WHITEHEAD FILTRATIONS FOR COMPUTATIONS IN TOPOLOGICAL HOCHSCHILD HOMOLOGY

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ABSTRACT. We discuss spectral sequences coming from Whitehead filtrations in the computation of topological Hochschild homology of ring spectra. Using cyclic invariance, this makes for simple computations of THH of connective rings R with coefficients in discrete ring spectra.[a] In particular, we show how to use this to compute $\mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2)$, and $\mathrm{THH}(\mathrm{tmf}, \mathbb{Z}_{(2)})$, where tmf denotes the $\mathbb{E}_\infty$ ring spectrum of topological modular forms. Then, we obtain a description of $\mathrm{THH}(\ell/v_1^n)$ in terms of $\mathrm{THH}(\ell, \ell/v_1^n)$, where the latter can be computed by results of [AHL09]. We next explain how the methods of this computation generalize to give us information about $\mathrm{THH}(\mathrm{cofib}(x^k : \Sigma^{k|x|}R \to R))$ for R and $\mathrm{cofib}(x^k)$ suitably structured connective ring spectra, $k > 1$, and $x \in \pi_*(R)$ an arbitrary element in positive degree. Finally, we examine the general framework to describe the topological Hochschild homology of 2-local connective self-conjugate K-theory, ksc_2.

[a]Where discrete means Eilenberg-MacLane.

CONTENTS

§1. INTRODUCTION

Topological Hochschild homology was introduced by Bökstedt in 1985 as a generalization of ordinary Hochschild homology to general homotopy coherent ring spectra, which has lead to many recent advances in algebraic K-theory [NS18], which in turn, lead to the recent disproof of Ravenel's telescope conjecture [Bur+23]. The aim of this paper is to discuss spectral sequences arising from Whitehead filtrations as a means to compute topological Hochschild homology (possibly with coefficients) over connective ring spectra. When combined with cyclic invariance, this recovers the Brun spectral sequence, and we get nice comparison maps for computing THH with coefficients.

For an $\mathbb{E}_1$-ring spectrum R, the topological Hochschild homology of R is defined as $\mathrm{THH}(R) = R \otimes_{R\otimes_{\mathbb{S}} R^{op}} R$, and the topological Hochschild homology with coefficients in a R-bimodule M is defined as $\mathrm{THH}(R, M) = R \otimes_{R\otimes_{\mathbb{S}} R^{op}} M$. Assuming that R is connective, we can apply the Whitehead filtration to R to get a filtered $R \otimes_{\mathbb{S}} R^{op}$-module spectrum $\tau_{\geq *}R$, to which we can then apply the functor $R \otimes_{R\otimes_{\mathbb{S}} R^{op}} - : \mathrm{Fil}(R \otimes_{\mathbb{S}} R^{op} - \mathrm{Mod}_L) \to \mathrm{Fil}(R - \mathrm{Mod}_L)$, giving us a filtered R-module spectrum. This gives rise to a spectral sequence with signature

$$E_1^{s,t} = \mathrm{THH}_{-s}(R, \pi_t(R)) \implies \mathrm{THH}_{t-s}(R),$$

where we consider $\pi_t(R)$ as a discrete $R \otimes_{\mathbb{S}} R^{op}$-module in degree zero. We get a similar spectral sequence for $\mathrm{THH}(R, M)$ whenever R is a connective $\mathbb{E}_1$-ring, and M an R-bimodule. In section 2, we discuss some basic results on spectral sequences of this type, to be used in the rest of the paper.

In section 3, we will use the results of section 2 in order to compute $\mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2)$, and then combining the Brun spectral sequence with the Bockstein spectral sequence, we compute $\mathrm{THH}(\mathrm{tmf}, \mathbb{Z}_{(2)})$, using this result. The main theorem of section 3 is the computation:

Theorem 1.1.

$$\mathrm{THH}_*(\mathrm{tmf}, \mathbb{Z}_{(2)}) = \begin{cases} \mathbb{Z}_{(2)} & \text{if } * = 0, 9, 13, 22 \\ \mathbb{Z}/2^k\mathbb{Z} & \text{if } * = 2r^{k+3} - 1, 2^{k+3}r - 1 + 9, 2^{k+3}r - 1 + 13, 2^{k+3}r - 1 + 22, \\ 0 & \text{otherwise}, \end{cases}$$

for all $k > 0$ and r odd.

Section 4 uses the results of section 2, together with the spectral sequences constructed by Lee-Levy [LL23] in order to compute the topological Hochschild homology of quotients of ℓ, $\mathrm{THH}(\ell/v_1^n)$, where ℓ is the mod p Adams summand for some fixed odd prime p, and $v_1 \in \pi_2(\ell)$ generates $\pi_*(\ell)$ as a polynomial algebra over $\mathbb{Z}_{(p)}$. The methods used in this section extend to prove:

Theorem 1.2. *Suppose that R is a connective $\mathbb{E}_m$-ring spectrum for some $m \geq 3$, and $x \in \pi_*(R)$ is a positive degree class such that, for some fixed $k > 1$, $R/x^k := \mathrm{cofib}(x^k : R \to R)$ admits an $\mathbb{E}_3$-algebra structure. Then, we have an equivalence of $\tau_{\leq 0}R$-modules $\mathrm{THH}(R/x^k, \pi_0(R)) \simeq \mathrm{THH}(R, \pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R) \otimes_R R/x^k} \pi_0(R))$.*

Under these same hypotheses, this result allows us to construct a spectral sequence with signature

$$E_1^{*,*} = \mathrm{THH}_{-*}(R, \pi_*(R/x^k)) \otimes_{\pi_0(R)} \Big(\bigoplus_{i \geq 0} \pi_0(R) \cdot a_i\Big) \implies \mathrm{THH}_*(R/x^k),$$

where a_i is a class in bidegree $(-i(2k|x|+2), 0)$. The differentials on the classes in $\mathrm{THH}_{-*}(R, \pi_*(R/x^k))$ are determined by the spectral sequence $\mathrm{THH}_{-*}(R, \pi_*(R/x^k)) \implies \mathrm{THH}_*(R, R/x^k)$, so in order to understand this spectral sequence, one must only understand what the differentials do to the classes a_i.

As another sample application, we compute the topological Hochschild homology of 2-local self-conjugate K-theory ksc_2 in section §6.

Notation/Conventions

- We will say "category" to refer to ∞-categories.
- Our filtered objects and spectral sequences will be as in [LL23], in particular, the d_r differential will go from $E_r^{s,t}$ to $E_r^{s+r+1,t+r}$.
- p will denote a fixed odd prime.
- ℓ will denote the mod p Adams summand, and tmf will denote the spectrum of topological modular forms.
- When dealing with divided power algebras, we will use the notation $x^{(k)}$ for the class "$\frac{x^k}{k!}$."

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§2. Filtered Objects and Spectral Sequences

We recall the definitions of filtered objects, following the description in [LL23]. For a stable category $\mathcal{C}$, we can associate the stable categories $\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}_{\leq}^{op}, \mathcal{C})$ and $\mathcal{C}^{gr} := \mathrm{Fun}(\mathbb{Z}^{ds,op}, \mathcal{C})$, of filtered objects and graded objects in $\mathcal{C}$, respectively. Here $\mathbb{Z}_{\leq}$ denotes the poset category on $\mathbb{Z}$ with the $\leq$ order, and $\mathbb{Z}^{ds}$ denotes the discrete category with objects the integers. For $c \in \mathcal{C}$, we have the object $c^{n,m} \in \mathrm{Fil}(\mathcal{C})$, where $c^{n,m} = \Sigma^n c^{0,n+m}$, and $c^{0,i}$ is defined as the image of c under the left adjoint to $ev_i : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ (i.e., $(c^{0,i})_j = c$ for $j \leq i$, with the identity map as transition maps, and is 0 for $j > i$). If $\mathcal{C}$ is symmetric monoidal, $\mathrm{Fil}(\mathcal{C})$ and $\mathcal{C}^{gr}$ inherit symmetric monoidal structures by Day convolution, with unit $1 = 1^{0,0}$, where 1 is the unit in $\mathcal{C}$. In this case, define bigradings on the homotopy groups of a filtered object by $\pi_{n,m}(A) = \pi_0 \mathrm{Hom}_{\mathrm{Fil}(\mathcal{C})}(1^{n,m}, A)$. There is an element $\tau : 1^{0,-1} \to 1^{0,0}$, which when tensored with any filtered object A, gives a morphism of filtered objects, which is the filtration map $A_{i+1} \to A_i$ in degree i. Modules over the cofiber of τ are identified with graded objects, and for any filtered object A, the cofiber sequence $A \otimes 1^{0,-1} \xrightarrow{\tau} A \to A^{gr} := A/\tau$ gives rise to an exact couple. Out of this arises a spectral sequence with signature (using the grading conventions from [LL23]):

$$E_1^{s,t} = \pi_{t-s,s} A^{gr} \implies \pi_{t-s}(A[\tau^{-1}]),$$

where $A[\tau^{-1}]$ is the "underlying" object of A obtained by inverting τ.

We now introduce the main spectral sequences that we will be using in this paper. Consider a connective $\mathbb{E}_1$-ring R, and an R-bimodule M, i.e., a left $R \otimes_{\mathbb{S}} R^{op}$-module. Working in the category $\mathrm{Fil}(\mathrm{LMod}_{R\otimes_{\mathbb{S}} R^{op}})$, we can take $\tau_{\geq *} M$ to be the Whitehead filtration on M, i.e., the filtered spectrum with mth graded piece $\tau_{\geq m} M$, with maps $\tau_{\geq m} M \to \tau_{\geq m-1} M$ the obvious ones. Applying the functor $\mathrm{THH}(R; -) : \mathrm{Fil}(\mathrm{LMod}_{R\otimes_{\mathbb{S}} R^{op}}) \to \mathrm{Fil}(\mathrm{LMod}_R)$, we can then form the associated spectral sequence with signature $E_1^{s,t} = \mathrm{THH}_{-s}(R, \pi_t(R)) \Rightarrow \mathrm{THH}_{t-s}(R)$, where we are treating $\pi_t(R)$ as a discrete $R \otimes_{\mathbb{S}} R^{op}$-module concentrated in degree 0. We find that:

Lemma 2.1. *The filtration on* $\mathrm{THH}(R; M)$ *induced by the above construction is complete.*

Proof. Since $\mathrm{THH}(R; -)$ is right t-exact, the nth filtered piece of $\mathrm{THH}(R; M)$ is n-connective, and the result follows. ◻

We recall that static modules over $R \otimes_{\mathbb{S}} R^{op}$ in degree 0 are exactly the modules which live in the heart of $\mathrm{LMod}_{R\otimes_{\mathbb{S}} R^{op}}$, and are thus in bijection with $\pi_0(R \otimes_{\mathbb{S}} R^{op}) \simeq \pi_0(R) \otimes_{\mathbb{Z}} \pi_0(R)^{op}$-modules. It is clear that this spectral sequence is functorial in M, and we also note that:

Proposition 2.2. *Let R and S be connective $\mathbb{E}_1$-ring spectra, and $f : R \to S$ a $\mathbb{E}_1$-ring map. If M is an R-bimodule, then the natural map* $\mathrm{THH}(R, M) \to \mathrm{THH}(S, (S \otimes S^{op} \otimes_{R\otimes R^{op}} M)$ *induces a map on the associated spectral sequences.*

Proof. It suffices to show that this holds on the level of filtered objects. Using the fact that we have natural factorizations $\tau_{\geq n} M \to (S \otimes S^{op}) \otimes_{R\otimes R^{op}} \tau_{\geq n} M \to \tau_{\geq n}(S \otimes S^{op}) \otimes_{R\otimes R^{op}} M$, we find that we have a map of filtered objects $\mathrm{THH}(R, \tau_{\geq \bullet} M) \to \mathrm{THH}(S, (S \otimes S^{op}) \otimes_{R\otimes R^{op}} \tau_{\geq \bullet} M) \to \mathrm{THH}(S, \tau_{\geq \bullet}(S \otimes S^{op}) \otimes_{R\otimes R^{op}} M)$, giving the claim. ◻

To make use of the spectral sequences above, it would be useful to have a point of comparison so we can start getting some differentials, and to know about the multiplicative structure. On the side of multiplicative structure, we have:

Lemma 2.3. *Suppose R is a connective $\mathbb{E}_n$-ring spectrum for some $n \geq 4$. Then the above spectral sequence is multiplicative whenever M is an $\mathbb{E}_2$-R-algebra.*

Proof. Suppose R is a $\mathbb{E}_n$-algebra for some $n \geq 4$. In particular, $R \otimes_{\mathbb{S}} R^{op}$ is an $\mathbb{E}_n$-algebra as well, and the canonical multiplication map $R \otimes_{\mathbb{S}} R^{op} \to R$ is an $\mathbb{E}_{n-1}$-algebra map. Thus, the functor $R \otimes_{R \otimes_{\mathbb{S}} R^{op}} - : \mathrm{LMod}_{R \otimes_{\mathbb{S}} R^{op}} \to \mathrm{LMod}_R$ is $\mathbb{E}_{n-2}$-monoidal, and in particular, $\mathbb{E}_2$-monoidal. Since the Whitehead filtration is compatible with multiplicative structure, this gives the claim. □

Next, we want to be able to find other useful spectral sequences to compare these to that will allow us to figure out some of their differentials. This is where cyclic invariance comes in. We work here in the case where R is an $\mathbb{E}_\infty$-ring, but the setup should work more generally with minimal modifications:

Proposition 2.4. *For R an $\mathbb{E}_1$-ring, and S a $\mathbb{E}_1$-R-algebra, we have an equivalence* $\mathrm{THH}(R, S) \simeq \mathrm{THH}(S, S \otimes_R S)$*. If R and S are both $\mathbb{E}_\infty$-algebras, then this is an equivalence of $\mathbb{E}_\infty$-algebras.*

Proof. See [LL23] where this equivalence is proven for R and S $\mathbb{E}_1$-algebras. The multiplicative identification for the $\mathbb{E}_\infty$ case follows from the string of equivalences $\mathrm{THH}(R, S) \simeq R \otimes_{R \otimes R} S \simeq R \otimes_{R \otimes R} ((S \otimes S) \otimes_{S \otimes S} S) \simeq (R \otimes_{R \otimes R} (S \otimes S)) \otimes_{S \otimes S} S \simeq (S \otimes_R S) \otimes_{S \otimes S} S \simeq \mathrm{THH}(S, S \otimes_R S)$, where each of these equivalences can be viewed as rewriting the same colimit in $\mathbb{E}_\infty$-rings, giving the identification as $\mathbb{E}_\infty$-rings. □

When we apply the Whitehead filtration to $S \otimes_R S$ in a situation as above, the resulting spectral sequence has been termed the Brun spectral sequence, introduced by Brun in [Bru00] and studied by Höning in [Hö20].

For R an arbitrary connective $\mathbb{E}_n$-ring spectrum, this already gives us a lot to compare R with. Namely, it is clear that $\mathrm{THH}(R, R \otimes_{\mathbb{S}} R^{op}) \simeq R$, so applying the Whitehead filtration to the coefficients produces a spectral sequence $E_1^{s,t} = \mathrm{THH}_{-s}(R, \pi_t(R)) \Rightarrow \pi_{t-s}(R)$. Furthermore, the natural map $R \otimes R^{op} \to R$ provides a map on spectral sequences which is surjective on the $\mathbb{E}_1$-page. Thus, if one could determine the structure of the spectral sequence associated to $\mathrm{THH}(R, R \otimes R^{op})$, one could compute $\mathrm{THH}(R)$, although the structure of the former can get quite complicated in general.

§3. Topological Hochschild Homology of tmf with Coefficients

In this section, we apply the tools developed in section 2 to study the topological Hochschild homology of connective topological modular forms with coefficients in $\mathbb{F}_2$ and $\mathbb{Z}_{(2)}$. These computations, as well as those with other coefficients, were studied independently by Bruner-Rognes in a work in progress [BR14]. We begin with $\mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2)$, where cyclic invariance makes this computation easy. As a precursor, we analyze $\mathrm{THH}(\mathbb{F}_2, \mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2) \simeq \mathbb{F}_2$:

Example 3.1. We have a spectral sequence $E_1^{s,t} = \mathrm{THH}_{-s}(\mathbb{F}_2, \pi_t(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2)) \Rightarrow \mathbb{F}_2$. By Bökstedt's computation of $\mathrm{THH}(\mathbb{F}_2)$, we know that the signature of this spectral sequence is

$$E_1^{s,t} = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u] \implies \mathbb{F}_2,$$

where ξ_i has bidegree $(0, 2^i - 1)$, and u has bidegree $(-2, 0)$. Since this spectral sequence converges to $\mathbb{F}_2$, we find that u must map to ξ_1 on the E_1-page, leaving $\mathbb{F}_2[\xi_2, \xi_3, \dots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u^2]$ on the E_2-page. Repeating inductively, we find that $u^{2^{n-1}}$ maps to ξ_n on the E_{2^n-1}-page for degree reasons, leaving $\mathbb{F}_2[\xi_{n+1}, \xi_{n+2}, \dots] \otimes_{\mathbb{F}_2} \mathbb{F}_2[u^{2^n}]$ on the E_{2^n}-page, and these differentials together with multiplicativity, account for all of the nonzero differentials in the spectral sequence.

Example 3.2. In a similar fashion, we can determine the differentials in the spectral sequence associated to $\mathrm{THH}(\mathbb{F}_p, \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p) \simeq \mathbb{F}_p$, which has signature

$$E_1^{s,t} = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes \Lambda[\tau_0, \tau_1, \dots] \otimes \mathbb{F}_p[u] \implies \mathbb{F}_p$$

, where $|\xi_i| = (0, 2(p^i - 1))$, $|\tau_i| = (0, 2p^i - 1)$, and $|u| = (-2, 0)$. We find that the differentials are determined on multiplicative generators (on respective pages) by $d_1(u) = \tau_0$, $d_{2p-3}(u^{p-1}\tau_0) = \xi_1$, and generally, $d_{2p^i-1}(u^{p^i}) = \tau_i$, $d_{2p^{i-1}(p-1)}(u^{p^{i-1}(p-1)}\tau_{i-1}) = \xi_i$

Example 3.3. Now, we can compare with the analogous spectral sequence for tmf, recalling that $\pi_*(\mathbb{F}_2 \otimes_{\mathrm{tmf}} \mathbb{F}_2) = \mathbb{F}_2[\xi_1, \xi_2, \xi_3]/(\xi_1^8, \xi_2^4, \xi_3^2)$ as a quotient of the dual Steenrod algebra. The signature of this spectral sequence is

$$E_1^{s,t} = \mathbb{F}_2[u] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\xi_1, \xi_2, \xi_3]/(\xi_1^8, \xi_2^4, \xi_3^2) \implies \mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2).$$

By the comparison map from $\mathrm{THH}(\mathbb{F}_2, \mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2)$, we find that u maps to ξ_1 on the E_1-page, leaving

$$E_2^{s,t} \simeq \mathbb{F}_2[u^2] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\xi_2, \xi_3]/(\xi_2^4, \xi_3^2) \otimes_{\mathbb{F}_2} \Lambda[\xi_1^7 u],$$

u^2 maps to ξ_2 on the E_3-page, leaving

$$E_4^{s,t} \simeq \mathbb{F}_2[u^4] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\xi_3]/(\xi_3^2) \otimes_{\mathbb{F}_2} \Lambda[\xi_1^7 u, \xi_2^3 u^2],$$

and u^4 maps to ξ_3 on the E_7 page, leaving us with

$$E_8^{s,t} = E_\infty^{s,t} = \mathbb{F}_2[u^8] \otimes_{\mathbb{F}_2} \Lambda(u\xi_1^7, u^2\xi_2^3, u^4\xi_3),$$

on the E_8-page, where the spectral sequence must degenerate for degree reasons since u^8 cannot hit any nonzero class. There are no multiplicative extension problems for degree reasons, so we find that

$$\pi_* \mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2) \simeq \pi_* \mathrm{THH}(\mathbb{F}_2, \mathbb{F}_2 \otimes_{\mathrm{tmf}} \mathbb{F}_2) \simeq \mathbb{F}_2[u^8] \otimes_{\mathbb{F}_2} \Lambda(u\xi_1^7, u^2\xi_2^3, u^4\xi_3)$$

is the tensor product of a polynomial algebra with a generator in degree 16 with an exterior algebra on classes in degrees 9, 13, and 15. For future reference, let us write $\pi_*(\mathrm{THH}(\mathrm{tmf}, \mathbb{F}_2)) = \mathbb{F}_2[\alpha] \otimes_{\mathbb{F}_2} \Lambda(\lambda_1, \lambda_2, \lambda_3)$, with $|\alpha| = 16$, $|\lambda_1| = 9$, $|\lambda_2| = 13$, and $|\lambda_3| = 15$.

Before we compute $\mathrm{THH}(\mathrm{tmf}, \mathbb{Z}_{(2)})$, let's warm up by redoing the well-known computation for $\mathrm{THH}(\ell, \mathbb{Z}_{(p)})$ at an odd prime p:

Example 3.4. Note that in a fashion similar to Example 3.3, using $\pi_*(\mathbb{F}_p \otimes_\ell \mathbb{F}_p) \simeq \Lambda[\tau_0, \tau_1]$, we find that

$$\mathrm{THH}_*(\ell, \mathbb{F}_p) \simeq \mathbb{F}_p[u^{p^2}] \otimes_{\mathbb{F}_p} \Lambda[u^{p-1}\xi_1, u^{p(p-1)}\xi_2].$$

Since $\pi_*(\ell) \simeq \mathbb{Z}_{(p)}[v_1]$, with $|v_1| = 2$, it is easy to see that $\pi_*(\mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)}) \simeq \Lambda_{\mathbb{Z}_{(p)}}[\rho]$, with $|\rho| = 2p-1$. The E_1-page of our Whitehead spectral sequence for $\mathrm{THH}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)})$ is given by $\mathrm{THH}_*(\mathbb{Z}_{(p)}) \otimes \Lambda[\rho]$, with ρ in bidegree $(0, 2p-1)$, and $a \in \mathrm{THH}_i(\mathbb{Z}_{(p)})$ in bidegree $(-i, 0)$.

Since we only have one copy of $\mathbb{F}_p$ in $\pi_{2p-1}(\mathrm{THH}(\ell, \mathbb{F}_p))$, we find that there must be a multiplicative extension between ρ and the $\mathbb{Z}/p\mathbb{Z}$ class in $\mathrm{THH}_{2p-1}(\mathbb{Z}_{(p)})$. By examining the locations where elements of $\mathrm{THH}(\ell, \mathbb{F}_p)$ are nonzero, and working inductively, we find that for all $r > 1$, there is a nonzero differential on the class $\mathrm{THH}_{2rp-1}(\mathbb{Z}_{(p)})$, which ends up giving us that

$$\mathrm{THH}_*(\ell, \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } * = 0, 2p-1 \\ \mathbb{Z}/p^k\mathbb{Z} & \text{if } * = 2rp^{k+1} - 1, 2rp^{k+1} - 1 + 2p - 1 \text{ with } k > 0, \gcd(r,k) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The case for $\mathrm{THH}(\mathrm{tmf}, \mathbb{Z}_{(2)})$ is slightly more complicated. First, we need to know $\pi_*(\mathbb{Z}_{(2)} \otimes_{tmf} \mathbb{Z}_{(2)})$. To compute this, we start by looking at $\pi_*(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf})$.

Proposition 3.5. *We have* $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf} \simeq \mathbb{Z}_{(2)}[\xi_1^8] \otimes^{\mathbb{L}}_{\mathbb{Z}_{(2)}} \mathbb{Z}_{(2)}[\xi_2^4] \otimes^{\mathbb{L}}_{\mathbb{Z}_{(2)}} \bigotimes^{\mathbb{L}}_{\mathbb{Z}_{(2)}, i>2} \mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i)$ *as a* $\mathbb{Z}_{(2)}$*-module.*

Proof. We know, for instance by [Mat15], that $\pi_*(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathrm{tmf}) \simeq \mathbb{F}_2[\xi_1^8, \xi_2^4, \xi_3^2, \xi_4, \xi_5, \ldots]$, identified as a subalgebra of the dual Steenrod algebra via the algebra map $\mathrm{tmf} \to \mathbb{F}_2$. Next, note that $\mathbb{F}_2 \otimes_{\mathbb{S}} \mathrm{tmf} \simeq \mathrm{cofib}(2 : \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf} \to \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf})$, and we have a commutative diagram of ring spectra:

$$\begin{array}{ccc} \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf} & \longrightarrow & \mathbb{F}_2 \otimes_{\mathbb{S}} \mathrm{tmf} \\ \downarrow & & \downarrow \\ \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)} & \longrightarrow & \mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2. \end{array}$$

The long exact sequence associated to the cofiber sequence tells us that we must have torsion-free classes in $\pi_8(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf})$ and $\pi_{12}(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf})$ mapping to ξ_1^8 and ξ_2^4, respectively. Since $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)} \simeq \bigotimes^{\mathbb{L}}_{\mathbb{Z}_{(2)}, i>0} \mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i)$, we find that the class in degree 8 must map to ξ_1^8 under the left vertical map, and the class in degree 12 must map to ξ_2^4 under the same map, and by abuse, we will give these classes in $\pi_*(\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf})$ the same name. For $i \geq 1$, the fact ξ_i^2 is 2-torsion in $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)}$ reflects the fact that, under taking the cofiber of 2 on this spectrum, the class in degree $2(2^i - 1) + 1 = 2^{i+1} - 1$ arising from this torsion is ξ_{i+1}. Since we don't have any classes in $\pi_*(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathrm{tmf})$ of the form $\xi_2 \xi_1^k$ or $\xi_3 \xi_2^k$, the powers of ξ_1^8 and ξ_2^4 form a polynomial algebra over $\mathbb{Z}_{(p)}$, giving the first two tensor factors above. The other tensor factors agree with those in $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)}$, which follows from examining the classes in $\pi_*(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathrm{tmf})$ and using commutativity of the above diagram. Since $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf}$ is 2-local, and we don't have any more classes showing up on the cofiber of 2, this determines $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf}$, as desired. □

From this, we can deduce:

Proposition 3.6.

$$\pi_*(\mathbb{Z}_{(2)} \otimes_{\mathrm{tmf}} \mathbb{Z}_{(2)}) \simeq \begin{cases} \mathbb{Z}_{(2)} & \text{if } * = 0, 22, \\ \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } * = 9, 13, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } * = 2, 4, 8, 10, 11, 12, 17, 19, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } * = 6, 15 \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Note that $\mathbb{S} \to \mathbb{Z}_{(2)}$ factors as $\mathbb{S} \to \mathrm{tmf} \to \mathbb{Z}_{(2)}$. Thus, we have a diagram of pushout squares of $\mathbb{E}_\infty$-ring spectra:

$$\begin{array}{ccccc} \mathbb{S} & \longrightarrow & \mathrm{tmf} & \longrightarrow & \mathbb{Z}_{(2)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_{(2)} & \longrightarrow & \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf} & \longrightarrow & \mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)} \\ & & \downarrow & & \downarrow \\ & & \mathbb{Z}_{(2)} & \longrightarrow & \mathbb{Z}_{(2)} \otimes_{\mathrm{tmf}} \mathbb{Z}_{(2)}, \end{array}$$

i.e., $\mathbb{Z}_{(2)} \otimes_{\mathrm{tmf}} \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)} \otimes_{\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf}} (\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)})$. This square splits up further by writing $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathrm{tmf}$ and $\mathbb{Z}_{(2)} \otimes_{\mathbb{S}} \mathbb{Z}_{(2)}$ as an infinite (derived) tensor product of $\mathbb{Z}_{(2)}$-algebras, as in Proposition 3.5. For $i \geq 3$, since $\mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i^2) \to \mathbb{Z}_{(2)}[\xi_i^2]/(2\xi_i^2)$ is an equivalence, this only contributes a $\mathbb{Z}_{(2)}$ to the final tensor product, and can be ignored. We are left to compute

$$\mathbb{Z}_{(2)}[\xi_1^2]/(2\xi_1^2) \otimes_{\mathbb{Z}_{(2)}[\xi_1^8]} \mathbb{Z}_{(2)}$$

and

$$\mathbb{Z}_{(2)}[\xi_2^2]/(2\xi_2^2) \otimes_{\mathbb{Z}_{(2)}[\xi_2^4]} \mathbb{Z}_{(2)}.$$

For the first case, note that

$$\mathbb{Z}_{(2)}[\xi_1^2]/(2\xi_1^2) \simeq \Sigma^2 \mathbb{Z}_{(2)}[\xi_1^8]/2 \oplus \Sigma^4 \mathbb{Z}_{(2)}[\xi_1^8]/2 \oplus \Sigma^6 \mathbb{Z}_{(2)}[\xi_1^8]/2 \oplus T$$

as a $\mathbb{Z}[\xi_1^8]$-module, where $T = \mathrm{cofib}(2\xi_1^8 : \sigma^8 \mathbb{Z}_{(2)}[\xi_1^8] \to \mathbb{Z}_{(2)}[\xi_1^8])$. It is then easy to compute with these free resolutions that

$$\mathbb{Z}_{(2)}[\xi_1^2]/(2\xi_1^2) \otimes_{\mathbb{Z}_{(2)}[\xi_1^8]} \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)} \oplus \Sigma^2 \mathbb{Z}/2\mathbb{Z} \oplus \Sigma^4 \mathbb{Z}/2\mathbb{Z} \oplus \Sigma^6 \mathbb{Z}/2\mathbb{Z} \oplus \Sigma^9 \mathbb{Z}_{(2)}.$$

Similar considerations give us

$$\mathbb{Z}_{(2)}[\xi_2^2]/(2\xi_2^2) \otimes_{\mathbb{Z}_{(2)}[\xi_2^4]} \mathbb{Z}_{(2)} \simeq \mathbb{Z}_{(2)} \oplus \Sigma^6 \mathbb{Z}/2\mathbb{Z} \oplus \Sigma^{13} \mathbb{Z}_{(2)}.$$

Computing the derived tensor product over $\mathbb{Z}_{(2)}$ of these two $\mathbb{Z}_{(2)}$-modules gives the desired result. □

With this in hand, we can finally compute $\mathrm{THH}_*(\mathrm{tmf}, \mathbb{Z}_{(2)})$:

Theorem 3.7 (Theorem 1.1)**.**

$$\mathrm{THH}_*(\mathrm{tmf}, \mathbb{Z}_{(2)}) = \begin{cases} \mathbb{Z}_{(2)} & \text{if } * = 0, 9, 13, 22 \\ \mathbb{Z}/2^k\mathbb{Z} & \text{if } * = 2r^{k+3} - 1, 2^{k+3}r - 1 + 9, 2^{k+3}r - 1 + 13, 2^{k+3}r - 1 + 22, \\ 0 & \text{otherwise}, \end{cases}$$

for all $k > 0$ and r odd.

Proof. First, note that, by Proposition 3.6, the Brun spectral sequence for $\mathrm{THH}(\mathbb{Z}_{(2)}, \mathbb{Z}_{(2)} \otimes_{\mathrm{tmf}} \mathbb{Z}_{(2)})$ has E_1-page with $\mathrm{THH}_*(\mathbb{Z}_{(2)})$ in degrees $(-*, 0), (-*, 22)$, $\mathrm{THH}_*(\mathbb{Z}_{(2)}, \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2\mathbb{Z})$ in degrees $(-*, 9), (-*, 13)$, $\mathrm{THH}_*(\mathbb{Z}_{(2)}, \mathbb{Z}/2\mathbb{Z})$ in degrees $(-*, 2), (-*, 4)$, $(-*, 8), (-*, 10)$, $(-*, 11), (-*, 12)$, $(-*, 17), (-*, 19)$, $\mathrm{THH}_*(\mathbb{Z}_{(2)}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ in degrees $(-*, 6)$ and $(-*, 15)$, and is zero otherwise.

This spectral sequence has a lot of terms, so would be difficult to deal with on its own. Instead, we examine also the Bockstein spectral sequence, with E_1-page $\mathrm{THH}_*(\mathrm{tmf}, \mathbb{F}_2)[\tilde{v_0}]$, with $|\tilde{v_0}| = (1, 1)$. Since every term of the Brun spectral sequence in negative s degree is torsion, the $\mathbb{Z}_{(2)}$-classes in degrees 0,9,13, and 22 must survive to the E_∞-page, giving the respective classes in $\mathrm{THH}_*(\mathrm{tmf}, \mathbb{Z}_{(2)})$. This is reflected in our Bockstein spectral sequence by the fact that our $\tilde{v_0}$-towers on the classes 1, λ_1, λ_2 and $\lambda_1\lambda_2$ must survive the spectral sequence. Thus, we must have that λ_3 is a permanent cycle. Since the spectral sequence is multiplicative, we either have that it has degenerated, or else $d_1(\alpha) = \tilde{v_0}\lambda_3$. The Brun spectral sequence shows that $\pi_{15}(\mathrm{THH}(\mathrm{tmf}, \mathbb{Z}_{(2)}))$ is finite, so we cannot be in the first case, and thus $d_1(\alpha) = \tilde{v_1}\lambda_3$, and $\alpha\lambda_3$ cannot support any differentials since anything it could hit is either a permanent cycle, or was killed off on the first page.

Inductively, we find that $\alpha^{2^n}\lambda_3$ cannot support any differentials, and the only multiplicative generator that can support a differential on or past the E_{n+2}-page, is $\alpha^{2^{n+1}}$, which can only possibly support the differential $d_{n+2}(\alpha^{2^{n+1}}) = \tilde{v_0}^{n+2}\alpha^{2^n}\lambda_3$. If we did not have this differential, the spectral sequence would be degenerate, but again, the Brun spectral sequence shows this cannot be the case, so we have $d_{n+2}(\alpha^{2^{n+1}}) = \tilde{v_0}^{n+2}\alpha^{2^n}\lambda_3$ for all n. By looking at the cofiber of 2, we find that $\mathrm{THH}_*(\mathrm{tmf}, \mathbb{Z}_{(2)})$ is either $\mathbb{Z}_{(2)}$ or finite cyclic in any given degree, which determines the multiplicative extensions in the Bockstein spectral sequence. The classes in degree $2^{k+3}r - 1$ are the classes $\alpha^{r2^{k-1}}\lambda_3$, and the other classes in the second item come from multiplying by λ_1, λ_2 and $\lambda_1\lambda_2$. □

§4. THH of Quotients of ℓ

In this section, we compute the topological Hochschild homology of quotients ℓ/v_1^n of ℓ, in terms of the topological Hochschild homology of ℓ, which was computed by Angeltveit-Hill-Lawson in [AHL09]. We will show that $\pi_* \operatorname{THH}(\ell/v_1^n) \simeq \pi_* \operatorname{THH}(\ell, \ell/v_1^n) \otimes_{\mathbb{Z}_{(p)}} \Gamma[x]$, the tensor product of the topological Hochschild homology of ℓ with coefficients in ℓ/v_1^n with a divided power algebra on a generator x in degree $2n(p-1)+2$. We start by computing $\operatorname{THH}(\ell/v_1^n, \mathbb{Z}_{(p)})$.

Lemma 4.1. *For $n > 1$, we have that $\mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)} \simeq (\mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} (\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)})$ as $\mathbb{E}_\infty$-algebras.*

Proof. We have a diagram of $\mathbb{E}_\infty$-rings where all squares are pushouts:

$$\begin{array}{ccccc}
\ell & \longrightarrow & \ell/v_1^n & \longrightarrow & \mathbb{Z}_{(p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_{(p)} & \longrightarrow & \mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n & \longrightarrow & \mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)} \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{Z}_{(p)} & \longrightarrow & \mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)}.
\end{array}$$

The map $\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n \to \mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)}$ factors over $\tau_{\leq 2(p-1)+1}\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n \simeq \mathbb{Z}_{(p)}$, so that the cospan $\mathbb{Z}_{(p)} \leftarrow \mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n \to \mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)}$ may be rewritten as the tensor product over $\mathbb{Z}_{(p)}$ of the cospans $\mathbb{Z}_{(p)} \leftarrow \mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n \to \mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)} \leftarrow \mathbb{Z}_{(p)} \to \mathbb{Z}_{(p)} \otimes_\ell \mathbb{Z}_{(p)}$. Since colimits commute, we find that the pushout of our original square, $\mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)}$ is isomorphic to the pushout of our first cospan tensored over $\mathbb{Z}_{(p)}$ with the pushout of our second cospan, which is precisely $(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} (\mathbb{Z}_{(p)} \otimes_{\ell/v_1^n} \mathbb{Z}_{(p)})$, as claimed. □

Next, we examine $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)}$. By an easy computation, its homotopy groups are

$$\pi_*(\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n) = \begin{cases} \mathbb{Z}_{(p)} & \text{if } * = 0, 2n(p-1)+1, \\ 0 & \text{otherwise.} \end{cases}$$

In order to figure out the multiplicative structure of $\pi_*(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)})$, we must figure out the structure of $\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n$ as an algebra. We may note that this is a $\mathbb{E}_\infty$-$\mathbb{Z}_{(p)}$-algebra, which is a square zero extension of $\mathbb{Z}_{(p)}$ in $\mathbb{E}_\infty$-$\mathbb{Z}_{(p)}$-algebras by [Lur17] Corollary 7.4.1.27. Since $\mathbb{Z}_{(p)}$ is initial in $\operatorname{CAlg}(\mathbb{Z}_{(p)})$, $\mathbb{L}_{\mathbb{Z}_{(p)}}^{\operatorname{CAlg}(\mathbb{Z}_{(p)})}$ vanishes, so there is a unique such square zero extension of $\mathbb{Z}_{(p)}$.

Since we have a dga model for an $\mathbb{E}_\infty$-$\mathbb{Z}_{(p)}$-algebra with the prescribed homotopy groups, we can take the tensor product $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)}$ with respect to this dga model, and it will tell us the structure of $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)}$ as an $\mathbb{E}_1$-algebra, and in particular will tell us the multiplicative structure of the homotopy groups. But it is easy to show that this just returns a ring with homotopy groups $\pi_*(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)}) = \Gamma[x]$ with x in degree $2n(p-1)+2$, as desired.

Lemma 4.2. $\operatorname{THH}_*(\ell/v_1^n, \mathbb{Z}_{(p)}) \simeq \operatorname{THH}_*(\ell, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \pi_*(\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}_{(p)} \otimes_\ell \ell/v_1^n} \mathbb{Z}_{(p)})$.

Proof. This follows from

$$\begin{aligned}\mathrm{THH}(\ell/v_1^n,\mathbb{Z}_{(p)}) &\simeq \mathrm{THH}(\mathbb{Z}_{(p)},\mathbb{Z}_{(p)}\otimes_{\ell/v_1^n}\mathbb{Z}_{(p)})\\ &\simeq \mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\mathbb{S}}\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}\otimes_{\ell/v_1^n}\mathbb{Z}_{(p)})\\ &\simeq \mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\mathbb{S}}\mathbb{Z}_{(p)}}((\mathbb{Z}_{(p)}\otimes_{\ell}\mathbb{Z}_{(p)})\otimes_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\ell}\ell/v_1^n}\mathbb{Z}_{(p)}))\\ &\simeq (\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\mathbb{S}}\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}\otimes_{\ell}\mathbb{Z}_{(p)}))\otimes_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\ell}\ell/v_1^n}\mathbb{Z}_{(p)})\\ &\simeq \mathrm{THH}(\ell,\mathbb{Z}_{(p)})\otimes_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}\otimes_{\mathbb{Z}_{(p)}\otimes_{\ell}\ell/v_1^n}\mathbb{Z}_{(p)}).\end{aligned}$$

□

Now, we are finally in a position to analyze the spectral sequence associated to $\mathrm{THH}(\ell/v_1^n)$. We have that $E_1^{s,t}=\mathbb{Z}_{(p)}[v_1]/v_1^n\otimes_{\mathbb{Z}_{(p)}}\mathrm{THH}_*(\ell,\mathbb{Z}_{(p)})\otimes_{\mathbb{Z}_{(p)}}\Gamma[x]$, where v_1 has bidegree $(0,2(p-1))$, x has bidegree $(-(2n(p-1)+2),0)$, and $\mathrm{THH}_s(\ell,\mathbb{Z}_{(p)})$ lives in degree $(-s,0)$. There is a comparison map ρ from the spectral sequence associated to $\mathrm{THH}(\ell,\ell/v_1^n)$ to this one, which determines many of the differentials. In fact,

Theorem 4.3. *All of the differentials vanish on the classes coming from* $\Gamma[x]$ *in the* $\mathrm{THH}(\ell/v_1^n)$ *spectral sequence. In particular, the comparison map* ρ*, together with multiplicativity, determine all of the differentials in this spectral sequence, and we have that* $\mathrm{THH}_*(\ell/v_1^n)\simeq\mathrm{THH}_*(\ell,\ell/v_1^n)\otimes_{\mathbb{Z}_{(p)}}\Gamma[x]$*, where* x *is in degree* $2n(p-1)+2$.

Proof. First, we note that in the case $n\neq 1 \mod p^2$, the theorem can be proven by looking only at the spectral sequences we have already constructed. To prove this theorem in general, we use the filtrations coming from [LL23]. Explicitly, we work over the filtered modules over the $\mathbb{E}_\infty$-algebra $\tau_{\geq *}(\ell/v_1^n)$ in filtered spectra, where $\tau_{\geq *}(\ell/v_1^n)$ denotes ℓ/v_1^n with the Whitehead filtration. We can then apply the construction of THH to get the filtered spectrum $\mathrm{THH}(\tau_{\geq *}(\ell/v_1^n))$, with underlying spectrum $\mathrm{THH}(\ell/v_1^n)$, and associated graded $\mathrm{THH}(\mathbb{Z}_{(p)}[\tilde{v_1}]/(\tilde{v_1}^n))$, where $|\tilde{v_1}|=2n(p-1)+2$. To understand the E_1-page of this spectral sequence, we use the following lemma, the proof of which is adapted from lemma 4.1 in [LL23]:

Lemma 4.4. *Suppose* k *is a discrete ring, and* R *is a connective (possibly graded)* $\mathbb{E}_2$-k*-algebra with* $\pi_*(R)=k[x]/x^n$*, on some class* x *in positive even degree, and* R *admits an* $\mathbb{E}_2$*-algebra map from a ring* S *with* $\pi_*(S)=k[x]$*. Then, we have an equivalence of (graded)* $\mathbb{E}_1$-k*-algebras* $\mathrm{THH}(R)=\mathrm{THH}(k)\otimes_k\mathrm{HH}(R/k)$.

Proof of lemma. We have, as in 4.1 of [LL23] $k[x]=k\otimes_{\mathbb{S}}\mathbb{S}[x]$. Now, as an $\mathbb{E}_1$-algebra $R=k[x]\otimes_{k[x^n]}k\simeq(k\otimes_{\mathbb{S}}\mathbb{S}[x])\otimes_{k\otimes_{\mathbb{S}}\mathbb{S}[x^n]}(k\otimes_{\mathbb{S}}\mathbb{S})\simeq k\otimes_{\mathbb{S}}\mathbb{S}[x]/x^n$, where $\mathbb{S}[x]/x^n$ denotes $\mathbb{S}[x]\otimes_{\mathbb{S}[x^n]}\mathbb{S}$. Since THH commutes with tensor products, there are equivalences of (graded) spectra,

$$\mathrm{THH}(R)\simeq\mathrm{THH}(k)\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{S}[x]/x^n)\simeq\mathrm{THH}(k)\otimes_k k\otimes_{\mathbb{S}}\mathrm{THH}(\mathbb{S}[x]/x^n)\simeq\mathrm{THH}(k)\otimes_k\mathrm{HH}((k[x]/x^n)/k).$$

$k[x]/x^n=\tau_{\leq n|x|-1}k[x]$, so that R inherits a canonical $\mathbb{E}_2$-k-algebra structure as this truncation. We can give x a new (positive) grading 1, to make $\mathbb{S}[x]$ a nonnegatively graded $\mathbb{E}_2$-ring spectrum, so an $\mathbb{E}_2$-algebra in $\mathrm{Sp}^{\mathbb{Z}^{ds}_{\geq 0}}$. There is a thick $\otimes$-ideal $\mathcal{I}$ of $\mathrm{Sp}^{\mathbb{Z}^{ds}_{\geq 0}}$ generated by elements concentrated in grading $\geq n$. Quotienting out this $\otimes$-ideal gives a symmetric monoidal functor $\mathrm{Sp}^{\mathbb{Z}^{ds}_{\geq 0}}\to\mathrm{Sp}^{\mathbb{Z}^{ds}_{\geq 0}}/\mathcal{I}$, whose right adjoint is then lax symmetric monoidal by [Lur17] Corollary 7.3.2.7. Composing these two functors gives a functor which sends our graded $\mathbb{S}[x]$ to a graded $\mathbb{E}_2$-algebra with underlying $\mathbb{E}_2$-ring $\mathbb{S}[x]/x^n$, as desired. Using this grading, we can get, from the

$\mathbb{E}_2$-map in $\mathrm{Sp}^{\mathbb{Z}^{ds}_{\geq 0}}$, $\mathbb{S}[x] \to k[x]$. Applying the endofunctor we just described, we get a $\mathbb{E}_2$-map $\mathbb{S}[x]/x^n \to k[x]/x^n$, which upgrades our isomorphism above to an $\mathbb{E}_2$-algebra isomorphism, which ensures the induced map on THH is an isomorphism of $\mathbb{E}_1$-algebras. □

We wish to apply this in our case. By [LL23] lemma 2.6, the t-structure on graded spectra given by saying $x_\bullet$ is connective if x_i is mi-connective for some m is compatible with the multiplicative structure. In particular, choosing m sufficiently large ($m > 2n(p-1)+2$), we get a t-structure on graded spectra such that $\tau_{\geq 0}(\pi_*(\ell/v_1^n)) = \mathbb{Z}_{(p)}$ concentrated in degree 0, which shows that $\pi_*(\ell/v_1^n)$ is a graded $\mathbb{E}_\infty$-$\mathbb{Z}_{(p)}$-algebra. Now, we can apply the above theorem to get the E_1-page of the spectral sequence as

$$E_1^{s,t} = \mathrm{THH}_*(\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathrm{HH}_*((\mathbb{Z}_{(p)}[\tilde{v_1}]/\tilde{v_1}^n)/\mathbb{Z}_{(p)}).$$

A standard calculation shows that

$$\mathrm{HH}_*((\mathbb{Z}_{(p)}[\tilde{v_1}]/\tilde{v_1}^n)/\mathbb{Z}_{(p)}) \simeq \Lambda_{\mathbb{Z}_{(p)}}[d\tilde{v_1}] \otimes \Gamma_{\mathbb{Z}_{(p)}}[\sigma^2\tilde{v_1}^n] \otimes \mathbb{Z}_{(p)}[\tilde{v_1}]/(\tilde{v_1}^n),$$

where $\sigma^2\tilde{v_1}^n$ is a class in bidegree $(-2, 2n(p-1))$, and dv_1 is a class in bidegree $(-1, 2(p-1))$.

The terms coming from $\mathrm{THH}_n(\mathbb{Z}_{(p)})$ live in bidegree $(-n, 0)$, and $\tilde{v_1}$ lives in bidegree $(0, 2(p-1))$. Examining the above spectral sequence, we find that $\sigma^2(\tilde{v_1}^n)$ is the only class in total degree $(-2, 2n(p-1))$ and nothing lives in degree $(0, 2n(p-1)+1)$. $\sigma^2(\tilde{v_1}^n)$ must vanish under the differentials on every page, so this is the same class corresponding to x in our other spectral sequence. Now, it is clear that for all classes a, with bigrading $|a| = (s,t)$, we have that $t \leq -n(p-1)s + 2(n-1)(p-1)$ for s even, and $t \leq -n(p-1)(s+1) + 2n(p-1)$ for s odd. Further, t is maximized with respect to s for $s \leq 0$ even by $\tilde{v_1}^{n-1}x^{(-\frac{s}{2})}$, and for s odd by $d\tilde{v_1}\tilde{v_1}^{n-1}x^{(-\frac{s+1}{2})}$. In particular, any differential off of $x^{(k)}$ on the E_r-page would have to hit a class in bidegree $(-2k+r+1, 2kn(p-1)+r)$. But, we have that, for r odd, $-2k+r+1 \geq -2k+2$, and then every class with $t > 2n(p-1)(k-1) + 2(n-1)(p-1)$ vanishes in this s degree. In particular, the target of d_r on $x^{(k)}$ is 0. If r is even, then $-2k+r+1 \geq -2k+3$, and thus if $t > 2n(p-1)(k-2) + 2n(p-1) = 2n(p-1)(k-1)$, the classes in degree $(-2k+r+1, t)$ vanish, and again, the target of $d_r((\sigma^2(\tilde{v_1}^n))^{(k)})$ vanishes. We have shown that the $(\sigma^2(\tilde{v_1})^n)^{(k)}$, are all permanent cycles, and then so are the $x^{(k)}$ from the first spectral sequence.

Note that there are no other nonzero terms in the above spectral sequence with total degree $k(2n(p-1)+2)$, and higher filtration degree than $(\sigma^2(\tilde{v_1}^n))^{(k)}$. Thus, there can be no nontrivial multiplicative extensions supported on these classes, and $x \mapsto \sigma^2(v_1^n)$ determines a map of graded commutative $\mathbb{Z}_{(p)}$-algebras $\Gamma_{\mathbb{Z}_{(p)}}[x] \to \mathrm{THH}_*(\ell/v_1^n)$, with x a class in degree $2n(p-1)+2$. This gives us a map of graded commutative $\mathbb{Z}_{(p)}$-algebras

$$\mathrm{THH}_*(\ell, \ell/v_1^n) \otimes_{\mathbb{Z}_{(p)}} \Gamma_{\mathbb{Z}_{(p)}}[x] \to \mathrm{THH}_*(\ell/v_1^n).$$

Since the E_1-page of the Whitehead spectral sequence for $\mathrm{THH}(\ell/v_1^n)$ is multiplicatively generated by the image of $\mathrm{THH}_*(\ell, \pi_*(\ell/v_1^n))$ and the classes $x^{(k)}$ (which have just been shown to be permanent cycles), all of the nontrivial differentials appearing in this spectral sequence come from the map $\mathrm{THH}_*(\ell, \tau_{\geq *}\ell/v_1^n) \to \mathrm{THH}_*(\ell/v_1^n, \tau_{\geq *}\ell/v_1^n)$ together with the Leibniz rule. In particular, it follows that the algebra map $\mathrm{THH}_*(\ell, \ell/v_1^n) \otimes_{\mathbb{Z}_{(p)}} \Gamma_{\mathbb{Z}_{(p)}}[x] \to \mathrm{THH}_*(\ell/v_1^n)$ must be an isomorphism, as claimed. □

§5. The General Case

We remark that many of the constructions in the last section admit a generalization.

Theorem 5.1. *Suppose that R is a connective $\mathbb{E}_m$-ring spectrum for some $m \geq 3$, and $x \in \pi_*(R)$ is a positive degree class such that, for some fixed $k > 1$, $R/x^k := \mathrm{cofib}(x^k : R \to R)$ admits an $\mathbb{E}_3$-algebra structure. Then, we have an equivalence of $\tau_{\leq 0}R$-modules* $\mathrm{THH}(R/x^k, \pi_0(R)) \simeq \mathrm{THH}(R, \pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R))$.

Proof. Suppose we have a connective $\mathbb{E}_m$-ring spectrum R such that we have a class x and an integer k with the desire properties. We then have that, by Proposition 2.2, $\mathrm{THH}(R/x^k, \pi_0(R)) \simeq \mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{R/x^k} \pi_0(R))$. Now, we have equivalences

$$\begin{aligned}
\pi_0(R) \otimes_{R/x^k} \pi_0(R) &\simeq (\pi_0(R) \otimes \pi_0(R)) \otimes_{R/x^k \otimes R/x^k} R/x^k \\
&\simeq (\pi_0(R) \otimes \pi_0(R)) \otimes_{R/x^k \otimes R/x^k} (R/x^k \otimes_R R/x^k) \otimes_{(R/x^k \otimes_R R/x^k)} R/x^k \\
&\simeq ((\pi_0(R) \otimes \pi_0(R)) \otimes_{R/x^k \otimes R/x^k} ((R/x^k \otimes R/x^k) \otimes_{R\otimes R} R)) \otimes_{(R/x^k \otimes_R R/x^k)} R/x^k \\
&\simeq ((\pi_0(R) \otimes \pi_0(R)) \otimes_{R\otimes R} R) \otimes_{(R/x^k \otimes_R R/x^k)} R/x^k \\
&\simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{(R/x^k \otimes_R R/x^k)} R/x^k \\
&\simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)\otimes_R R/x^k} (\pi_0(R) \otimes_R R/x^k) \otimes_{(R/x^k \otimes_R R/x^k)} R/x^k \\
&\simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R).
\end{aligned}$$

Now, by assumption, x^k is such that $R/x^k \to \pi_0(R)$ factors as $R/x^k \to R/x^{k-1} \to \pi_0(R)$, so, $\pi_0(R) \otimes_R R/x^k \to \pi_0(R) \otimes_R \pi_0(R)$ factors as $\pi_0(R) \otimes_R R/x^k \to \pi_0(R) \otimes_R R/x^{k-1} \to \pi_0(R) \otimes_R \pi_0(R)$. $\pi_0(R) \otimes_R R/x^{k-1}$ has homotopy groups given by

$$\pi_*(\pi_0(R) \otimes_R R/x^{k-1}) = \begin{cases} \pi_0(R) & \text{if } * = 0,\ (k-1)|x| + 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\pi_0(R) \otimes_R R/x^k \to \pi_0(R) \otimes_R R/x^{k-1}$ factors over $\tau_{\leq (k-1)|x|+1}(\pi_0(R) \otimes_R R/x^k) \simeq \pi_0(R)$. This implies that $\pi_0(R) \otimes_R \pi_0(R)$, as a right $\pi_0(R) \otimes_R R/x^k$-module, is equivalent to $(\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} \pi_0(R)$, with the induced right module structure on $\pi_0(R)$. Thus,

$$\begin{aligned}
\pi_0(R) \otimes_{R/x^k} \pi_0(R) &\simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R) \\
&\simeq ((\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} \pi_0(R)) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R) \\
&\simeq (\pi_0(R) \otimes_R \pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R)).
\end{aligned}$$

This derived tensor product in $\pi_0(R)$-modules can be computed as a tensor product on underlying modules, since the second module is flat (in fact free). Indeed, from the cofiber sequence $\Sigma^{k|x|}R \xrightarrow{x^k} R$, we can tensor this with $\pi_0(R)$ to find that $\pi_0(R) \otimes_R R/x^k \simeq \pi_0(R) \oplus \Sigma^{k|x|+1}\pi_0(R)$ as a $\pi_0(R)$-module. We then have a periodic resolution of $\pi_0(R)$ from this class in degree 2, which allows us to see that $\pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R) \simeq \bigoplus_{r\geq 0} \Sigma^{r(k|x|+2)}\pi_0(R)$ as a $\pi_0(R)$-module.

Now, we apply the Whitehead filtration to $\pi_0(R) \otimes_{R/x^k} \pi_0(R)$, and examine the spectral sequence associated to $\mathrm{THH}(\pi_0(R), \tau_{\geq *}(\pi_0(R) \otimes_{R/x^k} \pi_0(R)))$. Note that this spectral sequence is multiplicative, since by assumption, R/x^k is an $\mathbb{E}_3$-algebra, so that the maps $R/x^k \to \pi_0(R)/x^k \simeq \pi_0(R)$ are $\mathbb{E}_3$-algebra maps, which implies $(\pi_0(R) \otimes_{R/x^k} \pi_0(R))$ is an $\mathbb{E}_2$-$\pi_0(R)$-algebra. We have a map

$$\mathrm{THH}(\pi_0(R), \tau_{\geq *}(\pi_0(R) \otimes_R \pi_0(R))) \to \mathrm{THH}(\pi_0(R), \tau_{\geq *}(\pi_0(R) \otimes_{R/x^k} \pi_0(R)))$$

which descends to a map on the associated spectral sequences. By what we said above, the E_1-page of the target is multiplicatively generated by the classes in the image of this map, together with classes generating copies of $\pi_0(R)$ in degrees $(0, r(k|x|+2))$ for $r > 0$. Since there are no nonzero classes in bidegree (s,t) for $s > 0$, so the differentials vanish on these classes, and there are no multiplicative extension problems between them. The map from $\mathrm{THH}(\pi_0(R), \tau_{\geq *}(\pi_0(R) \otimes_R \pi_0(R)))$ determine the multiplicative extension problems on the classes in its image, and this determines all of the multiplicative extension problems, since any nonzero class a in the image of this map multiplies with any nonzero class b coming from $(\pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R))$ to a nonzero class.

This establishes the claim on the level of homotopy groups. For the full claim, note that we have a map $\mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{R/x^k} \pi_0(R)) \to \mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R)) \to \pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R) \simeq \bigoplus_{r\geq 0} \Sigma^{r(k|x|+2)}\pi_0(R)$, which admits a splitting $\varphi : \bigoplus_{r\geq 0} \Sigma^{r(k|x|+2)}\pi_0(R) \to \mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{R/x^k} \pi_0(R))$. Since $\mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{R/x^k} \pi_0(R))$ admits a $\mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_R \pi_0(R))$-module structure coming from the natural map, φ extends to a map $\mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_R \pi_0(R)) \otimes_{\tau_{\geq 0}R} \pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R) \to \mathrm{THH}(\pi_0(R), \pi_0(R) \otimes_{R/x^k} \pi_0(R))$, which provides our desired equivalence. ◻

Remark 5.2. As the proof indicates, we can replace x^k by any class x in even positive degree such that R/x is an $\mathbb{E}_3$-algebra, and such that $(R/x \otimes_R \pi_0(R)) \to (\pi_0(R) \otimes_R \pi_0(R))$ factors over $(R/x \otimes_R \pi_0(R)) \to \tau_{\leq 0}((R/x \otimes_R \pi_0(R))) \simeq \pi_0(R)$.

Corollary 5.3. *Let R, x and k be as in Theorem 5.1. Then, the E_1-page of the spectral sequence coming from* $\mathrm{THH}(R/x^k, \tau_{\geq *}(R/x^k))$ *converging to* $\mathrm{THH}(R/x^k)$ *is isomorphic to the E_1-page of the spectral sequence coming from* $\mathrm{THH}(R, \tau_{\geq *}(R/x^k))$ *tensored over $\pi_0(R)$ with $\bigoplus_{r\geq 0} \tau_{\geq 0}R \cdot a_r$, where a_r is a class in bidegree $(-r(k|x|+2), 0)$.*

Proof. This follows from the proposition together with the fact that if M is any $\pi_0(R)$-module, then

$$\begin{aligned}\mathrm{THH}(R/x^k, M) &\simeq \mathrm{THH}(R/x^k, \pi_0(R)) \otimes_{\pi_0(R)} M \\ &\simeq \mathrm{THH}(R, \pi_0(R)) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R)) \otimes_{\pi_0(R)} M \\ &\simeq \mathrm{THH}(R, M) \otimes_{\pi_0(R)} (\pi_0(R) \otimes_{\pi_0(R)\otimes_R R/x^k} \pi_0(R)).\end{aligned}$$

◻

Remark 5.4. We don't know whether or not this isomorphism extends to $\mathrm{THH}_*(R/x^k) \simeq \mathrm{THH}_*(R, R/x^k) \otimes \pi_*(R/x^k \otimes_{R/x^k \otimes_R R/x^k} R/x^k)$ in general, although this does seem to hold in many cases.

Together with the map from $\mathrm{THH}(R, \tau_{\geq *}(R/x^k))$, this means that in nice cases, if we understand $\mathrm{THH}(R, R/x^k)$, we need only understand the differentials on the classes a_r in order to understand $\mathrm{THH}(R/x^k)$.

§6. The THH of ksc_2

Using the results of section §5, we will compute $\mathrm{THH}(\mathrm{ksc}_2)$ and show that it fits into the same overall framework as above. Recall that self-conjugate K-theory, ksc is the $\mathbb{E}_\infty$-ring defined as the cofiber $\mathrm{cofib}(\eta^2 : \Sigma^2 \mathrm{ko} \to \mathrm{ko})$, or alternatively defined as the connective cover of the $\mathbb{Z}$-homotopy fixed points of the periodic complex K-theory spectrum KU, where $\mathbb{Z}$ acts as ψ^{-1}. We have:

Theorem 6.1. $\mathrm{THH}_*(\mathrm{ksc}_2) \simeq \mathrm{THH}_*(\mathrm{ko}_2, \mathrm{ksc}_2) \otimes_{\mathbb{Z}_{(2)}} \Gamma[\sigma^2\eta^2]$, *where* $\sigma^2\eta^2$ *is a class in degree* 4.

Proof. Since $\mathrm{ksc}_2 = \mathrm{cofib}(\eta^2 : \Sigma^2 \mathrm{ko}_2 \to \mathrm{ko}_2)$, the results of §5 give us a spectral sequence with signature

$$E_1^{s,t} = \mathrm{THH}_*(\mathrm{ko}_2, \pi_*(\mathrm{ksc}_2)) \otimes \Gamma[\sigma^2\eta^2] \implies \mathrm{THH}_*(\mathrm{ksc}_2).$$

Our goal is to show that the classes coming from $\Gamma[\sigma^2\eta^2]$ are permanent cycles. By similar results to §2, we can recover the known fact that $\mathrm{THH}_*(\mathrm{ko}_2, \mathbb{Z}_{(2)})$ is $\mathbb{Z}_{(2)}$ in degrees 0 and 5; $\mathbb{Z}/2^k\mathbb{Z}$ in degrees $r2^{k+2} - 1$ and $r2^{k+2} - 1 + 5$ for $r > 0$ odd; and is zero otherwise. Similarly,

$$\mathrm{THH}_*(\mathrm{ko}_2, \mathbb{F}_2) \simeq \mathbb{F}_2[u^4] \otimes \Lambda[u\xi_1^3, u^2\xi_2].$$

In particular, when we run the Whitehead spectral sequence for $\mathrm{THH}(\mathrm{ksc}_2)$, the class $\sigma^2\eta^2$ in bidegree $(-4, 0)$ cannot hit anything for degree reasons, and is thus a permanent cycle. In order to prove the theorem, it suffices to see that the classes $(\sigma^2\eta^2)^{(2^n)}$ do not hit any 2-torsion classes in the spectral sequence, since then the differentials on the divided power classes will all be trivial.

Similar to Theorem 4.3, we start with the filtration of ko_2 from [LL23] Definition 2.12. This has associated graded given by $\mathbb{Z}_{(2)}[v_1^2, \eta]/(2\eta)$, where η is in filtration degree 2, and v_1^2 is in filtration degree 4. Taking the cofiber of η^2 on ko_2^{fil} gives a filtered spectrum ksc_2^{fil} with underlying spectrum ksc_2, and associated graded with $\pi_{*,*}(\mathrm{ksc}_2^{gr}) = \mathbb{Z}_{(2)}[v_1^2, \eta, \rho]/(2\eta, \eta^2, \rho\eta, \rho^2)$, where ρ is in topological degree 3 and filtration degree 4. We wish to understand what $\mathrm{THH}(\mathrm{ksc}_2^{gr})$ looks like. To accomplish this task, we start by examining $\mathrm{THH}(\mathrm{ko}_2^{gr}, \mathbb{Z}_{(2)})$. Recall that $\pi_*(\mathrm{ko}_2^{gr}/2) = \mathbb{F}_2[v_1, \eta]$, and $\pi_*(\mathrm{ko}_2^{gr}/\eta) = \mathbb{Z}_{(2)}[v_1]$, coming from the fact ko_2^{gr}/η is the associated graded for the Whitehead filtration on ku_2. It follows that

$$\begin{aligned}\mathbb{Z}_{(2)} \otimes_{\mathrm{ko}_2^{gr}} \mathbb{Z}_{(2)} &\simeq \mathbb{Z}_{(2)} \otimes_{\mathrm{ko}_2^{gr}} \mathrm{ko}_2^{gr}/\eta \otimes_{\mathrm{ko}_2^{gr}/\eta} \mathbb{Z}_{(2)} \\ &\simeq \Lambda_{\mathbb{Z}_{(2)}}[\sigma\eta] \otimes_{\mathrm{ko}_2^{gr}/\eta} \mathbb{Z}_{(2)}.\end{aligned}$$

Using that $\mathbb{Z}_{(2)} = (\mathrm{ko}_2^{gr}/\eta)/v_1$, and noting that v_1 takes 1 to $2\sigma\eta$, we find that

$$\pi_*(\mathbb{Z}_{(2)} \otimes_{\mathrm{ko}_2^{gr}} \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}[\sigma\eta, \sigma v_1^2]/(\sigma\eta\sigma v_1^2, 2\sigma\eta, (\sigma\eta)^2, (\sigma v_1^2)^2).$$

Running the Brun spectral sequence for $\mathrm{THH}(\mathrm{ko}_2^{gr}, \mathbb{Z}_{(2)})$ shows that the only class that may appear in topological degree ≤ 4 in $\mathrm{THH}(\mathrm{ko}_2^{gr}, \mathbb{Z}_{(2)})$ is a $\mathbb{Z}/2\mathbb{Z}$ class in topological degree 3 and filtration degree 0 (which we will temporarily denote by α if it survives); and the class $\sigma\eta$ from above.

In the spectral sequence associated to the Whitehead filtration on the $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra ksc_2^{gr}, the only possible classes in topological degree 4 in $\mathrm{THH}(\mathrm{ksc}_2^{gr})$ are $\sigma^2\eta^2$, in filtration degree 4, and $\alpha\eta$, in topological degree 4 and filtration degree 2. Furthermore, for all $x \in \mathrm{THH}(\mathrm{ksc}_2^{gr})$, $\deg^{top}(x) \geq \deg^{fil}(x) - 1$.

These results tell us that in the spectral sequence for our filtration, with E_1-page $\mathrm{THH}_*(\mathrm{ksc}_2^{gr})$, the class $\sigma^2\eta^2$, which has (s,t)-degree $(0,4)$, cannot support any nonzero differentials, and is a permanent cycle. To see that this $\sigma^2\eta^2$ corresponds to the class with the same name in $\mathrm{THH}(\mathrm{ksc})$, we have to see that there are no multiplicative extensions. But the only possible multiplicative extension is with the class $\alpha\eta$, should it have survived. But if α survived to $\mathrm{THH}(\mathrm{ksc}_2^{gr}, \mathbb{Z}_{(2)})$, it would also survive to $\mathrm{THH}(\mathrm{ksc}_2^{gr})$ for degree reasons, giving a class in (s,t)-degree $(-3,0)$ on the E_1-page. Then there would be no classes for α to hit, and α would either give a new $\mathbb{Z}/2\mathbb{Z}$ class in $\mathrm{THH}_3(\mathrm{ksc}_2)$, or would sit in a multiplicative extension with ρ. Since the map $\mathrm{ksc}_2 \to \mathrm{THH}(\mathrm{ksc}_2)$ splits, the second case cannot happen, and the Whitehead spectral sequence for $\mathrm{THH}(\mathrm{ksc}_2)$ shows that the first case cannot happen, so in fact α does not survive to $\mathrm{THH}(\mathrm{ksc}_2^{gr})$, and there are no possible multiplicative extensions for $\sigma^2\eta^2$, showing that this is the desired class. Now, if all of the classes in $\Gamma[\sigma^2\eta^2]$ survive to $\mathrm{THH}(\mathrm{ksc}_2^{gr})$, then $(\sigma^2\eta^2)^{(2^n)}$ sits in (s,t)-degree $(0, 2^{n+2})$, and thus cannot support any differentials, giving us the desired classes $\Gamma[\sigma^2\eta^2]$ in $\mathrm{THH}(\mathrm{ksc}_2)$. We therefore shift our focus to proving this fact.

Lemma 6.2. *The classes $\Gamma[\sigma^2\eta^2] \subseteq \mathrm{THH}_*(\mathrm{ksc}_2^{gr}, \mathbb{Z}_{(2)})$ can be lifted to divided power algebra classes on $\sigma^2\eta^2$ in $\mathrm{THH}_*(\mathrm{ksc}_2^{gr})$.*

Proof of Lemma. We start with the filtered spectrum $\mathbb{Z}_{(2)}^{fil}$, given in the 2-adic filtration, with underlying spectrum $\mathbb{Z}_{(2)}$ and associated graded $\mathbb{F}_2[\tilde{v_0}]$. Tensoring this with the $\mathbb{E}_\infty$-$\mathbb{Z}_{(2)}$-algebra ksc_2^{gr} gives us a filtered spectrum with associated graded $\mathbb{F}_2[v_1, \eta, \tilde{v_0}]/(\eta^2)$, where v_1, η are in graded degree 0, and $\tilde{v_0}$ is in graded degree 1. This filtration gives rise to the 2-Bockstein spectral sequence

$$E_1^{s,t} = \mathrm{THH}_*(\mathbb{F}_2[\eta, v_1, \tilde{v_0}]/(\eta^2)) \implies \mathrm{THH}_*(\mathrm{ksc}_2^{gr}).$$

By monoidality of THH, we have

$$\mathrm{THH}(\mathbb{F}_2[\eta, v_1, \tilde{v_0}]/\eta^2) \simeq \mathrm{THH}(\mathbb{F}_2[\eta]/\eta^2) \otimes_{\mathrm{THH}(\mathbb{F}_2)} \mathrm{THH}(\mathbb{F}_2[v_1]) \otimes_{\mathbb{F}_2} \mathrm{THH}(\mathbb{F}_2[\tilde{v_0}]).$$

Using that $\mathbb{F}_2[\eta]/\eta^2$ is a square zero extension of $\mathbb{F}_2$ in $\mathbb{E}_\infty$-$\mathbb{F}_2$-algebras, we find that

$$\mathrm{THH}_*(\mathbb{F}_2[\eta]/\eta^2) = \mathrm{THH}_*(\mathbb{F}_2) \otimes \mathbb{F}_2[\eta]/\eta^2 \otimes \Lambda[\sigma\eta] \otimes \Gamma[\sigma^2\eta^2].$$

From this, it follows that

$$\mathrm{THH}_*(\mathbb{F}_2[\eta, v_1, \tilde{v_0}]/\eta^2) = \mathrm{THH}_*(\mathbb{F}_2) \otimes \mathbb{F}_2[\eta, v_1, \tilde{v_0}]/\eta^2 \otimes \Lambda[\sigma\eta, \sigma v_1, \sigma\tilde{v_0}] \otimes \Gamma[\sigma^2\eta^2].$$

In the 2-Bockstein spectral sequence, all of the above multiplicative generators have t-degree 0 except for $\tilde{v_0}$, in (s,t)-degree $(1,1)$, and $\sigma\tilde{v_0}$, in (s,t)-degree $(0,1)$. We begin by examining the class $\sigma^2\eta^2$ in degree $(-4,0)$. Almost all of the classes in the $\tilde{v_0}$-tower on this class must survive the spectral sequence in order to give the $\mathbb{Z}_{(2)} \cdot \sigma^2\eta^2$ class in degree 4 of $\mathrm{THH}_*(\mathrm{ksc}_2^{gr})$. We find that $\sigma^2\eta^2$ cannot support any differentials, since any nontrivial differentials would kill the entire tower. First, let's study this spectral sequence a little bit more:

We claim that if a is a class on the E_k-page with $a \neq 0$, but $\tilde{v_0}a = 0$, then $\deg^t(a) \leq k-1$. For $k = 1$, this is vacuous, since there is no $\tilde{v_0}$-torsion on the E_1-page of this spectral sequence. We proceed by induction. Suppose that $a \neq 0$ is a class on the E_k-page with $\tilde{v_0}a = 0$, and $\deg^t(a) \geq k$. The fact that $\tilde{v_0}a = 0$ means that at some point earlier in the spectral sequence, say on the E_{k-i}-page ($i > 0$), we had a class b with $d_{k-i}(b) = \tilde{v_0}a$. b then necessarily has t-degree $i + 1 > 1$. In particular, $\tilde{v_0}$ must divide b for degree reasons, so $b = c\tilde{v_0}$, for some class c (or more accurately, comes from a class on E_1 divisible by $\tilde{v_0}$, and by our inductive hypothesis, there cannot be a differential taking c to a nonzero class which multiplies with $\tilde{v_0}$ to zero). Now, $d(c) \neq a$, but $d(\tilde{v_0}c) = d(b) = \tilde{v_0}a$, so that $a - d(c) \neq 0$, but $\tilde{v_0}(a - d(c)) = 0$. Since $a - d(c)$ is a $\tilde{v_0}$-torsion class on E_{k-i} with t-degree $k \geq k - i$, it must be 0 by induction, so that $a = d(c)$, contradicting the choice of a.

This claim implies that if the differential of any class in t-degree 0 or 1 is nontrivial, then the entire $\tilde{v_0}$-tower on that class dies. Let n be the smallest natural such that $(\sigma^2\eta^2)^{(2^n)}$ does not live in $\mathrm{THH}_*(\mathrm{ksc}_2^{gr})$. In particular, we must have that the entire $\tilde{v_0}$-tower on the analogous class in the mod 2 Bockstein must vanish. Since $(\sigma^2\eta^2)^{(2^{n-1})}$ squares to a torsion-free class, there must be a nonvanishing $\tilde{v_0}$ tower in total degree $2^{n-1} \cdot 4 = 2^{n+1}$ in the Bockstein spectral sequence. However, considering the map from the spectral sequence associated to $\mathrm{THH}(\mathbb{Z}_{(2)}^{fil})$ shows that the classes divisible by u and $\sigma\tilde{v_0}$ are all $\tilde{v_0}$-torsion. In order for η to be 2-torsion, we need a differential to hit $\eta\tilde{v_0}$, and this can be checked to come from $\sigma\eta$. Thus, the only classes that can contribute to a nonvanishing $\tilde{v_0}$-tower are the $(\sigma^2\eta^2)^{(2^k)}$, for $k < n$, powers of v_1, and σv_1. Since no power of v_1 divides any element of $\Gamma[\sigma^2\eta^2]$, the only contribution can come from σv_1 and the $(\sigma^2\eta^2)^{(2^k)}$. But the total degree of $\sigma v_1 \cdot \prod_{k<n}(\sigma^2\eta^2)^{(2^k)}$ is 2^{n+1}, but the tower we need is in total degree 2^{n+2}, and thus must come from $(\sigma^2\eta^2)^{(2^n)}$! This shows that all of our $(\sigma^2\eta^2)^{(2^n)}$ classes have to survive this Bockstein spectral sequence, proving that they survive to give the divided power classes in $\mathrm{THH}_*(\mathrm{ksc}_2^{gr})$ that we were looking for. □

□

Remark 6.3. $\mathrm{THH}_*(\mathrm{ko}_2, \mathrm{ksc}_2)$ can be computed as a graded abelian group from the work of [AHL09], noting that they prove that η^2 acts as zero on $\overline{\mathrm{THH}}_*(\mathrm{ko}_2)$, which determines

$$\overline{\mathrm{THH}}(\mathrm{ko}_2, \mathrm{ksc}_2) \simeq \mathrm{cofib}(\eta^2 : \Sigma^2\overline{\mathrm{THH}}(\mathrm{ko}_2) \to \overline{\mathrm{THH}}(\mathrm{ko}_2))$$

up to extension problems. Since the only classes in $\bar{\mathrm{THH}}_*(\mathrm{ko}_2)$ in odd degrees are copies of $\mathbb{Z}_{(2)}$ living in what [AHL09] call F^{ko}, there can only be nontrivial extension problems if the map from

$$\mathrm{THH}_{5+4n}(\mathrm{ko}_2) \to \mathrm{THH}_{5+4n}(\mathrm{ko}_2, \mathrm{ksc}_2)$$

is not surjective on the torsion-free parts. However, we know from the computations in [AHL09] 7.2-7.3 that

$$\mathrm{THH}_{5+4n}(\mathrm{ko}_2) \to \mathrm{THH}_{5+4n}(\mathrm{ko}_2, \mathrm{ku}_2)$$

induces an isomorphism on the torsion-free part, and this map factors as

$$\mathrm{THH}_{5+4n}(\mathrm{ko}_2) \to \mathrm{THH}_{5+4n}(\mathrm{ko}_2, \mathrm{ksc}_2) \to \mathrm{THH}_{5+4n}(\mathrm{ko}_2, \mathrm{ku}_2).$$

Thus, we find that, as a graded abelian group,

$$\mathrm{THH}_*(\mathrm{ko}_2, \mathrm{ksc}_2) \simeq \mathrm{THH}_*(\mathrm{ko}_2) \oplus \mathrm{THH}_{*-3}(\mathrm{ko}_2).$$

Combined with the above, we have completely determined $\mathrm{THH}_*(\mathrm{ksc}_2)$.

References

[AHL09] Vigleik Angeltveit, Michael Hill, and Tyler Lawson. Topological Hochschild homology of l and ko. 2009. arXiv: 0710.4368 [math.AT].

[AK21] Gabriel Angelini-Knoll. "On topological Hochschild homology of the K(1)-local sphere". In: Journal of Topology 14.1 (2021), pp. 258–290.

[AR02] Christian Ausoni and John Rognes. "Algebraic K-theory of topological K-theory". In: Acta Math. 188.1 (2002), pp. 1–39. ISSN: 0001-5962. DOI: 10.1007/BF02392794. URL: https://doi-org.libproxy.mit.edu/10.1007/BF02392794.

[AR05] Vigleik Angeltveit and John Rognes. "Hopf algebra structure on topological Hochschild homology". In: Algebr. Geom. Topol. 5 (2005), pp. 1223–1290. ISSN: 1472-2747.

[Ang08] Vigleik Angeltveit. "Topological Hochschild homology and cohomology of A_∞ ring spectra". In: Geometry & Topology 12.2 (2008), pp. 987–1032.

[BM93] Marcel Bökstedt and Ib Madsen. "Topological cyclic homology of the integers". In: (1993).

[BR14] Robert R. Bruner and John Rognes. "Topological Hochschild homology of topological modular forms". In: (2014). In Preparation.

[Bru00] M. Brun. "Topological Hochschild homology of Z/pn". In: Journal of Pure and Applied Algebra 148.1 (2000), pp. 29–76. ISSN: 0022-4049. DOI: https://doi.org/10.1016/S0022-4049(98)00131-5. URL: https://www.sciencedirect.com/science/article/pii/S0022404998001315.

[Bur+23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. K-theoretic counterexamples to Ravene 2023. arXiv: 2310.17459 [math.AT].

[DM94] Bjorn Ian Dundas and Randy McCarthy. "Stable K-theory and topological Hochschild homology". In: Annals of Mathematics 140.3 (1994), pp. 685–701.

[Hes94] Lars Hesselholt. "Stable topological cyclic homology is topological Hochschild homology". In: Asterisque 226 (1994), pp. 175–192.

[Hön21] Eva Höning. "The topological Hochschild homology of algebraic K-theory of finite fields". In: Annals of K-Theory 6.1 (2021), pp. 29–96.

[Hö20] Eva Höning. "On the Brun spectral sequence for topological Hochschild homology". In: Algebraic & Geometric Topology 20.2 (2020), pp. 817–863. DOI: 10.2140/agt.2020.20.817. URL: https://doi.org/10.2140\%2Fagt.2020.20.817.

[LL23] David Jongwon Lee and Ishan Levy. Topological Hochschild homology of the image of j. 2023. arXiv: 2307.04248 [math.AT].

[Lur17] Jacob Lurie. "Higher Algebra". In: (2017). available online. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.

[MS93] James E McClure and RE Staffeldt. "On the topological Hochschild homology of bu, l". In: American Journal of Mathematics 115.1 (1993), pp. 1–45.

[Mat15] Akhil Mathew. The homology of tmf. 2015. arXiv: 1305.6100 [math.AT].

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: Acta Math. 221.2 (2018), pp. 203–409. ISSN: 0001-5962,1871-2509. DOI: 10.4310/ACTA.2018.v221.n2.a1. URL: https://doi.org/10.4310/ACTA.2018.v221.n2.a1.