WHITEHEAD FILTRATIONS FOR COMPUTATIONS IN TOPOLOGICAL HOCHSCHILD HOMOLOGY

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ABSTRACT. We discuss spectral sequences coming from Whitehead filtrations in the computation of topological Hochschild homology of ring spectra. Using cyclic invariance, this makes for simple computations of THH of connective ring spectra R with coefficients in discrete ring spectra.^{*a*} In particular, we show how to use this to compute THH(tmf, \mathbb{F}_2), and THH(tmf, $\mathbb{Z}_{(2)}$), where tmf denotes the \mathbb{E}_{∞} ring spectrum of topological modular forms. Then, we obtain a description of THH(ℓ/v_1^n) in terms of THH($\ell, \ell/v_1^n$), where the latter can be computed by results of [AHL09]. We next explain how the methods of this computation generalize to give us information about THH(cofib($x^k : \Sigma^{k|x|}R \to R$)) for R and cofib(x^k) suitably structured connective ring spectra, k > 1, and $x \in \pi_*(R)$ an arbitrary element in positive degree. Finally, we examine the general framework to describe the topological Hochschild homology of 2-local connective self-conjugate K-theory, ksc₂.

^aWhere discrete means Eilenberg-MacLane.

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§1. INTRODUCTION

Topological Hochschild homology was introduced by Bökstedt in 1985 as a generalization of ordinary Hochschild homology to general ring spectra, which has lead to many recent advances in algebraic K-theory [NS18], which in turn, lead to the recent disproof of Ravenel's telescope conjecture [Bur+23]. The aim of this paper is to discuss spectral sequences arising from Whitehead filtrations as a means to compute topological Hochschild homology (possibly with coefficients) of connective ring spectra. In tandem with these filtrations, we will heavily utilize the fact that topological Hochschild homology enjoys a cyclic invariance property, called the Dennis–Waldhausen Morita argument in [BM12, Proposition 6.2]. The cyclic invariance property states that if we have a morphism $f : R \to S$ of \mathbb{E}_1 -ring spectra, then we have an equivalence

$\operatorname{THH}(R, S) \simeq \operatorname{THH}(S, S \otimes_R S).$

Applying the Whitehead filtration together with this property recovers the Brun spectral sequence, and we get nice comparison maps for computing THH with coefficients.

For an \mathbb{E}_1 -ring spectrum *R*, the topological Hochschild homology of *R* is defined as

$$\mathrm{THH}(R) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} R,$$

and the topological Hochschild homology with coefficients in a R-bimodule M is similarly given by

$$\mathrm{THH}(R,M) = R \otimes_{R \otimes_{\mathbb{S}} R^{op}} M.$$

Assuming that *R* is connective, we can apply the Whitehead filtration to *R* to get a filtered left $R \otimes_{\mathbb{S}} R^{op}$ -module spectrum $\tau_{\geq *} R$. Applying the functor

$$R \otimes_{R \otimes_{\mathbb{S}} R^{op}} - : \operatorname{Fil}(\operatorname{LMod}_{R \otimes_{\mathbb{S}} R^{op}}) \to \operatorname{Fil}(\operatorname{LMod}_{R})$$

allows us to turn this into a filtered left R-module spectrum. This gives rise to a spectral sequence with signature

$$E_1^{s,t} = \text{THH}_{-s}(R, \pi_t(R)) \implies \text{THH}_{t-s}(R),$$

which we will refer to as the Whitehead spectral sequence. There is a similar Atiyah-Hirzebruch style spectral sequence computing THH(R, M) whenever R is a connective \mathbb{E}_1 -ring spectrum, and M an R-bimodule. These spectral sequences were considered in the case when R is an \mathbb{E}_{∞} ring by Hönig [Hö20]. Our main contribution is the extension to the \mathbb{E}_n -algebra case for $n < \infty$. In section 2, we discuss some basic results on spectral sequences of this type, which will then be applied in the remainder of the paper.

In section 3, we show how to use the tools we develop in order to compute THH(tmf, \mathbb{F}_2). With this in hand, by combining the Brun spectral sequence with a Bockstein spectral sequence, we are able to compute THH(tmf, $\mathbb{Z}_{(2)}$). The main theorem of section 3 is the computation:

Theorem 1.1. We have

$$\text{THH}_{*}(\text{tmf}, \mathbb{Z}_{(2)}) = \begin{cases} \mathbb{Z}_{(2)} & \text{if } * = 0, 9, 13, 22 \\ \mathbb{Z}/2^{k} \mathbb{Z} & \text{if } * = 2r^{k+3} - 1, 2^{k+3}r - 1 + 9, 2^{k+3}r - 1 + 13, 2^{k+3}r - 1 + 22 \\ 0 & \text{otherwise}, \end{cases}$$

for all k > 0 and r odd.

Section 4 combines the Atiyah-Hirzebruch spectral sequences with a May-type spectral sequence, as constructed in [Ang15] (see also [AKS18],[Kee20]), in order to compute the topological Hochschild homology of quotients of ℓ , THH (ℓ/v_1^n) . Here, ℓ denotes the mod p Adams summand for some fixed odd prime p, and $v_1 \in \pi_{2(p-1)}(\ell)$ generates $\pi_*(\ell)$ as a polynomial algebra over $\mathbb{Z}_{(p)}$. As we use numerous times throughout, $\ell/v_1^n \simeq \tau_{\leq 2n(p-1)-1}\ell$ inherits an $\mathbb{E}_{\infty} - \ell$ -algebra by [Lur17, Proposition 7.1.3.15]. The methods discussed in this section extend to prove:

Theorem 1.2. Suppose that R is a connective \mathbb{E}_m -ring spectrum for some $m \ge 4$, and $x \in \pi_*(R)$ is a positive degree class such that, for some fixed k > 1, there is an \mathbb{E}_3 -R-algebra S, such that the unique algebra map $R \to S$ fits into a fiber sequence

$$\Sigma^{k|x|} R \xrightarrow{x^k} R \to S.$$

Then, there is an equivalence of $\pi_0(R)$ -modules

$$\operatorname{THH}(S, \pi_0(S)) \simeq \operatorname{THH}(R, \pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)).$$

Under the same hypotheses, this result allows us to construct a spectral sequence with signature

$$E_1^{*,*} = \operatorname{THH}_{-*}(R, \pi_*(S)) \otimes_{\pi_0(R)} (\bigoplus_{i \ge 0} \pi_0(R) \cdot a_i) \implies \operatorname{THH}_*(S),$$

where a_i is a class in bidegree (-i(2k|x|+2), 0). There is a map to this spectral sequence from one with signature

$$\operatorname{THH}_{-*}(R, \pi_*(S)) \Longrightarrow \operatorname{THH}_*(R, S).$$

Thus, in cases where we have a Leibniz rule on our spectral sequence, the computation reduces to understanding topological Hochschild homology of R with coefficients in a quotient, as well as how differentials act on the classes a_i .

As a final sample application, we compute the topological Hochschild homology of 2-local self-conjugate K-theory ksc₂ in section §6.

Notation/Conventions

- We will use the term "category" to mean ∞-category in the sense of Lurie [Lur08].
- We write *p* to denote a fixed odd prime.
- The notation ℓ will denote the mod p Adams summand, and tmf will denote the spectrum of topological modular forms.
- The divided power algebra on a class x over a ring R will be denoted $\Gamma_R[x]$, and is defined by generators $x^{(n)}$ for $n \in \mathbb{Z}_{>0}$ with relations $x^{(n)}x^{(m)} = \frac{(n+m)!}{n!m!}x^{(n+m)}$.
- Classical Hochschild homology over a base ring R will be denoted by HH(-/R).

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§2. FILTERED OBJECTS AND SPECTRAL SEQUENCES

We review the theory of filtered objects developed in [Lur17, §1.2.2], [BHS22, §Appendix B], simultaneously setting our conventions. Let C be a stable category. To C, one can associate the category Fil(C) := Fun($\mathbb{Z}_{\leq}^{op}, C$) of filtered objects in C, i.e. the category of \mathbb{Z} -indexed collections of objects $c_i \in C$ together with maps $c_i \rightarrow c_{i-1}$. The category of filtered objects in C is equipped with an auto-equivalence (1) : $C \rightarrow C$, with $c(1)_n = c_{n-1}$, and a natural transformation $\tau : (-1) \implies id$ given in degree n by the map $c_{n+1} \rightarrow c_n$ induced by the filtration. Taking the cofiber of τ induces a functor from Fil(C) to the category of graded objects $Gr(C) := Fun(\mathbb{Z}^{op}, C)$ of functors from the discrete category \mathbb{Z}^{op} to C. The object $cofib(\tau : c_{\bullet}(-1) \rightarrow c_{\bullet})$ will be denoted c_{\bullet}^{gr} , and termed the associated graded object of c_{\bullet} .

To any $c \in C$ and $i \in \mathbb{Z}$, we associate an object $c^i \in Fil(C)$ such that

$$c_n^i = \begin{cases} 0 & \text{if } i > n, \\ c & \text{otherwise,} \end{cases}$$

with transition maps being 0 when the source is zero and the identity map otherwise. Suppose that there is an object $1 \in C$ which defines a notion of (\mathbb{Z} -graded) homotopy groups via

$$\pi_n(c) \coloneqq \pi_0(\operatorname{Hom}_{\mathcal{C}}(\Sigma^n 1, c)).$$

Then one can define bigraded homotopy groups for filtered objects of C via the formula

$$\pi_{n,m}(c) \coloneqq \pi_0(\operatorname{Hom}_{\operatorname{Fil}(\mathcal{C})}((\Sigma^n 1)^m, c)).$$

The fiber sequence $c(-1) \rightarrow c \rightarrow c^{gr}$ yields an exact couple, giving rise to a spectral sequence with signature

$$E_1^{s,t} \coloneqq \pi_{t-s,t}(c^{gr}) \implies \pi_{t-s}(\varinjlim c_{\bullet}),$$

which converges to the stated target if $\lim_{t \to \infty} c_{\bullet}$ vanishes. With this grading convention, the d_r -differential has bigrading (r + 1, r).

We now introduce the main spectral sequences that we will be using in this paper. Consider a connective \mathbb{E}_1 -ring R, and an R-bimodule M, i.e., a left $R \otimes_{\mathbb{S}} R^{op}$ -module. Working in the category Fil($\mathrm{LMod}_{R\otimes_{\mathbb{S}} R^{op}}$), we can take $\tau_{\geq *}M$ to be the Whitehead filtration on M, i.e., the filtered spectrum

$$\ldots \rightarrow \tau_{\geq m} M \rightarrow \tau_{\geq m-1} M \rightarrow \ldots$$

Applying the functor THH(R; -): $\text{Fil}(\text{LMod}_{R \otimes_{\mathbb{S}} R^{op}}) \rightarrow \text{Fil}(\text{LMod}_R)$, we get a filtered object $\text{THH}(R, \tau_{\geq *}M)$.

Definition 2.1. The *Atiyah-Hirzebruch spectral sequence* with coefficients in *M* is the spectral sequence associated to $\text{THH}(R, \tau_{>*}M)$, which has signature

$$E_1^{s,t} = \operatorname{THH}_{-s}(R, \pi_t(M)) \Rightarrow \operatorname{THH}_{t-s}(R, M).$$

In the special case M = R, we will refer to the resulting spectral sequence as the Whitehead spectral sequence.

In the above definition, $\pi_t(M)$ is treated as a discrete left $R \otimes_{\mathbb{S}} R^{op}$ -module concentrated in degree 0.

Note:

Lemma 2.2. Let *R* be a connective \mathbb{E}_1 -ring spectrum and *M* an *R*-bimodule. The filtered object THH(*R*; $\tau_{\geq*}M$) is complete, so the Atiyah-Hirzebruch spectral sequence converges.

Proof. Since THH(R; -) is right t-exact, the *n*th filtered piece of $\text{THH}(R; \tau_{\geq *}M)$ is *n*-connective, and the result follows.

We recall that discrete modules over $R \otimes_{\mathbb{S}} R^{op}$ in degree 0 are exactly the modules which live in the heart of $LMod_{R \otimes_{\mathbb{S}} R^{op}}$, and are thus in bijection with $\pi_0(R \otimes_{\mathbb{S}} R^{op}) \simeq \pi_0(R) \otimes_{\mathbb{Z}} \pi_0(R)^{op}$ modules. It is clear that this spectral sequence is functorial in M, and we also note that:

Proposition 2.3. Let R and S be connective \mathbb{E}_1 -ring spectra, and $f : R \to S$ a \mathbb{E}_1 -ring map. If M is an R-bimodule, then the natural map $\text{THH}(R, M) \to \text{THH}(S, (S \otimes_{\mathbb{S}} S^{op} \otimes_{R \otimes_{\mathbb{S}} R^{op}} M)$ induces a map on the associated Atiyah-Hirzebruch spectral sequences.

Proof. It suffices to show that we get a map on the level of filtered objects. Note that we have a natural factorization

$$\tau_{\geq n}M \to (S \otimes_{\mathbb{S}} S^{op}) \otimes_{R \otimes_{\mathbb{S}} R^{op}} \tau_{\geq n}M \to \tau_{\geq n}((S \otimes_{\mathbb{S}} S^{op}) \otimes_{R \otimes_{\mathbb{S}} R^{op}} M).$$

Applying the functor THH(R, -) and applying the cyclic invariance property gives an equivalence of filtered objects

$$\operatorname{THH}(R, \tau_{>*}M) \longrightarrow \operatorname{THH}(S, (S \otimes_{\mathbb{S}} S^{op}) \otimes_{R \otimes_{\mathbb{S}} R^{op}} \tau_{>*}M).$$

The morphism induced by the second map above gives a map

$$\operatorname{THH}(S, (S \otimes_{\mathbb{S}} S^{op}) \otimes_{R \otimes_{\mathbb{S}} R^{op}} \tau_{\geq *} M) \to \operatorname{THH}(S, \tau_{\geq *}(S \otimes_{\mathbb{S}} S^{op}) \otimes_{R \otimes_{\mathbb{S}} R^{op}} M),$$

and the composite provides our desired map of filtered objects.

To make use of the spectral sequences above, it would be useful to have a point of comparison so we can start getting some differentials, and to know about the multiplicative structure. On the side of multiplicative structure, we have:

Lemma 2.4. Suppose R is a connective \mathbb{E}_n -ring spectrum for some $n \ge 4$. Then the Atiyah-Hirzebruch spectral sequence with coefficients in M is multiplicative whenever M is an \mathbb{E}_2 -Ralgebra.

Proof. Suppose *R* is a \mathbb{E}_n -algebra for some $n \ge 4$. In particular, $R \otimes_{\mathbb{S}} R^{op}$ is an \mathbb{E}_n -algebra as well, and the canonical multiplication map $R \otimes_{\mathbb{S}} R^{op} \to R$ is an \mathbb{E}_{n-1} -algebra map, as given any $R \in \operatorname{Alg}_{\mathbb{E}_1}(\mathcal{C})$ for some symmetric monoidal ∞ -category \mathcal{C} (here we take $\mathcal{C} = \operatorname{Alg}_{\mathbb{E}_{n-1}}(\operatorname{Sp})$, and use [Lur17, Theorem 5.1.2.2]), there is a map $R \otimes R^{op} \simeq R \otimes R \to R$ in \mathcal{C} determined by the \mathbb{E}_1 -multiplication on R. It follows from [Lur17, Proposition 7.1.2.6] that the functor $R \otimes_{R \otimes_{\mathbb{S}} R^{op}} \to \operatorname{LMod}_R \operatorname{is} \mathbb{E}_{n-2}$ -monoidal, and in particular, \mathbb{E}_2 -monoidal. Since the Whitehead filtration is compatible with multiplicative structure, this gives the claim.

Next, we want to be able to find other useful spectral sequences to compare these to that will allow us to figure out some of their differentials. This is where cyclic invariance comes in. We work here in the case where R is an \mathbb{E}_{∞} -ring, but the setup should work more generally with minimal modifications:

Proposition 2.5. For R an \mathbb{E}_1 -ring, and S a \mathbb{E}_1 -R-algebra, we have an equivalence $\text{THH}(R, S) \simeq \text{THH}(S, S \otimes_R S)$. If R and S are both \mathbb{E}_{∞} -algebras, then this is an equivalence of \mathbb{E}_{∞} -algebras.

Proof. This follows from [BM12, Proposition 6.2] and [Hö20, Lemma 4.8, Remark 4.10].

Definition 2.6. When we take the Atiyah-Hirzebruch spectral sequence with coefficients in $S \otimes_R S$ in a situation as above, the resulting spectral sequence has been termed the *Brun spectral sequence*, introduced by Brun in [Bru00] and studied by Höning in [Hö20].

For *R* an arbitrary connective \mathbb{E}_n -ring spectrum, this already gives us a lot to compare *R* with. Namely, it is clear that THH($R, R \otimes_{\mathbb{S}} R^{op}$) $\simeq R$, so applying the Whitehead filtration to the coefficients produces a spectral sequence $E_1^{s,t} = \text{THH}_{-s}(R, \pi_t(R)) \Rightarrow \pi_{t-s}(R)$. Furthermore, the natural map $R \otimes_{\mathbb{S}} R^{op} \to R$ provides a map on spectral sequences which is surjective on the E_1 -page. Thus, if one could determine the structure of the spectral sequence associated to THH($R, R \otimes_{\mathbb{S}} R^{op}$), one could compute THH(R), although the structure of the former can get quite complicated in general.

A similar style spectral sequence that will feature prominently in §6 is the May-type spectral sequence, studied in detail in [Ang15], [AKS18], and [Kee20]. If *R* is a connective \mathbb{E}_n -ring spectrum, then $\tau_{\geq *}R$ is an \mathbb{E}_n -algebra object in filtered spectra. In particular, one can apply the construction of topological Hochschild homology to $\tau_{\geq *}R$ internally to filtered spectra, so get a filtered spectrum THH($\tau_{\geq *}R$), which allows us to define:

Definition 2.7. The *May-type spectral sequence* for a connective \mathbb{E}_n -ring spectrum (resp. with coefficients in a bimodule M) is the spectral sequence associated to the filtered object THH $(\tau_{\geq *}R)$ (resp. THH $(\tau_{\geq *}R, \tau_{\geq *}M)$).

The associated graded takes the form of the topological Hochschild homology of $\pi_* R$, which is an $\mathbb{E}_n - \mathbb{Z}$ -algebra, making it somewhat more amenable to computations. We will also consider some slight variants of this spectral sequence later.

Definition 2.8. The *Bockstein algebra* $\mathbb{Z}_{(2)}^{bok}$ as the filtered $\mathbb{E}_{\infty} - \mathbb{Z}_{(2)}$ -algebra with $(\mathbb{Z}_{(2)}^{bok})_i = \mathbb{Z}_{(2)}$ for all *i*, where the transition map $(\mathbb{Z}_{(2)}^{bok})_i \to (\mathbb{Z}_{(2)}^{bok})_{i-1}$ is the identity if $i \leq 0$, and multiplication by 2 if i > 0.

Given any $\mathbb{E}_{\infty} - \mathbb{Z}_{(2)}$ -algebra R, we can consider the filtered algebra $R^0 \otimes_{\mathbb{Z}_{(2)}^0} \mathbb{Z}_{(2)}^{bok}$, whose associated spectral sequence will be termed the *Bockstein spectral sequence*. This is a multiplicative spectral sequence which computes $\pi_*(R_2^{\wedge})$.

To conclude this section, we introduce some notation for specific *t*-structures on the filtered derived category, generalizing constructions of Beilinson, and investigated in [BHS22] (in this particular form, the definition is a special case of [LL23, Definition 2.4]).

Definition 2.9. Suppose that C is the category of (left) modules over some connective \mathbb{E}_1 -ring spectrum (e.g. C = Sp), and denote by τ the canonical *t*-structure on C defined as in [Lur17, Proposition 7.1.1.13]. Given any $r \in \mathbb{Q}_{\geq 0}$, there is a *t*-structure τ^r on Fil(C) (resp. Gr(C)) uniquely determined by stipulating that $x_{\bullet} \in \text{Fil}(C)$ (resp. $x_{\bullet} \in \text{Gr}(C)$) is connective if and only if x_i is an [ri]-connective object of C for all $i \in \mathbb{Z}$. In either case, the connective cover $\tau_{\geq}^r(x_{\bullet})$ has *i*th component $\tau_{\geq [ri]}x_i$.

By [LL23, Lemma 2.6], if C is the category of modules over an \mathbb{E}_{∞} -ring, then these *t*-structures defined above are compatible with the multiplicative structure.

§3. TOPOLOGICAL HOCHSCHILD HOMOLOGY OF tmf with COEFFICIENTS

In this section, we study the topological Hochschild homology of connective topological modular forms with coefficients in \mathbb{F}_2 and $\mathbb{Z}_{(2)}$. These computations, as well as those with other coefficients, were studied independently by Bruner-Rognes in a work in progress [BR14]. We begin with THH(tmf, \mathbb{F}_2), where the cyclic invariance condition will do most of the heavy lifting. As a precursor, we analyze $\mathbb{F}_2 \simeq \text{THH}(\mathbb{S}, \mathbb{F}_2) \simeq \text{THH}(\mathbb{F}_2, \mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2)$:

Example 3.1. Consider the spectral sequence

$$E_1^{s,t} = \operatorname{THH}_{-s}(\mathbb{F}_2, \pi_t(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2)) \Rightarrow \mathbb{F}_2.$$

By Bökstedt's computation of $\text{THH}(\mathbb{F}_2)$, we know that the signature of this spectral sequence is

$$E_1^{s,t} = \mathbb{F}_2[\xi_1,\xi_2,\xi_3,\ldots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u] \implies \mathbb{F}_2,$$

where ξ_i has bidegree $(0, 2^i - 1)$, and u has bidegree (-2, 0). The differentials must vanish on the classes ξ_i for degree reasons, so by the Leibniz rule, to understand the d_1 differential, we only need to understand how it acts on the class u. Since this spectral sequence converges to \mathbb{F}_2 , the class u cannot be a permanent cycle, and the only differential that can kill it is $d_1(u) = \xi_1$. This leaves

$$E_2 = \mathbb{F}_2[\xi_2, \xi_3, \ldots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u^2].$$

Inductively, we find that $E_k = E_{k+1}$ if $k \neq 2^n - 1$,

$$E_{2^n} = \mathbb{F}_2[\xi_{n+1}, \xi_{n+2}, \ldots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u^{2^n}],$$

and $d_{2^n-1}(u^{2^{n-1}}) = \xi_n$.

Example 3.2. In a similar fashion, we can determine the differentials in the spectral sequence associated to $\text{THH}(\mathbb{F}_p, \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p) \simeq \mathbb{F}_p$, which has signature

$$E_1^{s,t} = \mathbb{F}_p[\xi_1,\xi_2,\ldots] \otimes \Lambda[\tau_0,\tau_1,\ldots] \otimes \mathbb{F}_p[u] \implies \mathbb{F}_p[u]$$

where $|\xi_i| = (0, 2(p^i - 1)), |\tau_i| = (0, 2p^i - 1)$, and |u| = (-2, 0). The differentials in this spectral sequence are entirely determined by the Leibniz rule and the rules

$$d_{2p^{i}-1}(u^{p^{i}}) = \tau_{i}, \qquad d_{2p^{i-1}(p-1)}(u^{p^{i-1}(p-1)}\tau_{i-1}) = \xi_{i}.$$

Example 3.3. We can now compare with the Brun-Atiyah-Hirzebruch spectral sequence for tmf. Recalling that

$$\pi_*(\mathbb{F}_2 \otimes_{\mathrm{tmf}} \mathbb{F}_2) = \mathbb{F}_2[\xi_1, \xi_2, \xi_3] / (\xi_1^8, \xi_2^4, \xi_3^2)$$

as a quotient of the dual Steenrod algebra, our spectral sequence takes the form

$$E_1^{s,t} = \mathbb{F}_2[u] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\xi_1,\xi_2,\xi_3]/(\xi_1^8,\xi_2^4,\xi_3^2) \implies \text{THH}(\text{tmf},\mathbb{F}_2).$$

By the comparison map from the spectral sequence considered in Example 3.1, we find that $d_1(u) = \xi_1$, leaving

$$E_2^{s,t} \simeq \mathbb{F}_2[u^2] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\xi_2,\xi_3]/(\xi_2^4,\xi_3^2) \otimes_{\mathbb{F}_2} \Lambda[\xi_1^7 u].$$

Similarly, u^2 maps to ξ_2 on the E_3 -page, leaving

$$E_4^{s,t} \simeq \mathbb{F}_2[u^4] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\xi_3]/(\xi_3^2) \otimes_{\mathbb{F}_2} \Lambda[\xi_1^7 u, \xi_2^3 u^2],$$

and finally u^4 maps to ξ_3 on the E_7 page, giving

$$E_8^{s,t} = E_\infty^{s,t} = \mathbb{F}_2[u^8] \otimes_{\mathbb{F}_2} \Lambda(u\xi_1^7, u^2\xi_2^3, u^4\xi_3).$$

At this point, the spectral sequence must degenerate, as u^8 cannot hit any nonzero class. There are no multiplicative extension problems for degree reasons, and we find that

$$\pi_* \operatorname{THH}(\operatorname{tmf}, \mathbb{F}_2) \simeq \pi_* \operatorname{THH}(\mathbb{F}_2, \mathbb{F}_2 \otimes_{\operatorname{tmf}} \mathbb{F}_2) \simeq \mathbb{F}_2[u^8] \otimes_{\mathbb{F}_2} \Lambda(u\xi_1^7, u^2\xi_2^3, u^4\xi_3)$$

is the tensor product of a polynomial algebra with a generator in degree 16 with an exterior algebra on classes in degrees 9, 13, and 15. For future reference, let us write

$$\pi_*(\mathrm{THH}(\mathrm{tmf},\mathbb{F}_2)) = \mathbb{F}_2[\alpha] \otimes_{\mathbb{F}_2} \Lambda(\lambda_1,\lambda_2,\lambda_3),$$

with $|\alpha| = 16$, $|\lambda_1| = 9$, $|\lambda_2| = 13$, and $|\lambda_3| = 15$. We remark that is also a direct consequence of [AR05, Theorem 6.2(b)].

Before we compute THH(tmf, $\mathbb{Z}_{(2)}$), let's warm up by reproving Angelveit-Hill-Lawson's [AHL09] computation of THH($\ell, \mathbb{Z}_{(p)}$) at an odd prime *p*:

Example 3.4. In a fashion similar to Example 3.3, using $\pi_*(\mathbb{F}_p \otimes_{\ell} \mathbb{F}_p) \simeq \Lambda[\tau_0, \tau_1]$, one computes

$$\mathrm{THH}_*(\ell,\mathbb{F}_p)\simeq\mathbb{F}_p[u^{p^2}]\otimes_{\mathbb{F}_p}\Lambda[u^{p-1}\tau_0,u^{p(p-1)}\tau_1].$$

Since $\pi_*(\ell) \simeq \mathbb{Z}_{(p)}[v_1]$, with $|v_1| = 2(p-1)$, it is easy to see that $\pi_*(\mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)}) \simeq \Lambda_{\mathbb{Z}_{(p)}}[\rho]$, with $|\rho| = 2p-1$. Thus, the E_1 -page of our Brun-Atiyah-Hirzebruch spectral sequence computing THH $(\ell, \mathbb{Z}_{(p)})$ is given by

$$E_1 = \operatorname{THH}_*(\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \Lambda[\rho],$$

with ρ in bidegree (0, 2p - 1), and $a \in \text{THH}_i(\mathbb{Z}_{(p)})$ in bidegree (-i, 0).

The \mathbb{F}_p -vector space $\pi_{2p-1}(\text{THH}(\ell, \mathbb{F}_p))$ is one dimensional, which implies that there must be a multiplicative extension between ρ and the $\mathbb{Z}/p\mathbb{Z}$ class in $\text{THH}_{2p-1}(\mathbb{Z}_{(p)})$. By examining the locations where elements of $\text{THH}(\ell, \mathbb{F}_p)$ are nonzero, and working inductively, we find that for all r > 1, there is a nonzero differential on the class $\text{THH}_{2rp-1}(\mathbb{Z}_{(p)})$, which ends up giving us that

$$\text{THH}_{*}(\ell, \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } * = 0, 2p - 1 \\ \mathbb{Z}/p^{k} \mathbb{Z} & \text{if } * = 2rp^{k+1} - 1, 2rp^{k+1} - 1 + 2p - 1 \text{ with } k > 0, \gcd(r, k) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The case for THH(tmf, $\mathbb{Z}_{(2)}$) is only slightly more complicated. First, we need to know $\pi_*(\mathbb{Z}_{(2)} \otimes_{tmf} \mathbb{Z}_{(2)})$. To compute this, we start by looking at $\pi_*(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \text{tmf})$.

Proposition 3.5. There is an equivalence

$$\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf} \simeq \mathbb{Z}_{(2)}[\xi_1^8] \otimes_{\mathbb{Z}_{(2)}}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_2^4] \otimes_{\mathbb{Z}_{(2)}}^{\mathbb{L}} \bigotimes_{\mathbb{Z}_{(2)}, i>2}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i^2)$$

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of $\mathbb{E}_1 - \mathbb{Z}_{(2)}$ -algebras.

Proof. We know, for instance by [Mat15], that

$$\pi_*(\mathbb{F}_2 \otimes_{\mathbb{S}} \operatorname{tmf}) \simeq \mathbb{F}_2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \xi_5, \ldots],$$

identified as a subalgebra of the dual Steenrod algebra via the algebra map tmf $\rightarrow \mathbb{F}_2$. Next, note that

$$\mathbb{F}_2 \otimes_{\mathbb{S}} \operatorname{tmf} \simeq \operatorname{cofib}(2 : \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf} \to \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf}),$$

and we have a commutative diagram of ring spectra:

$$\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf} \longrightarrow \mathbb{F}_{2} \otimes_{\mathbb{S}} \operatorname{tmf} \\ \downarrow \qquad \qquad \downarrow \\ \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)} \longrightarrow \mathbb{F}_{2} \otimes_{\mathbb{S}} \mathbb{F}_{2}.$$

The long exact sequence associated to the cofiber sequence tells us that we must have torsion-free classes $a_1 \in \pi_8(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \text{tmf})$ and $a_2 \in \pi_{12}(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \text{tmf})$ mapping to ξ_1^8 and ξ_2^4 , respectively. By the equivalence

$$\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)} \simeq \bigotimes_{\mathbb{Z}_{(2)}, i > 0}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i),$$

one finds that a_1 must map to ξ_1^8 under the left vertical map. Similarly, a_2 must map to ξ_2^4 under this map, so by abuse of notation, we rename " $a_1 =: \xi_1^8$," " $a_2 =: \xi_2^4$." For $i \ge 1$, the fact ξ_i^2 is 2-torsion in $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)}$ reflects the fact that, under taking the cofiber of 2 on this spectrum, the class in degree $2(2^i - 1) + 1 = 2^{i+1} - 1$ arising from this torsion is ξ_{i+1} . The maps $\mathbb{Z}_{(2)}[\xi_i^2] \to \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \text{tmf for } i \ge 3$ factor over $\mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i^2)$, which follows from examining the classes in $\pi_*(\mathbb{F}_2 \otimes_{\mathbb{S}} \text{tmf})$ and using commutativity of the above diagram. Since $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \text{tmf}$ is 2-local, and we don't have any more classes showing up on the cofiber of 2, this allows us to conclude that there is an equivalence on the level of modules as stated.

As there are no classes in $\pi_*(\mathbb{F}_2 \otimes_{\mathbb{S}} \operatorname{tmf})$ of the form $\xi_2 \xi_1^k$ or $\xi_3 \xi_2^k$, the powers of ξ_1^8 and ξ_2^4 form a polynomial algebra over $\mathbb{Z}_{(2)}$, which implies the map from the free $\mathbb{E}_1 - \mathbb{Z}_{(2)}$ -algebra on generators in these degrees, $\mathbb{Z}_{(2)}\langle\xi_1^8,\xi_2^4\rangle$, to $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf}$ factors over $\mathbb{Z}_{(2)}[\xi_1^8] \otimes_{\mathbb{Z}_{(2)}}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_2^4]$, and the map from this quotient has injective image on π_* . We now note that there is an \mathbb{E}_1 -algebra map

$$\mathbb{Z}_{(2)}\langle \xi_1^8, \xi_2^4, \xi_3^2, \xi_4^2 \dots \rangle \to \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf}.$$

Using that the maps

$$\mathbb{Z}_{(2)}[2\xi_i^2] \to \mathbb{Z}_{(2)}\langle \xi_1^8, \xi_2^4, \xi_3^2, \xi_4^2 \dots \rangle \to \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf},$$

and

$$\mathbb{Z}_{(2)}[\xi_i^2\xi_j^2 - \xi_j^2\xi_i^2] \to \mathbb{Z}_{(2)}\langle \xi_1^8, \xi_2^4, \xi_3^2, \xi_4^2 \dots \rangle \to \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \operatorname{tmf},$$

factor over $\mathbb{Z}_{(2)}$, we can take \mathbb{E}_1 -quotients, and pass to a filtered colimit to conclude the result.

Now, one computes:

Proposition 3.6. The homotopy groups of $\mathbb{Z}_{(2)} \otimes_{\text{tmf}} \mathbb{Z}_{(2)}$ are given by

$$\pi_{*}(\mathbb{Z}_{(2)} \otimes_{\text{tmf}} \mathbb{Z}_{(2)}) \simeq \begin{cases} \mathbb{Z}_{(2)} & \text{if } * = 0, 22, \\ \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } * = 9, 13, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } * = 2, 4, 8, 10, 11, 12, 17, 19, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } * = 6, 15 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that the unit map $\mathbb{S} \to \mathbb{Z}_{(2)}$ factors as $\mathbb{S} \to \text{tmf} \to \mathbb{Z}_{(2)}$. Thus, there is a diagram of pushout squares of \mathbb{E}_{∞} -ring spectra:



That is to say,

$$\mathbb{Z}_{(2)} \otimes_{\mathrm{tmf}} \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf}} (\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)}).$$

By proposition 3.5, one obtains

$$\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \text{tmf}} \left(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)} \right) \simeq \left(\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)}[\xi_1^8]}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_1^2] / (2\xi_1^2) \right) \otimes_{\mathbb{Z}_{(2)}}^{\mathbb{L}} \left(\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)}[\xi_2^4]}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_2^2] / (2\xi_2^2) \right) \\ \otimes_{\mathbb{Z}_{(2)}}^{\mathbb{L}} \bigotimes_{\mathbb{Z}_{(2)}, i \ge 3}}^{\mathbb{L}} \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)}[\xi_i^2] / (2\xi_i^2)}^{\mathbb{Z}} \mathbb{Z}_{(2)}[\xi_i^2] / (2\xi_i^2).$$

This is equivalent to computing

$$\left(\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)}[\xi_1^8]}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_1^2] / (2\xi_1^2)\right) \otimes_{\mathbb{Z}_{(2)}}^{\mathbb{L}} \left(\mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)}[\xi_2^4]}^{\mathbb{L}} \mathbb{Z}_{(2)}[\xi_2^2] / (2\xi_2^2)\right),$$

which gives the desired result.

With this in hand, we can finally compute $\text{THH}_*(\text{tmf}, \mathbb{Z}_{(2)})$:

Theorem 3.7 (Theorem 1.1).

$$\text{THH}_{*}(\text{tmf}, \mathbb{Z}_{(2)}) = \begin{cases} \mathbb{Z}_{(2)} & \text{if } * = 0, 9, 13, 22 \\ \mathbb{Z}/2^{k} \mathbb{Z} & \text{if } * = 2r^{k+3} - 1, 2^{k+3}r - 1 + 9, 2^{k+3}r - 1 + 13, 2^{k+3}r - 1 + 22, \\ 0 & \text{otherwise}, \end{cases}$$

for all k > 0 and r odd.

Proof. First, note that, by proposition 3.6, the Brun-Atiyah-Hirzebruch spectral sequence for THH(tmf, $\mathbb{Z}_{(2)}$) has E_1 -page with THH_{*}($\mathbb{Z}_{(2)}$) in degrees (-*,0), (-*,22), THH_{*}($\mathbb{Z}_{(2)}$, $\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2\mathbb{Z}$) in degrees (-*,9), (-*,13), THH_{*}($\mathbb{Z}_{(2)}$, $\mathbb{Z}/2\mathbb{Z}$) in degrees (-*,2), (-*,4), (-*,8), (-*,10), (-*,11), (-*,12), (-*,17), (-*,19), THH_{*}($\mathbb{Z}_{(2)}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$) in degrees (-*,6) and (-*,15), and is zero otherwise.

This spectral sequence has a lot of terms, so would be difficult to deal with on its own. Instead, we examine also the Bockstein spectral sequence, with E_1 -page THH_{*}(tmf, $\mathbb{F}_2)[\tilde{v}_0]$, with $|\tilde{v}_0| = (1, 1)$, converging to the 2-adic completion of THH_{*}(tmf, $\mathbb{Z}_{(2)})$. The Bockstein spectral sequence arises from the filtered $\mathbb{E}_{\infty}(\mathbb{Z}_2)$ -algebra given by THH(tmf, $\mathbb{Z}_{(2)})^{(0)} \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_{(2)}^{bok}$, and is in particular multiplicative. Since every term of the Brun spectral sequence in negative *s* degree is torsion, the $\mathbb{Z}_{(2)}$ -classes in degrees 0,9,13, and 22 must survive to the E_{∞} -page, giving the respective classes in THH_{*}(tmf, $\mathbb{Z}_{(2)})$. This is reflected in our Bockstein spectral sequence by the fact that our \tilde{v}_0 -towers on the classes 1, λ_1 , λ_2 and $\lambda_1\lambda_2$ must survive the spectral sequence. Thus, we must have that λ_3 is a permanent cycle. Since the spectral sequence is multiplicative, we either have that it has degenerated, or else $d_1(\alpha) = \tilde{v_0}\lambda_3$. The Brun-Atiyah-Hirzebruch spectral sequence shows that π_{15} (THH(tmf, $\mathbb{Z}_{(2)})$) is finite, so we cannot be in the first case, and thus $d_1(\alpha) = \tilde{v_1}\lambda_3$, and $\alpha\lambda_3$ cannot support any differentials since anything it could hit is either a permanent cycle, or was killed off on the first page.

Inductively, we find that $\alpha^{2^n} \lambda_3$ cannot support any differentials, and the only multiplicative generator that can support a differential on or past the E_{n+2} -page, is $\alpha^{2^{n+1}}$, which can only possibly support the differential $d_{n+2}(\alpha^{2^{n+1}}) = \tilde{v_0}^{n+2}\alpha^{2^n}\lambda_3$. If we did not have this differential, the spectral sequence would be degenerate, but again, the Brun-Atiyah-Hirzebruch spectral sequence shows this cannot be the case, so we have $d_{n+2}(\alpha^{2^{n+1}}) = \tilde{v_0}^{n+2}\alpha^{2^n}\lambda_3$ for all *n*. By looking at the cofiber of 2, we find that THH_{*}(tmf, $\mathbb{Z}_{(2)}$) is either $\mathbb{Z}_{(2)}$ or finite cyclic in any given degree, which determines the multiplicative extensions in the Bockstein spectral sequence. The classes in degree $2^{k+3}r - 1$ are the classes $\alpha^{r2^{k-1}}\lambda_3$, and the other classes in the second item come from multiplying by λ_1 , λ_2 and $\lambda_1\lambda_2$.

§4. THH of Quotients of ℓ

In this section, we compute the topological Hochschild homology of quotients ℓ/v_1^n of ℓ , in terms of the topological Hochschild homology of ℓ , which was computed by Angeltveit-Hill-Lawson in [AHL09]. We will show that $\pi_* \text{THH}(\ell/v_1^n) \simeq \pi_* \text{THH}(\ell, \ell/v_1^n) \otimes_{\mathbb{Z}_{(p)}} \Gamma[x]$, the tensor product of the topological Hochschild homology of ℓ with coefficients in ℓ/v_1^n with a divided power algebra on a generator x in degree 2n(p-1) + 2. We start by computing $\text{THH}(\ell/v_1^n, \mathbb{Z}_{(p)})$.

Lemma 4.1. For n > 1, we have that

$$\mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)} \simeq (\mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} (\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)})$$

as \mathbb{E}_{∞} -algebras.

Proof. We have a diagram of \mathbb{E}_{∞} -rings where all squares are pushouts:

$$\begin{array}{cccc} \ell & \longrightarrow \ell/v_1^n & \longrightarrow \mathbb{Z}_{(p)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_{(p)} & \longrightarrow \mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n & \longrightarrow \mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)} \\ & & \downarrow & & \downarrow \\ \mathbb{Z}_{(p)} & \longrightarrow \mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)} \end{array}$$

The map $\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n \to \mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)}$ factors over $\tau_{\leq 2(p-1)+1} \left(\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n\right) \simeq \mathbb{Z}_{(p)}$, so that the cospan $\mathbb{Z}_{(p)} \leftarrow \mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n \to \mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)}$ may be rewritten as the tensor product over $\mathbb{Z}_{(p)}$ of the cospans $\mathbb{Z}_{(p)} \leftarrow \mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n \to \mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)} \leftarrow \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)}$. Since colimits commute, we find that the pushout of our original square, $\mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)}$, is isomorphic to the pushout of our first cospan tensored over $\mathbb{Z}_{(p)}$ with the pushout of our second cospan, which is precisely $(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} (\mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)})$, as claimed. \Box

Next, we examine $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)}$. By an easy computation, its homotopy groups are

$$\pi_*(\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } * = 0, 2n(p-1)+1, \\ 0 & \text{otherwise.} \end{cases}$$

By [DLR18, Proposition 2.1], this $\mathbb{E}_{\infty} - \mathbb{Z}_{(p)}$ -algebra is a trivial square-zero extension of $\mathbb{Z}_{(p)}$. A standard then argument shows that there is an identification of graded commutative rings $\pi_*(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)}) = \Gamma[x]$ with *x* in degree 2n(p-1) + 2, as desired.

Lemma 4.2. THH_{*}($\ell/v_1^n, \mathbb{Z}_{(p)}$) \simeq THH_{*}($\ell, \mathbb{Z}_{(p)}$) $\otimes_{\mathbb{Z}_{(p)}} \pi_*(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)}).$

Proof. By cyclic invariance, we have

$$\operatorname{THH}(\ell/v_1^n,\mathbb{Z}_{(p)})\simeq\operatorname{THH}(\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}\otimes_{\ell/v_1^n}\mathbb{Z}_{(p)}).$$

Expanding this out and applying Lemma 4.1, we get

$$\begin{aligned} \operatorname{THH}(\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}\otimes_{\ell/v_{1}^{n}}\mathbb{Z}_{(p)}) &\simeq \mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\mathbb{S}}\mathbb{Z}_{(p)}} \left(\mathbb{Z}_{(p)}\otimes_{\ell/v_{1}^{n}}\mathbb{Z}_{(p)}\right) \\ &\simeq \mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\mathbb{S}}\mathbb{Z}_{(p)}} \left(\left(\mathbb{Z}_{(p)}\otimes_{\ell}\mathbb{Z}_{(p)}\right)\otimes_{\mathbb{Z}_{(p)}} \left(\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\ell}\ell/v_{1}^{n}}\mathbb{Z}_{(p)}\right)\right) \end{aligned}$$

Finally, by rearranging the order of the tensor products, one concludes

$$\begin{aligned} \operatorname{THH}(\ell/v_1^n, \mathbb{Z}_{(p)}) &\simeq \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)}}} \left(\left(\mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)} \right) \otimes_{\mathbb{Z}_{(p)}} \left(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)} \right) \right) \\ &\simeq \left(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)}}} \left(\mathbb{Z}_{(p)} \otimes_{\ell} \mathbb{Z}_{(p)} \right) \right) \otimes_{\mathbb{Z}_{(p)}} \left(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)} \right) \\ &\simeq \operatorname{THH}(\ell, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \left(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_{\ell} \ell/v_1^n} \mathbb{Z}_{(p)} \right). \end{aligned}$$

Now, we are finally in a position to analyze the Whitehead spectral sequence for $\text{THH}(\ell/v_1^n)$. To start, note that

$$E_1^{s,t} = \mathbb{Z}_{(p)}[v_1]/v_1^n \otimes_{\mathbb{Z}_{(p)}} \operatorname{THH}_*(\ell, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \Gamma[x],$$

with v_1 in bidegree (0, 2(p-1)), x in bidegree (-(2n(p-1)+2), 0), and $\text{THH}_s(\ell, \mathbb{Z}_{(p)})$ living in degree (-s, 0). There is a comparison map ρ from the Brun-Atiyah-Hirzebruch spectral sequence for $\text{THH}(\ell, \ell/v_1^n)$ to this one, which determines many of the differentials. In fact,

Theorem 4.3. All of the differentials vanish on the classes coming from $\Gamma[x]$ in the THH (ℓ/v_1^n) spectral sequence. In particular, the comparison map ρ , together with the Leibniz rule, determine all of the differentials in the Whitehead spectral sequence, yielding

$$\operatorname{THH}_{*}(\ell/v_{1}^{n}) \simeq \operatorname{THH}_{*}(\ell, \ell/v_{1}^{n}) \otimes_{\mathbb{Z}_{(p)}} \Gamma[x],$$

for some class x in degree 2n(p-1) + 2.

Proof. In the case $n \neq 1 \mod p^2$, the theorem can be proven by looking only at the spectral sequences we have already constructed.

To prove this theorem in general, we use the May-type spectral sequence, combined with the following lemma, whose proof is adapted from [LL23, Lemma 4.1]:

Lemma 4.4. Suppose k is a discrete ring, and R is a connective (possibly graded) \mathbb{E}_2 -k-algebra with $\pi_*(R) = k[x]/x^n$, on some class x in positive even degree, and R admits an \mathbb{E}_2 -algebra map from a ring S with $\pi_*(S) = k[x]$. Then, we have an equivalence of (graded) \mathbb{E}_1 -k-algebras THH(R) = THH(k) \otimes_k HH(R/k).

Proof of lemma. We have $k[x] = k \otimes_{\mathbb{S}} \mathbb{S}[x]$. Now, as an \mathbb{E}_1 -algebra

$$R = k[x] \otimes_{k[x^n]} k \simeq (k \otimes_{\mathbb{S}} \mathbb{S}[x]) \otimes_{k \otimes_{\mathbb{S}} \mathbb{S}[x^n]} (k \otimes_{\mathbb{S}} \mathbb{S}) \simeq k \otimes_{\mathbb{S}} \mathbb{S}[x]/x^n,$$

where $S[x]/x^n$ denotes $S[x] \otimes_{S[x^n]} S$. Since THH commutes with tensor products, there are equivalences of (graded) spectra,

$$\mathrm{THH}(R) \simeq \mathrm{THH}(k) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[x]/x^n) \simeq \mathrm{THH}(k) \otimes_k k \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[x]/x^n) \simeq \mathrm{THH}(k) \otimes_k \mathrm{HH}((k[x]/x^n)/k).$$

Since $k[x]/x^n = \tau_{\leq n|x|-1}k[x]$, *R* inherits a canonical \mathbb{E}_{2} -*k*-algebra structure as this truncation. We can give *x* a new (positive) grading 1, to make $\mathbb{S}[x]$ a nonnegatively graded \mathbb{E}_{2} -ring spectrum, so an \mathbb{E}_{2} -algebra in $\mathrm{Sp}^{\mathbb{Z}_{\geq 0}^{ds}}$. There is a thick \otimes -ideal \mathcal{I} of $\mathrm{Sp}^{\mathbb{Z}_{\geq 0}^{ds}}$ generated by elements concentrated in grading $\geq n$. Quotienting out this \otimes -ideal gives a symmetric monoidal functor $\mathrm{Sp}^{\mathbb{Z}_{\geq 0}^{ds}} \to \mathrm{Sp}^{\mathbb{Z}_{\geq 0}^{ds}}/\mathcal{I}$, whose right adjoint is then lax symmetric monoidal by [Lur17, Corollary 7.3.2.7]. Composing these two functors gives a functor which sends our graded $\mathbb{S}[x]$ to a graded \mathbb{E}_{2} -algebra with underlying \mathbb{E}_{2} -ring $\mathbb{S}[x]/x^n$, as desired. Using this grading, we can get, from the \mathbb{E}_{2} -map in $\mathrm{Sp}^{\mathbb{Z}_{\geq 0}^{ds}}$, a map $\mathbb{S}[x] \to k[x]$. Applying the endofunctor we just described, we get an \mathbb{E}_{2} -algebra map $\mathbb{S}[x]/x^n \to k[x]/x^n$, which upgrades our isomorphism above to an \mathbb{E}_{2} -algebra. We wish to apply this in our case. Choosing $m \in \mathbb{Z}_{\geq 0}$ sufficiently large (m > 2n(p-1)+2), we get a *t*-structure τ^m on graded spectra such that $\tau^m_{\geq 0}(\pi_*(\ell/v_1^n)) = \mathbb{Z}_{(p)}$ concentrated in degree 0, which shows that $\pi_*(\ell/v_1^n)$ is a graded $\mathbb{E}_{\infty} \cdot \mathbb{Z}_{(p)}$ -algebra. Now, we can apply the above theorem to write the E_1 -page of the May-type spectral sequence as

$$E_1^{s,t} = \mathrm{THH}_*(\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathrm{HH}_*((\mathbb{Z}_{(p)}[\tilde{v_1}]/\tilde{v_1}^n)/\mathbb{Z}_{(p)}).$$

A standard calculation¹ shows that

$$\operatorname{HH}_{*}((\mathbb{Z}_{(p)}[\tilde{v_{1}}]/\tilde{v_{1}}^{n})/\mathbb{Z}_{(p)}) \simeq \Lambda_{\mathbb{Z}_{(p)}}[\sigma \tilde{v_{1}}] \otimes_{\mathbb{Z}_{(p)}} \Gamma_{\mathbb{Z}_{(p)}}[\sigma^{2} \tilde{v_{1}}^{n}] \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\tilde{v_{1}}]/(\tilde{v_{1}}^{n}),$$

where $\sigma^2 \tilde{v_1}^n$ is a class in bidegree (-2, 2n(p-1)), and σv_1 is a class in bidegree (-1, 2(p-1)).

The terms coming from $\text{THH}_n(\mathbb{Z}_{(p)})$ live in bidegree (-n, 0), and $\tilde{v_1}$ lives in bidegree (0, 2(p-1)). One notes that $\sigma^2(\tilde{v_1}^n)$ is the only class in total degree (-2, 2n(p-1)) and nothing lives in degree (0, 2n(p-1)+1). Thus, $\sigma^2(\tilde{v_1}^n)$ must vanish under the differentials on every page, and this is the same class corresponding to x in the Brun-Atiyah-Hirzebruch spectral sequence.

Next, note that for all classes *a*, with bigrading |a| = (s, t), we have that $t \le -n(p-1)s + 2(n-1)(p-1)$ for *s* even, and $t \le -n(p-1)(s+1) + 2n(p-1)$ for *s* odd. Furthermore, *t* is maximized with respect to *s* for $s \le 0$ even by $\tilde{v_1}^{n-1}x^{(-\frac{s}{2})}$, and for *s* odd by

 $\sigma \tilde{v_1} \tilde{v_1}^{n-1} x^{\left(-\frac{s+1}{2}\right)}$. In particular, any differential off of $x^{(k)}$ on the E_r -page would have to hit a class in bidegree (-2k + r + 1, 2kn(p-1) + r). However, for r odd, $-2k + r + 1 \ge -2k + 2$, and thus every class with t > 2n(p-1)(k-1) + 2(n-1)(p-1) vanishes in this s degree. Therefore, the target of d_r on $x^{(k)}$ is 0. If r is even, then $-2k + r + 1 \ge -2k + 3$, and thus if t > 2n(p-1)(k-2)+2n(p-1) = 2n(p-1)(k-1), the classes in degree (-2k+r+1, t) vanish, and again, the target of $d_r((\sigma^2(\tilde{v_1}^n))^{(k)})$ vanishes. We have shown that the $(\sigma^2(\tilde{v_1})^n)^{(k)}$, are all permanent cycles, and then so are the $x^{(k)}$ from the Brun-Atiyah-Hirzebruch spectral sequence.

Note that there are no other nonzero terms in the May-type spectral sequence with total degree k(2n(p-1)+2), and higher filtration degree than $(\sigma^2(\tilde{v_1}^n))^{(k)}$. Thus, there can be no nontrivial multiplicative extensions supported on these classes, and $x \mapsto \sigma^2(v_1^n)$ determines a map of graded commutative $\mathbb{Z}_{(p)}$ -algebras $\Gamma_{\mathbb{Z}_{(p)}}[x] \to \text{THH}_*(\ell/v_1^n)$, with x a class in degree 2n(p-1)+2. This gives us a map of graded commutative $\mathbb{Z}_{(p)}$ -algebras

$$\operatorname{THH}_{*}(\ell, \ell/v_{1}^{n}) \otimes_{\mathbb{Z}_{(p)}} \Gamma_{\mathbb{Z}_{(p)}}[x] \to \operatorname{THH}_{*}(\ell/v_{1}^{n}).$$

Since the E_1 -page of the Whitehead spectral sequence for $\text{THH}(\ell/v_1^n)$ is multiplicatively generated by the image of $\text{THH}_*(\ell, \pi_*(\ell/v_1^n))$ under ρ and the classes $x^{(k)}$ (which have just been shown to be permanent cycles), all of the nontrivial differentials appearing in this Whitehead spectral sequence arise from the map $\text{THH}_*(\ell, \tau_{\geq *}\ell/v_1^n) \rightarrow \text{THH}_*(\ell/v_1^n, \tau_{\geq *}\ell/v_1^n)$ together with the Leibniz rule. In particular, it follows that the algebra map $\text{THH}_*(\ell, \ell/v_1^n) \otimes_{\mathbb{Z}_{(p)}} \Gamma_{\mathbb{Z}_{(p)}}[x] \rightarrow \text{THH}_*(\ell/v_1^n)$ must be an isomorphism, as claimed.

Remark 4.5. The notations σx and $\sigma^2 x$ for certain classes *x* were used above, which deserves an explanation. Let *C* be a presentably symmetric monoidal stable category, and let *R* be an \mathbb{E}_1 -algebra object in *C*. In [HW22, §A], a map is constructed

$$\Sigma \operatorname{cofib}(1_{\mathcal{C}} \to R) \to \operatorname{THH}_{\mathcal{C}}(R),$$

where we denote for the moment $\text{THH}_{\mathcal{C}}(R)$ the topological Hochschild homology internal to the category \mathcal{C} . For a class $w \in \pi_*(R)$, we denote by σw the image of this class under the composite

¹For instance, one can use that $\operatorname{HH}((\mathbb{Z}_{(p)}[x]/x^n)/\mathbb{Z}_{(p)}) \simeq \operatorname{HH}(\mathbb{Z}_{(p)}[x]/\mathbb{Z}_{(p)}) \otimes_{\operatorname{HH}(\mathbb{Z}_{(p)}[x^n]/\mathbb{Z}_{(p)})} \mathbb{Z}_{(p)}.$

 $\Sigma R \to \Sigma \operatorname{cofib}(1_{\mathcal{C}} \to R) \to \operatorname{THH}_{\mathcal{C}}(R), \sigma w \in \pi_{*+1}(\operatorname{THH}_{\mathcal{C}}(R)).$ If $x \in \pi_*(1_{\mathcal{C}})$ has a lift to a class $\tilde{x} \in \pi_{*+1}(\operatorname{cofib}(1_{\mathcal{C}} \to R))$, then we write $\sigma^2 x$ for the image of \tilde{x} under $\Sigma \operatorname{cofib}(1_{\mathcal{C}} \to R) \to \operatorname{THH}_{\mathcal{C}}(R)$, which lives in $\pi_{*+2}(\operatorname{THH}_{\mathcal{C}}(R))$. Although the class denoted $\sigma^2 \tilde{v_1}^n$ above does not actually arise in this way, it is lifted from a class in $\operatorname{THH}_{\mathbb{Z}_{(p)}[\tilde{v_1}]-Mod}(\mathbb{Z}_{(p)}[\tilde{v_1}]/(\tilde{v_1}^n))$, which is constructed in this fashion, so we abusively denote the class $\sigma^2 \tilde{v_1}^n$. These naming conventions will make a reprise in the sequel.

§5. The General Case

We remark that many of the constructions in the last section admit a generalization.

Theorem 5.1. Suppose that R is a connective \mathbb{E}_m -ring spectrum for some $m \ge 4$, and $x \in \pi_*(R)$ is a positive degree class such that, for some fixed k > 1, there is an \mathbb{E}_3 -R-algebra S, such that the unique algebra map $R \to S$ fits into a fiber sequence of R-modules

$$\Sigma^{k|x|} R \xrightarrow{x^k} R \to S.$$

Then, there is an equivalence of $\pi_0(R)$ *-modules*

$$\operatorname{THH}(S,\pi_0(S)) \simeq \operatorname{THH}(R,\pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)).$$

Proof. By proposition 2.5, using that $\pi_0(R) = \pi_0(S)$,

$$\mathrm{THH}(S,\pi_0(R))\simeq\mathrm{THH}(\pi_0(R),\pi_0(R)\otimes_S\pi_0(R)).$$

There is a string of equivalences

$$\pi_{0}(R) \otimes_{S} \pi_{0}(R) \simeq (\pi_{0}(R) \otimes_{\mathbb{S}} \pi_{0}(R)) \otimes_{S \otimes_{\mathbb{S}} S} S$$

$$\simeq (\pi_{0}(R) \otimes_{\mathbb{S}} \pi_{0}(R)) \otimes_{S \otimes_{\mathbb{S}} S} (S \otimes_{R} S) \otimes_{(S \otimes_{R} S)} S$$

$$\simeq ((\pi_{0}(R) \otimes_{\mathbb{S}} \pi_{0}(R)) \otimes_{S \otimes_{\mathbb{S}} S} ((S \otimes_{\mathbb{S}} S) \otimes_{R \otimes_{\mathbb{S}} R} R)) \otimes_{(S \otimes_{R} S)} S$$

$$\simeq ((\pi_{0}(R) \otimes_{\mathbb{S}} \pi_{0}(R)) \otimes_{R \otimes_{\mathbb{S}} R} R) \otimes_{(S \otimes_{R} S)} S$$

$$\simeq (\pi_{0}(R) \otimes_{R} \pi_{0}(R)) \otimes_{(S \otimes_{R} S)} S$$

$$\simeq (\pi_{0}(R) \otimes_{R} \pi_{0}(R)) \otimes_{\pi_{0}(R) \otimes_{R} S} (\pi_{0}(R) \otimes_{R} S) \otimes_{(S \otimes_{R} S)} S$$

$$\simeq (\pi_{0}(R) \otimes_{R} \pi_{0}(R)) \otimes_{\pi_{0}(R) \otimes_{R} S} \pi_{0}(R).$$

The first, third, fourth, fifth, and seventh equivalences hold by [Lur17, Theorem 5.1.4.10]. The second equivalence follows by simple rewriting *S* as $(S \otimes_R S) \otimes_{S \otimes_R S} S$, and the fact that $S \otimes_S S \rightarrow S$ factors over $S \otimes_R S \rightarrow S$. Finally, the sixth equivalence uses that $S \otimes_R S \rightarrow \pi_0(R) \otimes_R \pi_0(R)$ factors as $S \otimes_R S \rightarrow \pi_0(R) \otimes_R S \rightarrow \pi_0(R) \otimes_R \pi_0(R)$.

Now, by the assumption $S \simeq R/x^k$ as *R*-modules, it follows that the map $S \to \pi_0(S) = \pi_0(R)$ factors over some *R*-module map $S \to R/x^{k-1} \to \pi_0(R)$. Thus, $\pi_0(R) \otimes_R S \to \pi_0(R) \otimes_R \pi_0(R)$ factors as $\pi_0(R) \otimes_R S \to \pi_0(R) \otimes_R R/x^{k-1} \to \pi_0(R) \otimes_R \pi_0(R)$. Since $\pi_0(R) \otimes_R R/x^{k-1}$ has homotopy groups given by

$$\pi_*(\pi_0(R) \otimes_R R/x^{k-1}) = \begin{cases} \pi_0(R) & \text{if } * = 0, \ (k-1)|x|+1, \\ 0 & \text{otherwise,} \end{cases}$$

we see that $\pi_0(R) \otimes_R S \to \pi_0(R) \otimes_R R/x^{k-1}$ factors over $\tau_{\leq (k-1)|x|+1}(\pi_0(R) \otimes_R S) \simeq \pi_0(R)$. This implies that $\pi_0(R) \otimes_R \pi_0(R)$, as a right $\pi_0(R) \otimes_R S$ -module, is equivalent to $(\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} \pi_0(R)$, with the induced right module structure on $\pi_0(R)$. Thus,

$$\pi_0(R) \otimes_S \pi_0(R) \simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)$$
$$\simeq ((\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} \pi_0(R)) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)$$
$$\simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)).$$

The first two equivalences follow by our discussion above, and the final equivalence uses that $\pi_0(R) \otimes_R S \to \pi_0(R) \otimes_R \pi_0(R)$ factors over $\pi_0(R)$. This derived tensor product in $\pi_0(R)$ -modules can be computed as a tensor product on underlying modules, since the second module

is flat (in fact free). Indeed, from the cofiber sequence $\Sigma^{k|x|}R \xrightarrow{x^k} R \to S$, we can tensor this with $\pi_0(R)$ to find that $\pi_0(R) \otimes_R S \simeq \pi_0(R) \oplus \Sigma^{k|x|+1} \pi_0(R)$ as a $\pi_0(R)$ -module. We then have a periodic resolution of $\pi_0(R)$ from this class in degree k|x| + 1, which allows us to see that

$$\pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R) \simeq \bigoplus_{r \ge 0} \Sigma^{r(k|x|+2)} \pi_0(R)$$

as a $\pi_0(R)$ -module.

Now, we apply the Whitehead filtration to $\pi_0(S) \otimes_S \pi_0(S)$, and examine the Brun-Atiyah-Hirzebruch spectral sequence for THH($S, \pi_0(S)$). Note that this spectral sequence is multiplicative, since by assumption, S is an \mathbb{E}_3 -R-algebra, so that the maps $S \to \pi_0(S) = \tau_{<0}S$ are \mathbb{E}_3 -algebra maps. This implies that $(\pi_0(S) \otimes_S \pi_0(S))$ is an \mathbb{E}_2 - $\pi_0(R)$ -algebra by [Lur17, Proposition 7.1.2.6]. We have a map

$$\operatorname{THH}(\pi_0(R), \tau_{\geq *}(\pi_0(R) \otimes_R \pi_0(R))) \to \operatorname{THH}(\pi_0(S), \tau_{\geq *}(\pi_0(S) \otimes_S \pi_0(S)))$$

which descends to a map on the associated Brun-Atiyah-Hirzebruch spectral sequences. Since $\pi_0(S) \simeq \pi_0(R)$ are \mathbb{E}_{∞} -rings, lemma 2.4 applies and both of these Atiyah-Hirzebruch spectral sequences are multiplicative. The E_1 -page of the target is multiplicatively generated by the classes in the image of this map, together with classes generating copies of $\pi_0(R)$ in degrees (0, r(k|x|+2)) for r > 0. Since there are no nonzero classes in bidegree (s, t) for s > 0, the differentials vanish on these classes, and there are no multiplicative extension problems between them. The map from $\text{THH}(\pi_0(R), \tau_{\geq *}(\pi_0(R) \otimes_R \pi_0(R)))$ determines the extension problems on the classes in its image, and this determines all of the additive extension problems, since any nonzero class a in the image of this map multiplies with any nonzero class b coming from $(\pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R))$ to a nonzero class.²

This establishes the claim on the level of homotopy groups. For the full claim, note that we have a map

$$\mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_S \pi_0(R)) \to \mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)) \to \pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R),$$

which admits a splitting $\varphi : \bigoplus_{r \ge 0} \Sigma^{r(k|x|+2)} \pi_0(R) \to \text{THH}(\pi_0(R), \pi_0(R) \otimes_S \pi_0(R))$. Since $\text{THH}(\pi_0(R), \pi_0(R) \otimes_S \pi_0(R))$ admits a $\text{THH}(\pi_0(R), \pi_0(R) \otimes_R \pi_0(R))$ -module structure coming from the natural map, φ extends to a map

 $\operatorname{THH}(\pi_0(R), \pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} \pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R) \to \operatorname{THH}(\pi_0(R), \pi_0(R) \otimes_S \pi_0(R)),$

which provides our desired equivalence.

Remark 5.2. As

Remark 5.2. As the proof indicates, we can replace
$$x^k$$
 by any class x in positive degree such that R/x admits some \mathbb{E}_3 - R -algebra, and such that $(R/x \otimes_R \pi_0(R)) \rightarrow (\pi_0(R) \otimes_R \pi_0(R))$ factors over $(R/x \otimes_R \pi_0(R)) \rightarrow \tau_{\leq 0}((R/x \otimes_R \pi_0(R))) \simeq \pi_0(R)$.

Corollary 5.3. Let R, x and S be as in Theorem 5.1. Then, the E_1 -page of the Whitehead spectral sequence computing THH(S) is isomorphic to the E_1 -page of the Atiyah-Hirzebruch spectral sequence computing THH(R, S) tensored over $\pi_0(R)$ with $\bigoplus_{r>0} \pi_0(R) \cdot a_r$, where a_r is a class in bidegree (-r(k|x|+2), 0).

²In fact, we have maps of underlying graded $\pi_0(S)$ -algebras $\text{THH}_*(R, \pi_0(R)) \rightarrow \text{THH}_*(S, \pi_0(S))$, and $\pi_*(\pi_0(R) \otimes_{\pi_0(R) \otimes_R S} \pi_0(R)) \to \text{THH}_*(S, \pi_0(S))$, which extends to an algebra homomorphism on their graded tensor product, which Theorem 5.1 shows to be an isomorphism, showing there are no multiplicative extension problems either.

Proof. This follows from the string of equivalences

$$\begin{aligned} \operatorname{THH}(S,\pi_*(R)) &\simeq \operatorname{THH}(S,\pi_0(R)) \otimes_{\pi_0(R)} \pi_*(R) \\ &\simeq \operatorname{THH}(R,\pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R)\otimes_R S} \pi_0(R)) \otimes_{\pi_0(R)} \pi_*(R) \\ &\simeq \operatorname{THH}(R,\pi_*(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R)\otimes_R S} \pi_0(R)), \end{aligned}$$

where Theorem 5.1 was used to get the second equivalence.

Remark 5.4. We don't know whether or not

$$\operatorname{THH}_*(S) \simeq \operatorname{THH}_*(R, S) \otimes_{\pi_0(S)} \pi_*(\pi_0(S) \otimes_{\pi_0(S) \otimes_R S} S)$$

in general, although this does seem to hold in many cases.

Together with the map from $\text{THH}(R, \tau_{\geq *}(S))$, this means that to understand THH(S), we need only understand THH(R, S), and how the differentials act on the classes a_r .

§6. The THH of ksc₂

Using the results of section §5, we will compute THH(ksc₂) and show that it fits into the same overall framework. Recall (cf. [Bou90]) that connective self-conjugate *K*-theory, ksc is the \mathbb{E}_{∞} -ring defined as the connective cover of the homotopy fixed points of KU for the action of \mathbb{Z} through complex conjugation ψ^{-1} . In particular, since this \mathbb{Z} action factors over a $\mathbb{Z}/2\mathbb{Z}$ -action, there is a natural \mathbb{E}_{∞} -algebra map ko $\simeq \tau_{\geq 0} \operatorname{KU}^{h\mathbb{Z}/2\mathbb{Z}} \to \tau_{\geq 0} \operatorname{KU}^{h\mathbb{Z}} \simeq$ ksc. This sits naturally in a cofiber sequence Σ^2 ko $\xrightarrow{\eta^2}$ ko \to ksc, which is what allows for the methods developed in §5 to apply.

To begin, we define a filtration similar to what has been studied in [Ang15], [LL23].

Definition 6.1. There is an *equivariant May-style filtration* on ksc₂, given by the filtered \mathbb{E}_{∞} -ring ksc₂^{fil} := $\tau_{\geq 0}^{1/2}((\tau_{\geq *} \operatorname{ku}_2)^{h\mathbb{Z}})$. Similarly, one can define ko₂^{fil} := $\tau_{\geq 0}^{1/2}((\tau_{\geq *} \mu k u_2)^{hC_2})$.

To begin, let's compute the associated graded objects for these equivariant May-style filtrations:

Lemma 6.2. We have

$$\pi_{*,*}(\mathrm{ko}_2^{gr}) = \mathbb{Z}_{(2)}[v_1^2,\eta]/(2\eta),$$

with $|\eta| = (1,2)$, $|v_1| = (0,4)$. The cofiber of η^2 on this filtered spectrum gives the underlying filtered spectrum ksc₂^{fil}, and thus

$$\pi_{*,*}(\mathrm{ksc}_2^{gr}) = \mathbb{Z}_{(2)}[v_1^2, \eta, \rho]/(2\eta, \eta^2, \rho\eta, \rho^2),$$

with $|\rho| = (1, 4)$.

Proof. Consider the filtered \mathbb{E}_{∞} -ring $\tau_{\geq *}$ ku₂. The associated graded is given simply by a polynomial ring $\mathbb{Z}_{(2)}[v_1]$, with $|v_1| = (0, 2)$. Since taking homotopy fixed points commutes with passing to the associated graded, we have

$$((\tau_{\geq *} \operatorname{ku}_2)^{hC_2})^{gr} = (\pi_* \operatorname{ku}_2)^{hC_2}.$$

Computing the homotopy fixed points, it follows that

$$\pi_{*,*}((\pi_* \operatorname{ku}_2)^{hC_2}) = \mathbb{Z}_{(2)}[v_1^2, x, y]/(2x, 2y, y^2 - xv_1^2),$$

where |x| = (2, 0) generates $\mathbb{Z}_{(2)}^{hC_2}$ in filtration degree 0, and |y| = (1, 2) generates the homotopy fixed points of $\mathbb{Z}_{(2)} \cdot v_1$ as a $\mathbb{Z}_{(2)}^{hC_2}$ -module. By the argument in [LL23, Lemma 2.16], it suffices to apply $\tau_{\geq 0}^{1/2}$ to the associated graded. Since, for $i \geq 0$ $\tau_{\geq 0}^{1/2}(x)_i = \tau_{\geq \lfloor i/2 \rfloor}(x_i)$, one sees that

$$\mathrm{ko}_{2}^{gr} = \tau_{\geq 0}^{1/2}((\pi_{*} \,\mathrm{ku})^{hC_{2}}) = \mathbb{Z}_{(2)}[v_{1}^{2}, y]/(2y),$$

proving the first claim.

For ksc₂, we proceed similarly, using that the homotopy fixed points for the trivial action are given by $\mathbb{Z}_{(2)}^{h\mathbb{Z}} \simeq \Lambda(w)$ on a class w in $\pi_{-1}(\mathbb{Z}_{(2)}^{h\mathbb{Z}})$. For the antipodal action, one can compute that $(\mathbb{Z}_{(2)}[t^{\pm 1}]/(t+1))^{h\mathbb{Z}} \simeq \Sigma^{-1}\mathbb{Z}/2\mathbb{Z}$. Putting these together,

$$\pi_{*,*}((\pi_* \operatorname{ku}_2)^{h\mathbb{Z}}) = \mathbb{Z}_{(2)}[v_1^2, w, y]/(w^2, 2y, y^2, wy),$$

with |w| = (1, 0), |y| = (1, 2).

In order to see that $\tau_{\geq 0}^{1/2}$ commutes with taking the associated graded for ksc₂, one must compute $\pi_{*,*}(\tau_{\geq *} \operatorname{ku}_2)^{h\mathbb{Z}}$. For this, note that our filtered object has the form

$$\dots \xrightarrow{\sim} \tau_{\geq 3} \operatorname{ku}_2 \to \tau_{\geq 2} \operatorname{ku}_2 \xrightarrow{\sim} \tau_{\geq 1} \operatorname{ku}_2 \to \operatorname{ku}_2 \xrightarrow{\sim} \dots$$

The homotopy fixed point spectral sequences computing $\pi_*(\tau_{\geq n} \operatorname{ku}_2^{h\mathbb{Z}})$ degenerate at the E_2 -page for all *n*, which yields:

$$\pi_{*,*}((\tau_{\geq *} \operatorname{ku}_2)^{h\mathbb{Z}}) = \mathbb{Z}_{(2)}[\tau, z, \tau^{-2}\eta, \tau^{-4}v_1^2]/(2\tau^{-2}\eta, (\tau^{-2}\eta)^2, z^2, z\tau^{-2}\eta),$$

with $|\tau| = (-1, -1)$, |z| = (1, 0), $|\tau^{-2}\eta| = (1, 2)$, $|\tau^{-4}v_1^2| = (0, 4)$. As there are no classes in bidegrees (n + 1, 2n + 1) for $n \ge 0$, and there is no τ -torsion, it follows that

$$ksc_2^{gr} = \tau_{\geq 0}^{1/2}((\pi_* ku_2)^{h\mathbb{Z}})$$

By the computation above, this has homotopy groups

$$\pi_{*,*}(\mathrm{ksc}_2^{gr}) = \mathbb{Z}_{(2)}[v_1^2,\rho,\eta]/(\eta^2,2\eta,\rho^2,\eta\rho),$$

where ρ comes from the class previously denoted $v_1^2 w$, so $|\rho| = (1,4)$, and η comes from the class previously denoted by *y*, sitting in bidegree (1,2).

Corollary 6.3. The canonical map of graded \mathbb{E}_{∞} -rings $\mathrm{ko}_2^{gr} \to \mathrm{ksc}_2^{gr}$ fits into a fiber sequence of ko_2^{gr} -modules:

$$\Sigma^2 \operatorname{ko}_2^{gr} \xrightarrow{\eta^2} \operatorname{ko}_2^{gr} \to \operatorname{ksc}_2^{gr}$$
.

Proof. Note first that η^2 vanishes under this map, so we get some map from the cofiber of η^2 to $\operatorname{ksc}_2^{gr}$. The easiest way to see that this is an equivalence is to work modulo 2, where, by modifying the computations of Lemma 6.2, one would find that the algebras $\operatorname{ko}_2^{gr}/2$ and $\operatorname{ksc}_2^{gr}/2^3$ are given by $\mathbb{F}_2[v_1,\eta]$ and $\mathbb{F}_2[v_1,\eta]/(\eta^2)$, respectively, which makes the computation clear.

Returning now to the main goal of computing $THH(ksc_2)$, we arrive at the main theorem of this section

Theorem 6.4. There is an isomorphism

$$\Gamma HH_*(ksc_2) \simeq THH_*(ko_2, ksc_2) \otimes_{\mathbb{Z}_{(2)}} \Gamma[\sigma^2 \eta^2],$$

with $\sigma^2 \eta^2$ a class in degree 4.

By corollary 5.3, the Whitehead spectral sequence computing $THH_*(ksc_2)$ has signature which can be identified with

$$E_1^{s,t} = \mathrm{THH}_{-s}(\mathrm{ko}_2, \pi_t(\mathrm{ksc}_2)) \otimes_{\mathbb{Z}_{(2)}} \Gamma[x] \Longrightarrow \mathrm{THH}_{t-s}(\mathrm{ksc}_2),$$

with the divided power class x in bidegree (-4, 0). Using similar methods as in §2, one can recover the fact from [AHL09] that THH_{*}(ko₂, $\mathbb{Z}_{(2)}$) is $\mathbb{Z}_{(2)}$ in degrees 0 and 5; $\mathbb{Z}/2^k\mathbb{Z}$ in degrees $r2^{k+2} - 1$ and $r2^{k+2} - 1 + 5$ for r > 0 odd; and is zero otherwise. Furthermore,

$$\mathrm{THH}_*(\mathrm{ko}_2,\mathbb{F}_2)\simeq\mathbb{F}_2[u^4]\otimes\Lambda[u\xi_1^3,u^2\xi_2].$$

³Both associated graded algebras were $\mathbb{E}_{\infty} - \mathbb{Z}$ -algebras, so the cofiber of 2 has a canonical \mathbb{E}_{∞} -algebra structure on the associated gradeds, even though ko₂/2 itself cannot support any algebra structure.

In particular, when we run the Whitehead spectral sequence for THH(ksc₂), the class $\sigma^2 \eta^2$ in bidegree (-4,0) cannot hit anything for degree reasons, and is thus a permanent cycle, so too then are all powers of $\sigma^2 \eta^2$. The claim will reduce to showing that *x* and all of its divided powers are permanent cycles. To aid in this endeavor, we investigate the equivariant May-type spectral sequence arising from THH(ksc₂^{*il*}), with signature

$$E_1^{s,t} = \pi_{*,*} \operatorname{THH}(\operatorname{ksc}_2^{gr}) \implies \operatorname{THH}_*(\operatorname{ksc}_2).$$

Proposition 6.5. There is an isomorphism of graded algebras

$$\operatorname{THH}_*(\operatorname{ksc}_2^{gr}) \simeq \operatorname{THH}_*(\operatorname{ko}_2^{gr}, \operatorname{ksc}_2^{gr}) \otimes_{\mathbb{Z}_{(2)}} \Gamma[\sigma^2 \eta^2],$$

with $\sigma^2 \eta^2$ a class in degree 4.

Proof. As a consequence of corollary 5.3 and corollary 6.3, the Whitehead spectral sequence computing $\text{THH}_*(\text{ksc}_2^{gr})$ has signature

$$E_1 = \mathrm{THH}_*(\mathrm{ko}_2^{gr}, \pi_*(\mathrm{ksc}_2^{gr})) \otimes_{\mathbb{Z}_{(2)}} \Gamma[\sigma^2 \eta^2] \implies \mathrm{THH}_*(\mathrm{ksc}_2^{gr}).$$

It suffices to show that the classes in $\Gamma[\sigma^2 \eta^2]$ are permanent cycles, since then there will be an algebra map $\text{THH}_*(\text{ko}_2^{gr}, \text{ksc}_2^{gr}) \otimes_{\mathbb{Z}_{(2)}} \Gamma[\sigma^2 \eta^2] \to \text{THH}_*(\text{ksc}_2^{gr})$ inducing the desired isomorphism.

To see this, we will work with the Bockstein spectral sequence associated to the filtered \mathbb{E}_{∞} - \mathbb{Z} -algebra THH(ksc₂^{gr} $\otimes_{\mathbb{Z}} \mathbb{Z}_{(2)}^{bok}$). This spectral sequence has signature

$$E_1^{s,t} = \mathrm{THH}_*(\mathbb{F}_2[\eta, v_1, \tilde{v_0}]/(\eta^2)) \implies \mathrm{THH}_*(\mathrm{ksc}_2^{gr})_2^{\wedge}.$$

By monoidality of THH, we have

$$\operatorname{THH}(\mathbb{F}_{2}[\eta, v_{1}, \tilde{v_{0}}]/\eta^{2}) \simeq \operatorname{THH}(\mathbb{F}_{2}[\eta]/\eta^{2}) \otimes_{\operatorname{THH}(\mathbb{F}_{2})} \operatorname{THH}(\mathbb{F}_{2}[v_{1}]) \otimes_{\operatorname{THH}(\mathbb{F}_{2})} \operatorname{THH}(\mathbb{F}_{2}[\tilde{v_{0}}]).$$

Using that $\mathbb{F}_2[\eta]/\eta^2$ is a square zero extension of \mathbb{F}_2 in \mathbb{E}_{∞} - \mathbb{F}_2 -algebras, we find that

$$\mathrm{THH}_*(\mathbb{F}_2[\eta]/\eta^2) = \mathrm{THH}_*(\mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\eta]/\eta^2 \otimes \Lambda[\sigma\eta] \otimes_{\mathbb{F}_2} \Gamma[\sigma^2\eta^2].$$

From this, it follows that

$$\mathrm{THH}_{*}(\mathbb{F}_{2}[\eta, v_{1}, \tilde{v_{0}}]/\eta^{2}) = \mathrm{THH}_{*}(\mathbb{F}_{2}) \otimes_{\mathbb{F}_{2}} \mathbb{F}_{2}[\eta, v_{1}, \tilde{v_{0}}]/\eta^{2} \otimes_{\mathbb{F}_{2}} \Lambda[\sigma\eta, \sigma v_{1}, \sigma\tilde{v_{0}}] \otimes_{\mathbb{F}_{2}} \Gamma[\sigma^{2}\eta^{2}].$$

In the 2-Bockstein spectral sequence, the multiplicative generators have bidegrees $|\tilde{v_0}| = (1, 1)$, $|\sigma \tilde{v_0}| = (0, 1)$, $|\sigma \eta| = (-2, 0)$, $|\sigma v_1| = (-3, 0)$, $|\eta| = (-1, 0)$, $|v_1| = (-2, 0)$, $\sigma^2 \eta^2 = (-4, 0)$, and the class *u* in bidegree (-2, 0) which generates THH_{*}(\mathbb{F}_p) as a polynomial algebra. We begin by examining the class $\sigma^2 \eta^2$ in degree (-4, 0). Almost all of the classes in the $\tilde{v_0}$ -tower on the class $\sigma^2 \eta^2$ must survive the spectral sequence in order to give the $\mathbb{Z}_{(2)} \cdot \sigma^2 \eta^2$ class in degree 4 of THH_{*}(ksc₂^{gr}). One finds that $\sigma^2 \eta^2$ cannot support any differentials, since any nontrivial differential on this class would kill the entire tower. We make use of the following lemma

Lemma 6.6. If a is a class on the E_k -page of the Bockstein spectral sequence computing THH(ksc₂^{gr})₂^{\wedge}, with $a \neq 0$, but $\tilde{v_0}a = 0$, then deg^t(a) $\leq k - 1$.

Proof. For k = 1, this is vacuous, since there is no \tilde{v}_0 -torsion on the E_1 -page of this spectral sequence. We proceed by induction. Suppose that $a \neq 0$ is a class on the E_k -page with $\tilde{v}_0 a = 0$, and deg^t(a) $\geq k$. The fact that $\tilde{v}_0 a = 0$ means that at some point earlier in the spectral sequence, say on the E_{k-i} -page (i > 0), we had a class b with $d_{k-i}(b) = \tilde{v}_0 a$. b then necessarily has

t-degree i + 1 > 1. In particular, $\tilde{v_0}$ must divide *b* for degree reasons, so $b = c\tilde{v_0}$, for some class *c* (or more accurately, comes from a class on E_1 divisible by $\tilde{v_0}$, and by our inductive hypothesis, there cannot be a differential taking *c* to a nonzero class which multiplies with $\tilde{v_0}$ to zero). Now, $d(c) \neq a$, but $d(\tilde{v_0}c) = d(b) = \tilde{v_0}a$, so that $a - d(c) \neq 0$, but $\tilde{v_0}(a - d(c)) = 0$. Since a - d(c) is a $\tilde{v_0}$ -torsion class on E_{k-i} with *t*-degree $k \geq k - i$, it must be 0 by induction, so that a = d(c), contradicting the choice of *a*.

This lemma implies that if the differential of any class in *t*-degree 0 or 1 is nontrivial, then the entire $\tilde{v_0}$ -tower on that class dies. Let *n* be the smallest natural such that $(\sigma^2 \eta^2)^{(2^n)}$ does not live in THH_{*}(ksc₂^{gr}). In particular, we must have that the entire $\tilde{v_0}$ -tower on the analogous class in the mod 2 Bockstein must vanish. Since $(\sigma^2 \eta^2)^{(2^{n-1})}$ squares to a torsion-free class, there must be a nonvanishing $\tilde{v_0}$ tower in total degree $2 \cdot 2^{n-1} \cdot 4 = 2^{n+2}$ in the Bockstein spectral sequence. Considering the map from the spectral sequence associated to THH($\mathbb{Z}_{(2)}^{fil}$) shows that the classes divisible by *u* and $\sigma \tilde{v_0}$ are all $\tilde{v_0}$ -torsion. In order for η to be 2-torsion, we need a differential to hit $\eta \tilde{v_0}$, and this can be checked to come from $\sigma \eta$. Thus, the only classes that can contribute to a nonvanishing $\tilde{v_0}$ -tower are multiples of the classes $(\sigma^2 \eta^2)^{(2^k)}$, for k < n, powers of v_1 , and σv_1 . Since no power of v_1 divides any element of $\mathbb{Z}_{(2)}[\sigma^2 \eta^2]$, the only contribution can come from σv_1 and the $(\sigma^2 \eta^2)^{(2^k)}$. The total degree of $\sigma v_1 \cdot \prod_{k < n} (\sigma^2 \eta^2)^{(2^k)}$ is 2^{n+1} , but the tower we need is in total degree 2^{n+2} , and thus must come from $(\sigma^2 \eta^2)^{(2^n)}$! This shows that all of our $(\sigma^2 \eta^2)^{(2^n)}$ classes have to survive this Bockstein spectral sequence, proving that they survive to give the divided power classes in THH_*(ksc_2^{gr}) that we were looking for.

Proof of Theorem 6.4. By proposition 6.5, the equivariant May-type spectral sequence has E_1 -page

THH_{*}(ko^{gr}₂, ksc^{gr}₂)
$$\otimes_{\mathbb{Z}_{(2)}} \Gamma[\sigma^2 \eta^2].$$

The class $\sigma^2 \eta^2$ sits in (s,t)-degree (0,4), and the only classes in THH_{*}(ko₂^{gr}, ksc₂^{gr}) with positive s-degree are the classes ρ and η , in degrees (1,4) and (1,2), respectively. Thus, the classes in $\Gamma[\sigma^2 \eta^2]$ are permanent cycles. By comparing with the map THH_{*}(ksc₂^{gr}) \rightarrow THH_{*}(ksc₂^{gr}, $\mathbb{Z}_{(2)}$), we find that $\sigma^2 \eta^2$ reduces to the same class originally termed x in the Whitehead spectral sequence. In particular, this analysis shows that the classes $\Gamma[x]$ in the Whitehead spectral sequence for THH(ksc₂) must be permanent cycles as well, so that the E_{∞} -page of the Whithead spectral sequence is the tensor product of the E_{∞} -page of the Atiyah-Hirzebruch spectral sequence computing THH(ko₂, ksc₂) with the divided power algebra $\Gamma[x]$. We find that there is an induced map of graded-commutative $\mathbb{Z}_{(2)}$ -algebras

$$\operatorname{THH}_{*}(\operatorname{ko}_{2},\operatorname{ksc}_{2})\otimes_{\mathbb{Z}_{(2)}}\Gamma[\sigma^{2}\eta^{2}] \to \operatorname{THH}_{*}(\operatorname{ksc}_{2}),$$

which, from the computation of the Whitehead spectral sequence for $THH(ksc_2)$, must be an isomorphism.

Remark 6.7. THH_{*}(ko₂, ksc₂) can be computed as a graded abelian group from the work of [AHL09], noting that they prove that η^2 acts as zero on THH_{*}(ko₂), which determines

$$\overline{\text{THH}}(\text{ko}_2, \text{ksc}_2) \simeq \text{cofib}(\eta^2 : \Sigma^2 \overline{\text{THH}}(\text{ko}_2) \rightarrow \overline{\text{THH}}(\text{ko}_2))$$

up to extension problems. Since the only classes in $T\bar{H}H_*(ko_2)$ in odd degrees are copies of $\mathbb{Z}_{(2)}$ living in what [AHL09] call F^{ko} , there can only be nontrivial extension problems if the map from

$$\text{THH}_{5+4n}(\text{ko}_2) \rightarrow \text{THH}_{5+4n}(\text{ko}_2, \text{ksc}_2)$$

is not surjective on the torsion-free parts. However, we know from the computations in [AHL09, §7.2-7.3] that

$$\text{THH}_{5+4n}(\text{ko}_2) \rightarrow \text{THH}_{5+4n}(\text{ko}_2, \text{ku}_2)$$

induces an isomorphism on the torsion-free part, and this map factors as

$$\text{THH}_{5+4n}(\text{ko}_2) \rightarrow \text{THH}_{5+4n}(\text{ko}_2, \text{ksc}_2) \rightarrow \text{THH}_{5+4n}(\text{ko}_2, \text{ku}_2).$$

Thus, we find that, as a graded abelian group,

 $\text{THH}_*(\text{ko}_2, \text{ksc}_2) \simeq \text{THH}_*(\text{ko}_2) \oplus \text{THH}_{*-3}(\text{ko}_2).$

Combined with the above, we have completely determined $THH_*(ksc_2)$.

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