SCHEMES AND THE RIEMANN-ROCH THEOREM

OWEN KOLEAN

Abstract. The purpose of this paper is to build up to the notion of a scheme, and then use the scheme's properties and geometric behavior to prove the Riemann-Roch theorem, which has many important applications in algebraic geometry (such as providing a formula for computing the Hilbert polynomial of line bundles on a curve). This paper assumes familiarity with classical algebraic geometry (i.e. varieties), and will bridge an understanding of varieties to the more generalized concept of a scheme.

Contents

1. Sheaves

We first must define the sheaf, which will be a necessary ingredient for the definition of a scheme. A sheaf is made up of collections of functions assigned to open subsets of a topological space, and will therefore allow for the tracking of local data on a topological space.

Definition 1.1. Let X be a topological space. A presheaf $\mathcal{F}$ of Abelian groups on X consists of the data:

Date: August 21, 2023.

- For every open subset $U \subset X$, an abelian group $\mathcal{F}(U)$.
  Note: The elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.
- For every inclusion $V \subset U$ of open subsets of X, a morphism of abelian groups $p_{UW} = p_{VW} \circ p_{UV}$ (Restriction mapping).

With the additional conditions...

- $\mathcal{F}(\emptyset) = 0$, where $\emptyset$ is the empty set,
- p_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
- if $W \subset V \subset U$ are three open subsets, then $p_{UW} = p_{VW} \circ p_{UV}$

Now that we have defined the notion of a presheaf, we can add on a couple of extra conditions to get the definition of a sheaf which, roughly speaking, is a presheaf with sections that are determined by local data.

Definition 1.2. A sheaf is a presheaf $\mathcal{F}$ on a topological space X that satisfies the following additional conditions:

(III, Identity Axiom): if U is an open set, $\{V_i\}$ an open covering of U, and $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0\ \forall i$, then $s = 0$.

(IV, Gluability Axiom): if U is an open set, $\{V_i\}$ an open covering of U, and if we have $s_i \in \mathcal{F}(V_i)\ \forall i$, with the property that $\forall i, j$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i.

Definition 1.3. Let $\mathcal{F}$ be a presheaf on a topological space X, and let P be a point of X. Then we define the stalk $\mathcal{F}_p$ of $\mathcal{F}$ at P to be the direct limit of groups $\mathcal{F}(U)$ for all open sets U containing P.

The stalk gives a way of isolating the behavior of a sheaf around a specific point. Note that the limit of open sets around p is necessary, as points are not open sets, so we take increasingly small open sets containing the point to best describe the sheaf's behavior at the point.

Definition 1.4. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space X. Then a morphism $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U, such that whenever $V \subset U$ is an inclusion, the diagram below is commutative.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle p_{UV}}\downarrow & & \downarrow{\scriptstyle p'_{UV}} \\ \mathcal{F}(V) & \xrightarrow[\varphi(V)]{} & \mathcal{G}(V) \end{array}$$

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, the same definition holds. Additionally, an isomorphism is a morphism with a two sided inverse.

Example 1.5. Suppose X is a topological space with $p \in X$, and S is a set. Let $i_p : p \to X$ be the inclusion mapping. Then $i_{p,*}S$ defined below is called the skyscraper sheaf:

$$i_{p,*}S(U) = \begin{cases} S & p \in U \\ \{e\} & p \notin U \end{cases}$$

This sheaf is called the skyscraper sheaf, as the picture of the sheaf looks like a skyscraper at the point p.

Proposition 1.6. *The skyscraper sheaf is a sheaf.*

Proof. We will describe $i_{p,*}S(U)$ as a sheaf of functions $f : U \to S$ such that $f(u) = e$ for all $u \neq p$, where the restriction mappings are the usual restrictions of functions.

Suppose U is open, and $\{U_i\}_{i \in I}$ is an open cover of U. We first show that the identity axiom holds. Suppose we have $f_1, f_2 \in \mathcal{F}(U)$ and $p_{UU_i} f_1 = p_{UU_i} f_2$ $\forall i \in I$. Suppose for a contradiction that $f_1 \neq f_2$ or that the exist $t \in U_j$, $j \in I$ such that $f_1(t) \neq f_2(t)$. Then $f_1|_{u_i} \neq f_2|_{u_i}$, a contradiction.

We next prove that the gluability axiom holds. Let $\{U_i\}_{i \in I}$ be an open cover of U. Given $f_i \in \mathcal{F}(U_i)$ for all i such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall i, j \in I$, we can uniquely glue together a function $f : U \to S$ that agrees on these intersections. Then clearly $f(u) = e$ for all $u \neq p$. Hence, $f \in \mathcal{F}(U)$.

□

2. The Road to Schemes

This section of the paper will consist of the construction of schemes. When said and done, the scheme will contain three pieces: The set, the topology, and the structure sheaf. In the construction of the scheme, these pieces will be covered in this order.

2.1. **The Set.**

Definition 2.1. Let A be a ring. The spectrum of A, denoted $SpecA$, is the set of prime ideals of A (denoted $[p]$ for a prime ideal p). Elements $a \in A$ will be called functions on $SpecA$, and their value at a point $[p]$ will be $a(mod p)$.

Example 2.2. The function 11 on $Spec\mathbb{Z}$ takes on values 4 at [7] and 1 at [2].

Example 2.3. $Spec\mathbb{C}[x, y]$ consists of prime ideals of the form $[(x - a, y - b)]$, $a, b \in \mathbb{C}$, which exist in $1-1$ correspondence with elements $(a, b) \in \mathbb{A}^2_{\mathbb{C}}$.

Hence, constructing the complex plane is possible by considering prime ideals of the 2-dimensional polynomial ring with coefficients in the complex numbers. Note that the same does not hold for $\mathbb{R}$, as $\mathbb{R}$ is not an algebraically closed field, and thus its polynomial ring contains polynomials of higher degrees which are irreducible.

2.2. **The Topology.** We will define the Zariski topology in terms of closed subsets of $SpecA$ for an arbitrary ring A. Let S be a subset of A. Define the vanishing set of S by $\mathbb{V}(S) := \{[\mathfrak{p}] \in SpecA | S \subset \mathfrak{p}\}$. We claim that these closed sets form a topology on A

Proof. We first prove that $\emptyset$ and $SpecA$ are open subsets of $SpecA$. Note that this follows from the fact that $\mathbb{V}(A) = \emptyset$ and $\mathbb{V}((0)) = SpecA$.

The next step is to show that if $\mathfrak{a}$ and $\mathfrak{b}$ are prime ideals of A, then $\mathbb{V}(\mathfrak{ab}) = \mathbb{V}(\mathfrak{a}) \cup \mathbb{V}(\mathfrak{b})$. Let $\mathfrak{p} \in \mathbb{V}(\mathfrak{a}) \cup \mathbb{V}(\mathfrak{b})$. Without loss of generality, we assume $\mathfrak{p} \in \mathbb{V}(\mathfrak{a})$. Then $\mathfrak{a} \subset \mathfrak{p}$, which implies $\mathfrak{ab} \subset \mathfrak{p}$. For the other direction, suppose, $\mathfrak{p} \in \mathbb{V}(\mathfrak{ab})$, or that $\mathfrak{ab} \subset \mathfrak{p}$. Suppose, for a contradiction, that $\mathfrak{a}$ is not contained in $\mathfrak{p}$. Then $\exists a \in \mathfrak{a}$ such that $a \notin \mathfrak{p}$. But then for any arbitrary $b \in \mathfrak{b}$ it must be the case that $ab \in \mathfrak{p}$, which is a prime ideal, so $b \in \mathfrak{p}$. This implies $\mathfrak{b} \subset \mathfrak{p}$.

Lastly, we prove that if $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $\mathbb{V}(\sum \mathfrak{a}_i) = \bigcap \mathbb{V}(\mathfrak{a}_i)$. This follows from the fact that any prime ideal $\mathfrak{p}$ contains $\sum \mathfrak{a}_i$ if and only if $\mathfrak{p}$ contains each $\mathfrak{a}_i$, due to the definition of $\sum \mathfrak{a}_i$ as being the smallest ideal containing all of the ideals $\mathfrak{a}_i$.

□

Definition 2.4. If $f \in A$ define the distinguished open set $D(f) := \{[\mathfrak{p}] \in SpecA | f \notin \mathfrak{p}\}$.

Definition 2.5. Let X be a topological space. A point $p \in X$ is a closed point if $\{p\}$ is a closed subset of X.

Recall from Example 2.3 that each point of the complex plane is a closed point, as each corresponds to an irreducible polynomial, and thus a prime ideal.

2.3. **The Structure Sheaf and Definition of a Scheme.** We will now define the notion of a structure sheaf, which will act on distinguished open subsets of $SpecA$ for some ring A. Define $\mathcal{O}_{SpecA}(D(f))$ to be the localization of A at the multiplicative set of functions that do not vanish outside of $\mathbb{V}(f)$. As a brief example, note that $\mathcal{O}_{SpecA}(\emptyset) = \{0\}$

Definition 2.6. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and the structure sheaf $\mathcal{O}_X$ on X. A ringed space is a locally ringed space if the stalk $\mathcal{O}_{X,P}$ is a local ring for each point $P \in X$.

Definition 2.7. An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of some ring. A scheme is a locally ringed space in which every point in X has an open neighborhood U such that the topological space U, together with the restriction sheaf $\mathcal{O}_{X|U}$, is an affine scheme.

Example 2.8. Let k be an algebraically closed field. Define $\mathbb{A}^1_k := Speck[x]$ to be the affine line.

Note that closed points of $\mathbb{A}^1_k$ are in $1-1$ correspondence with elements of k, as each element of k corresponds to an irreducible degree one polynomial, which are the prime ideals of $Speck[x]$.

Example 2.9. We can extend Example 2.8 to n dimensions to get $\mathbb{A}^n_k := Speck[x_1, ..., x_n]$. Interestingly, this scheme is homeomorphic to $\mathbb{A}^n$, the affine variety.

3. Schemes

3.1. **Projective Schemes.** Projective schemes can be constructed by gluing together affine schemes, but we will approach their construction in a different way given the difficulty of keeping track of gluing. This construction of schemes will be more algebraic, and will focus on carving them out of graded rings.

Definition 3.1. A $\mathbb{Z}$-graded ring is a ring $S_\bullet = \bigoplus_{n\in\mathbb{Z}} S_n$ where multiplication sends $S_n \times S_m$ to S_{n+m}. Additionally, elements of S_n are called homogeneous elements of $S_\bullet$ and the subset $S_+ := \bigoplus_{i>0} S_i \subset S_\bullet$ is called the irrelevant ideal.

Definition 3.2. Similar to how we constructed $SpecA$ by considering its points and then imbuing a topology and a structure sheaf, we will now construct $ProjS_\bullet$ in an analogous way:

-The Set: The points of $ProjS_\bullet$ will be the homogeneous prime ideals of $S_\bullet$ not containing the irrelevant ideal S_+.

-The Topology: If T is a set of homogeneous elements of $S_\bullet$ of positive degree, define $\mathbb{V}(T) \subset ProjS_\bullet$ to be the set of homogeneous prime ideals containing T but not S_+. This vanishing set will be the closed set for the topology. Additionally, the projective distinguished open subsets are defined, for homogeneous f, as $D(f) := ProjS_\bullet \backslash \mathbb{V}(f)$.

-The Structure Sheaf: The structure sheaf sends each distinguished open subset $D(f)$ to the subset $Spec(((S_\bullet)_f)_0)$, which represents the rational functions with homogeneous numerator and some power of f as the denominator, which has the same degree as the numerator. In other words, they are the zero graded pieces of the graded ring $S_\bullet$, localized at f.

Definition 3.3. A scheme of the form $ProjS_\bullet$, where $S_\bullet$ is a finitely generated graded ring over A, is called a projective A-scheme.

Example 3.4. Let k be an algebraically closed field. Then the variety $\mathbb{P}^n_k$ is homeomorphic to the subspace of closed points of $Projk[x_0, ..., x_n]$.

3.2. **Sheaves of Modules.**

Definition 3.5. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules (also known as an $\mathcal{O}_X$-module), is a sheaf $\mathcal{F}$ on X such that for each open set $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X$-module, and for each inclusion of open sets $V \subset U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Definition 3.6. A sheaf of ideals on X is a sheaf of modules $\mathcal{J}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open U, $\mathcal{J}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Definition 3.7. An $\mathcal{O}_X$-module $\mathcal{F}$ is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$-module. Additionally, the rank of $\mathcal{F}$ is the number of copies of the structure sheaf needed. An invertible sheaf is a locally free sheaf of rank one.

3.3. **Quasicoherent Sheaves.** In this section, we will briefly define what a quasicoherent sheaf is. Much like how you glue together rings to form a scheme, constructing a quasicoherent sheaf involves gluing together modules over those rings. The concept of a quasicoherent sheaf will be necessary for when we later define the Čech cohomology and look at the geometry of schemes.

Definition 3.8. Let M be an A-module. Define the sheaf $\tilde{M}$ such that for distinguished open subsets $D(f)$, $\tilde{M}(D(f))$ is the localization of M at the set of functions which do not vanish outside of f (i.e. $1, f, f^2, ...$).

Definition 3.9. Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if X can be covered by affine open subsets U_i such that for each affine subset $U_i = SpecA_i \subset X$, $\mathcal{F}|_{SpecA_i} \cong \tilde{M_i}$ for some A_i-module M. Further, we say $\mathcal{F}$ is coherent if M_i is a finitely generated A_i-module.

4. The Geometry of Schemes

4.1. **Divisors.** In this section, we will define two types of divisors, which are useful for studying the intrinsic geometry of a scheme. We will first define Weil divisors (which come with a clearer geometric intuition) and then move on to Cartier divisors (which can be used for more general schemes).

Definition 4.1. Let X be a scheme. A Weil divisor is a formal $\mathbb{Z}$-linear combination of codimension 1 irreducible closed subsets of X. Additionally, only a finite number of the integer coefficients of the closed subsets can be non-zero.

A Weil divisor can also be written in the form $\sum_{Y \subset X} n_Y[Y]$, where X is codimension 1.

Example 4.2. We will later define a curve for the statement of the Riemann-Roch Theorem, but for now you can think of it as a dimension 1 scheme with some added conditions. In this case, the Weil divisors for a curve would be linear combinations of points, as codimension 1 subsets of a 1-dimensional scheme would be 0-dimensional.

Definition 4.3. Let X be a scheme. For each open affine subset $U = SpecA$, let S denote the set of elements of A which are not zero divisors, and let $K(U)$ be the localization of A by the multiplicative system S. $K(U)$ is called the total quotient ring of A. For open U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}$ we call the sheaf of total quotient rings of $\mathcal{O}$. We denote by $\mathcal{K}^*$ the sheaf of invertible elements in the sheaf of rings $\mathcal{K}$. Similarly, $\mathcal{O}^*$ is the sheaf of invertible elements in $\mathcal{O}$.

Definition 4.4. A Cartier Divisor on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. A Cartier divisor on X can be described by giving an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each i, j, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$.

Definition 4.5. A Cartier divisor on a scheme X is effective if it can be represented by $\{(U_i, f_i\}$, where all the $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. We then define the associated subscheme of codimension 1, Y, to be the closed subscheme defined by the sheaf of ideals $\mathcal{J}$, which is locally generated by f_i.

4.2. **Invertible Sheaves.**

Definition 4.6. Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{L}(D)$ of the sheaf of total quotient rings $\mathcal{K}$ by taking $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i. We call $\mathcal{L}(D)$ the sheaf associated to D.

Proposition 4.7. *Let X be a scheme. Then:*

(a) For any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a bijection between Cartier divisors on X and invertible subsheaves of $\mathcal{H}$.

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

Proof. (a) Let D be a Cartier divisor. Then D can be described as $\{(U_i, f_i)\}$ such that for each $f_i \in \Gamma(U_i, \mathcal{K}^*)$, the map $\mathcal{O}_{U_i} \to \mathcal{L}(D)|_{U_i}$ defined by $1 \mapsto f_i^{-1}$ is an isomorphism. Note that because $\mathcal{L}(D)$ can be covered by open sets $\{U_i\}$ such that $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}_{U_i}$, $\mathcal{L}(D)$ is locally free sheaf of rank 1, so it is an invertible sheaf.

Further, we can recover the Cartier divisor D from $\mathcal{L}(D)$ by taking f_i on U_i to be the inverse of a local generator of $\mathcal{L}(D)$.

(b) Let D_1 and D_2 be Cartier divisors. Then for each f_i, g_i that locally define D_1 and D_2 respectively, it follows that $f_i^{-1} g_i$ locally generates $\mathcal{L}(D_1 - D_2)$, which implies $\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1) \cdot \mathcal{L}(D_2)^{-1}$, which is isomorphic to the tensor product $\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

□

Proposition 4.8. *Let D be an effective Cartier divisor on a scheme X, and let Y be the associated locally principal closed subscheme. Then $\mathcal{J}_Y \cong \mathcal{L}(-D)$.*

Proof. $\mathcal{L}(-D)$ is locally generated by f_i. Note that by the definition of $\mathcal{J}_Y$ being locally generated by f_i as well, we have that $\mathcal{J}_Y \cong \mathcal{L}(-D)$. □

Definition 4.9. A scheme X is integral if for all open $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.

Before we can present the definition of a curve, we must lastly define the condition of separatedness for a scheme. This condition is analagous to the Hausdorff condition for manifolds which, through its exclusion of spaces like the real line with doubled origin, proves to be extremely useful.

Definition 4.10. A morphism $\pi : X \to Y$ is separated if the diagonal morphism $\delta_\pi : X \to X \times_Y X$ is a closed embedding.

If the morphism π defined above has the property of separatedness, then the scheme Y also has the property of separatedness. This formality of defining separatedness of schemes by the morphisms between them may seem awkward, but is an instance of a trend established by Grothendiek to describe properties of schemes not by the objects themselves, but by mappings between them.

It can be shown that all morphisms between affine schemes are separated. Additionally, as the above exposition would suggest, the line with doubled origin is not separated.

Definition 4.11. Let k be an algebraically closed field. A curve over k is an integral separated scheme X of finite type over k, of dimension 1. If X is proper over k, we say that X is complete.

For this paper, we will additionally add the conditions that a curve is also complete and nonsingular over k.

4.3. **The Čech Cohomology.**

Definition 4.12. Suppose X is a quasi-compact and separated topological space. Additionally, suppose $\mathcal{F}$ is a quasicoherent sheaf, and $\mathcal{U} = \{U_i\}_{i=1}^n$ is a finite collection of affine open sets covering X. Then define the Čech complex as...

$$0 \to \prod_{|I|=1, I\subset\{1,\dots,n\}} \mathcal{F}(\bigcap_{i\in I} U_i) \to \dots$$
$$\to \prod_{|I|=i, I\subset\{1,\dots,n\}} \mathcal{F}(\bigcap_{i\in I} U_i) \to \prod_{|I|=i+1, I\subset\{1,\dots,n\}} \mathcal{F}(\bigcap_{i\in I} U_i) \to \dots$$

where we denote $\alpha_{i_0,\dots,i_p} \in \mathcal{F}(U_{i_0,\dots,i_p})$, and we define the coboundary map by setting $(d\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k \alpha_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}}$.

Proposition 4.13. *The Čech complex is a complex.*

Proof. In order to show the Čech complex is actually a complex, we have to show that $d^2 = 0$.

$$\begin{aligned}
(d(ds))_{i_0,\dots,i_{p+2}} &= \sum_{k=0}^{p+2}(-1)^k (ds)_{i_0,\dots,\hat{i}_k,\dots,i_{p+2}}|_{U_{i_0,\dots,i_{p+2}}} \\
&= \sum_{k=0}^{p+2}(-1)^k (\sum_{l=0, l\neq k}^{p+2} (-1)^l s_{i_0,\dots,\hat{i}_l,\dots,\hat{i}_k,\dots,i_{p+2}}|_{U_{i_0,\dots,\hat{i}_k,\dots,i_{p+2}}})|_{U_{i_0,\dots,i_{p+2}}} \\
&= \sum_{k=0}^{p+2}(-1)^k (\sum_{l=0}^{k-1}(-1)^{k+1} s_{i_0,\dots,\hat{i}_l,\dots,\hat{i}_k,\dots,i_{p+2}} + \sum_{l=k+1}^{p+2} (-1)^{k+(l-1)} s_{i_0,\dots,\hat{i}_k,\dots,\hat{i}_l,\dots,i_{p+2}}) \\
&= 0
\end{aligned}$$

□

Definition 4.14. for the open cover $\mathcal{U}$ as defined above, define $H^i_{\mathcal{U}}(X, \mathcal{F})$ to be the ith cohomology group of the Čech complex defined above.

Note that because X was chosen to be a quasicompact and separated topological space, $H^i_{\mathcal{U}}(X, \mathcal{F})$ is independent of the choice of covering, so we will instead use the notation $H^i(X, \mathcal{F})$ to refer to the ith cohomology group of the complex.

Proposition 4.15. *Let X be a projective variety over a field k. Then $H^0(X, \mathcal{O}_X) = k$*

Proof. Omitted

□

Definition 4.16. Let X be a projective scheme over a field k, and let $\mathcal{F}$ be a coherent sheaf on X. We define the Euler characteristic of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum(-1)^i dim_k H^i(X, \mathcal{F}).$$

Proposition 4.17. *Suppose $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of coherent sheaves on a projective k-scheme X. Then $\chi(X, \mathcal{G}) = \chi(X, \mathcal{F}) + \chi(X, \mathcal{H})$.*

Proof. Omitted.

□

Definition 4.18. Let X be a projective scheme of dimension r over a field k. We define the arithmetic genus p_a of X by

$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$.

Note: The arithmetic genus is also known as the genus, and which is denoted g.

Lemma 4.19. *Let X be a nonsingular curve over k with a function field K. Then X is projective if and only if X is complete.*

Proof. Omitted □

Proposition 4.20. *Let X be a curve. Then $p_a(X) = dim_k H^1(X, \mathcal{O}_X)$.*

Proof. Note that since X is complete (we've defined a curve to include completeness), it is also projective (Lemma 4.19). Thus, the Euler characteristic is:

$$\begin{aligned}\chi(X) &= dim_k H^0(X, \mathcal{O}_X) - dim_k H^1(X, \mathcal{O}_X) \\ &= dim_k k - dim_k H^1(X, \mathcal{O}_X) \text{ (Proposition 4.13)} \\ &= 1 - dim_k H^1(X, \mathcal{O}_X).\end{aligned}$$

It then follows that

$$\begin{aligned}p_a(X) &= (-1)^r(\chi(\mathcal{O}_X) - 1) \\ &= (-1)^1(1 - dim_k H^1(X, \mathcal{O}_X) - 1) \\ &= dim_k H^1(X, \mathcal{O}_X).\end{aligned}$$

□

5. The Riemann-Roch Theorem

This last section is dedicated to the proof of the Riemann-Roch Theorem. In order to prove the result, we must establish a couple final definitions and state a major result known as the Serre Duality (Lemma 5.5). The proof of the Serre duality is very challenging and beyond the scope of this paper, but a proof can be found in Vakil's class notes (which will be linked in the references below).

Definition 5.1. Suppose A is a B-algebra. Define $\Omega_{A/B}$ to be the finite A-linear combination of da for $a \in A$, such that:

- $da + da' = d(a + a')$
- $d(aa') = ada' + a'da$
- $db = 0$ for $b \in \phi(B)$, where $\phi : B \to A$ is the morphism that comes with A being a B-algebra.

Example 5.2. Let $A = k[x, y, z]$, $B = k$. Then $5z^2xdz + 3yxdy \in \Omega_{A/B}$, and $d(5z^2x) = 5z^2dx + 10zxdz$.

Definition 5.3. Let X be a nonsingular variety over k and $n = dimX$. Define the canonical sheaf of X to be $\omega_X := \wedge^n \Omega_{X/k}$

Definition 5.4. Two divisors are linearly equivalent if their difference is a divisor of a rational function. Any divisor in the linear equivalence class that corresponds to the canonical sheaf is called a canonical divisor, denoted K.

Lemma 5.5. *Let X be a projective nonsingular variety of equidimension 1 over an algebraically closed field k. Then for any locally free sheaf $\mathcal{F}$ on X, there are natural isomorphisms,*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^* \otimes \omega_X)*.$$

Proof. Omitted □

Theorem 5.6. *Let D be a divisor on a curve X of genus g. Then*

$$dim_k H^0(X, \mathcal{L}(D)) - dim_k H^0(X, \mathcal{L}(K-D)) = degD + 1 - g.$$

Proof. Given the isomorphism established in the Proposition 4.7, it follows that the divisor $K-D$ corresponds to the invertible sheaf $\omega_X \otimes \mathcal{L}(D)^*$.

Additionally, X is a curve, so it is complete and thus projective (Lemma 4.19), so we can use the Serre duality (Lemma 5.5) to conclude that $H^0(X, \omega_X \otimes \mathcal{L}(D)^*)$ is dual to $H^1(X, \mathcal{L}(D))$ as a vector space. Hence, we are trying to show that $\chi(\mathcal{L}(D)) = dimH^0(X, \mathcal{L}(D)) - dimH^1(X, \mathcal{L}(D)) = degD + 1 - g$, where $\chi(\mathcal{L}(D))$ is the Euler characteristic.

first, suppose $D = 0$. To prove Riemann-Roch for this case, we are trying to show that $dimH^0(X, \mathcal{O}_X) - dimH^1(X, \mathcal{O}_X) = 0 + 1 - g$. Note that $H^0(X, \mathcal{O}_X) = k$ (Proposition 4.15), and $dimH^1(X, \mathcal{O}_X) = p_a(X) = g$ (Proposition 4.20). This completes the case there $D = 0$.

The next step is to show that for any arbitrary divisor D, the Riemann-Roch theorem holds for D if and only if it holds to $D + P$, where P is an arbitrary point. The reason for this being the more general case has to do with the structure of Weil divisors, as Weil divisors on curves are finite $\mathbb{Z}$-linear combinations of points, so it is possible to get from $D = 0$ to any divisor by adding a finite number of points.

Then consider P as a closed subscheme of X, whose sheaf is the skyscraper sheaf (Example 1.5), which we will denote $k(P)$. Then its ideal sheaf is isomorphic to $\mathcal{L}(-P)$ (Proposition 4.8). We can then construct the exact sequence

$0 \to \mathcal{L}(-P) \to \mathcal{O}_X \to k(P) \to 0$, which when tensored with $\mathcal{L}(D+P)$ gives

$0 \to \mathcal{L}(D) \to \mathcal{L}(D+P) \to k(P) \to 0$. Then given the additivity of the Euler characteristic on short exact sequences (Proposition 4.17) and the fact that $\chi(k(P)) = 1$, we get $\chi(\mathcal{L}(D+P)) = \chi(\mathcal{L}(D)) + 1$.

Combine this with the fact that $deg(D+P) = degD + 1$, and the proof is complete, as assuming the forward direction, $\chi(\mathcal{L}(D)) = degD + 1 - g$ becomes $\chi(\mathcal{L}(K-D)) = deg(K-D) + 1 - g$ through substitution, and likewise for the backwards direction.

□

Acknowledgments

I'm extremely grateful for my advisor, Livia Xu, for guiding me every step of the way by introducing me to interesting topics, giving me challenging problems to advance my understanding of the material, and for looking over my paper. I would also like to thank Peter May for organizing this great program, and for giving me the opportunity to participate in it.

References

[1] https://math.stanford.edu/ vakil/216blog/FOAGnov1817public.pdf
[2] https://math.stanford.edu/ vakil/0506-216/216class5354.pdf
[3] Hartshorne, Robin. Algebraic Geometry. Springer, 2010.
[4] https://pi.math.cornell.edu/ hatcher/AT/AT.pdf