LERAY-HOPF WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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ABSTRACT. The Navier-Stokes equations describe the motion of an idealized fluid. In this paper, we introduce the Helmholtz-Weyl decomposition and present a weak formulation of the Navier-Stokes equations along with its properties. We then build up to a proof of the existence of global-in-time Leray-Hopf weak solutions. We focus on the case of a torus, where solutions are periodic in space.

CONTENTS

1. INTRODUCTION: PHYSICAL DERIVATION OF INCOMPRESSIBLE MODEL

The equations of motion of an incompressible, Newtonian fluid were first proposed in 1822 by the French engineer Navier, before Stokes rederived them in 1845. Since their conception, the Navier-Stokes equations have posed challenges to multiple areas of mathematics. In particular, there is no consensus on whether solutions "break down" at finite time. In his seminal 1934 work [1], Leray pioneered the now general concept of a "weak solution" to a partial differential equation, before the development of distribution theory by Schwartz in 1950 and even the formal introduction of Sobolev spaces in 1936. Subsequently, Hopf [2] extended his results from the whole space to an arbitrary open domain. Leray's work demonstrates the deep connection between functional analysis techniques and the study of partial differential equations.

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This paper owes much to Robinson, Rodrigo and Sadowski's comprehensive textbook [3]. We choose the torus for simplicity because it is compact and has no boundary. In Section 3, we recall definitions and results from the functional analysis textbook [4]. Then, we discuss the Helmholtz-Weyl decomposition which leads to a Galerkin approximation similar to Hopf's method. We state Leray's definition of weak solutions in Section 5. The paper concludes with the proof of existence of weak solutions in three dimensions in Section 7.

We first state the Navier-Stokes equations, which describe the flow of an incompressible homogeneous fluid in a domain Ω submitted to a body force field f. Given a divergence-free initial condition v_0, we seek a vector field v and a scalar field p that satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v - \nu \Delta v + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{(boundary condition) on } \partial\Omega, \\ v(0) = v_0 & \text{(initial condition).} \end{cases} \tag{1.1}$$

In this section, we present an introduction to the model of fluid mechanics based on Chapter I of Boyer and Fabrie's monograph [5] as well as Chapter II of Lemarié-Rieusset's survey [6]. We include the physical meaning of various terms in the derivation of (1.1). More details on the physical aspects of hydrodynamics can be found in the book by Landau and Lifschitz [7].

In order to define the macroscopic quantities of a fluid system, we assume the description of a continuous medium, where the characteristic lengths are much larger than the particles' mean free paths. Hence, properties like density, pressure, temperature, and velocity vary in a continuous manner.

We also adopt an Eulerian coordinate system with a fixed reference frame, rather than a Lagrangian system that follows a fluid particle through its motion. Given a quantity $f(t, X)$, we want to describe the temporal variation of f for a particle which passes through position X at time t. We apply the chain rule and the identity $\frac{\partial X}{\partial t} = v$ for the material derivative.

Definition 1.2. The *material derivative* of f is given by $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \nabla f$.

We are ready to describe the transport theorem, which describes the temporal variation of a scalar quantity integrated over a fluid element.

Theorem 1.3. *(Transport Theorem) For any C^1 function f with respect to the variables $(t, X) \in \mathbb{R} \times \mathbb{R}^3$, we have*

$$\frac{d}{dt} \int_{\Omega_t} f(t, X) dX = \int_{\Omega_t} \left(\frac{\partial f}{\partial t} + \operatorname{div}(f v) \right) dX = \int_{\Omega_t} \left(\frac{Df}{Dt} + f \operatorname{div}(v) \right) dX.$$

When $F(t, X)$ is a vector-valued quantity, we get

$$\frac{d}{dt} \int_{\Omega_t} F(t, X) dX = \int_{\Omega_t} \left(\frac{\partial F}{\partial t} + \operatorname{div}(F \otimes v) \right) dX = \int_{\Omega_t} \left(\frac{DF}{Dt} + F \cdot \operatorname{div}(v) \right) dX,$$

where $\otimes$ denotes the tensor product between two vector fields.

Proof. Chapter I.2 of [5]. We use the change of variables formula. □

Since the mass of fluid contained in some arbitrary volume Ω_t always remains unchanged, we can apply Theorem 1.3 to the density $\rho(t, X)$, which gives the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = \frac{D\rho}{Dt} + \rho \operatorname{div} v = 0. \tag{1.4}$$

In the case of an incompressible fluid, (1.4) reduces to $\operatorname{div} v = 0$ because the density ρ is constant.

Next, we apply Newton's second law to a moving parcel of fluid, which states that the rate of change of its total momentum is equal to the sum of the external forces applied to it. The linear momentum density is ρv, while the force density at time t and position X is $F(t, X)$, which gives

$$\frac{d}{dt}\int_{\Omega_t} \rho v dX = \int_{\Omega_t} F dX.$$

We apply Theorem 1.3 again to obtain

$$\int_{\Omega_t} \left(\frac{D\rho v}{Dt} + \rho v \cdot \operatorname{div} v - F \right) dX = 0.$$

Since this holds for an arbitrary fluid element Ω_t, we subtract (1.4) to arrive at the *linear momentum equation* in its local form

$$\frac{D\rho v}{Dt} + \rho v \cdot \operatorname{div} v = F \iff \rho \frac{Dv}{Dt} = F. \tag{1.5}$$

The linear momentum equation is more commonly written as

$$\rho\left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = F. \tag{1.6}$$

Finally, we examine the force density term F. If there are no external forces, we can decompose F into two components: $F = f_p + f_{\text{visc}}$.

(a) The force due to the pressure is $f_p = -\nabla p$.
(b) The force due to the viscosity is $f_{\text{visc}} = \mu \Delta v + \lambda \nabla(\operatorname{div} v)$, where μ is the dynamic viscosity, η is the volume viscosity and $\lambda = \mu + \eta$. This relation holds for *Newtonian* fluids and originates from the expression of the viscous stress tensor as a function of the strain rate tensor (see Chapter I.4 of [5]).

Because we have $\operatorname{div} v = 0$ for an incompressible fluid, we divide (1.6) by ρ for

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v = -\nabla \frac{p}{\rho} + \frac{\mu}{\rho} \Delta v.$$

We replace p with its scaled version and take the kinematic viscosity $\nu = \frac{\mu}{\rho}$ to get (1.1) as desired.

2. Fourier expansion on the torus

We can express a function on the torus in terms of a Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k e^{ik \cdot x}. \tag{2.1}$$

Because u is real-valued, the coefficients satisfy $\hat{u}_k = \overline{\hat{u}_{-k}}$ for all $k \in \mathbb{Z}^3$. We compute the Fourier coefficients $\hat{u}_k$ via the integral

$$\hat{u}_k = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} e^{-ik\cdot x} u(x) dx,$$

which holds by the orthogonality of the basis. We characterize functions in $L^2(\mathbb{T}^3)$ by their Fourier coefficients because a function u belongs to $L^2(\mathbb{T}^3)$ if and only if the sum $\sum |\hat{u}_k|^2$ is finite.

Proposition 2.2. *Let $(C_N)_{N\in\mathbb{N}} \subset \mathbb{Z}^3$ be a sequence such that $C_1 \subset C_2 \subset \dots$ and $\bigcup_{N=1}^{\infty} C_n = \mathbb{Z}^3$. Then for every $u \in L^p(\mathbb{T}^3)$ and $1 \leq p < \infty$, we have*

$$||u - \sum_{k\in C_N} \hat{u}_k e^{ik\cdot x}||_{L^p} \to 0 \text{ as } N \to \infty$$

if and only if the map $u \mapsto \sum_{k\in C_N} \hat{u}_k e^{ik\cdot x}$ is uniformly bounded from $L^p(\mathbb{T}^3)$ into $L^p(\mathbb{T}^3)$.

Proof. The forward direction follows from the uniform boundedness principle (Theorem 2.2 of [4]), while the backward direction uses the density of trigonometric polynomials in $L^p(\mathbb{T}^3)$. □

3. Function spaces

This section describes the analysis tools for later sections. In particular, we provide an introduction to the theory of Sobolev spaces, which allow us to apply the machinery of functional analysis to partial differential equations. We adopt the following notation:

- $C(\Omega)$ is the space of continuous functions on Ω.
- $C^k(\Omega)$ is the space of all k-times continuously differentiable functions.
- $C_c^k(\Omega)$ is the space of functions in $C^k(\Omega)$ with compact support.
- For $1 \leq p < \infty$, $L^p(\Omega)$ is the Lebesgue space of measurable functions where
$$||f||_{L^p(\Omega)} = \left(\int_\Omega |f|^p dx \right)^{1/p} < \infty.$$
$L^2(\Omega)$ is a Hilbert space with inner product $\langle u, v\rangle := \int_\Omega u(x) \cdot v(x) dx$. We refer to this space if we omit a subscript because we use it very frequently.
- $L^\infty(\Omega)$ is the space of essentially bounded functions on Ω, i.e. measurable functions where
$$||f||_{L^\infty(\Omega)} = \text{ess sup}_\Omega |f| < \infty.$$
- $L^p_{\text{loc}}(\Omega)$ consists of functions that are contained in $L^p(K)$ for every compact subset K of Ω.

We also derive two results from Hölder's inequality (see Theorem 4.6 of [4]).

Lemma 3.1. *Assume $1 \leq p, q, r < \infty$ and let Ω be a measurable set in $\mathbb{R}^n$. If we have $u \in L^p(\Omega), v \in L^q(\Omega)$ and $w \in L^r(\Omega)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then uvw belongs to $L^1(\Omega)$ and*

$$\int_\Omega |u(x)v(x)w(x)| dx \leq ||u||_{L^p} ||v||_{L^q} ||w||_{L^r}.$$

In particular, we have $||uvw||_{L^1} \leq ||u||_{L^6} ||v||_{L^2} ||w||_{L^3}$.

Lemma 3.2. *(Lebesgue interpolation) Assume $1 \leq p, q, r < \infty$ and let Ω be a measurable set in $\mathbb{R}^n$. If u belongs to $L^p(\Omega) \cap L^q(\Omega)$, then we have $u \in L^r(\Omega)$ with*

$$||u||_{L^r} \leq ||u||_{L^p}^{\alpha} ||u||_{L^q}^{1-\alpha}, \text{ where } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

In particular, we have $||u||_{L^3} \leq ||u||_{L^2}^{1/2} ||u||_{L^6}^{1/2}$.

We also cite two important theorems that ensure compactness. Compact sets play a vital role in the existence mechanisms we use for subsequent sections.

Theorem 3.3. *(Arzela-Ascoli) Let X be a Banach space and (u_n) be a sequence of functions in $C([0,T]; X)$ such that*

(1) for each $t \in [0,T]$ there exists a compact set $K(t) \subset X$ such that for every $n \in \mathbb{N}$ we have $u_n(t) \in K(t)$;
(2) the functions u_n are equicontinuous: for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$

$$|s - t| \leq \delta \implies ||u_n(s) - u_n(t)||_X \leq \epsilon.$$

Then there exists a subsequence (u_{n_k}) and a function $u \in C([0,T]; X)$ such that

$$u_{n_k} \to u \text{ in } C([0,T]; X).$$

Proof. Theorem 11.28 of [8]. □

Theorem 3.4. *(Banach-Alaoglu) If X is a separable Banach space, then any bounded sequence in X^* has a weakly-* convergent subsequence.*

Proof. Theorem 3.16 of [4]. □

Corollary 3.5. *If X is a reflexive Banach space, then any bounded sequence in X has a weakly convergent subsequence.*

3.1. **Sobolev spaces $W^{k,p}$.** We motivate Sobolev spaces in line with Chapter 5 of [9] by introducing the concept of a weak derivative. Given a test function $\varphi \in C_c^\infty$, we apply the integration by parts formula for $u \in C^1(\Omega)$ to obtain

$$\int_\Omega u \varphi_i dx = -\int_\Omega u_i \varphi dx. \tag{3.6}$$

We generalize this observation to higher derivatives when we apply (3.6) repeatedly. For $k \in \mathbb{N}$ and α a multiindex of order k, we take $u \in C^k(\Omega)$ to get

$$\int_\Omega u D^\alpha \varphi dx = (-1)^k \int_\Omega D^\alpha u \varphi dx. \tag{3.7}$$

The left hand side of (3.7) makes sense only if u is locally summable, while we can replace $D^\alpha u$ with some locally summable function g.

Definition 3.8. Suppose that $g, u \in L^1_{\text{loc}}$. We say that g is the *αth weak derivative* of u, written $D^\alpha u = g$ if we have

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega g(x) \varphi(x) dx$$

for all test functions $\varphi \in C_C^\infty$.

We can verify that a weak derivative is unique up to a set of measure zero.

Definition 3.9. Let $1 \le p \le \infty$ and $k \in \mathbb{N}$. The *Sobolev space* $W^{k,p}(\Omega)$ comprises all locally summable functions u such that for each multiindex α with $|\alpha| \le k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$. We define its norm as

$$||u||_{W^{k,p}}(\Omega) := \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \sum_{|\alpha|\le k} \text{ess sup}_\Omega |D^\alpha u| & \text{if } p = \infty. \end{cases}$$

Definition 3.10. The space $W_0^{k,p}$ is the closure of C_c^∞ in the $W^{k,p}$ norm.

Proposition 3.11. *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space and $W^{k,2}(\Omega)$ is a Hilbert space with inner product and norm*

$$\langle u, v \rangle_{W^{k,2}} := \sum_{0 \le |\alpha| \le k} \langle \partial^\alpha u, \partial^\alpha v \rangle, ||u||^2_{W^{k,2}} := \sum_{0 \le |\alpha| \le k} ||\partial^\alpha u||^2.$$

We will concentrate on the L^2-based Sobolev spaces $W^{k,2}(\mathbb{T}^3)$ and $W_0^{k,2}(\mathbb{T}^3)$ for the torus. We have

$$||u||^2_{W^{1,2}(\mathbb{T}^3)} = ||u||^2_{L^2(\mathbb{T}^3)} + ||\nabla u||^2_{L^2(\mathbb{T}^3)} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)|\hat{u}_k|^2.$$

Moreover, $W^{s_2,2}(\mathbb{T}^3)$ is a subset of $W^{s_1,2}(\mathbb{T}^3)$ if $s_2 \ge s_1$. Because we consider the torus, we can take without loss of generality *homogeneous spaces* where functions have zero mean. We denote them as $\dot{L}^2(\mathbb{T}^3)$ and $\dot{W}^{s,2}(\mathbb{T}^3)$ respectively.

Definition 3.12. We set $\dot{\mathbb{Z}}^3 = \mathbb{Z} \setminus \{(0,0,0)\}$. The *homogeneous space* $\dot{L}^2(\mathbb{T}^3)$ consists of all functions $u \in L^2(\mathbb{T}^3)$ such that $\int_{\mathbb{T}^3} u(x)dx = 0$, i.e. the Fourier coefficient $\hat{u}_0$ is zero. Hence, u takes the form

$$\sum_{k \in \dot{\mathbb{Z}}^3} \hat{u}_k e^{ik\cdot x} \text{ with } \sum_{k \in \dot{\mathbb{Z}}^3} |\hat{u}_k|^2 < \infty.$$

The *homogeneous Sobolev space* $\dot{W}^{s,2}(\mathbb{T}^3)$ is defined as

$$\dot{W}^{s,2}(\mathbb{T}^3) := W^{s,2}(\mathbb{T}^3) \cap \dot{L}^2(\mathbb{T}^3).$$

For functions in $\dot{W}^{s,2}(\mathbb{T}^3)$, the L^2 part of the norm always vanishes. We redefine the norm as

$$||u||^2_{\dot{W}^{s,2}} := (2\pi)^3 \sum_{k \in \dot{\mathbb{Z}}^3} |k|^{2s} |\hat{u}_k|^2.$$

We also obtain the following simplifications for $\dot{W}^{1,2}$ and $\dot{W}^{2,2}$

$$||u||^2_{\dot{W}^{1,2}} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k|^2 |\hat{u}_k|^2 = (2\pi)^3 \sum_{k \in \dot{\mathbb{Z}}^3} |k\hat{u}_k|^2 = ||\nabla u||^2 \text{ and}$$

$$||u||^2_{\dot{W}^{2,2}} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k|^4 |\hat{u}_k|^2 = (2\pi)^3 \sum_{k \in \dot{\mathbb{Z}}^3} ||k|^2 \hat{u}_k|^2 = ||\Delta u||^2.$$

Next, we want to know if a function $u \in W^{1,p}$ automatically belongs to other spaces. Through estimates called Sobolev inequalities, we obtain embeddings of various Sobolev spaces into others. The proofs of the following two theorems can be found in Chapters 9.3-9.4 of [4] and Chapters 5.6-5.8 of [9].

Theorem 3.13. *(Sobolev-Gagliardo-Nirenberg) If we have $1 \leq p < n$, then we have a continuous embedding*

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \text{ where } p^* = np/(n-p).$$

Furthermore, there exists a constant $c > 0$ such that

$$||u||_{L^{p^*}} \leq c||Du||_{L^p}.$$

Theorem 3.14. *(Poincaré inequality) For $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{T}^3)$ such that $\int_{\mathbb{T}^3} u = 0$, there exists C such that $||u||_{L^p} \leq C||\nabla u||_{L^p}$.*

Definition 3.15. Let X and Y be Banach spaces such that $X \subset Y$. We say that X is *compactly embedded* into Y, written $X \subset\subset Y$ if for all $x \in X$, $||x||_Y \leq C||x||_X$ for some $C > 0$ and each bounded sequence in X has compact closure in Y.

Theorem 3.16. *(Sobolev embedding) If $0 \leq s < 3/2$, then $W^{s,2}(\Omega) \subset L^{6/(3-2s)}(\Omega)$ and there exists $c_s > 0$ such that*

$$||u||_{L^{6/(3-2s)}} \leq c_s||u||_{W^{s,2}} \text{ for all } u \in W^{s,2}(\Omega).$$

Proof. Suppose that $u \in W^{s,2}(\Omega)$ with $0 \leq s < \frac{3}{2}$. For all multiindices α such that $|\alpha| = s$, we have $D^\alpha u \in L^2(\Omega)$. Then for all β such that $|\beta| = s - 1$, we apply Theorem 3.13 for

$$||D^\beta u||_{L^{p^*}} \leq c||D^\alpha u||_{L^p} \leq c^*||u||_{W^{s,2}}.$$

We get $u \in W^{s-1}, p^*(\Omega)$. We repeat this process inductively to obtain

$$u \in W^{s-2,p^{**}}(\Omega), \text{ where} \frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}.$$

Eventually after s steps, we conclude

$$u \in W^{0,p^{\overbrace{* * \cdots *}^{s \text{ times}}}}(\Omega) = W^{0,r}(\Omega) = L^r(\Omega),$$

where we have $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$. □

3.2. **Bochner spaces.** We can extend the notions of measurability and integrability to maps of the form $f : (0,T) \to X$, where X is a Banach space. We find solutions to the Navier-Stokes equations within these Bochner spaces.

Definition 3.17. Let $s_n : (0,T) \to X$ be simple functions of the form $\sum_{i=1}^m \chi_{E_i} u_i$ where E_i are Lebesgue measurable and $u_i \in X$. A function $f : (0,T) \to X$ is *measurable* if $f(t) = \lim_{n\to\infty} s_n(t)$ for almost all $t \in [0,T]$.

Definition 3.18. Let X be a Banach space. The *Bochner space* $L^p(0,T;X)$ comprises all measurable functions for which the following norm is finite

$$||f||_{L^p(0,T,;X)} := \begin{cases} \left(\int_a^b ||f(s)||_X^p ds \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \text{ess sup}_{s\in(0,T)}||f(s)||_X & \text{if } p = \infty. \end{cases}$$

If X is a Banach space, then the space $L^p(0,T;X)$ is complete.

When X^* is separable, we can easily identify the dual of $L^p(0,T;X)$.

Theorem 3.19. *Assume that X is a Banach space with X^* separable. Let $1 \le p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual space of $L^p(0,T;X)$ is $L^q(0,T;X^*)$ with*

$$\langle f, u \rangle_{L^q(0,T;X^*)\times L^p(0,T;X)} := \int_0^T \langle f(t), u(t)\rangle_{X^*\times X} dt.$$

Proof. Chapter IV, Theorem 1 of [10]. □

Definition 3.20. We say that $u \in L^1(0,T;X)$ has *weak time derivative* $\partial_t u \in L^1(0,T;X)$ if for all $\varphi \in C_c^\infty(0,T)$, we have

$$\int_0^T u(s)\partial_t \varphi(s) ds = -\int_0^T \partial_t u(s)\varphi(s) ds. \tag{3.21}$$

Lemma 3.22. *Suppose that $u, g \in L^1(0,T;X)$. Then the following are equivalent*

- *we have $\partial_t u = g$ in the sense of Definition 3.20,*
- *there exists $\xi \in X$ such that $u(t) = \xi + \int_0^t g(s)ds$ for almost every $t \in [0,T]$,*
- *for every $f \in X^*$, we have $\frac{d}{dt}\langle f, u\rangle = \langle f, g\rangle$, where the time derivative is taken in a weak sense.*

Proof. Chapter 3, Lemma 1.1 of [11]. □

Corollary 3.23. *Suppose that $u, g \in L^1(0,T;X)$ and that for every $f \in X^*$ and $0 \le t_1 \le t_2 \le T$, we have*

$$\langle f, u(t_2)\rangle - \langle f, u(t_1)\rangle = \int_{t_1}^{t_2} \langle f, g(s)\rangle ds. \tag{3.24}$$

Then we have $\partial_t u = g$ in the sense of Definition 3.20.

4. Helmholtz-Weyl decomposition

We introduce the notation $\mathbb{L}^2 := [L^2]^3$ and $\mathbb{W}^{s,2} := [W^{s,2}]^3$ for legibility. In this section, we aim to show that

$$\mathbb{L}^2 = H \oplus G,$$

where H is the space of divergence-free functions and G is the space of gradients of functions in $W^{1,2}$. In other words, any $u \in L^2$ can be written uniquely as $u = h + \nabla g$, where we have $\nabla h = 0$ while g belongs to $W^{1,2}$. We also introduce the Leray projector $\mathbb{P}$ and Stokes operator A.

We decompose only the homogeneous space $\dot{\mathbb{L}}^2(\mathbb{T}^3)$ from Definition 3.12 since we will include the zero-average condition for the torus. We begin by defining the spaces H and G in the context of the three-dimensional torus $\mathbb{T}^3$. If u is given by $u(x) = \hat{u}_k e^{ik\cdot x}$, then we compute its divergence as $\operatorname{div} u(x) = i(k \cdot \hat{u}_k)e^{ik\cdot x}$. An arbitrary function u is divergence free if and only if $\hat{u}_k$ is orthogonal to k.

Definition 4.1. We define the spaces $H(\mathbb{T}^3)$ and $G(\mathbb{T}^3)$ as

$$H(\mathbb{T}^3) := \left\{ u \in \dot{\mathbb{L}}^2(\mathbb{T}^3) : u = \sum_{k\in\mathbb{Z}^3} \hat{u}_k e^{ik\cdot x}, \hat{u}_{-k} = \overline{\hat{u}_k} \text{ and } k\cdot \hat{u}_k = 0 \text{ for all } k \in \dot{\mathbb{Z}}^3 \right\},$$

$$G(\mathbb{T}^3) := \left\{ u \in \dot{\mathbb{L}}^2(\mathbb{T}^3) : u = \nabla g, \text{ where } g \in \dot{W}^{1,2}(\mathbb{T}^3) \right\}.$$

Since we will subsequently consider limits of smooth functions too, we call a function $u \in L^1(\Omega)$ *weakly divergence free* if $\langle u, \nabla\varphi\rangle = 0$ for every $\varphi \in C_c^\infty(\Omega)$.

Lemma 4.2. *If we have $u \in H(\mathbb{T}^3)$, then the function u is weakly divergence free and $\langle u, \nabla\varphi\rangle = 0$ for all $\varphi \in W^{1,2}(\mathbb{T}^3)$.*

Proof. We assume without loss of generality that $\varphi(x) = e^{-ik\cdot x}$ for some $k \in \mathbb{Z}^3$ because any $\varphi \in W^{1,2}(\mathbb{T}^3)$ can be approximated arbitrarily closely by finite linear combinations of such functions. We take $u(x) = \sum_{l\in\dot{\mathbb{Z}}^3} \hat{u}_l e^{il\cdot x}$, which gives

$$\langle u, \nabla\varphi\rangle = \int_{\mathbb{T}^3} \hat{u}_k e^{ik\cdot x} \cdot \nabla e^{-ik\cdot x} = -i\int_{\mathbb{T}^3} \hat{u}_k \cdot k.$$

This expression vanishes because we have $\hat{u}_k \cdot k = 0$. $\square$

Corollary 4.3. *The spaces $H(\mathbb{T}^3)$ and $G(\mathbb{T}^3)$ are orthogonal, i.e. $\langle h, \nabla g\rangle = 0$ for every $h \in H(\mathbb{T}^3)$ and $\nabla g \in G(\mathbb{T}^3)$.*

We proceed to the main result of this section: the Helmholtz-Weyl decomposition on the torus.

Theorem 4.4. *The space $\dot{\mathbb{L}}^2(\mathbb{T}^3)$ can be written as $\dot{\mathbb{L}}^2(\mathbb{T}^3) = H(\mathbb{T}^3) \oplus G(\mathbb{T}^3)$, i.e. every function $u \in \dot{\mathbb{L}}^2(\mathbb{T}^3)$ can be written uniquely as $u = h + \nabla g$, where the vector-valued function h belongs to $H(\mathbb{T}^3)$ and the scalar function g belongs to $\dot{W}^{1,2}(\mathbb{T}^3)$. Moreover, the functions h and ∇g are orthogonal in $\mathbb{L}^2(\mathbb{T}^3)$.*

Proof. We take some arbitrary $u \in \dot{\mathbb{L}}^2(\mathbb{T}^3)$ given by $u(x) = \sum_{k\in\dot{\mathbb{Z}}^3} \hat{u}_k e^{ik\cdot x}$. We can express each Fourier coefficient as $\hat{u}_k = \alpha_k k + w_k \in \mathbb{C}$, where $w_k \cdot k = 0$ and $\alpha_k = \hat{u}_k \cdot \frac{k}{|k|^2}$. We also have

$$|\hat{u}_k|^2 = |\alpha_k|^2|k|^2 + |w_k|^2. \tag{4.5}$$

We now take $g(x) := \sum_{k\in\dot{\mathbb{Z}}^3}(-i\alpha_k)e^{ik\cdot x}$ and $h(x) := \sum_{k\in\dot{\mathbb{Z}}^3} w_k e^{ik\cdot x}$. We verify

$$\begin{aligned} h(x) + \nabla g(x) &= \sum_{k\in\dot{\mathbb{Z}}^3} \left((-i\alpha_k)(ik)e^{ik\cdot x} + w_k e^{ik\cdot x}\right) \\ &= \sum_{k\in\dot{\mathbb{Z}}^3} (\alpha_k k + w_k)e^{ik\cdot x} = u(x). \end{aligned}$$

Moreover, h is weakly divergence free because $w_k \cdot k = 0$ for every $k \in \dot{\mathbb{Z}}^3$. We have

$$\begin{aligned} ||\nabla g||_{\mathbb{L}^2(\mathbb{T}^3)} &= ||g||^2_{\dot{W}^{1,2}(\mathbb{T}^3)} = \sum_{k\in\dot{\mathbb{Z}}^3} |\alpha_k|^2|k|^2, \\ ||h||^2_{\mathbb{L}^2(\mathbb{T}^3)} &= \sum_{k\in\dot{\mathbb{Z}}^3} |w_k|^2, \text{ and} \\ ||u||^2_{\dot{\mathbb{L}}^2(\mathbb{T}^3)} &= ||g||^2_{\dot{W}^{1,2}(\mathbb{T}^3)} + ||h||^2_{\dot{\mathbb{L}}^2(\mathbb{T}^3)}, \end{aligned}$$

where the last equality follows from (4.5). Since both norms $||g||_{\dot{W}^{1,2}(\mathbb{T}^3)}$ and $||h||^2_{\dot{L}^2(\mathbb{T}^3)}$ are finite, we deduce $h \in \dot{\mathbb{L}}^2(\mathbb{T}^3)$ and $g \in \dot{W}^{1,2}(\mathbb{T}^3)$.

We claim that this representation is unique. If $u = h_1 + \nabla g_1 = h_2 + \nabla g_2$, then Corollary 4.3 tells us

$$||h_1 - h_2 + \nabla g_1 - \nabla g_2||^2 = 0 \implies ||h_1 - h_2||^2 + ||\nabla g_1 - \nabla g_2||^2 = 0,$$

which yields $h_1 = h_2$. Since g_1 and g_2 both have zero mean and we have $\nabla g_1 = \nabla g_2$, we conclude $g_1 = g_2$. $\square$

Corollary 4.6. *If we have $u \in \dot{\mathbb{W}}^{s,2}(\mathbb{T}^3)$ for $s > 0$, then we get $h \in \dot{\mathbb{W}}^{s,2}(\mathbb{T}^3)$ and $g \in \dot{W}^{s+1,2}(\mathbb{T}^3)$.*

Proof. Suppose that $u \in W^{s,2}(\mathbb{T}^3)$. Then we multiply (4.5) by $|k|^{2s}$ for

$$|\hat{u}_k|^2|k|^{2s} = |\alpha_k|^2|k|^{2(s+1)} + |w_k|^2|k|^{2s}.$$

We can rewrite this as $||u||^2_{\dot{\mathbb{W}}^{s,2}} = ||g||^2_{\dot{W}^{s+1,2}} + ||h||^2_{\dot{\mathbb{W}}^{s,2}}$. □

We are now ready to introduce the Leray projector onto the space of divergence-free functions. The Leray projector is well defined due to the Helmholtz-Weyl decomposition.

Definition 4.7. The *Leray projector* $\mathbb{P} : \dot{\mathbb{L}}^2(\mathbb{T}^3) \to H(\mathbb{T}^3)$ is given by $\mathbb{P}u = v$ if $u = v + \nabla w$, where $v \in H(\mathbb{T}^3)$ and $\nabla w \in G(\mathbb{T}^3)$.

Based on the proof of Theorem 4.4, we can even compute the Leray projector explicitly in the case of the torus. If u belongs to $\dot{\mathbb{L}}^2(\mathbb{T}^3)$ and $u(x) = \sum_{k\in\dot{\mathbb{Z}}^3} \hat{u}_k e^{ik\cdot x}$, then we write its Leray projection as

$$\mathbb{P}u(x) = \sum_{k\in\dot{\mathbb{Z}}^3} \left(\hat{u}_k - \frac{\hat{u}_k \cdot k}{|k|^2}\right) e^{ik\cdot x}.$$

Lemma 4.8. *The Leray projector $\mathbb{P}$ on the torus commutes with any derivative. For all $u \in \dot{\mathbb{W}}^{1,2}(\mathbb{T}^3)$, we have*

$$\mathbb{P}(\partial_j u) = \partial_j(\mathbb{P}u), j = 1, 2, 3.$$

Proof. We assume without loss of generality that $u(x) = \hat{u}_k e^{ik\cdot x}$ because both $\mathbb{P}$ and ∂_j are continuous. We verify $\mathbb{P}\partial_j(u) = \mathbb{P}(ik_j u) = ik_j\mathbb{P}u = \partial_j\mathbb{P}u$. □

4.1. **Stokes operator.** We narrow our scope from all divergence-free functions to those contained in $\mathbb{W}^{1,2}$. This new space $V(\mathbb{T}^3)$ is a subset of $H(\mathbb{T}^3)$ because we impose a greater degree of regularity.

Definition 4.9. We define $V(\mathbb{T}^3) := H(\mathbb{T}^3) \cap \mathbb{W}^{1,2}(\mathbb{T}^3)$. This space is equipped with the norm $||\cdot||_V = ||\cdot||_{\mathbb{W}^{1,2}}$.

Definition 4.10. The *Stokes operator* A on the domain $D(A) := V \cap \mathbb{W}^{2,2}$ is defined by

$$Au := -\mathbb{P}\Delta u.$$

In the case of the torus, Lemma 4.8 tells us $Au = -\mathbb{P}\Delta u = -\Delta\mathbb{P}u = -\Delta u$. We can thus write the Stokes operator explicitly as

$$Au(x) = \sum_{k\in\dot{\mathbb{Z}}^3} |k|^2 \hat{u}_k e^{ik\cdot x}. \tag{4.11}$$

We can also introduce the following regularity estimate for the torus.

Theorem 4.12. *Given $m \in \mathbb{N}$ and $u \in V(\mathbb{T}^3)$ such that $Au \in \mathbb{W}^{m,2}$, we have $||u||_{W^{m+2,2}} \leq ||Au||_{W^{m,2}}$.*

Proof. We apply the formula given by (4.11) to get

$$\begin{aligned} ||u||^2_{\mathbb{W}^{m+2,2}} &= ||u||^2_{\dot{\mathbb{W}}^{m+2,2}} \\ &= (2\pi)^3 \sum_{k\in\dot{\mathbb{Z}}^3} |k|^{2(m+2)}|\hat{u}_k|^2 \\ &= \left|\left| \sum_{k\in\dot{\mathbb{Z}}^3} |k|^2 \hat{u}_k e^{ik\cdot x} \right|\right|^2_{\dot{\mathbb{W}}^{m,2}} \\ &= ||Au||^2_{\dot{\mathbb{W}}^{m,2}} \leq ||Au||^2_{\mathbb{W}^{m,2}}, \end{aligned}$$

where we have the first equality because we have $u \in \dot{\mathbb{L}}^2(\mathbb{T}^3)$ and $\int_{\mathbb{T}^3} u = 0$. □

Theorem 4.13. *There exists a family of functions $\mathcal{N} = \{a_1, a_2, a_3, \dots\}$ such that*

(i) $\mathcal{N}$ is an orthonormal basis in $H(\mathbb{T}^3)$ and $V(\mathbb{T}^3)$.

(ii) $a_j \in D(A) \cap C^\infty(\mathbb{T}^3)$ are eigenfunctions of the Stokes operator, i.e. we have $Aa_j = \lambda_j a_j$ for all $j \in \mathbb{N}$ with

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_j \leq \dots, \text{ and } \lambda_j \to \infty.$$

Proof. We rewrite the Fourier expansion in an appropriate form. For a vector $k \in \mathbb{Z}^3$, we pick vectors m_k and m_{-k} such that $m_k \perp k, m_{-k} \perp k$, and $m_k \perp m_{-k}$. Since $\hat{u}_k \cdot k = 0$, this implies $\hat{u}_k + \hat{u}_{-k} \in \operatorname{span}\{m_k, m_-k\}$ and $i(\hat{u}_k - \hat{u}_{-k}) \in \operatorname{span}\{m_k m_{-k}\}$. Then we get

$$\begin{aligned} u(x) &= \sum_{k\in\dot{\mathbb{Z}}^3} \hat{u}_k e^{ik\cdot x} \\ &= \frac{1}{2}\sum_{k\in\dot{\mathbb{Z}}^3} \hat{u}_k e^{ik\cdot x} + \frac{1}{2}\sum_{-k\in\dot{\mathbb{Z}}^3} \hat{u}_k e^{-ik\cdot x} \\ &= \sum_{k\in\dot{\mathbb{Z}}^3} \frac{1}{2}(\hat{u}_k + \hat{u}_{-k})\cos(k\cdot x) + \sum_{k\in\dot{\mathbb{Z}}^3} \frac{1}{2} i(\hat{u}_k - \hat{u}_{-k})\sin(k\cdot x) \\ &= \sum_{k\in\dot{\mathbb{Z}}^3} (a_k m_k + b_k m_{-k})\cos(k\cdot x) + \sum_{k\in\dot{\mathbb{Z}}^3} (c_k m_k + d_k m_{-k})\sin(k\cdot x) \\ (*) \qquad &= \sum_{k\in\dot{\mathbb{Z}}^3} (a_k m_k + b_{-k} m_k)\cos(k\cdot x) + \sum_{k\in\dot{\mathbb{Z}}^3} (c_k m_k - d_{-k} m_k)\sin(k\cdot x) \\ &= \sum_{k\in\dot{\mathbb{Z}}^3} \alpha_k m_k \cos(k\cdot x) + \sum_{k\in\dot{\mathbb{Z}}^3} \beta_k m_k \sin(k\cdot x), \end{aligned}$$

where (*) holds because cosine is even and sine is odd. We obtain an orthonormal basis of $H(\mathbb{T}^3)$ with smooth eigenfunctions when we normalize the vectors $m_k \cos(k\cdot x)$ and $m_k \sin(k\cdot x)$ in $\mathbb{L}^2(\mathbb{T}^3)$.

We compute the eigenvalues $\lambda_k = |k|^2$ for $k \in \dot{\mathbb{Z}}^3$ because we have

$$\begin{aligned} Am_k \cos(k\cdot x) &= -m_k \sum_{j\in[3]} \partial_j^2 \cos(k\cdot x) = |k|^2 m_k \cos(k\cdot x), \\ Am_k \sin(k\cdot x) &= -m_k \sum_{j\in[3]} \partial_j^2 \sin(k\cdot x) = |k|^2 m_k \sin(k\cdot x). \end{aligned}$$

It follows that the set of eigenvalues is positive and unbounded.

Finally, we show that $\mathcal{N}$ is also orthonormal in $V(\mathbb{T}^3)$. Suppose that $a_i, a_j \in \mathcal{N}$. Then we have

$$\langle \nabla a_i, \nabla a_j\rangle = \langle a_i, -\Delta a_j\rangle = \langle \mathbb{P} a_i, -\Delta a_j\rangle = \langle a_i, -\mathbb{P}\Delta a_j\rangle = \langle a_i, A a_j\rangle = \lambda_{a_j}\langle a_i, a_j\rangle.$$

We conclude that $\mathcal{N}$ is also orthogonal in $\mathbb{W}^{1,2}$. □

We can define the powers of the Stokes operator in a straightforward way too.

Definition 4.14. Let $u \in V(\mathbb{T}^3)$ be given by $u = \sum_{j=1}^{\infty} \tilde{u}_j a_j$, where $a_j \in \mathcal{N}$ and $\tilde{u}_j \in \mathbb{R}$. For $\alpha \geq 0$ we define

$$A^{\alpha} u := \sum_{j=1}^{\infty} \lambda_j^{\alpha} \tilde{u}_j a_j,$$

where the domain $D(A^{\alpha})$ contains all functions u such that $\sum_{j=1}^{\infty} \lambda_j^{2\alpha} |\tilde{u}_j|^2$ is finite. We also have $D(A^{s/2}) = \dot{W}^{s,2}(\mathbb{T}^3)$ and $||u||_{\dot{W}^{s,2}} = ||A^{s/2} u||_{L^2}$.

5. Leray's weak formulation

We present the construction of Leray's weak solutions to the Navier-Stokes equations. We first consider the appropriate space of test functions.

Definition 5.1. The space of test functions $\mathcal{D}_{\sigma}$ on the space-time domain $\Omega\times[0,\infty)$ is given by

$$\mathcal{D}_{\sigma} := \Big\{ \varphi \in C_c^{\infty}(\Omega \times [0,\infty)) : \operatorname{div} \varphi(t) = 0 \text{ for all } t \in [0,\infty) \Big\}.$$

If $u(x,t)$ and $p(x,t)$ form a strong solution to the Navier-Stokes equation

$$\partial_t u - \nu \Delta u + (u\cdot\nabla) u + \nabla p = 0, \tag{5.2}$$

we multiply both sides by an arbitrary test function $\varphi \in \mathcal{D}_{\sigma}$ and integrate over $\Omega \times [0,s)$ for some time s to find

$$\int_0^s \Big(\langle \partial_t u, \varphi\rangle - \nu\langle \Delta u, \varphi\rangle + \langle u\cdot\nabla u, \varphi\rangle + \langle \nabla p, \varphi\rangle \Big) = 0. \tag{5.3}$$

Here, we notice via the integration by parts formula (in time and space respectively) that we have

$$\int_0^s \langle \partial_t u, \varphi\rangle = \langle u(s), \varphi(s)\rangle - \langle u(0), \varphi(0)\rangle - \int_0^s \langle u, \partial_t \varphi\rangle,$$
$$\langle \Delta u, \varphi\rangle = -\langle \nabla u, \nabla\varphi\rangle, \text{ and } \langle \nabla p, \varphi\rangle = -\langle p, \operatorname{div}\varphi\rangle = 0.$$

Thus, (5.3) reduces to the following form for all $\varphi \in \mathcal{D}_{\sigma}$ and all times $s \geq 0$

$$\int_0^{\infty} \Big(-\langle u, \partial_t\varphi\rangle + \nu\langle \nabla u, \nabla\varphi\rangle + \langle u\cdot\nabla u, \varphi\rangle \Big) = \langle u(0), \varphi(0)\rangle - \langle u(s), \varphi(s)\rangle. \tag{5.4}$$

However, this class of weak solutions remains very broad. We impose an additional energy-based constraint that holds for all strong solutions. This condition involves the nonlinear term, which we can define as a trilinear form

$$b(u,v,w) := \langle (u\cdot\nabla)v, w\rangle \text{ on } \mathbb{W}^{1,2}\times\mathbb{W}^{1,2}\times\mathbb{W}^{1,2}.$$

We check the continuity of b via the Sobolev embedding $W^{1,2}(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$.

Lemma 5.5. *Let $u \in V$ and $v, w \in \mathbb{W}^{1,2}$. Then we have $\langle (u\cdot\nabla)v, w\rangle = -\langle (u\cdot\nabla)w, v\rangle$. In particular, we have $\langle (u\cdot\nabla)v, v\rangle = 0$.*

Proof. Assume without loss of generality that u, v, w are smooth, otherwise we can appeal to their density. We obtain

$$\begin{aligned} b(u,v,w) &= \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} u_i \frac{\partial v_j}{\partial x_i} w_j dx \\ &= -\sum_{i,j=1}^{3} \int_{\mathbb{T}^3} \frac{\partial u_i}{\partial x_i} v_j w_j dx - \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} u_i v_j \frac{\partial w_j}{\partial x_i} dx \\ &= -\int_{\mathbb{T}^3} (\operatorname{div} u)(v \cdot w) dx - b(u,w,v) = -b(u,w,v), \end{aligned}$$

because the divergence of u vanishes. Hence $\langle (u\cdot\nabla)v, v\rangle = -\langle (u\cdot\nabla)v, v\rangle$. $\square$

For a strong solution u of the Navier-Stokes equations, we take the L^2 inner product of (5.2) with u for

$$\frac{1}{2}\frac{d}{dt}||u||^2 + \nu||\nabla u||^2 + \langle (u\cdot\nabla)u, u\rangle = 0 \implies \frac{1}{2}\frac{d}{dt}||u||^2 + ||\nabla u||^2 = 0.$$

Finally, an integration in time yields

$$\frac{1}{2}||u(t)||^2 + \int_0^t ||\nabla u(s)||^2 ds = \frac{1}{2}||u(0)||^2 \tag{5.6}$$

for any positive t. Hence we expect that for $T > 0$, the quantities

$$\text{ess sup}_{0\le t\le T}||u(t)| \text{ and } \int_0^T ||\nabla u(s)||^2 ds$$

are finite. Thus, we look for weak solutions which satisfy both $u \in L^\infty(0,T;H)$ and $u \in L^2(0,T;V)$.

Definition 5.7. We say that a function u is a *Leray-Hopf weak solution* of the Navier-Stokes equation with initial condition $u_0 \in H(\mathbb{T}^3)$ if

(a) $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ for all $T > 0$, and

(b) u satisfies the following equation for all $\varphi \in \mathcal{D}_\sigma$ and almost every $s > 0$,

$$\int_0^s \Big(-\langle u, \partial_t\varphi\rangle + \nu\langle \nabla u, \nabla\varphi\rangle + \langle u\cdot\nabla u, \varphi\rangle \Big) = \langle u_0, \varphi(0)\rangle - \langle u(s), \varphi(s)\rangle. \tag{5.8}$$

We now examine a consequence of condition (a) of Definition 5.7.

Lemma 5.9. *If u is an element of $L^\infty(0,T;H)\cap L^2(0,T;V)$, then we have*

$$(u\cdot\nabla)u \in L^{4/3}(0,T;L^{6/5}).$$

Proof. We apply Hölder's inequality for

$$\begin{aligned} ||(u\cdot\nabla)u||_{L^{6/5}} &\le \left(\int |\nabla u|^{6/5}|u|^{6/5}\right)^{5/6} \\ &\le \left(\int |\nabla u|^2\right)^{1/2}\left(\int |u|^3\right)^{1/3} \\ &= ||\nabla u||_{L^2}||u||_{L^3} \\ &\le ||\nabla u||_{L^2}||u||_{L^2}^{1/2}||u||_{L^6}^{1/2} \quad \text{(by Lemma 3.2)} \\ &\le ||u||_{W^{1,2}}^{3/2}||u||^{1/2}. \end{aligned}$$

Therefore, we have $\int_0^T ||(u\cdot\nabla)u||_{L^{6/5}}^{4/3} \le \int_0^T ||u||_{W^{1,2}}^2 ||u||^{2/3} < \infty$. □

We proceed to consider condition (b) of Definition 5.7 more closely.

Definition 5.10. We introduce the space $C_{c,\sigma}^{\infty}$ of divergence-free functions with compact support,

$$C_{c,\sigma}^{\infty}(\Omega) := \{\varphi \in [C_c^{\infty}(\Omega)]^3 : \operatorname{div}\varphi = 0\}.$$

Lemma 5.11. *If u is a weak solution and ϕ belongs to $C_{c,\sigma}^{\infty}(\mathbb{T}^3)$, then we have*

$$\nu\int_{t_1}^{t_2}\langle\nabla u,\nabla\phi\rangle + \int_{t_1}^{t_2}\langle(u\cdot\nabla)u,\phi\rangle = \langle u(t_1),\phi\rangle - \langle u(t_2),\phi\rangle \tag{5.12}$$

for almost every $t_1 \ge 0$ and almost every $t_2 \ge t_1$.

Proof. We take some arbitrary $\phi \in C_{c,\sigma}^{\infty}(\mathbb{T}^3)$ and choose $\alpha \in C_c^{\infty}[0,\infty)$ such that $t_1 \le t \le t_2$ implies $\alpha(t) = 1$. We get

$$\begin{aligned}&\int_{t_1}^{t_2}\Big(\nu\langle\nabla u,\nabla\phi\rangle + \langle u\cdot\nabla u,\phi\rangle\Big)\\ &= \langle u_0,\varphi(0)\rangle - \langle u(t_2),\varphi(t_2)\rangle - \langle u_0,\varphi(0)\rangle + \langle u(t_1),\varphi(t_1)\rangle\end{aligned}$$

when we take the difference of (5.8) for $\varphi(x,t) = \alpha(t)\phi(x)$ between the two times $s = t_2$ and $s = t_1$. □

Furthermore, the weak solution u has a well-defined time derivative that belongs to a very large Bochner space.

Lemma 5.13. *If u is a weak solution, then $\partial_t u$ is an element of $L^{4/3}(0,T;V^*)$. We can modify u on a set of measure zero such that $u \in C([0,T];V^*)$ and (5.8) and (5.12) are satisfied for all $s > 0$ and all $t_1, t_2 > 0$ respectively.*

Proof. Lemma 5.11 tells us that for almost all $t > 0$ and all $\phi \in C_{c,\sigma}^{\infty}(\mathbb{T}^3)$, we have

$$\langle u(t),\phi\rangle - \langle u_0,\phi\rangle = \int_0^t \langle g,\phi\rangle_{V^*\times V},$$

where $g \in V^*$ is given by $\langle g,\phi\rangle_{V^*\times V} := -\nu\langle\nabla u,\nabla\phi\rangle - \langle(u\cdot\nabla)u,\phi\rangle$. Then g is a bounded linear functional on V because we have

$$\begin{aligned}|\langle\nabla u,\nabla\phi\rangle| &\le ||\nabla u||||\nabla\phi||, \text{ and}\\ |\langle(u\cdot\nabla)u,\phi\rangle| &\le ||(u\cdot\nabla)u||_{L^{6/5}}||\phi||_{L^6}\\ &\le c||(u\cdot\nabla)u||_{L^{6/5}}||\nabla\phi||,\end{aligned}$$

due to the Sobolev embedding $W^{1,2}(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$. From Lemma 5.9, we obtain

$$\begin{aligned}||g||_{V^*}^{4/3} &\le c||(u\cdot\nabla)u||_{L^{6/5}}^{4/3} + \nu||\nabla u||^{4/3}\\ &\le c||(u\cdot\nabla)u||_{L^{6/5}}^{4/3} + \nu(1 + ||\nabla u||^2) \in L^1(0,T).\end{aligned}$$

It follows that $g \in L^{4/3}(0,T;V^*)$ and Corollary 3.23 gives us $\partial_t u \in L^{4/3}(0,T;V^*)$. By Lemma 3.22, the function u is (almost everywhere) an absolutely continuous function from $[0,T]$ into V^*.

To prove that (5.12) holds for all $t_1, t_2 > 0$, we notice that for all $t_1, t_2 \in (0,T]$ we can find sequences of times $s_n \to t_1$ and $z_n \to t_2$ such that

$$\nu\int_{s_n}^{z_n}\langle\nabla u,\nabla\phi\rangle + \int_{s_n}^{z_n}\langle(u\cdot\nabla)u,\phi\rangle = \langle u(s_n),\phi\rangle - \langle u(z_n),\phi\rangle.$$

The left hand side is continuous with respect to s_n and z_n, while we also have

$$\langle u(s_n), \phi\rangle \to \langle u(t_1), \phi\rangle \text{ and } \langle u(z_n), \phi\rangle \to \langle u(t_2), \phi\rangle$$

as n approaches ∞ due to $u \in C([0,T], V^*)$. A similar argument yields (5.8). □

From this point forth, we assume that we have modified every weak solution u so that it lies in $C([0,T]; V^*)$ and satisfies (5.8) and (5.12).

Theorem 5.14. *Weak solutions are L^2-weakly continuous in time, i.e.*

$$\lim_{t\to t_0} \langle u(t), v\rangle = \langle u(t_0), v\rangle$$

for all t_0 and $v \in L^2(\mathbb{T}^3)$. This ensures that as $t \to 0^+$, we have the weak convergence $u(t) \rightharpoonup u_0$. Moreover, for a given weak solution u the value of $u(t)$ is uniquely determined for every time $t > 0$.

Proof. We take $v \in \dot{L}^2(\mathbb{T}^3)$ because u has zero average and $\langle u(t), c\rangle = 0$ for every $c \in \mathbb{R}^3$. By the Helmholtz-Weyl decomposition of the torus, we write $v = h + \nabla g$ with $h \in H$ and $\nabla g \in G$. Because $u(t) \in H(\mathbb{T}^3)$ and ∇g are orthogonal, we obtain $\langle u(t), v\rangle = \langle u(t), h\rangle$. Thus, we only need to show that for all $h \in H(\mathbb{T}^3)$, we have

$$\lim_{t\to t_0} \langle u(t), h\rangle = \langle u(t_0, h)\rangle.$$

We assume without loss of generality that $h \in C^\infty_{c,\sigma}(\mathbb{T}^3)$, which is dense in $H(\mathbb{T}^3)$. Lemma 5.11 implies

$$|\langle u(t) - u(t_0), h\rangle| \le \nu \left| \int_t^{t_0} \langle \nabla u(t), \nabla h\rangle \right| + \left| \int_t^{t_0} \langle (u\cdot\nabla)u, h\rangle \right| \to 0$$

as t approaches zero. The second integral is well defined due to Lemma 5.9. □

5.1. **Alternative formulation.** We notice that it also might be difficult to verify condition (b) in Definition 5.7 for times $s > 0$. To this end, we introduce a different condition and prove that both are in fact equivalent. This condition also involves a different yet equivalent space of test functions.

Definition 5.15. The space $\tilde{\mathcal{D}}_\sigma(\mathbb{T}^3)$ is given by

$$\tilde{\mathcal{D}}_\sigma(\mathbb{T}^3) := \left\{ \varphi : \varphi = \sum_{k=1}^{N} \alpha_k(t) a_k(x), \alpha_k \in C^1_c([0,\infty)), a_k \in \mathcal{N}, N \in \mathbb{N} \right\},$$

where $\mathcal{N}$ is the basis of $H(\mathbb{T}^3)$ that comprises eigenfunctions of the Stokes operator A from Theorem 4.13. Note that we only take finite linear combinations of a_k.

Proposition 5.16. *If $u \in L^\infty(0,T; H(\mathbb{T}^3)) \cap L^2(0,T; V(\mathbb{T}^3))$ for all $T > 0$ then the following two statements are equivalent:*
(a) u satisfies (5.8) for all $\varphi \in \mathcal{D}_\sigma(\mathbb{T}^3)$.
(b) u satisfies (5.8) for all $\varphi \in \tilde{\mathcal{D}}_\sigma(\mathbb{T}^3)$.

Proof. $(a) \implies (b)$: We assume without loss of generality that $\phi \in \tilde{\mathcal{D}}_\sigma$ can be written as $\phi(x,t) = \alpha(t)a(x)$, where we have $a \in \mathcal{N}$ and $\alpha \in C^1_c([0,\infty))$.

We find a sequence of smooth functions $(\alpha_n)_{n\in\mathbb{N}}$ with compact support in $[0, s{+}1)$ such that $\alpha_n \to \alpha$ in $C^1([0,s])$. Since $C^\infty_{c,\sigma}(\mathbb{T}^3)$ is dense in $V(\mathbb{T}^3)$, we can also find a sequence of functions $(\varphi_n)_{n\in\mathbb{N}} \subset C^\infty_{c,\sigma}(\mathbb{T}^3)$ such that $\varphi_n \to a$ in $\mathbb{W}^{1,2}(\mathbb{T}^3)$.

For each n, we define the function $\psi_n(x,t) := \alpha_n(t)\varphi_n(x) \in \mathcal{D}_\sigma$. By our assumptions, ψ_n satisfies (5.8) for all $n \in \mathbb{N}$. Moreover, we already have

$$\psi_n \to \phi \in C([0,s];V) \text{ and } \partial_t\psi_n \to \partial_t\phi \text{ in } L^2(0,T;L^2(\mathbb{T}^3)). \tag{5.17}$$

We pass to the limit as n tends to infinity. (5.17) tells us that

$$\int_0^s \langle u, \partial_t\psi_n\rangle \to \int_0^s \langle u, \partial_t\phi\rangle, \int_0^s \langle \nabla u, \nabla\psi_n\rangle \to \int_0^s \langle \nabla u, \nabla\phi\rangle, \tag{5.18}$$

$$\langle u(0), \psi_n(0)\rangle \to \langle u(0), \phi(0)\rangle, \text{ and } \langle\langle u(s), \psi_n(s)\rangle \to \langle u(s), \phi(s)\rangle. \tag{5.19}$$

With the Sobolev embedding $W^{1,2}(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$, we also have $\psi_n \to \phi$ in $C([0,s];L^6)$, which means $\psi_n \to \phi$ in $L^4(0,s;L^6)$. Then Lemma 5.9 gives us

$$\left|\int_0^s \langle (u\cdot\nabla)u, \psi_n - \phi\rangle\right| \le ||(u\cdot\nabla)u||_{L^{4/3}(0,s;6/5)}||\varphi_n - \phi||_{L^4(0,s;L^6)} \to 0. \tag{5.20}$$

(b) $\implies$ *(a)*: Let $\phi \in \mathcal{D}_\sigma$. Then $\phi(t) \in \mathbb{W}^{k,2} \cap V$ for every $k \in \mathbb{N}$ and $t > 0$, so we can express ϕ in terms of the eigenfunctions $a_k \in \mathcal{N}$ by

$$\phi(x,t) = \sum_{k=1}^{\infty} a_k(x)c_k(t).$$

We take $\psi_n := \sum_{k=1}^{n} a_k(x)c_k(t) \in \tilde{\mathcal{D}}_\sigma$, which fulfils $\psi_n \to \phi$ in $C([0,s],V)$. As n approaches infinity, we achieve the following upper bounds

$$\begin{aligned}
\sup_{0\le t\le s} ||\phi(t) - \psi_n(t)||_V^2 &= \sup_{0\le t\le s} \left|\left| \sum_{k=n+1}^{\infty} c_k(t)a_k(x)\right|\right|_V^2 \\
&\le \sup_{0\le t\le s} \sum_{k=n+1}^{\infty} \frac{\lambda_k^2 c_k^2(t)}{\lambda_n} \text{ (because } \lambda_k \ge \lambda_n) \\
&\le \frac{1}{\lambda_n} \sup_{0\le t\le s} ||A\phi(t)||^2 \\
&\le \frac{1}{\lambda_n} \sup_{0\le t\le s} ||\phi(t)||_{W^{2,2}}^2,
\end{aligned}$$

which approaches zero as n tends to infinity. Because we have $\partial_t\psi_n \to \partial_t\phi$ in $L^2(0,T;L^2)$,(5.18), (5.19) and (5.20) hold as well. $\square$

Lemma 5.21. *Let u be a weak solution and take $\alpha \in W^{1,2}(0,T)$ and $a \in \mathcal{N}$. Then for almost all $s \in [0,T]$ and $\varphi(x,t) = \alpha(t)a(x)$, (5.8) holds.*

Proof. There exists a weak derivative $g = \frac{d\alpha}{dt} \in L^2(0,T)$ such that

$$\alpha(t) = c_0 + \int_0^t g(s)ds.$$

Given $\epsilon > 0$, let g_ϵ denote a mollification of g (see Appendix C of [9]). We define

$$\alpha_\epsilon(t) := c_0 + \int_0^t g_\epsilon(s)ds \text{ and } \varphi_\epsilon := \alpha_\epsilon(t)a(x).$$

Then the function φ_ϵ belongs to $\tilde{\mathcal{D}}_\sigma$. As ϵ approaches zero, we know that $\frac{d\alpha_\epsilon}{dt} - \frac{d\alpha}{dt} = g_\epsilon - g$ tends to zero in $L^2(0,T)$. As a result, we have

$$\sup_{0\le t\le T} |\alpha_\epsilon(t) - \alpha(t)| = \sup_{0\le t\le T}\left|\int_0^t (g_\epsilon - g)\right| \le \int_0^T |g_\epsilon - g| \to 0.$$

Hence the function φ_ϵ tends to φ in $C([0,T];W^{1,2})$. The claim follows from the forward direction of Proposition 5.16. ☐

Proposition 5.22. *A function $u \in L^\infty(0,T;H(\Omega) \cap L^2(0,T;V(\Omega)$ for all $T > 0$ and an initial condition u_0 is a weak solution if and only if for all $\varphi \in \tilde{\mathcal{D}}_\sigma$, we have*

$$\int_0^\infty \Big(-\langle u, \partial_t \varphi\rangle + \nu\langle \nabla u, \nabla\varphi\rangle + \langle (u\cdot\nabla)u, \varphi\rangle \Big) = \langle u_0, \varphi(0)\rangle. \tag{5.23}$$

Proof. ($\Rightarrow$) We take some arbitrary $\varphi \in \tilde{\mathcal{D}}_\sigma$. Proposition 5.16 tells us that (5.8) is valid for φ for almost all $s > 0$. Because φ has compact time support, we know $\varphi(s) = 0$ for sufficiently large s.

($\Leftarrow$) Let us take some time $s > 0$. We assume without loss of generality that $\varphi \in \tilde{\mathcal{D}}_\sigma$ is given by $\varphi(x,t) = \alpha(t)a_k(x)$, where $a_k \in \mathcal{N}$. For $h > 0$, we define a piecewise function by

$$\alpha_h(t) := \begin{cases} \alpha(t) & t \in [0,s), \\ \alpha(s)[1 - \frac{t-s}{h}] & t \in (s, s+h), \text{ and} \\ 0 & t \in [s+h, \infty). \end{cases}$$

We see that α_h is continuous with compact support because it decays linearly from s to $s+h$. We get a corresponding sequence of functions φ_n by

$$\varphi_n(x,t) = \alpha_{1/n}(t)a_k(x).$$

Due to $\alpha_{1/n} \in W^{1,2}$, Lemma 5.21 tells us that (5.23) is valid for all φ_n. We now pass it to the limit as n approaches infinity. For the second term, we have

$$\begin{aligned} \Big| \int_s^\infty \langle \nabla u, \nabla\varphi_n\rangle \Big| &\le \Big(\int_s^{s+1/n} ||\nabla u||^2 \Big)^{1/2} \Big(\int_s^{s+1/n} ||\nabla\varphi_n||^2 \Big)^{1/2} \\ &\le ||u||_{L^2(0,s+1;W^{1,2})} |\alpha(s)| ||\nabla a_k||_{L^2} n^{-1/2} \to 0. \end{aligned}$$

We apply a similar argument for the third term because

$$\begin{aligned} \Big| \int_s^\infty \langle \nabla u, \nabla\varphi_n\rangle \Big| &\le \Big(\int_s^{s+1/n} ||(u\cdot\nabla)u||_{L^{6/5}}^{4/3} \Big)^{3/4} \Big(\int_s^{s+1/n} ||\varphi_n||_{L^6}^4 \Big)^{1/4} \\ &\le ||(u\cdot\nabla)u||_{L^{4/3}(0,s+1;L^{6/5})} |\alpha(s)| ||\nabla a_k||_{L^6} n^{-1/4} \to 0. \end{aligned}$$

Since we have $\varphi_n = \varphi$ for times in $[0,s]$, we deduce

$$\int_0^\infty \Big(\nu\langle \nabla u, \nabla\varphi_n\rangle + \langle (u\cdot\nabla)u, \varphi_n\rangle \Big) \to \int_0^s \Big(\nu\langle \nabla u, \nabla\varphi\rangle + \langle (u\cdot\nabla)u, \varphi\rangle \Big).$$

We now examine the convergence of the time derivative in the first term

$$\begin{aligned} \int_0^\infty \langle u(t), \partial_t\varphi_n(t)\rangle dt &= \int_0^s \langle u(t), \partial_t\varphi_n(t)\rangle dt + \int_s^{s+\frac{1}{n}} \langle u(t), \partial_t\varphi_n(t)\rangle dt \\ &= \int_0^s \langle u(t), \partial_t\varphi_n(t)\rangle dt - n\alpha(s) \int_s^{s+\frac{1}{n}} \langle u(t), a_k(t)\rangle dt. \end{aligned}$$

By the Lebesgue Differentiation Theorem (see Appendix E, Theorem 6 of [9]), the last term on the right hand side converges to $\alpha(s)\langle u(s), a_k\rangle = \langle u(s), \varphi(s)\rangle$ whenever

s is a Lebesgue point of the function $\langle u(t), a_k\rangle$. Because there are only countably many functions $\langle u(t), a_k\rangle$, we have for almost all $s > 0$ and all $\varphi \in \tilde{\mathcal{D}}_\sigma$

$$\int_0^s \Big(-\langle u, \partial_t \varphi\rangle + \nu\langle \nabla u, \nabla \varphi\rangle + \langle u\cdot\nabla u, \varphi\rangle \Big) = \langle u_0, \varphi(0)\rangle - \langle u(s), \varphi(s)\rangle,$$

which meets condition (b) of Proposition 5.16. □

6. Galerkin approximations

We build a weak solution to the Navier-Stokes equations by constructing solutions within finite n-dimensional spaces before passing them to a limit. This is known more generally as the *Faedo-Galerkin method*. For further examples with second-order parabolic and hyperbolic equations, refer to Chapter 7.1.2 and Chapter 7.2.2 of [9] respectively. In our present case, we define the projection operator $P_n : L^2 \to H$ by

$$P_n u := \sum_{i=1}^n \langle u, a_i\rangle a_i, \text{ where } a_i \in \mathcal{N}.$$

We consider a sequence of approximate solutions u_n belonging to the finite-dimensional space P_nH spanned by the first n eigenfunctions of the Stokes operator A,

$$P_nH := \operatorname{span}\{a_1, a_2, \ldots, a_n\}, a_i \in \mathcal{N}.$$

We also see that $\langle P_n u, v\rangle = \langle u, P_n v\rangle$.

Definition 6.1. The nth order *Galerkin equation* corresponding to the Navier-Stokes equations is given by

$$\partial_t u_n + Au_n + P_n[(u_n\cdot\nabla)u_n] = 0, \tag{6.2}$$

$$u_n(0) = P_n u_0. \tag{6.3}$$

We call u_n the *Galerkin approximations.*

If we take the dot product of the Galerkin equation with some test function $\varphi \in \tilde{\mathcal{D}}_\sigma$, we can integrate in space and time for

$$\langle \partial_t u_n, \varphi\rangle + \langle \nabla u_n, \nabla\varphi\rangle + \langle (u_n\cdot\nabla)u_n, \varphi\rangle = 0 \text{ and} \tag{6.4}$$

$$-\int_0^\infty \langle u_n, \partial_t\varphi\rangle + \int_0^\infty \langle \nabla u_n, \nabla\varphi\rangle + \int_0^\infty \langle (u_n\cdot\nabla)u_n, \varphi\rangle = \langle u_0, \varphi(0)\rangle. \tag{6.5}$$

In Section 7, we aim to show that (6.5) holds even when we take the limit as n approaches infinity. To this end, we introduce the Aubin-Lions lemma which follows from Theorem 3.3.

Lemma 6.6. *(Aubin-Lions) Assume that $p, q > 1$ and*

$$||u_n||_{L^q(0,T;V)} + ||\partial_t u_n||_{L^p(0,T;V^*)} \le C \textit{ for all } n \in \mathbb{N},$$

where V^* denotes the dual space of V. Then there exists $u \in L^q(0, T; H)$ and a subsequence of u_n such that

$$u_{n_j} \to u \textit{ strongly in } L^q(0, T; H).$$

Proof. Given $j \in \mathbb{N}$, we consider the map $t \mapsto \langle u_n(t), a_j\rangle$, where $a_j \in \mathcal{N}$. From Lemma 3.22, this mapping is absolutely continuous almost everywhere on $[0,T]$ with weak derivative $\langle \partial_t u_n, a_j\rangle$. The inner product is well defined since we have $\partial_t u_n \in L^p(0,T;V^*)$ and $a_j \in V$. For almost all $s, s' \in [0,T]$, we get

$$\langle u_n(s), a_j\rangle = \langle u_n(s'), a_j\rangle + \int_{s'}^{s} \langle \partial_t u_n, a_j\rangle.$$

By the mean value theorem for integrals, there also exists some time $s^* \in [0,T]$ such that

$$\langle u_n(s^*), a_j\rangle = \frac{1}{T}\int_0^T \langle u_n, a_j\rangle.$$

Because we have $||a_j||_{L^2} = 1$ and

$$||a_j||_V^2 = \int_{\mathbb{T}^3} \nabla a_j \cdot \nabla a_j = -\int_{\mathbb{T}^3} a_j \cdot \Delta a_j = \lambda_j \int_{\mathbb{T}^3} |a_j|^2 = \lambda_j,$$

we obtain the following estimate

$$\begin{aligned}
\sup_{0\le s\le T} |\langle u_n(s), a_j\rangle| &\le |\langle u_n(s^*), a_j\rangle| + \left|\int_{s^*}^{s} \langle \partial_t u_n, a_j\rangle\right| \\
&\le \frac{1}{T}\int_0^T ||u_n||_{L^2}||a_j||_{L^2} + \int_0^T ||\partial_t u_n||_{V^*}||a_j||_V \\
(*)\qquad &\le \frac{1}{T}||u_n||_{L^q(0,T;H)}T^{1-1/q} + ||\partial_t u_n||_{L^p(0,T;V^*)}T^{1-1/p}\sqrt{\lambda_j} \\
(**)\qquad &\le \tilde{C}||u_n||_{L^q(0,T;V)}T^{-1/q} + ||\partial_t u_n||_{L^p(0,T;V^*)}T^{1-1/p}\sqrt{\lambda_j} \\
&\le C_1 + C_2\sqrt{\lambda_j}.
\end{aligned}$$

We apply Hölder's inequality and Theorem 3.19 for (*) as well as Theorem 3.14 for (**). We obtain the uniform bounds C_1, C_2 via our assumptions.

We verify that $P_k u_n = \sum_{j=1}^n \langle u_n, a_j\rangle a_j$ belongs to $C([0,T];H)$ with the bound

$$\sup_{0\le s\le T} ||P_k u_n(s)||_H \le \sum_{j=1}^{k}(C_1 + C_2\sqrt{\lambda_j}) \le k(C_1 + C_2\sqrt{\lambda_k}) \tag{6.7}$$

because (λ_j) is an increasing sequence.

We claim that for any given $k \in \mathbb{N}$, the sequence $(P_k u_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $C([0,t];P_kH)$. Since we have uniform bounds on the sequence $(P_k u_n)$ within a finite-dimensional space P_kH from (6.7), there is a compact set $K \subset P_kH$ such that the image $P_k u_n([0,T])$ is contained in K. We also have the equicontinuity of $(P_k u_n)$ with

$$\begin{aligned}
||P_k u_n(t_2) - P_k u_n(t_1)||_H &= \left|\left|\int_{t_1}^{t_2} \partial_t P_k u_n(s)ds\right|\right|_H \\
&\le \int_{t_1}^{t_2} ||\partial_t P_k u_n(s)||_H ds \\
&\le c_k \int_{t_1}^{t_2} ||\partial_t P_k u_n(s)||_{V^*} ds \text{ (equivalence of all norms)} \\
&\le \ c_k C|t_2 - t_1|^{1-1/p},
\end{aligned}$$

again via Hölder's inequality and Theorem 3.19. Then, Theorem 3.3 gives a desired convergent subsequence. By a diagonal method, we find and relabel at each step a subsequence (u_n) such that for every k, the sequence $(P_k u_n)_{n\in\mathbb{N}}$ converges in $L^q(0,T;H)$.

All that remains is to find a Cauchy subsequence of $(u_n)_{n\in\mathbb{N}}$ in $L^q(0,T;H)$. In other words, our goal is to show that for each $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have

$$\int_0^T ||u_n(s) - u_m(s)||^q ds < \epsilon.$$

We claim that for any $\delta > 0$, there exists some $k \in \mathbb{N}$ such that $n \geq k$ implies

$$\int_0^T ||P_k u_n(s) - u_n(s)||^q ds < \delta. \tag{6.8}$$

Since we have $||u_n||_V = ||A^{1/2} u_n||$, we know the constant C is at least

$$\int_0^T ||\nabla u_n||^q = \int_0^T \left(\sum_{j=1}^{\infty} \lambda_j |\langle u_n(s), a_j \rangle|^2 \right)^{\frac{q}{2}} ds \geq \int_0^T \left(\sum_{j=k+1}^{\infty} \lambda_j |\langle u_n(s), a_j \rangle|^2 \right)^{\frac{q}{2}} ds.$$

Because (λ_j) is increasing, we obtain

$$C \geq \lambda_{k+1}^{q/2} \int_0^T \left(\sum_{j=k+1}^{\infty} |\langle u_n(s), a_j \rangle|^2 \right)^{\frac{q}{2}} ds \geq \lambda_{k+1}^{q/2} \int_0^T ||P_k u_n(s) - u_n(s)||^q ds.$$

Given some $k \in \mathbb{N}$ such that (6.8) holds, we know $(P_k u_n)_{n\in\mathbb{N}}$ is Cauchy in $L^q(0,T;H)$. Then there exists some $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, we have

$$\int_0^T ||P_k u_n(s) - P_k u_m(s)||^q ds < \delta.$$

For $n, m \geq \max\{k, N_0\}$, we apply the triangle inequality for

$$\begin{aligned} &||u_n - u_m||_{L^q(0,T;H)} \\ &\leq ||u_n - P_k u_n||_{L^q(0,T;H)} + ||P_k u_n - P_k u_m||_{L^q(0,T;H)} + ||P_k u_m - u_m||_{L^q(0,T;H)} \\ &< 3\delta^{1/q}. \end{aligned}$$

We pick $\delta = \epsilon^q / 3^q$ to complete the proof. ☐

7. Convergence to Leray-Hopf weak solutions

We conclude with the proof of the existence of Leray-Hopf weak solutions. Given an initial condition $u_0 \in H(\mathbb{T}^3)$, we show that there exists a global-in-time weak solution u which satisfies the requirements of Proposition 5.22.

Theorem 7.1. *Given $u_0 \in H(\mathbb{T}^3)$, we know that solutions of the Galerkin equations exist locally in time.*

Proof. We consider a sequence of problems

$$\partial_t u_n + A u_n + P_n[(u_n \cdot \nabla) u_n] = 0 \tag{7.2}$$

$$u_n(0) = P_n u_0, \tag{7.3}$$

for functions u_n. We start with an ansatz of the form

$$u_n(x,t) = \sum_{k=1}^{n} c_k^n(t) a_k(x), a_k \in \mathcal{N},$$

where $\{c_k^n\}$ are scalar functions of time.

In order to determine u_n, we seek an expression for the functions $c_1^n, c_2^n, \dots, c_n^n$. To accomplish this, we take the inner product of (7.2) in L^2 with a_k. Since we have $\langle P_n v, a_k\rangle = \langle v, a_k\rangle$ for every $v \in L^2$ and $1 \le k \le n$, this yields

$$\sum_{j=1}^{n} \langle \dot{c}_j^n(t) a_j, a_k\rangle + \sum_{j=1}^{n} \langle c_j^n(t) A a_j, a_k\rangle + \sum_{i,j=1}^{n} \langle (c_i^n(t) a_i \cdot \nabla) c_j^n(t) a_j, a_k\rangle = 0.$$

Because a_k are orthonormal eigenfunctions of the Stokes operator in L^2, we obtain a system of n ordinary differential equations for $k \in [n]$,

$$\dot{c}_k^n(t) = -\lambda_k c_k^n(t) - \sum_{i,j=1}^{n} B_{ijk} c_i^n(t) c_j^n(t), \text{ where } B_{ijk} = \langle (a_i \cdot \nabla) a_j, a_k\rangle, \tag{7.4}$$

$$c_k^n(0) = \langle u_0, a_k\rangle, \tag{7.5}$$

where we derive initial conditions from the inner product of (7.3) with a_k. Since the right hand side of (7.4) is continuous and locally Lipschitz, we apply the Picard-Lindelöf Theorem (see Theorem 7.3 of [4] for details). Then there exists a unique solution on some time interval $[0, T_n)$. □

Theorem 7.6. *There exist uniform estimates on the solutions u_n from Theorem 7.1. Hence, they exist globally in time.*

Proof. We know that the approximate solutions u_n exist at least on some time interval $[0, T_n)$. Our goal is to show that the functions c_k^n do not blow up for all times $T_n < \infty$. Let $s \in (0, T_n)$. We take the inner product of (7.2) with $u_n(s)$ for

$$\langle \partial_t u_n(s), u_n(s)\rangle + \langle A u_n(s), u_n(s)\rangle + \langle P_n(u_n(s) \cdot \nabla) u_n(s), u_n(s)\rangle = 0.$$

We notice that $\langle \partial_t u_n(s), u_n(s)\rangle = \frac{1}{2}\frac{d}{dt}||u_n(s)||^2$ and

$$\begin{aligned}\langle A u_n(s), u_n(s)\rangle &= \langle -\mathbb{P}\Delta u_n, u_n\rangle = \langle -\Delta u_n, \mathbb{P} u_n\rangle \\ &= \langle -\Delta u_n, u_n\rangle = ||\nabla u_n(s)||^2.\end{aligned}$$

The nonlinear term vanishes because Lemma 5.5 tells us that

$$\langle P_n[(u_n \cdot \nabla) u_n], u_n\rangle = \langle (u_n \cdot \nabla) u_n, P_n u_n\rangle = \langle (u_n \cdot \nabla) u_n, u_n\rangle = 0.$$

Therefore, we achieve the following energy estimate for all $s > 0$

$$\frac{1}{2}\frac{d}{dt}||u_n(s)||^2 + ||\nabla u_n(s)||^2 = 0. \tag{7.7}$$

In other words, the L^2 norm of u_n decreases monotonically with time. Since $\sum_{k=1}^{n} |c_k^n(s)|^2 = ||u_n(s)||^2$, it follows that c_k^n do not blow up in finite time and hence $T_n = \infty$. Moreover, if we integrate (5.6) in time between 0 and any $t \in [0, \infty)$ we obtain

$$\frac{1}{2}||u_n(t)||^2 + \int_0^t ||\nabla u_n(s)||^2 ds = \frac{1}{2}||u_n(0)||^2 \le \frac{1}{2}||u_0||^2,$$

so $\sup_{t\geq 0}||u_n(t)||^2 \leq ||u_0||^2$ and $\int_0^\infty ||\nabla u_n(s)||^2 ds \leq \frac{1}{2}||u_0||^2$. Hence, the function u_n is uniformly bounded with

$$\sup_{t\geq 0}||u_n(t)||^2 + \int_0^\infty ||\nabla u_n(s)||^2 ds \leq 2||u_0||^2. \tag{7.8}$$

in both $L^\infty(0,\infty;H)$ and $L^2(0,\infty;V)$. □

We now want to find bounds on $\partial_t u_n$ so that we can apply Lemma 6.6.

Theorem 7.9. *For all $T > 0$, the time derivative $\partial_t u_n$ has a uniform bound in the Bochner space $L^{4/3}(0,T;V^*)$.*

Proof. Assume $\varphi \in V$. We take the L^2 inner product of the Galerkin equation (7.2) with φ to get

$$\langle \partial_t u_n, \varphi\rangle = -\langle Au_n, \varphi\rangle - \langle P_n[(u_n\cdot\nabla)u_n], \varphi\rangle \tag{7.10}$$
$$= -\langle Au_n, \varphi\rangle - \langle (u_n\cdot\nabla)u_n, P_n\varphi\rangle. \tag{7.11}$$

We find estimates for the norm $||\partial_t u_n||_{V^*}$ by considering the right hand side of (7.11). We begin with the first term

$$\begin{aligned}|\langle Au_n,\varphi\rangle| &= |\langle -\mathbb{P}\Delta u_n,\varphi\rangle| \leq |\langle -\Delta u_n, \mathbb{P}\varphi\rangle| \\ &= |\langle -\Delta u_n,\varphi\rangle| = |\langle \nabla u_n, \nabla\varphi\rangle| \\ &\leq ||\nabla u_n||||\varphi||_V.\end{aligned}$$

We can also express an upper bound for the second term:

$$\begin{aligned}|\langle (u_n\cdot\nabla)u_n, P_n\varphi\rangle| &\leq ||u_n||_{L^3}||\nabla u_n||_{L^2}||P_n\varphi||_{L^6} \text{ (from Lemma 3.1)} \\ &\leq ||u_n||_{L^2}^{1/2}||u_n||_{L^6}^{1/2}||\nabla u_n||_{L^2}||P_n\varphi||_V \text{ (from Lemma 3.2)} \\ &\leq c||u_n||_{L^2}^{1/2}||\nabla u_n||_{L^2}^{3/2}||\varphi||_V \text{ (because } W_0^{1,2}(\mathbb{T}^3)\subset L^6(\mathbb{T}^3)).\end{aligned}$$

Therefore we obtain

$$||\partial_t u_n||_{V^*} \leq ||\nabla u_n|| + c||u_n||^{1/2}||\nabla u_n||^{3/2}.$$

We arrive at the following bound for all $n\in\mathbb{N}$

$$\begin{aligned}\int_0^T ||\partial_t u_n(s)||_{V^*}^{4/3} ds &\leq \int_0^T ||\nabla u_n(s)||^{4/3} ds + c\int_0^T ||u_n(s)||^{2/3}||\nabla u_n(s)||^2 ds \\ &\leq T^{1/3}||u_n||_{L^2(0,T;V)}^{4/3} + c||u_n||_{L^\infty(0,T;H)}^{2/3}||u_n(s)||_{L^2(0,T;V)}^2 \\ &\leq T^{1/3}||u_0||^{4/3} + c||u_0||^{8/3},\end{aligned}$$

where the last inequality follows from Theorem 7.6. □

With our uniform bounds on u_n from Theorems 7.6 and 7.9, we proceed to extract a convergent subsequence. At each step, we consider a subsequence of the previous subsequence. We also relabel each subsequence as u_n for simplicity of notation.

Proposition 7.12. *There exists a sequence u_n of Galerkin approximations and a function u such that we have*

- $u_n \to u$ *strongly in* $L^2(0,T;H(\mathbb{T}^3))$ *by Lemma 6.6,*
- $u_n \rightharpoonup$ *weakly-⋆ in* $L^\infty(0,T;H(\mathbb{T}^3))$ *by Theorem 3.4*
- $\nabla u_n \rightharpoonup \nabla u$ *weakly in* $L^2(0,T;L^2(\mathbb{T}^3))$ *by Corollary 3.5.*

Finally, we conclude by showing that u is a Leray-Hopf weak solution of the Navier-Stokes equations.

Theorem 7.13. *The function u from Proposition 7.12 satisfies the requirements of Proposition 5.22.*

Proof. Our goal is to check that for any test function $\varphi \in \tilde{\mathcal{D}}_\sigma$ given by $\varphi(x,t) = \alpha(t)a_N(x)$, we have

$$-\int_0^\infty \langle u, \partial_t\varphi\rangle + \int_0^\infty \langle \nabla u, \nabla\varphi\rangle + \int_0^\infty \langle (u\cdot\nabla)u, \varphi\rangle = \langle u_0, \varphi(0)\rangle\rangle. \tag{7.14}$$

Since $\alpha(t) \in C_c^\infty([0,\infty))$, we can pick some $T > 0$ such that the support of α is contained in $[0,T)$. Hence we can simplify our goal to the following equality

$$-\int_0^T \langle u, \partial_t\varphi\rangle + \int_0^T \langle \nabla u, \nabla\varphi\rangle + \int_0^T \langle (u\cdot\nabla)u, \varphi\rangle = \langle u_0, \varphi(0)\rangle\rangle. \tag{7.15}$$

We know from the Galerkin approximations in (6.5) that for all $n \geq N$, the function u_n satisfies

$$-\int_0^T \langle u_n, \partial_t\varphi\rangle + \int_0^T \langle \nabla u_n, \nabla\varphi\rangle + \int_0^T \langle (u_n\cdot\nabla)u_n, \varphi\rangle = \langle u_0, \varphi(0)\rangle\rangle. \tag{7.16}$$

There are three terms to examine on the left hand side. We already have

$$\int_0^T \langle u_n, \partial_t\varphi\rangle \to \int_0^T \langle u, \partial_t\varphi\rangle \text{ and } \int_0^T \langle \nabla u_n, \nabla\varphi\rangle \to \int_0^T \langle \nabla u, \nabla\varphi\rangle.$$

All that remains is the third term on the left hand side of (7.16), i.e. we only need to show that $\int_0^T \langle (u_n\cdot\nabla)u_n, \varphi\rangle$ converges to $\int_0^T \langle (u\cdot\nabla)u, \varphi\rangle$ as n approaches infinity. We have

$$\begin{aligned}
&\left|\int_0^T \langle (u_n\cdot\nabla)u_n, \varphi\rangle - \int_0^T \langle (u\cdot\nabla)u, \varphi\rangle\right| \\
&\leq \left|\int_0^T \langle ((u_n - u)\cdot\nabla)u_n, \varphi\rangle\right| + \left|\int_0^T \langle u\cdot\nabla(u_n-u), \varphi\rangle\right| \\
&\leq C_\varphi \int_0^T ||u_n - u||||\nabla u_n|| + \sum_{i,j\in[3]} \left|\int_0^T \langle \partial_j(u_n-u)_i, u_j\varphi_i\rangle\right| \\
&\leq C_\varphi \left(\int_0^T ||u_n-u||^2\right)^{\frac{1}{2}} \left(\int_0^T ||\nabla u_n||^2\right)^{\frac{1}{2}} + \sum_{i,j\in[3]} \left|\int_0^T \langle \partial_j(u_n-u)_i, u_j\varphi_i\rangle\right|.
\end{aligned}$$

We have a uniform bound on the norm $\int_0^T ||\nabla u_n||^2$ from (7.8) and we know from Proposition 7.12 that as n approaches infinity, we have

$$\int_0^T ||u_n - u||^2 \to 0 \text{ and } \left|\int_0^T \langle \partial_j(u_n-u)_i, u_j\varphi_i\rangle\right| \to 0.$$

Therefore, the entire expression converges to zero as desired. □

Today, the global existence of regular (or strong) solutions as well as the uniqueness of weak solutions in three dimensions remain open problems which inspire active research. Without a uniqueness result for weak solutions, it is possible that there are Leray–Hopf weak solutions that do not coincide with those constructed from the Galerkin method.

As a final note, we mention a few relevant, known results:

- While we eliminated the pressure term with divergence-free test functions, the pressure p can be recovered in the absence of boundaries.
- Weak solutions of the Navier-Stokes equations are unique in two dimensions.
- Leray-Hopf weak solutions obey an energy inequality similar to (5.6): for all times $t > 0$, we have
$$\frac{1}{2}||u(t)||^2 + \int_0^t ||\nabla u(s)||^2 ds \leq \frac{1}{2}||u(0)||^2.$$
- If the initial condition u_0 belongs to $W^{1,2}$, then there even exists a strong solution in $L^\infty(0, T; W^{1,2})$ and $L^\infty(0, T; W^{2,2})$ for some $T > 0$, i.e. strong solutions exist locally in time.

Among the many books that present the above material in greater detail are the works by Robinson, Rodrigo and Sadowski [3] as well as Boyer and Fabrie [5].

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