BEHAVIOR OF GROWTH FOR FIRST PASSAGE PERCOLATION

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Abstract. This paper aims to provide an expository study of first passage percolation in probability theory by approaching it both theoretically and empirically. First passage percolation studies the set of reachable points in $\mathbb{Z}^d$ when the passage time for each edge is an independent identically distributed random variable. We want to show that the graph eventually grows linearly and is uniform in all directions. We will first show this property using measure theory and ergodicity. Subsequently, we will run computer simulation to obtain a more intuitive understanding of the problem.

Contents

1. Introduction

This paper studies the model of first passage percolation, one of the most classical areas of probability theory. It is a concept first invented by Hammersley and Welsh in 1965 [HW65] and studies the collection of points on a graph that can be reached within a given period of time given a fixed starting point where weights are assigned to edges. Most studies of first passage percolation, like all topics in probability theory, has a strong reliance on measure theory. However, the beauty of the first passage percolation lies in its simple definition and how straightforward some of its properties are. Therefore, this paper aims to provide a basic introduction to readers who are interested first passage percolation and probability, but are not yet equipped with the full knowledge of measure theory. It will do so by providing an

TABLE 1. Summary of Notations Used

$t(e)$	the passage time of an edge e
F	the distribution of every $t(e)$
$T(u, v)$	the travel time from u to v
$\tilde{B}(t)$	the set of reachable points within time t
$B(t)$	the continuous shape covering the reachable points within time t
$a_{m,n}$	point to point passage time
$b_{m,n}$	point to line passage time

intuitive explanation for the theories used and offering illustrative graphs for the properties in first passage percolation.

This paper is divided into five parts. First, we will introduce the general set up of first passage percolation and define the notations that will be used throughout the following sections. Since there are many different notations used, a table is provided for the reader to refer to when the definition of a certain notation is unclear. See table 1. Next, we will introduce some basic yet relevant concepts in measure theory in order to prove certain theorems for first passage percolation. In this part, since our target audience is pre-measure theory, we will place our emphasis on understanding the statement of the theorem rather than the proof. Thirdly, we will introduce two key properties of the first passage percolation model. We will show how they directly result from the theorems in measure theory which we introduced in the previous section. Following, we will demonstrate the two properties using computer simulations. We will simulate the graph for different time limits and different distribution. This will give us a very intuitive understanding of first passage percolation. Last but not least, we will introduce several applications of first passage percolation in a diverse range of fields for interested readers to look into.

2. General Set Up

A graph $\mathcal{G}$ of first passage percolation consists of a network of points consisting of roads with different passage time. This graph can be of any form, but for the simplicity of notation, we study the network in $\mathbb{Z}^d$. The demonstrative graphs in this article will be in $\mathbb{Z}^2$, but the same method applies to graphs of higher dimensions.

Let e be an edge in $\mathbb{Z}^d$. A part of a path is defined as an edge if it connects two points $u = (u_1, u_2, \cdots, u_d)$ and $v = (v_1, v_2, \cdots, v_d)$ such that u and v are adjacent, i.e.,

$$\sum_{i=1}^{d} |u_i - v_i| = 1$$

We assign to each edge a random variable $t(e)$ and define it as the *passage time* of an edge e. For all $e \in \mathbb{Z}^d$, $t(e)$ are independent and have the same distribution F where $F(0-) = 0$. This means that F doesn't take any negative value. For example, $t(e) \sim |Z|$, where Z is a standard normal variable. This means that the passage time of every edge is the absolute value of a standard normally distributed variable; or $t(e) \sim U(0, 1)$, which means that the passage time of every edge is a uniformly distributed variable between 0 and 1.

For vertices u, v of any graph $\mathcal{G}$ a path from u to v is an alternating sequence $(v_0, e_1, v_1, ...e_n, v_n)$ of vertices and edges of $\mathcal{G}$ such that v_i is adjacent on $\mathcal{G}$ to v_{i-1} and e_i is an edge between v_i and v_{i-1} for $i = 1, 2.....n$ (and $v_0 = u$, $v_n = v$). For a path $r = (v_0, e_1, v_1, ...e_n, v_n)$ in $\mathbb{Z}^d$, we define the passage time of r as

$$T(r) = \sum_{i=1}^{n} t(e_i).$$

The travel time from u to v is defined as

$$T(u, v) = \inf\{T(r) : r \text{ is a path from u to v}\}.$$

Finally, with $\mathbf{0}$ denoting the origin, we define the set of reachable points within a given time t as

$$\tilde{B}(t) = \{v \in \mathbb{Z}^d : T(\mathbf{0}, v) \leq t\}.$$

$\tilde{B}(t)$ then consists of all points which can be reached in time t from the origin. Our ultimate goal is to study the behavior for large time of $\tilde{B}(t)$. However, the set $\tilde{B}(t)$ is a set of discrete points, and to better study this problem, we want a continuous shape. Thus, we define a "fattened version" of $\tilde{B}$, the set

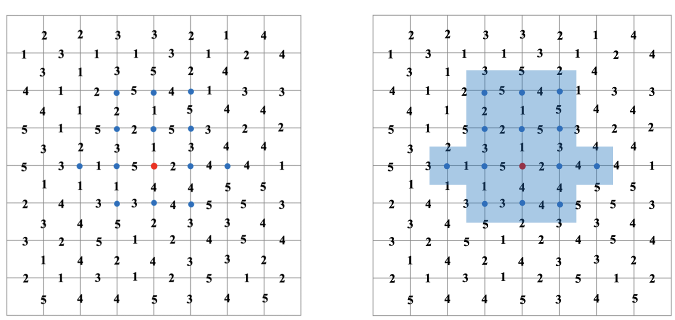
$$B(t) = \{v + \overline{U} : v \in \tilde{B}(t)\},$$

where $\overline{U}$ is the closed cube

$$\overline{U} = \{x = (x(1), ..., x(d)) : -\frac{1}{2} \leq x(i) \leq \frac{1}{2}, 1 \leq i \leq d\}$$

In figure 2, the blue points in the picture on the left represent the set $\tilde{B}(6)$ whereas the picture on the right represents the set $B(6)$, which could be more easily studied.

FIGURE 1. Comparison between $\tilde{B}(6)$ and $B(6)$



In the following sections, we wish to show by theory and by experiment that $B(t)$

- grows linearly, and
- has an asymptotic shape which is not random.

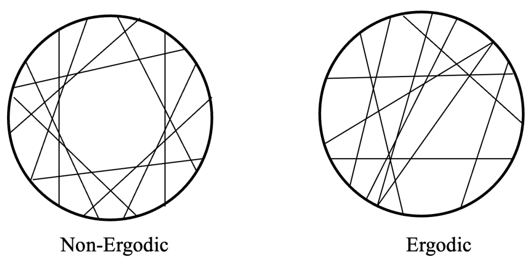
3. Ergodicity

In this section, we will begin by examining some theorems in measure theory that give us a basic idea of ergodicity, which will be used in the upcoming section in order to prove the qualities of first passage percolation.

First, we will introduce several definitions in measure theorem to set up the basic notation for the proof. Then, we will look at Birkhoff's Ergodic Theorem, which studies stationary sequences of random variables. It has its applications in the context of random walks [Lal]. Next, we will look at Kingman's Subadditive Ergodic Theorem. It is a variant and a generalization of Birkhoff's Ergodic Theorem. More importantly, we will look at it in the language of probability and sequences, which gives us a direct application to first passage percolation.

Remark 3.1. We would like to begin by explaining the intuition of ergodicity. Ergodicity describes the property of the path of a point in a moving system, either a dynamical system or a stochastic process, which will eventually visit all parts of the space that the system moves in. See figure 3.

Figure 2. Illustration of Ergodicity



Definition 3.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $T : \Omega \to \Omega$ be a measurable transformation. The transformation T is said to be measure-preserving if for every $A \in \mathcal{F}$,

$$P(T^{-1}(A)) = P(A)$$

We then call the triple $(\Omega, \mathcal{F}, P)$ a measure-preserving system. An invertible measure-preserving transformation is an invertible mapping $T : \Omega \to \Omega$ such that both T and T^{-1} are measure-preserving transformations.

Example 3.3. A rotation is a measure preserving transformation. Let $\Omega = \mathbb{T}^1 = \{z \in \mathbb{C}||z| = 1\}$ be the unit circle in the complex plane, and for any real number θ define $R_\theta : \Omega \to \Omega$ by $R_\theta(z) = e^{i\theta}z$. Thus, R_θ rotates Ω through an angle θ. Let λ be the normalized arclength measure on Ω. Then each R_θ is $\lambda-$measure-preserving.

Definition 3.4. If T is a measure-preserving transformation of $(\Omega, \mathcal{F}, P)$, then an event $A \in \mathcal{F}$ is said to be invariant if $T^{-1}A = A$, equivalently, if $\mathbf{1}_A = \mathbf{1}_A \circ T$. The collection $\mathcal{F}$ of all invariant events is the invariant σ-algebra. If the invariant σ-algebra I contains only events of probability 0 or 1 then the measure-preserving transformation T is said to be *ergodic*. Similar terminology is used for random variables: a random variable f is said to be invariant if $f = f \circ T$. It is easily seen that if T is ergodic, then every invariant function is almost surely constant.

Definition 3.5. We say a random sequence X_j is stationary if its joint probability distribution is invariant over time, i.e.,

$$\begin{aligned} & F_{X_n, X_{n+1}, \ldots, X_{n+N-1}}(x_n, x_{n+1}, \ldots, x_{n+N-1}) \\ = & F_{X_{n+k}, X_{n+k+1}, \ldots, X_{n+k+N-1}}(x_n, x_{n+1}, \ldots, x_{n+N-1}), \end{aligned} \tag{3.6}$$

Lemma 3.7. *Let $(\Omega, \mathcal{F}, P)$ be an ergodic, measure-preserving system. For any bounded measurable function f the functions*

$$A^* f := \limsup_{n\to\infty} A_n f$$

and

$$A_* f := \liminf_{n\to\infty} A_n f$$

with $A_n \in \mathcal{A}$ where $\mathcal{A}$ is an algebra $\mathcal{A}$, are invariant, and therefore, since T is ergodic, are constant.

Proof. We have

$$A_n f - A_n f \circ T = \frac{(f - f \circ T^n)}{n}.$$

We also know that f is bounded, so the difference converges to 0 as $n \to \infty$. Hence,the sequences $A_n f$ and $A_n f \circ T$ have the same limsup and liminf.[Hau17][Hau] □

Theorem 3.8 (Birkhoff 's Ergodic Theorem)**.** *If T is an ergodic, measure-preserving transformation of $(\Omega, \mathcal{F}, P)$ then for every random variable $X \in L^1$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} X \circ T^j = EX.$$

Theorem 3.9 (The Kingman's Subadditive Ergodic Theorem [Ste89])**.** *If T is a measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mu)$ and $\{g_n, 1 \le n < \infty\}$ is a sequence of integrable functions satisfying*

$$g_{n+m}(x) \le g_n(x) + g_m(T^n x),$$

then

$$\lim_{n\to\infty} \frac{g_n(x)}{n} = g(x) \ge -\infty$$

with probability one.

Remark 3.10. We will now write Kingman's Subadditive Ergodic Theorem in the language of sequence and probability, which gives us a more direct application to the first passage percolation model we wish to study.

Let $\{X_{mn}\}_{0\le m\le n}$ be a family of random variables which satisfies the conditions below.

(1) $X_{0n} \le X_{0m} + X_{mn}$ for all $0 < m < n$.
(2) The distribution of the sequences $\{X_{m+h,m+h+k}\}_{k\ge l}$ is the same for all $h \ge 0$.
(3) For each $k \ge 1$, the sequence $\{X_{nk,(n+1)k}\}_{n\ge 0}$ is stationary.
(4) $E[X_{01}^+] < \infty$

Then

$$\lim_{n\to\infty}\frac{1}{n}X_{0n}$$

exists or tends to $-\infty$ with probability one. If the stationary sequence $\{X_{nk,(n+1)k}\}_{n\geq 0}$ is ergodic, then the limit $\lim_{n\to\infty}\frac{1}{n}X_{0n}$ is constant with probability one and is equal to

$$\lim_{n\to\infty}\frac{1}{n}EX_{0n}=\inf_{n\geq 0}\frac{1}{n}EX_{0n}$$

If, in addition to the four conditions listed above one has

$$EX_{0n}^{-}\geq -A_n$$

for some $A<\infty$ then the convergence of $\lim_{n\to\infty}\frac{1}{n}X_{0n}$ also holds in L^1.[ADH17]

4. Time Constant

Theorem 4.1. *Assume that*

$$E\min\{t_1,\cdots,t_{2d}\}<\infty$$

where t_i are independent and identically distributed copies of $t(e)$. Then there exists a time constant $\mu(\mathbf{v_1})\in[0,\infty)$, where $\mathbf{v_1}$ is the unit vector in the direction of x_1, such that

$$\lim_{n\to\infty}\frac{T(0,n\mathbf{v_1})}{n}=\mu(\mathbf{v_1})$$

Historically, one began not with the study of all of $B(t)$, which is the continuous shape covering the reachable points within time t, but with its furthest point to the right on the first coordinate axis, i.e., the point $(x,0)\in B(t)$ with the greatest x. [Kes06]

Definition 4.2. We define $a_{m,n}$ as the point to point passage time where

$$a_{m,n}=T((m,0,\cdots,0),(n,0,\cdots,0))$$

and $b_{m,n}$ as the point to line passage time, or "point to hyper-plane passage time" in higher dimensions, where

$$b_{m,n}=\inf\{T((m,0,\cdots,0),(n,k_2,\cdots,k_d)):k_2,\cdots,k_d\in\mathbb{Z}\}$$

Specifically, we care about the two special cases for $a_{m,n}$ and $b_{m,n}$ where the point m is the origin, namely

$$a_{\mathbf{0},n}=T(\mathbf{0},(n,0,\cdots,0))$$

and

$$b_{\mathbf{0},n}=\inf\{T(\mathbf{0},(n,k_2,\cdots,k_d)):k_2,\cdots,k_d\in\mathbb{Z}\}$$

Theorem 4.3. *If*

$$E\min\{t_1,\cdots,t_{2d}\}<\infty \tag{4.4}$$

and

$$E\min\{t_1^d,\cdots,t_{2d}^d\}<\infty,<\infty \tag{4.5}$$

then there exists a constant $\mu=\mu(F,d)<\infty$, the so-called time constant, such that

$$\lim_{n\to\infty}\frac{1}{n}a_{0,n}=\lim_{n\to\infty}\frac{1}{n}b_{0,n}=\mu \tag{4.6}$$

with probability one and in L^1.

Remark 4.7. The requirement in (4.4) is not necessary for the existence of the time constant. We can define a time constant $\hat{\mu} < \infty$ [ADH17], but then one only obtains

$$\frac{1}{n}a_{0,n} \to \hat{\mu} \text{ in probability and } \frac{1}{n}b_{0,n} \to \hat{\mu} \text{ with probability 1.} \tag{4.8}$$

Proof. To prove this theorem, we only need to apply the Kingman's Subadditive Ergodic Theorem to the family of $\{a_{m,n}\}_{0<m\leq n}$ and $\{b_{m,n}\}_{0<m\leq n}$ in the first passage percolation model.

We will now check whether $\{a_{m,n}\}_{0<m\leq n}$ satisfies all the conditions in Remark 3.10 for the Kingman's Subadditive Ergodic Theorem.

The equivalence of condition (1) in Remark 3.10 is

$$a_{\mathbf{0},n} \leq a_{\mathbf{0},m} + a_{m,n}$$

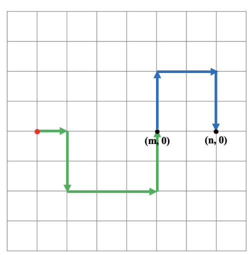
for all $0 < m < n$.

This is true by definition of a. Recall that

$$a_{\mathbf{0},n} = T(\mathbf{0},(n,0,\cdots,0)) = \inf\{T(r) : r \text{ is a path from } \mathbf{0} \text{ to } (n,0,\cdots,0))\}$$

Since the path from $\mathbf{0}$ to $(m,0,\cdots,0)$, then to $(n,0,\cdots,0)$ is obviously a path from $\mathbf{0}$ to $(n,0,\cdots,0)$, the infimum of the time traveling from $\mathbf{0}$ to $(n,0,\cdots,0)$ is less than or equal to the time traveling from $\mathbf{0}$ to $(m,0,\cdots,0)$ and $(m,0,\cdots,0)$ to $(n,0,\cdots,0)$ combined as demonstrated in figure 4. Thus, we have $a_{\mathbf{0},n} \leq a_{\mathbf{0},m} + a_{m,n}$.

FIGURE 3. Path from $\mathbf{0}$ to $(m,0,\cdots,0)$ then to$(n,0,\cdots,0)$



The equivalence of condition (2) in Remark 3.10 is

The distribution of the sequences $\{a_{m+h,m+h+k}\}_{k\geq 1}$ is the same for all $h \geq 0$.

We will write out a few terms to see how this is true.

For $h = 0$, we have the sequence

$$\{a_{m,m+k}\}_{k\geq 1} = \{a_{m,m+1}, a_{m,m+2}, a_{m,m+3}, a_{m,m+4}, \cdots\}. \tag{4.9}$$

For $h = 1$, we have the sequence

$$\{a_{m+1,m+1+k}\}_{k\geq 1} = \{a_{m+1,m+2}, a_{m+1,m+3}, a_{m+1,m+4}, a_{m+1,m+5}, \cdots\}. \tag{4.10}$$

For $h = 2$, we have the sequence

$$\{a_{m+2,m+2+k}\}_{k\geq 1} = \{a_{m+2,m+3}, a_{m+2,m+4}, a_{m+2,m+5}, a_{m+2,m+6}, \cdots\}. \tag{4.11}$$

Notice that the sequence in (4.10) is only the path in (4.9) shifted to its right by 1. Since every path e in the graph is independent identically distributed, the distribution doesn't change when we shift every point by 1. Similarly, the sequence in (4.11) is the path in (4.9) shifted 2 unit to its right, which also doesn't change the distribution. We can then use induction to prove that every sequence has the same distribution.

The equivalence of condition (3) in Remark 3.10 is

$$\text{For each } k \geq 1, \text{ the sequence } \{a_{nk,(n+1)k}\}_{n\geq 0} \text{ is stationary.}$$

Again, we will write out a few terms to form a proof by induction.

For $k = 1$, we have the sequence

$$\{a_{n,n+1}\}_{n\geq 0} = \{a_{0,1}, a_{1,2}, a_{2,3}, a_{3,4}, \cdots\}. \tag{4.12}$$

For $k = 2$, we have the sequence

$$\{a_{2n,2(n+1)}\}_{n\geq 0} = \{a_{0,2}, a_{2,4}, a_{4,6}, a_{6,8}, \cdots\}. \tag{4.13}$$

For $k = 3$, we have the sequence

$$\{a_{3n,3(n+1)}\}_{n\geq 0} = \{a_{0,3}, a_{3,6}, a_{6,9}, a_{9,12}, \cdots\}. \tag{4.14}$$

Recall the definition of stationary sequences at Definition 3.5. For (4.12) we have, for any N, $\{a_{0,1}, a_{1,2}, \cdots, a_{N-1,N}\}$ has the same distribution as $\{a_{k,k+1}, a_{k+1,k+2}, \cdots, a_{k+N-1,k+N}\}$.

The equivalence of condition (4) in Remark 3.10 is

$$E\{a_{01}^+\} < \infty$$

which is true by definition.

□

Theorem 4.15. *[Kes06] Let $\{X_{mn}\}_{0\leq m\leq n}$ be a family of random variables which satisfies the conditions (1)-(3) below.*

(1) $X_{0n} \leq X_{0m} + X_{mn}$ for all $0 < m < n$.
(2) The distribution of the sequences $\{X_{m+h,m+h+k}\}_{k\geq 1}$ is the same for all $h \geq 0$.
(3) $E\min\{t_1^d, \cdots, t_{2d}^d\} < \infty$ where $t_1, \cdots, t_{2d}$ are i.i.d. random variables with distribution F.

Then there exists a nonrandom convex set $B_0 \subset \mathbb{R}^d$, which is invariant under permutations of the coordinates and under reflections in the coordinate hyperplanes, has nonempty interior, and which is either compact or equals all of $\mathbb{R}^d$, and has the following property:

$$(1-\epsilon)B_0 \subset \frac{1}{t}B(t) \subset (1+\epsilon)B_0$$

eventually with probability 1. If $B_0 = \mathbb{R}^d$, then for all $\epsilon > 0$

$$\{x : |x| \leq \epsilon^{-1}\} \subset \frac{1}{t}B(t)$$

eventually with probability one. If condition (3) fails, then

$$\limsup_{v\to\infty} \frac{1}{|v|}T(\mathbf{0}, v) = \infty$$

with probability one.

Remark 4.16. The idea of the proof of Theorem 4.15 is to first use subadditivity to demonstrate the linear growth of $B(t)$ in a fixed rational direction. This implies that, with probability one, we have the right growth rate in a countable dense set of directions simultaneously. To obtain the full result from this, we need some bound which allows us to interpolate between these directions, insuring that the convergence occurs along all directions with probability one [ADH17].

There are different ways to implement this interpolation step. Here we sketch a method which first appeared in [Han13] for establishing the interpolative bound just mentioned. We want to introduce a lemma about the difference estimate and then define a concept of "good" vertex such that we can show further claims.

Lemma 4.17 (Difference estimate)**.** *Let $v \in \mathcal{M}$. Then there exists a constant $\mathcal{K} < \infty$ such that, for any $x \in \mathbb{Z}^d$,*

$$\mathbb{P}\left(\sup_{z\in\mathbb{Z}^d, z\neq x} \frac{T(x,z)}{||x-z||_1} < \mathcal{K}\right) > 0 \tag{4.18}$$

We will call an x in $\mathbb{Z}^d$ for which the event appearing in (4.18) occurs a "good" vertex.

Lemma 4.19. *Let $\zeta \in \mathbb{Z}^d \setminus \{0\}$. For a given realization of edge-weights, denote by (n_k) the sequence of natural numbers such that $n_k\zeta$ is a good vertex. Then with probability one, the sequence (n_k) is infinite and $\lim_k(\frac{n_{k+1}}{n_k}) = 1$.*

Proof. We know from the Ergodic theorem that the sequence (n_k) is infinite almost surely. Let B_m denote the event that $m\zeta$ is a good vertex. Then

$$\frac{k}{n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k} \mathbf{1}_{B_i}.$$

The right side converges to the probability in 4.18 by the ergodic theorem. Thus,

$$\frac{n_{k+1}}{n_k} = \left(\frac{n_{k+1}}{k+1}\right)\left(\frac{k}{n_k}\right)\left(\frac{k+1}{k}\right) \to 1$$

almost surely.

□

We will now prove Theorem 4.15.

Proof of Theorem 4.15. The following proof is adapted from [ADH17].

Let Ξ_1 denote the event that $\lim_n T(\mathbf{0}, nq)/n = \mu(q)$ for all $\mathbf{q}$ having rational coordinates; let Ξ_2 denote the event that for every $\zeta \in \mathbb{Z}^d$, the sequence (n_k) defined in Lemma 4.19 is infinite and that the ratio of successive terms tends to one. From here, the proof of Theorem 4.15 proceeds by contradiction. Assume the Shape Theorem does not hold. Then there exists $\delta > 0$ and a collection of edge-weight configurations $D(\delta)$ with $P(D\delta) > 0$ such that, for every outcome in $D\delta$, there are infinitely many vertices $x \in \mathbb{Z}^d$ with

$$|T(\mathbf{0}, x) - \mu(x)| > \delta||x||_1. \tag{4.20}$$

Since $\mathbb{P}(\Xi_1) = 1 = \mathbb{P}(\Xi_2)$, the event $D_\delta \cap \Xi_1 \cap \Xi_2$ contains some outcome ω; we claim that ω has contradictory properties. On outcome ω, there must exist a sequence $(x_i) \subset \mathbb{Z}^d$ satisfying the condition in (4.20). We can assume that $x_i/||x_i||_1$ converges to some y with $||y||_1 = 1$ by compactness of the unit sphere. Let $\delta' > 0$

be arbitrary; we will fix its value at the end of the proof. We first choose some large N such that $||x_n/||x_n||_1 - y||_1 < \delta'$ and such that

$$|\mu(x_n) - ||x_n||_1\mu(y)| < \delta||x_n||_1/2$$

for $n > N$. Using our assumption in (4.20), we have for $n > N$:

$$|T(0, x_n) - ||x_n||_1\mu(y)| < \delta||x_n||_1/2 \tag{4.21}$$

Next, we set up a sequence of approximating good vertices. We find some $z \in \mathbb{R}^d$, $||z||_1 = 1$ such that $||z - y||_1 < \delta'$, with the additional property that $z = x/M$ for some $x \in \mathbb{Z}^d$ and some positive integer M. This can be done because vectors with rational coordinates are dense in the unit sphere. On ω, there must exist a sequence (n_k) such that $n_k Mz$ is a good vertex and such that n_{k+1}/n_k tends to one. For any n, there exists a value of k such that

$$n_{k+1}M \geq ||x_n||_1 \geq n_k M;$$

denote this value by $k(n)$. Finally, fix $K > 0$ such that $n_{k+1} < (1 + \delta')n_k$ and

$$|\frac{T(0, n_k Mz)}{n_k M} - \mu(z)| < \delta'$$

for all $k > K$. We now let $n > N$ be large enough that $k(n) > K$.

Now, we will find the contradiction.

We have that $T(\mathbf{0}, ny) - n\mu(y)$ is of order n for infinitely many n. Since μ is a norm, $\mu(y)$ and $\mu(z)$ are arbitrarily close and since infinitely many of the $\{nz\}$ are good vertices, $T(\mathbf{0}, ny)$ and $T(\mathbf{0}, nz)$ are arbitrarily close. Thus $T(\mathbf{0}, nz) - n\mu(z)$ is large – but this is counter to the properties assumed for ω. To turn the above into a rigorous estimate, write k for $k(n)$ and expand

$$\begin{aligned} \left|\frac{T(\mathbf{0}, x_{n_k})}{||x_n||_1} - \mu(y)\right| = &\left|\frac{T(\mathbf{0}, x_n) - T(\mathbf{0}, n_k Mz)}{||x_n||_1} - \mu(y)\right| \\ &+ \frac{T(\mathbf{0}, n_k Mz)}{n_k Mz}\left(1 - \frac{n_k M}{||x_n||_1}\right) \\ &+ \left|\frac{T(\mathbf{0}, n_k Mz)}{n_k Mz} - \mu(z)\right| + |\mu(z) - \mu(y)| \end{aligned} \tag{4.22}$$

We wish to find an upper bound for each term on the right hand side of equation 4.22 in terms of δ', which gives us a overall upper bound of the left hand side of the equation.

(1) We know that $n > K$ and $k > K$, so we have

$$n_k M \leq ||x_n||_1 \leq (1 + \delta')n_k M,$$

$$||n_k My - n_k Mz||_1 \leq \delta' n_k M,$$

and

$$||\frac{x_n}{||x_n||_1 - ||y||_1}||_1 < \delta'.$$

Combining these three equations, we get

$$||x_n - n_k Mz||_1 \leq 2\delta'||x_n||_1.$$

Since we also know that $n_k Mz$ is a good vertex, we have

$$T(\mathbf{0}, x_n) - T(\mathbf{0}, n_k Mz) \leq ||||x_n - n_k Mz||_1 \leq 2\mathcal{K}\delta'||x_n||_1$$

(2) From the first term, we already know that $1 - \frac{n_k M}{||x_x||_1}$ is bounded above. Since we also know that $k > \mathcal{K}$, we get the overall bound for the second term
$$[\mu(z) + \delta'](1 - (1 + \delta')^{-1}).$$
(3) Since k is chosen to be greater than $\mathcal{K}$, this term is bounded above by δ'.
(4) There are two cases for μ. If μ is identically zero, this term is trivially zero. If μ is not identically zero, it is a norm on R^d and is thus bounded by the $||\cdot||_1$ norm:
$$c_L||\cdot||_1 \leq \mu(\cdot) \leq c_U||\cdot||_1.$$
Since $||z-y||_1 < \delta'$, given μ, there exists some constant C such that this term is bounded above by $C\delta'$.

Thus, given δ', there is some ϵ such that $\epsilon \to 0$ as $\delta' \to 0$, and that $\epsilon||x_n||_1$ is an upper bound for the left side of 4.22. However, since δ' is arbitrary, we can find some δ' such that $\epsilon||x_n||_1$ is less than the right hand side of 4.22. Thus, we have found a contradiction.

□

5. Simulation

In order to better understand the theorem, it is very helpful to visualize the result. It can be used to form conjectures or provide visual aids after we know the result. Computer simulation adds empirical evidence to the theorems we proved to help us have a more solid perception of the theorem. Simulations is widely used to study different characters of first passage percolation. Alm and Deijfen[AD14] studied the growth of first passage percolation on $\mathbb{Z}^2$ in a certain direction and the shape of its reachable set. They ran simulations on different distributions and studied the behavior of the estimated time constant. Here, we simulated the graph of first passage percolation using Dijsktra algorithm. Since we are interested in the behavior of reachable sets, we simulated the graph given different time constraints.

5.1. Dijkstra Algorithm.

In order to calculate the travel time from the origin to each point on the graph, we use Dijkstra algorithm to find the shortest travel time. Conceived by Edsger W. Dijkstra in 1956, Dijkstra algorithm finds the shortest paths between nodes in a weighted graph, which can be directly applied on the first passage percolation model to calculate the time needed to reach a certain point. Given a distribution, we will first generate the $t(e)$ for every edge, and then calculate the time needed to reach each point using Dijkstra algorithm. We will then filter all the points that are reachable within time t and generate a graph for $B(t)$ for that specific distribution and time limit.

The part of the code for Dijkstra Algorithm is adapted from the code written by Divyanshu Mehta. [Meh]

5.2. Simulation.

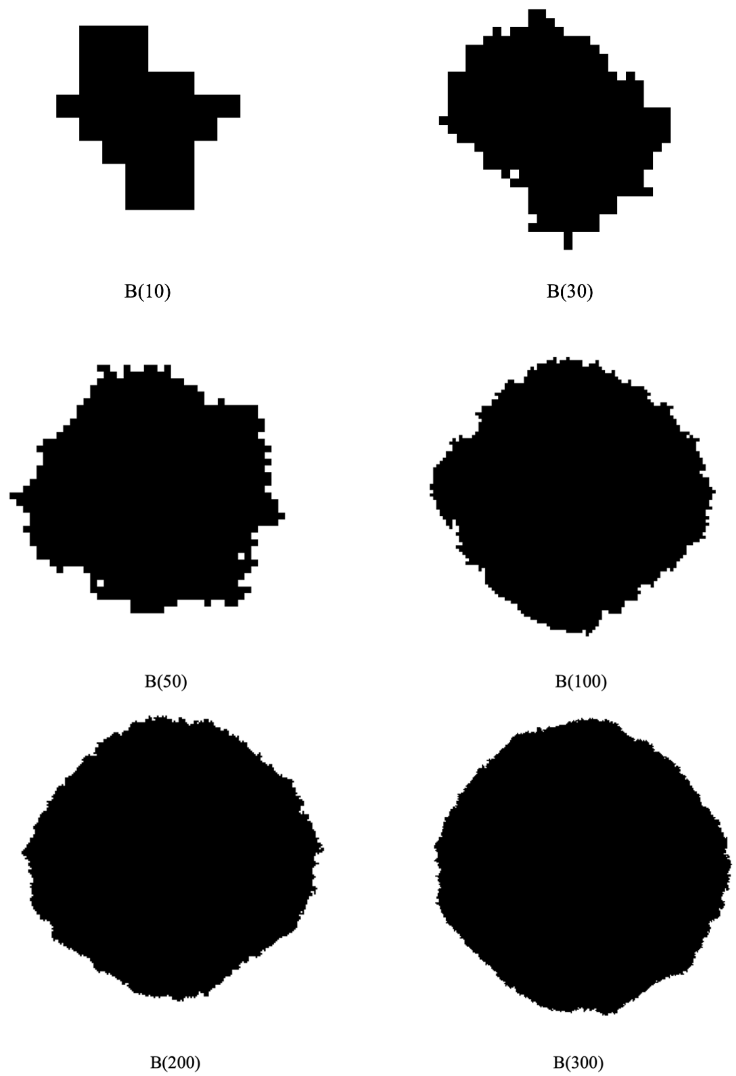
We will run the simulation for three different distributions in which one is discrete and two are continuous. We will run each distribution on 4 to 6 time limits that are deemed appropriate to demonstrate the evolution of the graph. Note that all the graphs are not to scale and we place emphasis on the shape of the graph.

5.2.1. *Random Integer.* First, we want to simulate the distribution of a discrete case where $t(e)$ takes its value with equal probability among $1, 2, 3, 4, 5$. This can be understood as rolling a fair dice with 5 sides, where each path takes the number on the dice to pass.

$$P(1) = 0.2, P(2) = 0.2, P(3) = 0.2, P(4) = 0.2, P(5) = 0.2$$

We take time $t = 10, 30, 50, 100, 200$, and 300 and simulate the graph for $B(t)$ as shown in figure 4.

FIGURE 4. Simulation for Random Integers 1 to 5

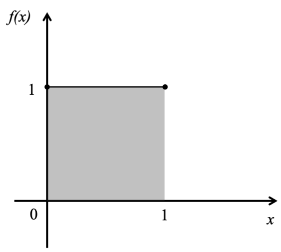


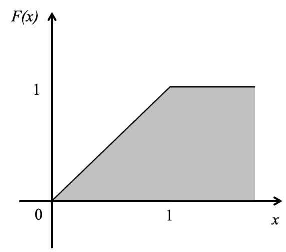
5.2.2. *Uniform Distribution.* Next, we want to examine the case when the distribution is continuous. We will simulate the graph for $t(e)$ takes its value on a uniform distribution on $[0,1]$, where

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

and

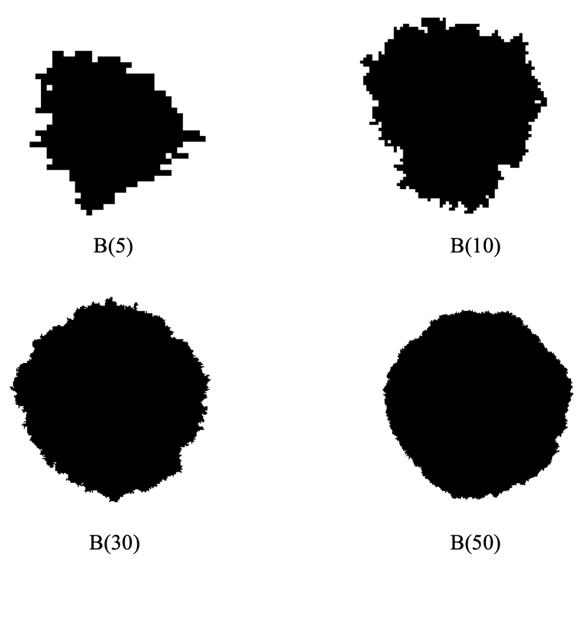
$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1 \end{cases}$$





We take time $t = 5, 10, 30$ and 50 and simulate the graph for $B(t)$ as shown in figure 5.

FIGURE 5. Graph for Uniform Distribution $[0,1]$

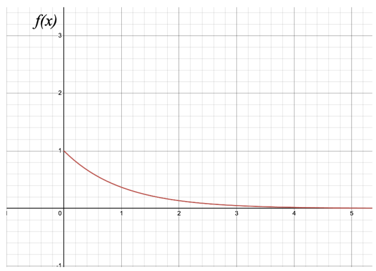


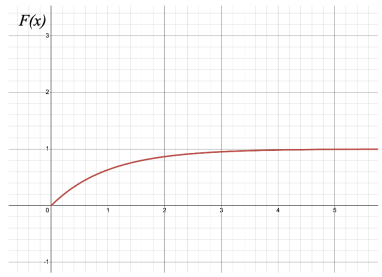
5.2.3. *Exponential.* Last but not least, we want to examine the case where $t(e)$ taking its value on a exponential distribution, where

$$f(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \geq 0 \end{cases}$$





Note that this case has its special application in the real world, namely the Eden growth model, which will be further discussed in section 6.3.

We take time $t = 10, 30, 50$ and 100 and simulate the graph for $B(t)$ as shown in figure 6.

5.3. **Summary.** We can clearly see the trend that the shape of $B(t)$ gets more and more smooth and circular as time increase in all three simulations, which is consistent with our theories.

6. Application

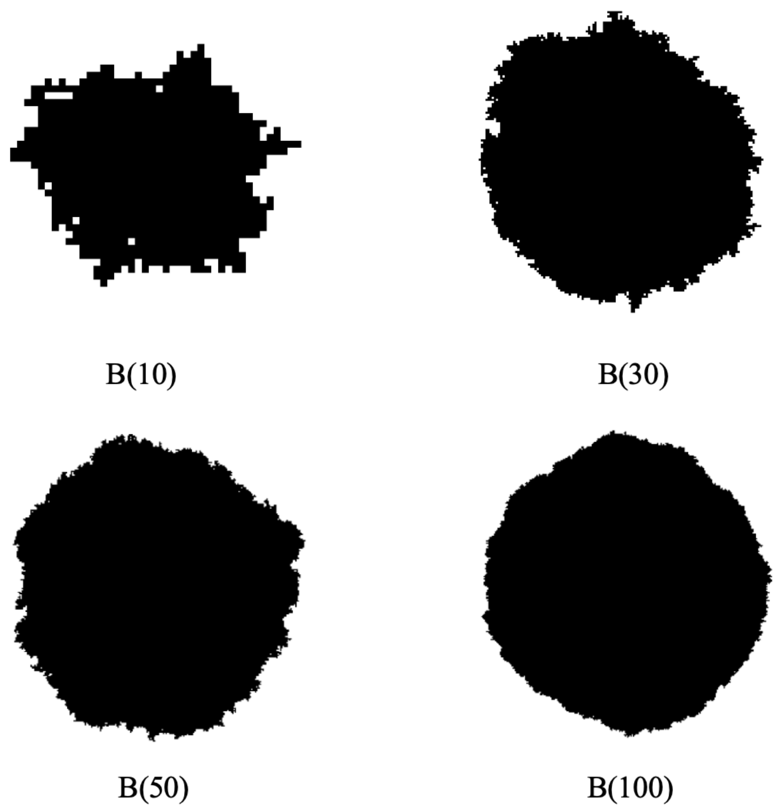
The first passage percolation model has various applications in different fields.

6.1. **Physics.** It is used in physics to study the traveling of particles given certain restrictions characterized by the traveling time in the first passage percolation model. V. M. Joshi studied how a particle travels on the plane square lattice from the origin to the points $(n, 0), n = 1, 2$, the path to $(n, 0)$ being subject to the restrictions $0 \leq x \leq n, 0 \leq y \leq 1$. [Jos77]

6.2. **Social Sciences.** The model can also be easily adapted to study social problems like rumor-spreading. Based on a model introduced by Maki and Thompson[Gan00], Aidan Sudbury showed that the proportion of the population never hearing the rumour converges in probability to 0.203 as the population size tends to ∞.[Sud85] https://www.overleaf.com/project/64b319ae7d4b3be6af80ae3c

6.3. **Biology.** First passage percolation also has its application in biology, specifically the Eden growth model. The Eden growth model studies the growth of specific types of clusters such as bacterial colonies and deposition of materials. [Ger18] These clusters grow by random accumulation of material on their boundary, which could be characterized by first passage percolation of the exponential distribution. One can easily notice the similarity between the graph of a study for Eden growth

Figure 6. Graph for Exponential Distribution



model (see figure 7 [Fer06]) and the simulated graphs for first passage percolation in section 5.2.3.

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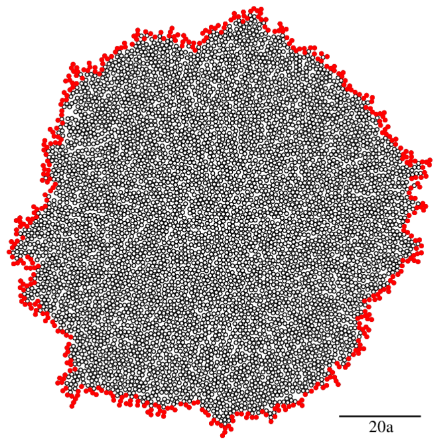
Figure 7. Demonstration for the Eden Growth Model



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