrm{colim}_n \left(\mathbf{G}\left[F^n\right]\right)(A)$. Our theorem is thus the consequence of the following two facts.

(1) The functor $\mathbf{G}_0|_{\mathcal{E}}$ can be extended to a formal group over R.
(2) For each $A \in \mathcal{E}$, we have a fiber sequence $\mathbf{G}_0(A) \to \mathbf{G}(A) \to \mathbf{G}(A^{\mathrm{red}})$.

The first fact is a Zariski-local problem. It suffices to justify that the underlying scheme is a formal hyperplane. Then we can take advantage of the simple representation of $\mathbf{G}[F^n]$ above. Take C_n as the dual coalgebra of $R[x_1, \cdots, x_d]/(x_1^{p^n}, \cdots, x_d^{p^n})$, then the underlying scheme of $\mathbf{G}[F^n]$ is $\mathrm{cSpec}(C_n)$. So the underlying scheme of $\mathbf{G}_0$ can be regarded as the cospectrum of the colimit of C_n that is isomorphic to $\Gamma^*_R(R^d)$.

As to the second fact, we need to return to Verschiebung maps. After similar processes, one can construct a fiber sequence $\mathbf{G}_0 \to \mathbf{G} \to H$ in $\mathrm{Fun}\left(\mathrm{CAlg}^{\mathrm{cn}}_R, \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}\right)$. In this diagram, for $A \in \mathcal{E}$, the isomorphism (3.18) implies $\mathbf{G}_0(A^{\mathrm{red}}) \simeq 0$.

$$\begin{tikzcd}
 & \mathbf{G}_0(A) \arrow[r] \arrow[d] & \mathbf{G}(A) \arrow[r] \arrow[d] & H(A) \arrow[d] \\
0 \arrow[r, "\sim"] & \mathbf{G}_0(A^{\mathrm{red}}) \arrow[r] & \mathbf{G}(A^{\mathrm{red}}) \arrow[r] & H(A^{\mathrm{red}})
\end{tikzcd}$$

If we can prove that $H(A) \simeq H(A^{\mathrm{red}})$, our theorem directly follows from diagram chasing. This isomphism holds because of the definition of H and (3.18). □

Now that formal groups are the derivatives of p-divisible groups, the notions of height and Hasse invariants can be translated naturally under this setting. But we will only give the definition of the nth Landweber idea in section 4 but omit the generalized Hasse invariants.

Definition 3.19 (Height)**.** Let R be an ordinary commutative ring, let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R, and let $[p] : \widehat{\mathbf{G}} \to \widehat{\mathbf{G}}$ be the map given by multiplication by p. For $n \geq 1$, we will say that $\widehat{\mathbf{G}}$ has height $\geq n$ if $p = 0$ in R and the map $[p]$ factors through the iterated relative Frobenius map $\widehat{\mathbf{G}} \xrightarrow{\varphi_{\widehat{\mathbf{G}}^n}} \widehat{\mathbf{G}}^{(p^n)}$ introduced in the proof above. We extend this terminology to the case $n = 0$ by declaring that all formal groups over R have height ≥ 0. In this case, we do not require that $p = 0$ in R. It coincides with the classical definition because when our formal group is coordinated, the iterated relavtive Frobenius map is $t \mapsto t^{p^n}$, and the decomposition of natural transformation $[p]$ is written as $[p](t) = h'(t^{p^n})$ for some h' in the sense of Lemma 2.10.

Remark 3.20. The motivation behind our construction is inherently straightforward. The Frobenius maps extracts these *torsion* points. When we deal with Lie groups, similar procedures make us obtain the data of Lie algebras instead of the original Lie groups.

Afterwards, we will introduce a technical decomposition for our future use. Its classical version is also a famous theorem in algebraic geometry. In order to avoid tedious but useless proof, I just list the definitions and properties here. Impatient readers can directly read Theorem 3.27 for the following sections. For detailed theories, see [10].

Definition-Theorem 3.21. A morphism of p-divisible groups $\mathbf{G} \to \mathbf{G}'$ over an $\mathbb{E}_\infty$-ring R is called *strict epimorphism* if and only it satisfies the following three equivalent conditions:

(1) For every finite abelian p-group M, the induced map $\mathbf{G}[M] \to \mathbf{G}'[M]$ is an epimorphism of finite flat group schemes over R.
(2) For each $m \geqslant 0$, the induced map $\mathbf{G}\left[p^m\right] \to \mathbf{G}'\left[p^m\right]$ is an epimorphism of finite flat group schemes over R.
(3) The induced map $\mathbf{G}[p] \to \mathbf{G}'[p]$ is an epimorphism of finite flat group schemes over R.

In addition, a morphism of p-divisible groups $\mathbf{G} \to \mathbf{G}'$ over an $\mathbb{E}_\infty$-ring R is called *monomorphism* if and only it satisfies the following four equivalent conditions:

(1) For every finite abelian p-group M, the induced map $\mathbf{G}[M] \to \mathbf{G}'[M]$ is a monomorphism of finite flat group schemes over R.
(2) For each $m \geqslant 0$, the induced map $\mathbf{G}\left[p^m\right] \to \mathbf{G}'\left[p^m\right]$ is a monomorphism of finite flat group schemes over R.
(3) The induced map $\mathbf{G}[p] \to \mathbf{G}'[p]$ is a monomorphism of finite flat group schemes over R.
(4) For every discrete R-algebra A, the induced map $\mathbf{G}(A) \to \mathbf{G}'(A)$ is a monomorphism of abelian groups.

Definition-Theorem 3.22. Let R be an $\mathbb{E}_\infty$-ring, let $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}, \mathrm{Nil}(p)}$ denote the category of connective (p)-torsion $\mathbb{Z}$-module spectra, and let $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}, \mathrm{Nil}(p)})$ denote the full subcategory spanned by those functors $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}, \mathrm{Nil}(p)}$ which are sheaves with respect to the finite flat topology. Suppose we are given a commutative diagram σ :

$$\begin{tikzcd}
\mathbf{G}' \arrow[r, "f"] \arrow[d] & \mathbf{G} \arrow[d, "g"] \\
0 \arrow[r] & \mathbf{G}''
\end{tikzcd}$$

in the $\mathcal{C}$. The following conditions are equivalent:

(1) The functors $\mathbf{G}$ and $\mathbf{G}''$ are p-divisible groups, the map g is a strict epimorphism of p-divisible groups, and the diagram σ is a pullback square in $\mathcal{C}$.
(2) The functors $\mathbf{G}'$ and $\mathbf{G}$ are p-divisible groups, the map f is a monomorphism of p-divisible groups, and σ is a pushout square in $\mathcal{C}$.
(3) The functors $\mathbf{G}'$ and $\mathbf{G}''$ are p-divisible groups and σ is a pushout square in $\mathcal{C}$.

In this case, we say that σ is a short exact sequence. Sometimes we will denote it by $0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$. According to the property above, this sequence is exact if and only if σ is not only a pullback diagram but also a pushout diagram, f is a monomorphism, and g is a strict epimorphism.

Our target is to introduce the essentially unique connected-étale sequence of p-divisible groups. These two concepts were introduced by Tate and other people to determine whether a formal group is given by a p-divisible groups.

Definition 3.23. Let R be a $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R, so that the functor $(\Omega^\infty \circ \mathbf{G}[p]) : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is corepresentable by a finite flat R-algebra A. We say that $\mathbf{G}$ is connected if the underlying map of topological spaces $|\operatorname{Spec}(A)| \to |\operatorname{Spec}(R)|$ bijective.

Definition 3.24. Let R be an adic $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. We will say that $\mathbf{G}$ is formally connected if $\mathbf{G}_{\pi_0(R)/I}$ is a connected p-divisible group over the commutative ring $\pi_0(R)/I$, where $I \subseteq \pi_0(R)$ is a finitely generated ideal of definition.

Remark 3.25. By [10, Theorem 2.3.12], we know that when dealing with adic $\mathbb{E}_\infty$-rings such that p is topologically nilpotent in $\pi_0(R)$, the category of formally connected p-divisible groups are fully faithfully embedded into the category of formal groups through identity components. Under this setting, we say that a formal group is p-divisible if and only if it lies in the essential image of identity component functor.

Definition-Theorem 3.26. Let R be an $\mathbb{E}_\infty$-ring and let $\mathbf{G}$ be a p-divisible group over R. The following conditions are equivalent:

(1) For every finite abelian p-group M, the functor
$$\Omega^\infty \mathbf{G}[M] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \quad A \mapsto \mathrm{Map}_{\mathrm{Mod}_\mathbb{Z}}(M, \mathbf{G}(A))$$
is corepresentable by an étale R-algebra.
(2) For each $n \geq 0$, the functor
$$\Omega^\infty \mathbf{G}\left[p^n\right] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \quad A \mapsto \mathrm{Map}_{\mathrm{Mod}_\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z}, \mathbf{G}(A))$$
is corepresentable by an étale R-algebra.
(3) The functor
$$\Omega^\infty \mathbf{G}[p] : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S} \quad A \mapsto \mathrm{Map}_{\mathrm{Mod}_\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbf{G}(A))$$
is corepresentable by an étale R-algebra.

At this time, $\mathbf{G}$ is called an étale p-divisible group.

In the study of algebraic groups, people already discovered that after quotienting the maximal connected component, one can get an étale quotient group. This analogous fact holds even in the field of Lie groups. We will then introduce the generalized form in the sense of p-divisible groups. Two features are that the identity component of the connected part is the same as that of the whole p-divisible group, and that the étale part is as easy as possible.

Theorem 3.27. *Let R be a complete adic $\mathbb{E}_\infty$-ring such that p is topologically nilpotent in $\pi_0(R)$ and $\mathbf{G}$ be a p-divisible group over R. Then the following two statements are equivalent:*

(1) There exists a short exact sequence of p-divisible groups $0 \to \mathbf{G}' \to \mathbf{G} \to \mathbf{G}'' \to 0$ such that the former one is formally connected and the latter one is étale.
(2) The identity component of $\mathbf{G}$ is p-divisible.

Besides, when these statements are right, the decomposition is essentially unique.

3.4. **Deformation theory of p-divisible groups.** Our main theorem here is the generalized version of Theorem 2.51. The improvements are mirrored in these aspects:

- We are focused on the deformation of p-divisible groups instead of the original 1-dimensional formal groups.
- We study more ring spectra instead of just perfect fields.
- The forms of deformations get more colorful.

Before the statement of our theorem, we need some notions first. They appear in the statement of Theorem 1.2, and endow our geometric objects with great regularity so that many nice properties hold.

Definition 3.28. Like the definition of deformation above, given a p-divisible group $\mathbf{G}_0$ over a commutative ring R_0 and an $\mathbb{E}_\infty$-ring A with a morphism $\rho_A : A \to R_0$, a deformation of $\mathbf{G}_0$ along ρ_A is a pair ($\mathbf{G}$,α), where $\mathbf{G}$ is a p-divisible group over A and $\alpha : \mathbf{G}_0 \simeq \mathbf{G}_{R_0}$ is an isomorphism from $\mathbf{G}_0$ to the extention of scalars of $\mathbf{G}$. The collection of such deformations is denoted by $\mathrm{Der}_{\mathbf{G}_0}(A, \rho_A)$, which is a homotopy fiber of $\mathrm{BT}^p(A)$ over $\mathrm{BT}^p(R_0)$.

Construction 3.29. Let $\mathbf{G}$ be a p-divisible group defined over a commutative ring R. Suppose that we are given a point $x \in |\operatorname{Spec}(R)|$ and a derivation $d : R \to \kappa(x)$, where $\kappa(x)$ denotes the residue field of R at x. Then the canonical map $\beta_0 : R \to \kappa(x)$ lifts to a ring homomorphism $\beta : R \to \kappa(x)[\epsilon]/(\epsilon^2)$, given by the formula $\beta(t) = \beta_0(t) + \epsilon dt$. Let $\mathbf{G}_d$ denote the p-divisible group over $\kappa(x)[\epsilon]/(\epsilon^2)$ obtained from $\mathbf{G}$ by extending scalars along β. Then $\mathbf{G}_d$ is a first-order deformation of the p-divisible group $\mathbf{G}_{\kappa(x)}$. If $d = 0$, then $\mathbf{G}_d$ is a trivial first order deformation of $\mathbf{G}_{\kappa(x)}$.

Definition 3.30. Let R be a commutative ring and let $\mathbf{G}$ be a p-divisible group over R. We will say that $\mathbf{G}$ is *nonstationary* if it satisfies the following condition: For every point $x \in |\operatorname{Spec}(R)|$ and every nonzero derivation $d : R \to \kappa(x)$, the p-divisible group $\mathbf{G}_d$ of the construction above is a nontrivial first-order deformation of $\mathbf{G}_{\kappa(x)}$.

The following theorem is also the enhancement of classical universal deformation. It's slightly technical, so we directly use it without further illustration.

Theorem 3.31. *Let R_0 be an oridinary Noetherian $\mathbb{F}_p$-algebra which is F-finite (that is, the Frobenius morphism $\varphi : R_0 \to R_0$ is finite) and let $\mathbf{G}_0$ be a nonstationary p-divisible group over R_0. Then there exists a morphism of connective $\mathbb{E}_\infty$-rings $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R_0$ and a deformation $\mathbf{G}$ of $\mathbf{G}_0$ along ρ with the following properties:*

- *The $\mathbb{E}_\infty$-ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ is Noetherian, the morphism ρ induces a surjection of commutative rings $\epsilon : \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0}) \to R_0$, and $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to the ideal $\ker(\epsilon)$.*
- *Let A be any Noetherian $\mathbb{E}_\infty$-ring equipped with a map $\rho_A : A \to R_0$ for which the underlying ring homomorphism $\epsilon_A : \pi_0(A) \to R_0$ is surjective and A is complete with respect to $\ker(\epsilon_A)$. Then extension of scalars induces an equivalence $\mathrm{Map}_{\mathrm{CAlg}_{R_0}}(R^{\mathrm{un}}_{\mathbf{G}_0}, A) \to \mathrm{Def}_{\mathbf{G}_0}(A, \rho_A)$.*

In particular, $\pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$ is exactly the classical universal deformation ring to which the scalar extention of $\mathbf{G}$ is exactly the classical universal deformation of $\mathbf{G}_0$. We will denote the classical universal deformation of $\mathbf{G}_0$ by $\mathbf{G}_{\mathbf{cl}}$.

Remark 3.32. Our strategy is similar to the one we use in the proof of Lemma 2.10, although we are dealing with more complicated problems. In differential geometry, differential forms are sections of cotangent bundles. But in algebraic geometry, one needs to introduce a reasonable definition of dualizing sheaves. The author of [10] preferes the language of dualizing lines because dualizing lines are easy and comprehensive for the special case of 1-dimensional formal groups.

4. Orientations

Similar to the case of complex-oriented spectra, a fixed (pre)orientation makes more data coherent. For example, we have a well-behaved formal group law and then develop its theory in section 2. As a result, better properties will emerge if we consider the *oriented* classifier and the oriented universal deformation. In the beginning of this section, we will generalize our theory of formal groups associated with a given complex periodic spectrum. Then we will develop the notion of orientation. Last but not the least, we will review the properties of Lubin-Tate theory for our main theorem.

4.1. Generalized Quillen formal groups.

Convention 4.1. We adopt the convention of notations in [10]. So we will denote by $C_*(X; A)$ the spectrum whose homotopy groups are A-homology groups of X. Besides, $C^*(X; A)$ is the associated spectrum of cohomology groups.

The following proposition doesn't touch any new ideas, and interested readers can try to prove it on their own or read the proof of [10, Theorem 4.1.11]. We need this proposition to define generalized Quillen formal groups.

Proposition 4.2. *Let A be a complex periodic $\mathbb{E}_\infty$-ring. Then $C_*(\mathbb{CP}^\infty; A)$ is a smooth coalgebra of dimension 1 over A.*

Construction 4.3 (The (generalized) Quillen formal group)**.** As we know, the category of finite sets with base point is equivalent to the category of free abelian groups of finite rank. If we denote the latter one by $\mathrm{Lat}^{\mathrm{op}}$, then we get a functor

$$\mathrm{Lat}^{\mathrm{op}} \to \mathrm{Hyp}(A) \quad M \mapsto \mathrm{cSpec}(C_*(K(M^\vee, 2), A)).$$

for a complex periodic $\mathbb{E}_\infty$-ring A since the proposition above and $\mathbb{CP}^\infty \simeq K(\mathbb{Z},2)$. It is an abelian object in $\mathrm{Hyp}(A)$, which is thus, by Remark 3.15, called the Quillen formal group and denoted by $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$. Sometimes we use the name, generalized Quillen formal group, to distinguish it from the classical Quillen formal group constructed by the complex orientation. Moreover, unwinding the definition, we know that when we extend the scalar of $\widehat{\mathbf{G}}_A^{\mathcal{Q}}$ to $\pi_0(A)$, this formal group is exactly the classical one. We will denote the classical one by $\widehat{\mathbf{G}}_A^{\mathcal{Q}_0}$.

4.2. **(Generalized) Orientations.** The complex orientations in classical cases are (noncanonical) bridges connecting formal groups and complex-oriented spectra. Abstractly speaking, an orientation is a recognition of a concrete formal group as the inherent one of some complex-oriented spectrum, and two orientations give us different but isomorphic formal groups. In this way, Theorem 4.12 is much more meaningful than it looks like, although we take a very indirect and abstract way define the generalized orientations. Besides, we will define the orientation classifier so as to fulfill the claim in Theorem 1.2.

To define the generalized orientations, we have to define preorientations for formal hyperplanes first. They are the direct generalizations of the classical theory.

Definition 4.4 (Preorientations)**.** Let R be an $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}_*$ be a pointed formal hyperplane over R. A preorientation of X is a map of pointed spaces

$$e : S^2 \to X(R).$$

We let $\mathrm{Pre}(X) = \Omega^2 X(R)$ denote the space of preorientations of X. A preoriented formal hyperplane is a pair (X, e), where X is a pointed formal hyperplane over R and $e \in \mathrm{Pre}(X)$ is a preorientation of X. In particular, a preorientation of a formal group is just the preorientation of its underlying formal hyperplane, and the notation $\mathrm{Pre}(\widehat{\mathbf{G}})$ thus makes sense.

Then the following series of definitions and constructions can be regarded as the correction of orientations, compared with preorientations, from the perspective of algebraic geometry. The dualizing line is the counterpart of the dualizing sheaf in ordinary algebraic geometry, and the linearization map can be seen as the differential.

Definition 4.5 (The dualizing line)**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(R)$, classified by an augmentation $\epsilon : \mathcal{O}_X \to R$. We let $\mathcal{O}_X(-\eta)$ denote the fiber of ϵ, which we regard as a module over $\mathcal{O}_X$. We let $\omega_{X,\eta}$ denote the tensor product $R \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\eta)$. We will refer to $\omega_{X,\eta}$ as the dualizing line of X at the point η.

For example, in oridinary algebraic geometry, we can consider the case that $X = \operatorname{Spec} A$, η is a maximal ideal $\mathfrak{m}$, and R is the residue field of $\mathfrak{m}$. Then the tensor product $R \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\eta)$ is identified with $A/\mathfrak{m} \otimes_A \mathfrak{m} \simeq \mathfrak{m}/\mathfrak{m}^2$, the Zariski cotangent space at $\mathfrak{m}$.

Construction 4.6 (The linearization map)**.** Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R equipped with a base point $\eta \in X(R)$. If

A is a connective $\mathbb{E}_\infty$-algebra over R, we obtain a canonical map

$$\begin{aligned}\Omega X(A) &\simeq \mathrm{Map}_{\mathrm{CAlg}_R}\left(R \otimes_{\mathcal{O}_X} R, A\right)\\ &\to \mathrm{Map}_{\mathrm{Mod}_R}\left(R \otimes_{\mathcal{O}_X} R, A\right)\\ &\xrightarrow{u} \mathrm{Map}_{\mathrm{Mod}_R}\left(\Sigma\left(\omega_{X,\eta}\right), A\right)\\ &\simeq \Omega\,\mathrm{Map}_{\mathrm{Mod}_R}\left(\omega_{X,\eta}, A\right).\end{aligned}$$

where we have the first isomorphism because the homotopy pullback of $* \to X(A) \leftarrow * \ \Omega X(A)$ is transformed into a homotopy pushout diagram in the side of rings: that is the reason for the tensor product. In addition, u is induced by the fiber sequence $\omega_{X,\eta} \to R \otimes_{\mathcal{O}_X} R \xrightarrow{m} R$ in [10, Proposition 4.2.8]. We will denote the composite map by

$$\mathfrak{L} : \Omega X(A) \to \Omega\,\mathrm{Map}_{\mathrm{Mod}_R}\left(\omega_{X,\eta}, A\right)$$

and refer to it as the linearization map associated with the pair (X, η).

Construction 4.7 (The Bott map)**.** Let R be an $\mathbb{E}_\infty$-ring, let X be a 1-dimensional formal hyperplane over R, equipped with a base point $\eta \in X(R)$ and the associated dualizing line $\omega_{X,\eta}$. Applying the construction above, we obtain a linearization map

$$\mathfrak{L} : \Omega X(R) \to \Omega\,\mathrm{Map}_{\mathrm{Mod}_R}\left(\omega_{X,\eta}, R\right).$$

Passing to loop spaces, we obtain a map

$$\mathrm{Pre}(X) \to \mathrm{Map}_{\mathrm{Mod}_R}\left(\omega_{X,\eta}, \Sigma^{-2}(R)\right)$$

For each preorientation $e \in \mathrm{Pre}(X)$, we denote its image under this map by $\beta_e : \omega_{X,\eta} \to \Sigma^{-2}(R)$. We will refer to β_e as the Bott map of e.

Definition 4.8. Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional formal hyperplane over R with a base point η. An orientation of X is a preorientation $e \in \mathrm{Pre}(X)$ for which the Bott map $\beta_e : \omega_{X,\eta} \to \Sigma^{-2}(R)$ is an equivalence. We let $\mathrm{OrDat}(X)$ denote the subcategory of $\mathrm{Pre}(X)$ spanned by the orientations of X.

Remark 4.9. Some readers may feel confused about the condition that the Bott map is an equivalence. In ordinary algebraic geometry, we discussed the dualizing sheaves of projective spaces. It satisfies the criterion that $\omega_{\mathbb{P}^n} \simeq \mathcal{O}(-n-1)$ by adjunction formulas in [7, Proposition 8.20]. Here, $\mathcal{O}(-n-1)$ means the ordinary Serre twisting operation. Because the oridinary way to twist a sheaf is fulfilled by the suspension, we can understand the definition of orientations that excludes the weird preorientations from the perspective of algebraic geometry.

The remaining task of this subsection is to prove the following theorems on representability of (pre)orientations in [10]. The first theorem interprets what the preorientation is in detail. The second theorem introduces the orientation classifier for our main theorem. At last, the third theorem reflects the meaning of the existence of an orientation, as is the corollary of the complex-orientation.

Theorem 4.10. *Let R be a complex periodic $\mathbb{E}_\infty$-ring, let $\widehat{\mathbf{G}}^{\mathcal{Q}}_R \in \mathrm{FGroup}(R)$ denote the Quillen formal group, and let $\widehat{\mathbf{G}}$ be any formal group over R. Then we have a canonical homotopy equivalence*

$$\mathrm{Pre}(\widehat{\mathbf{G}}) \simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}^{\mathcal{Q}}_R, \widehat{\mathbf{G}}).$$

Proof. Let C denote the image of $\widehat{\mathbf{G}}$ under the equivalence

$$\mathrm{FGroup}(R) \simeq \mathrm{Ab}(\mathrm{Hyp}(R)) \xleftarrow{\mathrm{cSpec}} \mathrm{Ab}\left(\mathrm{cCAlg}_R^{\mathrm{sm}}\right),$$

We then have canonical homotopy equivalences

$$\begin{aligned} &\mathrm{Pre}(\widehat{\mathbf{G}}) = \mathrm{Map}_{\mathcal{S}_*}(S^2, \Omega^\infty \widehat{\mathbf{G}}(R)) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathcal{S})}(\mathbb{CP}^\infty, \mathrm{Map}_{\mathrm{cCAlg}_R}(R, C)) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{cCAlg}_R)}\left(C_*\left(\mathbb{CP}^\infty; R\right), C\right) \\ &\simeq \mathrm{Map}_{\mathrm{Ab}(\mathrm{Hyp}(R))}\left(\mathrm{cSpec}\left(C_*\left(\mathbb{CP}^\infty; R\right)\right), \mathrm{cSpec}(C)\right) \\ &\simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \widehat{\mathbf{G}}). \end{aligned}$$

□

Theorem 4.11. *Let R be an $\mathbb{E}_\infty$-ring and let X be a 1-dimensional pointed formal hyperplane over R. Then there exists an $\mathbb{E}_\infty$-algebra $\mathfrak{O}_X$, called the orientation classifier, and an orientation $e \in \mathrm{OrDat}\left(X_{\mathfrak{O}_X}\right)$ which is universal in the following sense: for every object $R' \in \mathrm{CAlg}_R$, evaluation on e induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}_R}\left(\mathfrak{O}_X, R'\right) \to \mathrm{OrDat}\left(X_{R'}\right).$$

Proof. First, let's study the case of preorientations. Then the analogous functor is $\Omega^2 X$ according to the definition. We need to check its corepresentability. Since the looping functor is realized by the (homotopy) pullback of $* \to X(R') \leftarrow *$, we know that this functor can be corepresented by $R \otimes_{R \otimes_{\mathcal{O}_X} R} R$ by the universal property of tensor products of $\mathbb{E}_\infty$-ring spectra. Let's denote this ring spectrum by A.

Next, in order to get the orientation classifier, we should study the procedure of "inverting" the Bott map. In ordinary commutative algebra, one can easily finish it by introducing its inverse. Here, out of close reasoning, we will give the construction with respect to the Bott map of X_A, $\beta_e : \omega_{X_A,\eta} \to \Sigma^{-2}(A)$.

Let u denote the morphism $R \to \omega_{X_A \cdot \eta}^{-1} \otimes_R \Sigma^{-2}(A)$ induced by β_e. We define $A[\beta_e^{-1}]$ as the direct limit of the following sequence:

$$R \xrightarrow{u} \omega_{X_A \cdot \eta}^{-1} \otimes_R \Sigma^{-2}(A) \xrightarrow{u} (\omega_{X_A \cdot \eta}^{-1})^{\otimes 2} \otimes_R \left(\Sigma^{-2}(A)\right)^{\otimes 2} \xrightarrow{u} \cdots.$$

Then by [10, Proposition 4.3.17], for every $B \in \mathrm{CAlg}_R$, $\mathrm{Map}_{\mathrm{CAlg}_R}(A[\beta_e^{-1}], B)$ is contractible if $\beta_e \otimes_R B$ is an equivalence, and empty otherwise. This universal property of $A[\beta_e^{-1}]$ implies that $A[\beta_e^{-1}]$ satisfies the conditions of the orientation classifier. □

Theorem 4.12. *Let R be an $\mathbb{E}_\infty$-ring, let $\widehat{\mathbf{G}}$ be a 1-dimensional formal group over R, and let $e \in \mathrm{Pre}(\widehat{\mathbf{G}})$ be a preorientation of $\widehat{\mathbf{G}}$. Then e is an orientation if and only if the following condition are satisfied:*

(1) The $\mathbb{E}_\infty$-ring R is complex periodic.
(2) Let $f : \widehat{\mathbf{G}}_R^{\mathcal{Q}} \to \widehat{\mathbf{G}}$ denote the image of e under the homotopy equivalence $\mathrm{Pre}(\widehat{\mathbf{G}}) \simeq \mathrm{Map}_{\mathrm{FGroup}(R)}(\widehat{\mathbf{G}}_R^{\mathcal{Q}}, \widehat{\mathbf{G}})$, then f is an equivalence of formal groups over R.

Sketch of Proof. First, let's assume the existence of the orientation. Then the Bott map, as an isomorphism, implies that the dualizing line is an invertible R-module. So it is automatically weakly 2-periodic. For the complex orientation, we believe that the orientation of formal groups induces it after diagram chasing. So anyway, R is complex periodic and Quillen formal group is well-defined. This theorem thus follows from that whether f is an isomorphism can be identified with the same question of $f^*: \omega_{\widehat{\mathbf{G}}} \to \omega_{\widehat{\mathbf{G}}_R^{\mathcal{Q}}}$ because we study the case of hyperplanes, and that the Bott map can be decomposed into $\omega_{\widehat{\mathbf{G}}} \xrightarrow{f^*} \omega_{\widehat{\mathbf{G}}_R^{\mathcal{Q}}} \xrightarrow{\sim} \Sigma^{-2}(R)$. □

4.3. **Lubin-Tate spectra.** Lubin-Tate spectra give us a simple but essential case of our main theorem. Even the proof of the main theorem heavily depends on the similar claim about Lubin-Tate spectra, since the process of decomposing problems reduces the general case to that of Lubin-Tate spectra. Here, we tend to directly use the statement in [5], Theorem 4.15. One of the necessary lemmas is the following proposition.

Proposition 4.13 (Landweber ideal)**.** *Let R be a commutative ring and let $\widehat{\mathbf{G}}$ be a formal group of dimension 1 over R. Then, for each integer $n \geq 0$, there exists a finitely generated ideal $\mathfrak{I}_n^{\widehat{\mathbf{G}}} \subseteq R$, called the nth Landweber ideal, with the following property: a ring homomorphism $R \to R'$ annihilates $\mathfrak{I}_n^{\widehat{\mathbf{G}}}$ if and only if the formal group $\widehat{\mathbf{G}}_{R'}$ has height $\geq n$.*

In [5], Goerss-Hopkins-Miller enhanced the theorem about Lubin-Tate spectra, endowing it with an essentially unique $\mathbb{E}_\infty$-ring structure. We will state their theorem completely after this notation.

Notation 4.14. We define a category $\mathcal{FG}$ as follows:

- The objects of $\mathcal{FG}$ are pairs $(R, \widehat{\mathbf{G}})$, where R is a commutative ring and $\widehat{\mathbf{G}}$ is a 1-dimensional formal group over R.
- A morphism from $(R, \widehat{\mathbf{G}})$ to $(R', \widehat{\mathbf{G}}')$ in the category $\mathcal{FG}$ is a pair (f, α), where $f : R \to R'$ is a ring homomorphism and $\alpha : \widehat{\mathbf{G}}' \simeq \widehat{\mathbf{G}}_{R'}$ is an isomorphism of formal groups over R'.

Theorem 4.15 (Goerss-Hopkins-Miller)**.** *Let κ be a perfect field of characteristic $p > 0$ and let $\widehat{\mathbf{G}}_0$ be a 1-dimensional formal group of height $n < \infty$ over κ. Then there exists an even periodic $\mathbb{E}_\infty$-ring E, called the Lubin-Tate spectrum of $\widehat{\mathbf{G}}_0$, and an isomorphism*

$$\alpha : (\kappa, \widehat{\mathbf{G}}_0) \simeq (\pi_0(E)/\mathfrak{I}_n^E, \widehat{\mathbf{G}}_E^{\mathcal{Q}_n})$$

in the category $\mathcal{FG}$ with the following features:

- *The $\mathbb{E}_\infty$-ring E is even periodic and the composite map*
$$(\pi_0(E), \widehat{\mathbf{G}}_E^{\mathcal{Q}_0}) \to (\pi_0(E)/\mathfrak{I}_n^E, \widehat{\mathbf{G}}_E^{\mathcal{Q}_n}) \xrightarrow{\alpha^{-1}} (\kappa, \widehat{\mathbf{G}}_0)$$
exhibits the classical Quillen formal group $\widehat{\mathbf{G}}_E^{\mathcal{Q}_0}$ as a universal deformation of $\widehat{\mathbf{G}}_0$. In particular, $\pi_0(E)$ can be identified with the Lubin-Tate ring of $\widehat{\mathbf{G}}_0$ in Theorem 2.51.
- *The $\mathbb{E}_\infty$-ring E is $K(n)$-local. Moreover, for every complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring A, composition with α induces a homotopy equivalence*
$$\mathrm{Map}_{\mathrm{CAlg}}(E, A) \simeq \mathrm{Hom}_{\mathcal{FG}}((\kappa, \widehat{\mathbf{G}}_0), (\pi_0(A)/\mathfrak{I}_n^A, \widehat{\mathbf{G}}_A^{\mathcal{Q}_n}))$$
In particular, the mapping space $\mathrm{Map}_{\mathrm{CAlg}}(E, A)$ is discrete.

Their original proof took use of the perspective of moduli problems. They defined the moduli space $\mathrm{CAlg}(\mathrm{Sp}) \times_{\mathrm{CAlg}(\mathrm{hSp})} \{E\}$ and then calculated the obstruction of contractibility. The author of [10] used a different way. He claimed that the Lubin-Tate spectrum is exactly the $K(n)$-localization of the orientation classifier. This claim solves the second part of our theorem above. So we have to compute its homotopy groups to complete our proof. Both of these ideas are interesting, fruitful, and related to deformation—in [5], we deform our Lubin-Tate theory to its partial approximations in order to obtain an easier obstruction theory.

We have fully prepared for our final proof. So many mathematical properties proved or unproved will mix together.

4.4. Oriented deformation ring spectrum and proof of the main theorem.

Construction 4.16 (Oriented deformation ring spectrum)**.** Let R_0 be an ordinary Noetherian $\mathbb{F}_p$-algebra which is F-finite, let $\mathbf{G}_0$ be an ordinary nonstationary p-divisible group over R_0, and let $\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ be a universal deformation of $\mathbf{G}_0$ (see Theorem 3.31). We let $R^{\mathrm{or}}_{\mathbf{G}_0}$ denote an orientation classifier for the underlying formal group $\mathbf{G}^\circ$ (See Theorem 4.11). We will refer to $R^{\mathrm{or}}_{\mathbf{G}_0}$ as the oriented deformation ring spectrum of $\mathbf{G}_0$.

Using these notions defined throughout this article, we have a precise statement of Theorem 1.2. The strategy to prove it is that we decompose a general p-divisible group into an étale one and a connected one, and by dévissage, we finally need to check the case of perfect field, which is proved as the enhancement of Lubin-Tate spectra in [5].

Theorem 4.17 (Lurie)**.** *Let R_0 be an ordinary Noetherian $\mathbb{F}_p$-algebra and let $\mathbf{G}_0$ be an ordinary 1-dimensional p-divisible group over R_0. Assume that the Frobenius map $\varphi_{R_0} : R_0 \to R_0$ is finite and that $\mathbf{G}_0$ is nonstationary. Then there exists a universal deformation $\mathbf{G}$ of this p-divisible group defined over the spectral deformation ring $R^{\mathrm{un}}_{\mathbf{G}_0}$ and an oriented deformation ring spectrum $R^{\mathrm{or}}_{\mathbf{G}_0}$ such that:*

1. *$R^{\mathrm{or}}_{\mathbf{G}_0}$ is an even periodic $\mathbb{E}_\infty$-ring with the localization map $R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{or}}_{\mathbf{G}_0}$.*
2. *If we extend the scalar of $\mathbf{G}$ to $R^{\mathrm{or}}_{\mathbf{G}_0}$, the identity component of $\mathbf{G}_{R^{\mathrm{or}}_{\mathbf{G}_0}}$ will be the generalized Quillen formal group induced by the even periodic ring spectrum $R^{\mathrm{or}}_{\mathbf{G}_0}$.*
3. *$\pi_0(R^{\mathrm{or}}_{\mathbf{G}_0})$ is the same as the classical deformation ring of $\mathbf{G}_0$, and the identity component of the classical universal deformation is exactly the classical Quillen formal group induced by $R^{\mathrm{or}}_{\mathbf{G}_0}$. In other words, $R^{\mathrm{or}}_{\mathbf{G}_0}$ is constructed by a certain formal group in Landweber's way.*

Moreover, all of these constructions can be chosen to depend functorially on the pair $(R_0, \mathbf{G}_0)$.

Definition 4.18. Let R be an connective $\mathbb{E}_\infty$-ring, let X be a 1-dimensional pointed formal hyperplane over R, and let $\mathfrak{O}_X$ denote the orientation classifier of X. We will say that X is balanced if the following conditions are satisfied:

- The unit map $R \to \mathfrak{O}_X$ induces an isomorphism of commutative rings $\pi_0(R) \to \pi_0(\mathfrak{O}_X)$.
- The homotopy groups of $\mathfrak{O}_X$ are concentrated in even degrees.

We will say that a 1-dimensional formal group $\widehat{\mathbf{G}}$ is balanced if the underlying pointed formal hyperplane $X = \Omega^\infty \widehat{\mathbf{G}}$ is balanced.

I claim that in order to prove Theorem 4.17, we only need to prove this theorem:

Theorem 4.19. *Let R_0 be an ordinary F-finite Noetherian $\mathbb{F}_p$-algebra, let $\mathbf{G}_0$ be an ordinary nonstationary p-divisible group of dimension 1 over R_0, and let*

$\mathbf{G} \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}_0})$ *be its spectral universal deformation. Then the identity component* $\mathbf{G}^\circ$ *is a balanced generalized formal group over* $R^{\mathrm{un}}_{\mathbf{G}_0}$.

Proof of Theorem 4.17 from Theorem 4.19. Let $(R_0, \mathbf{G}_0)$ be as in the statement of Theorem 1.2. Let $\mathbf{G} \in \mathrm{BT}^p\left(R^{\mathrm{un}}_{\mathbf{G}}\right)$ denote the universal deformation of $\mathbf{G}_0$, and set $E = R^{\mathrm{or}}_{\mathbf{G}_0}$. By construction, the formal group $\mathbf{G}^\circ$ acquires an orientation after extending scalars from $R^{\mathrm{un}}_{\mathbf{G}_0}$ to E. It follows that E is complex periodic and that the $\mathbf{G}^\circ_E$ can be identified with the Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}}_E$ due to Theorem 4.12. It follows from Theorem 4.19 that E is even periodic and that the classical Quillen formal group $\widehat{\mathbf{G}}^{\mathcal{Q}_0}_E$ agrees with the identity component of the p-divisible group $\mathbf{G}_{\mathrm{cl}}$ obtained from $\mathbf{G}$ by extending scalars along the projection map $R^{\mathrm{un}}_{\mathbf{G}_0} \to \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$, which is the classical universal deformation of $\mathbf{G}_0$. □

Proof of Theorem 4.19. First, suppose that we have proved the Lubin-Tate case where $R_0\kappa$ is a perfect field of characteristic p and p-divisible group $\mathbf{G_0}$ is connected. Unwinding the definition, the property of being balanced is a local property. It means that it suffices to consider the localization at each maximal ideal $\mathfrak{m} \subseteq \pi_0(R^{\mathrm{un}}_{\mathbf{G}_0})$. Note that since $R^{\mathrm{un}}_{\mathbf{G}_0}$ is complete with respect to the kernel of the map $\pi_0\left(R^{\mathrm{un}}_{\mathbf{G}_0}\right) \to R_0$, we can write $\mathfrak{m}$ as the inverse image of a maximal ideal $\mathfrak{m}_0 \subseteq R_0$.

Let κ be any perfect extension field of $R_0/\mathfrak{m}_0$, and let $\mathbf{G}_1 = (\mathbf{G}_0)_\kappa$ be the p-divisible group obtained from $\mathbf{G}_0$ by extending scalars to κ. Using [10, Theorem 6.1.2], we obtain a flat map of spectral deformation rings $\rho : R^{\mathrm{un}}_{\mathbf{G}_0} \to R^{\mathrm{un}}_{\mathbf{G}_1}$. Moreover, the inverse image under ρ of the maximal ideal of $R^{\mathrm{un}}_{\mathbf{G}_1}$ is $\mathfrak{m}$, so that ρ induces a faithfully flat map $(R^{\mathrm{un}}_{\mathbf{G}_0})_{\mathfrak{m}} \to R^{\mathrm{un}}_{\mathbf{G}_1}$. By virtue of the descent property of faithfully flat morphisms, it will suffice to show that the formal group $\mathbf{G}^\circ_{R^{\mathrm{un}}_{\mathbf{G}_1}}$ is balanced. We may therefore replace $(R_0, \mathbf{G}_0)$ by $(\kappa, \mathbf{G}_1)$ and thereby reduce to proving this theorem in the special case where $R_0 = \kappa$ is a perfect field of characteristic p. In this case, the p-divisible group $\mathbf{G}_0$ admits a connected-étale sequence

$$0 \to \mathbf{G}'_0 \to \mathbf{G}_0 \to \mathbf{G}''_0 \to 0.$$

Let $R^{\mathrm{un}}_{\mathbf{G}'_0}$ be the spectral deformation ring of $\mathbf{G}'_0$ and let $\mathbf{G}' \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}'_0})$ be its universal deformation. As in the following proof of the assumption, we will observe that the formal group $\mathbf{G}^\circ$ can be obtained from $\mathbf{G}'^\circ$ by extending scalars along a comparison map $u : R^{\mathrm{un}}_{\mathbf{G}'_0} \to R^{\mathrm{un}}_{\mathbf{G}_0}$. Since u is flat by [10, Theorem 6.2.4], we are reduced to proving that the formal group $\mathbf{G}''$ is balanced because of descent, which is our assumption in the beginning of this proof.

As for our assumption, the same process, localization at some prime ideal, implies that it suffices to prove that $\mathbf{G}^\circ_{R^{\mathrm{un}}_{\mathbf{G}_2}}$ is a balanced formal group over $R^{\mathrm{un}}_{\mathbf{G}_2}$, where $\mathbf{G}_2$ is the extension of $\mathbf{G}$ to an algebraic closed field contained a residue field. Note that the p-divisible group $\mathbf{G}_2$ admits a connected-étale sequence

$$0 \to \mathbf{G}'_2 \xrightarrow{i_0} \mathbf{G}_2 \to \mathbf{G}''_2 \to 0.$$

Let $R^{\mathrm{un}}_{\mathbf{G}'_2}$ be the spectral deformation ring of $\mathbf{G}'_2$ and let $\mathbf{G}' \in \mathrm{BT}^p(R^{\mathrm{un}}_{\mathbf{G}'_2})$ be its universal deformation. Then we have a comparison map $u : R^{\mathrm{un}}_{\mathbf{G}'_2} \to R^{\mathrm{un}}_{\mathbf{G}_2}$, which is essentially characterized by the requirement that i_0 can be lifted to a monomorphism $\mathbf{G}'_{R^{\mathrm{un}}_{\mathbf{G}_2}} \to \mathbf{G}_{R^{\mathrm{un}}_{\mathbf{G}_2}}$ of p-divisible groups over $R^{\mathrm{un}}_{\mathbf{G}_2}$. In particular, the formal group $\mathbf{G}^\circ_{R^{\mathrm{un}}_{\mathbf{G}_2}}$ can be obtained from the formal group $\mathbf{G}'$ by extension of scalars along u. Since u is flat due to the same theorem above, it will suffice to show that the formal group $\mathbf{G}'$ is balanced. We can then do induction on the heights of prime ideals. Thus we can assume that we are dealing with the maximal ideal $\mathfrak{m}$. So we can directly consider the fiber of $R^{\mathrm{or}}_{\mathbf{G}_0} \to E := L_{K(n)} R^{\mathrm{or}}_{\mathbf{G}_0}$. We have known that it vanishes at every non-maximal ideal. Unwinding definition, we know that this fiber vanishes at $\mathfrak{m}$-completion. On the basis of [11, Proposition 7.3.1.7], this localization is actually an isomorphism. So our claim follows directly from the Theorem 4.15 and the observation in that subsection. □

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