A BRIEF DISCUSSION OF THE DIMENSION DROP CONJECTURE

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Abstract. We discuss the dimension drop conjecture and the background needed to understand it. This includes iterative function systems, existence and uniqueness of their attractors, exact overlaps, and Hausdorff dimension. We also discuss some recent progress made by Hochman on the conjecture, and briefly introduce the concept of entropy. We prove one corollary of Hochman's theorem involving algebraic parameters, and explain why it is not enough to prove the conjecture.

Contents

1. Introduction

In fractal geometry, one way to generate fractal sets is via an iterative function system. The dimension drop conjecture concerns the dimension of those fractal sets generated by iterative function systems.

Definition 1.1. An *iterative function system (IFS)* $\Phi = (\varphi_i)_{i\in\Lambda}$ is a finite collection of contractions on $\mathbb{R}$, where $\varphi_i(x) = r_i x + a_i$, with $|r_i| \leq 1$. The symbol Λ refers to an indexing set, so it can be thought of as $\{1, ..., m\}$. Note that all the φ_i are continuous.

Definition 1.2. We call a set X the *attractor* of the IFS Φ if it is invariant under Φ, meaning

$$X = \bigcup_{i\in\Lambda} \varphi_i(X).$$

Proposition 1.3. *Every IFS has an attractor that is unique.*

Proof. (We follow Falconer's proof, see [2]) Let $\Phi = (\varphi_i)_{i\in\Lambda}$ be an IFS. If E is a compact set, define

$$S(E) = \bigcup_{i=1}^{m} \varphi_i(E),$$

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and for $k \in \mathbb{N}$, define

$$S^k(E) = S(S^{k-1}(E)),$$

with $S^0(E) := S(E)$. We will show that the set

$$X = \bigcap_{k=1}^{\infty} S^k(E)$$

is the unique attractor of the IFS if E is a nonempty compact set such that $\varphi_i(E) \subset E$ for every $\varphi_i \in \Phi$. Such an E always exists because our IFS is made up of contractions, and there are finitely many, so there exists some radius r where the interval $[-r, r]$ suffices.

By our choice of E, $\varphi_i(E) \subset E$ for all $\varphi_i \in \Phi$, so $S(E) \subset E$. Similarly, because our φ_i are contractions, $S^k(E) \subset S^{k-1}(E)$. Thus, the sequence of $S^k(E)$ is decreasing.

Since all of the φ_i are continuous, iterating them preserves the compactness of E, so all of the $S^k(E)$ are also compact. Since $S^k(E)$ is a decreasing sequence of nonempty compact sets, they converge to a nonempty compact set, so their intersection X is compact and nonempty.

X is invariant, since $S(X) \subset X$ due to fact that $S^k(E)$ are decreasing, so $S(X) \subset S^k(E)$ for all k, implying that $S(X) \subset X$. For the other direction, applying S to any of the $S^k(E)$ increments k, so $X \subset S(X)$.

Next, we will show uniqueness of this attractor. To do this we will need to define a metric on the space of compact subsets of $\mathbb{R}$. Call this space $\mathscr{S}$. If $A, B \in \mathscr{S}$ then let

$$d(A, B) = \inf\{\delta : B \subset A^\delta \text{ and } A \subset B^\delta\}$$

where A^δ is the δ-parallel body of A, defined as

$$A^\delta = \bigcup_{x \in A} B(x, \delta).$$

This is called the *Hausdorff metric*, and it equips the space of compact subsets of $\mathbb{R}^n$ with a metric.

Note that if $M = \sup\{|x - y| : x \in A, y \in B\}$, then $B \subset A^M$ and $A \subset B^M$, since every element of B is within M of some element of A. Thus, $d(A, B) \leq M$.

Now, suppose A and B are both attractors of the same IFS. Consider

$$d\Big(S(A), S(B)\Big) = d\left(\bigcup_{i=1}^{m} \varphi_i(A), \bigcup_{i=1}^{m} \varphi_i(B)\right) \leq \max_{1 \leq i \leq m} d\Big(\varphi_i(A), \varphi_i(B)\Big).$$

The inequality follows because if $\delta = \max_{1 \leq i \leq m} d(\varphi_i(A), \varphi_i(B))$, then $\varphi_i(A) \subset \varphi_i(B)^\delta$ for all $1 \leq i \leq m$. This means that

$$\bigcup_{i=1}^{m} \varphi_i(A) \subset \varphi_i(B)^\delta \subset \left(\bigcup_{i=1}^{m} \varphi_i(B)\right)^\delta.$$

By an identical argument, replacing A with B, the inequality follows. Now suppose i is such that $d(\varphi_i(A)\varphi_i(B))$ is maximum. Because φ_i is a contraction, for all $x \in A, y \in B$,

$$|\varphi_i(x) - \varphi_i(y)| \leq r_i|x - y| \leq r_i \max_{\substack{x \in A, \\ y \in B}} |x - y|.$$

Thus,

$$\max_{1\le i\le m} d\Big(\varphi_i(A), \varphi_i(B)\Big) \le \max_{1\le i\le m} r_i \cdot d(A,B).$$

Now, if A and B are both attractors, $S(A) = A$ and $S(B) = B$, so we have

$$d(A,B) \le \max_{1\le i\le m} r_i \cdot d(A,B).$$

Since $|r_i| < 1$ for all $i \in \Lambda$, the above inequality can only be true if $d(A,B) = 0$, implying that $A = B$. Thus, the attractor of an IFS is unique. □

Now that we can be assured that an attractor always exists and is unique for any IFS, what can we say about attractors in general? In fact, it is often the case that the attractor of an IFS defines a fractal set. Fractal sets have the following properties: they contain detail at arbitrarily fine scales, they are sometimes self-similar, and their dimension can be fractional. Generally, they behave weirdly compared to objects from classical geometry. As such, we need stronger tools to study them, such as Hausdorff dimension.

The reason a more sophisticated notion of dimension is necessary to study fractal sets is that otherwise, there is no way to distinguish between null sets. What this means is if the ambient space is equipped with the wrong measure, lower dimensional sets always have measure 0, even when it is clear that they are distinct, and we are interested in distinguishing between them. For example, if we are trying to "measure" a line and a plane using cubes, they would both appear to be null sets because we can cover them using cubes that get arbitrarily thin. However, a line and a plane are not the same, and clearly the plane has more "thickness". Operating in the correct dimension distinguishes the objects of different dimension. With fractal sets especially, which often have fractional dimension, we want a way to find this dimension, which is defined as follows.

Definition 1.4. For a subset $X \subset \mathbb{R}^n$ and $\delta > 0$, we say $\{U_i\}$ is a δ-cover of X if

$$X \subset \bigcup U_i$$

and $|U_i| < \delta$ for all i, where $|U_i| := \sup\{|x-y| : x, y \in U_i\}$ is the diameter of U_i.

Definition 1.5. If $X \subset \mathbb{R}^n$, we define

$$\mathscr{H}^s_\delta = \inf \sum_{i=1}^{\infty} |U_i|^s$$

where the infimum is taken over all countable δ-covers of X. $\mathscr{H}^s_\delta$ defines an outer measure on $\mathbb{R}^n$.

Definition 1.6. For a subset $X \subset \mathbb{R}^n$, we define the *Hausdorff s-dimensional measure* of X as

$$\mathscr{H}^s(X) = \lim_{\delta\to 0} \mathscr{H}^s_\delta.$$

Since $\mathscr{H}^s_\delta$ increases as δ decreases, it is also equivalent to define $\mathscr{H}^s$ as the supremum over all δ of $\mathscr{H}^s_\delta$.

This defines a measure on $\mathbb{R}^n$, but since we are restricting our attention to the real line, note that on $\mathbb{R}$, $\mathscr{H}^1$ has an identical defintion to Lebesgue measure on the real line, and in general Hausdorff measure in $\mathbb{R}^n$ is equivalent to Lebesgue outer measure.

Proposition 1.7. *For every $X \subset \mathbb{R}^n$ there is a unique number s such that if $0 \leq t < s$, $\mathscr{H}^t(X) = \infty$, and if $t > s$, $\mathscr{H}^t = 0$. This s is called the Hausdorff dimension of X, or* $\dim X$.

Proof. If $s < t$,

$$\mathscr{H}^t_\delta(X) = \inf \sum_{i=1}^{\infty} |U_i|^t \leq \inf \sum_{i=1}^{\infty} |U_i|^s \cdot \delta^{t-s} = \delta^{t-s} \mathscr{H}^s_\delta(X).$$

Suppose $\mathscr{H}^t(X) > 0$. Then, $\mathscr{H}^s(X)$ must be infinite, since the term $\delta^{s-t} \to \infty$ as $\delta \to 0$. Now, suppose $\mathscr{H}^s(X) < \infty$. This implies that $\mathscr{H}^t(X) = 0$, since the term $\delta^{t-s} \to 0$ as $\delta \to 0$. □

All this implies that there must be some unique threshold where this transition from infinite to zero measure happens. This threshold is the Hausdorff dimension.

Returning to the attractor of an IFS, we have a result that relates the Hausdorff dimension of the attractor to the so-called similarity dimension, a number which depends only on the parameters of the IFS.

Definition 1.8. If X is the attractor of the IFS Φ, then the similarity dimension of X, $\dim_{\text{sim}}(X)$ is the unique number s satisfying

$$\sum_{i=1}^{m} r_i^s = 1,$$

where the r_i s are the contraction ratios of Φ.

Note that although the notation is $\dim_{\text{sim}}(X)$, the similarity dimension depends on the parameters of the IFS only. Since the same set X can be the attractor for different IFSs, this notation is slightly imprecise.

Proposition 1.9. *If X is the attractor of the IFS Φ, then*

$$\dim X \leq \dim_{\text{sim}} X. \tag{1.10}$$

Proof. Let $s = \dim_{\text{sim}} X$. We will show that $\mathscr{H}^s(X) < \infty$, so by definition of Hausdorff dimension, $\dim X \leq s$.

Since X is invariant under Φ, we can iterate the contractions, and X will still be invariant, so

$$X = \bigcup_{\mathbf{j} \in \Lambda^n} \varphi_{\mathbf{j}}(X),$$

where $\varphi_{\mathbf{j}} = \varphi_{i_1} \circ ... \circ \varphi_{i_n}$, for $\mathbf{j} = \{i_1, ..., i_n\} \in \Lambda^n$. Thus, we can cover X with the images of itself under $\varphi_{\mathbf{j}}$ for all $\mathbf{j} \in \Lambda^n$. Because the φ_i are contractions, for all $i \in \Lambda$, $|\varphi_i(X)| \leq r_i \cdot |X|$. If we compose them,

$$|\varphi_{\mathbf{j}}(X)| \leq r_{i_1} ... r_{i_n} \cdot |X|.$$

Then,

$$\mathscr{H}^s(X) \leq \lim_{n \to \infty} \sum_{\mathbf{j} \in \Lambda^n} (r_{i_1} ... r_{i_n} \cdot |X|)^s = \lim_{n \to \infty} |X| \Bigg(\sum_{i \in \Lambda} r_i^s \Bigg)^n,$$

with the equality following from Vieta's formulas. But s was chosen such that

$$\sum_{i \in \Lambda} r_i^s = 1,$$

so

$$\mathscr{H}^s(X) \leq |X| < \infty,$$

since X is compact. □

The dimension drop conjecture concerns the inequality in Proposition 1.9. When do we have equality in (1.10), and when is the inequality strict? Let us examine an example.

Example 1.11. One simple example of a fractal set arising from an IFS is the middle-third Cantor Set.

If our IFS is made up of the contractions $\varphi_1 = \frac{1}{3}x$ and $\varphi_2 = \frac{1}{3}x + \frac{2}{3}$, then its attractor is exactly the middle-third Cantor set F. This is evident from the fact that the self-similarities of the middle-third Cantor Set are exactly described by the two contractions of this IFS, so we have

$$F = \varphi_1(F) \cup \varphi_2(F),$$

with the union disjoint.

We also know that the Hausdorff dimension of F is $\log 2/\log 3$. Without getting into a rigorous calculation, notice that because our union above is disjoint, we can use finite additivity and scaling properties of s-dimensional Hausdorff measure, where $s = \log 2/\log 3$. Then,

$$\begin{aligned}
\mathscr{H}^s(F) &= \mathscr{H}^s(\varphi_1(F)) + \mathscr{H}^s(\varphi_2(F)) \\
&= \mathscr{H}^s\left(\frac{1}{3}F\right) + \mathscr{H}^s\left(\frac{1}{3}F + \frac{2}{3}\right) \\
&= 2\left(\frac{1}{3}\right)^s \mathscr{H}^s(F).
\end{aligned}$$

Then, assuming that $0 < \mathscr{H}^s(F) < \infty$, we can divide out to get

$$\begin{aligned}
\frac{1}{2} &= \left(\frac{1}{3}\right)^s \\
\implies s &= \frac{\log 2}{\log 3}.
\end{aligned}$$

This argument is non-rigorous because of our assumption that $\mathscr{H}^s(F)$ is non-zero and finite. A better calculation of $\dim F$ requires justification of this assumption by bounding $\mathscr{H}^s$.

However, the reason this heuristic argument works is that the IFS for the Cantor set satisfies a necessary separation assumption. What we are secretly doing above is calculating the similarity dimension of F, but because the IFS is sufficiently separated, we actually have that $\dim F = \dim_{\mathrm{sim}} F$. In the case of the Cantor set, the IFS is completely separated, meaning that the image of the attractor under every contraction in the IFS is disjoint. However, as long as the contractions in the IFS don't overlap "too much", we are guaranteed equality between the Hausdorff dimension and the similarity dimension of the attractor.

In general, $\dim X = \dim_{\mathrm{sim}} X$ if a condition called the open set condition is satisfied.

Definition 1.12. An IFS $(\varphi_i)_{i\in\Lambda}$ satsifies the open set condition if there exists a nonempty open set U such that for all $i, j \in \Lambda$,

$$\varphi_i(U) \subset U$$

and

$$\varphi_i(U) \cap \varphi_j(U) = \emptyset$$

if $i \neq j$.

The open set condition means that the contractions in Φ are sufficiently separated, i.e. they can overlap, but the overlaps have bounded multiplicity. The Cantor set satsifies the open set condition, as well as a special case called the strong separation condition, meaning that the images of each contraction are disjoint.

For IFSs not satisfying the open set condition, which have "too much" overlap, the dimension of their attractors becomes very difficult to calculate.

However, if there are so-called exact overlaps in the IFS, we know that the inequality in (1.10) is strict.

Definition 1.13. Let $\Phi = (\varphi_i)_{i\in\Lambda}$ be an IFS. The *n-th generation cylinders* of Φ are the set of compositions of n contractions in Φ. If $\mathbf{j} = \{i_1, ..., i_n\} \in \Lambda^n$, then

$$\varphi_{\mathbf{j}} := \varphi_{i_1} \circ ... \circ \varphi_{i_n}$$

is the cylinder corresponding to $\mathbf{j}$.

Definition 1.14. We say $\Phi = (\varphi_i)_{i\in\Lambda}$ has *exact overlaps* if there exist distinct $\mathbf{j}, \mathbf{k} \in \Lambda^n$ such that

$$\varphi_{\mathbf{j}} = \varphi_{\mathbf{k}}.$$

Note that if Φ has exact overlaps for some n, then exact overlaps exist for all sufficiently large n, because we can always add an identical sequence of any length to both $\mathbf{j}$ and $\mathbf{k}$ and they would still be distinct.

Proposition 1.15. *If $\Phi = (\varphi_i)_{i\in\Lambda}$ has exact overlaps, and X is the attractor of Φ, then $\dim X < \dim_{\mathrm{sim}} X$.*

Proof. There is some n such that there exist $\mathbf{j} \neq \mathbf{k} \in \Lambda^n$ with $\varphi_{\mathbf{j}} = \varphi_{\mathbf{k}}$. Define a new IFS as follows:

$$\Phi^n := (\varphi_{\mathbf{i}})_{\mathbf{i}\in\Lambda^n}.$$

This IFS contains all possible compositions of length n of the contractions of Φ. Because of the exact overlaps, we can chose Φ' which is a proper subset of Φ^n by simply discarding one of the overlapping cylinders. However, X is still the attractor of Φ', because we are just composing the functions from Φ.

Now, let s' be the similarity dimension coming from Φ', and let s be the similarity dimension coming from Φ. By definition, s' is such that

$$\sum_{\mathbf{i}\in\Lambda^n\setminus\{\mathbf{j}\}} (r_{i_1} \cdot ... \cdot r_{i_n})^{s'} = 1$$

But since we are excluding $\mathbf{j}$,

$$\sum_{\mathbf{i}\in\Lambda^n\setminus\{\mathbf{j}\}} (r_{i_1} \cdot ... \cdot r_{i_n})^{s'} < \sum_{\mathbf{i}\in\Lambda^n} (r_{i_1} \cdot ... \cdot r_{i_n})^{s'} = \left(\sum_{i\in\Lambda} r_i\right)^{s'},$$

by Vieta's formulas. Now, by definition of s,

$$\sum_{i\in\Lambda}(r_i)^s = 1 < \left(\sum_{i\in\Lambda} r_i\right)^{s'} < \sum_{i\in\Lambda}(r_i)^{s'},$$

implying $s' < s$, since $0 < r_i < 1$. Finally, by Proposition 1.9,

$$\dim X \leq s' < s,$$

so there is a dimension drop. □

The dimension of attractors of IFSs are relatively well-understood if there are either exact overlaps in the IFS, or the IFS satisfies the open set condition or other specific assumptions. However, in full generality, much less is understood if either exact overlaps or sufficient separation assumptions are not satisfied, and it is not even know whether or not the inequality in Proposition 1.9 is strict.

Conjecture 1.16. *(Dimension Drop) If X is the attractor of an IFS Φ, then $\dim X < \dim_{\mathrm{sim}} X$ if and only if there are exact overlaps.*

If the conjecture were true, then the dimension and the similarity dimension would be equal in all cases besides those with exact overlaps. Hochman's result does not prove the conjecture, but it does make some progress, and gives rise to a corollary that proves the conjecture if the IFS has parameters which are algebraic numbers.

2. Results of Hochman

We can now state the main result of Hochman's 2014 paper which lends some support to the conjecture. First, we need some notation.

Recall the definition of the cylinders of Φ. To characterize the existence of exact overlaps, define a distance between cylinders as follows:

Definition 2.1. For $\mathbf{i}, \mathbf{j} \in \Lambda^n$,

$$d(\mathbf{i}, \mathbf{j}) = \begin{cases} \infty & \text{if } r_{\mathbf{i}} \neq r_{\mathbf{j}} \\ |\varphi_{\mathbf{i}}(0) - \varphi_{\mathbf{j}}(0)| & \text{if } r_{\mathbf{i}} = r_{\mathbf{j}} \end{cases},$$

where $r_{\mathbf{i}}$ is the contraction ratio of $\varphi_{\mathbf{i}}$, i.e. $r_{\mathbf{i}} = r_{i_1} \cdot \ldots \cdot r_{i_n}$.

Note that if $d(\mathbf{i}, \mathbf{j}) = 0$, this means $\varphi_{\mathbf{i}}(0) = \varphi_{\mathbf{j}}(0)$ and $r_{\mathbf{i}} = r_{\mathbf{j}}$, implying that $\varphi_{\mathbf{i}} = \varphi_{\mathbf{j}}$. Now, we can characterize exact overlaps with the following:

Definition 2.2. For $n \in \mathbb{N}$, let

$$\Delta_n = \min\{d(\mathbf{i}, \mathbf{j}) : \mathbf{i}, \mathbf{j} \in \Lambda^n\}.$$

The reason this definition is useful is because there exist exact overlaps in our IFS if and only if $\Delta_n = 0$ for some n, that is, if there exists some n such that $d(\mathbf{i}, \mathbf{j}) = 0$ for some $\mathbf{i}, \mathbf{j} \in \Lambda^n$, meaning that $\varphi_{\mathbf{i}} = \varphi_{\mathbf{j}}$. This definition of Δ_n has to do with the existence of exact overlaps, and the main result of Hochman's paper links this definition with the strict inequality in (1.10), hence the dimension drop connection. Here is the main result:

Theorem 2.3. *(Hochman) If X is the attractor of an IFS, and if $\dim X < \dim_{\mathrm{sim}} X$, then*

$$\lim_{n\to\infty}\left(-\frac{1}{n}\log\Delta_n\right)=\infty.$$

This result is important is because it lends support to the conjecture by knocking out certain specific cases, and in fact giving rise to a corollary that proves the conjecture in the case that the parameters of the IFS are algebraic numbers.

However, the result is not enough to prove the conjecture, and in fact there are counterexamples. It was shown by Bárány and Käenmäky that there exist attractors of IFSs with super-exponential condensation, meaning $\lim\limits_{n\to\infty}(-\frac{1}{n}\log\Delta_n)=\infty$, but no exact overlaps. If such sets did not exist, then the conjecture would be proved, since super-exponential condensation would imply exact overlaps, so it would follow immediately from Hochman's result. However, this is not the case, so it is necessary to study overlaps in a more nuanced way.

To prove this result, Hochman used results about entropy, which is, loosely, a way to measure the "randomness" of a probability measure.

Definition 2.4. If μ is a probability measure and $\mathcal{E}$ is a countable partition, then the entropy of μ with respect to $\mathcal{E}$ is

$$H(\mu,\mathcal{E})=-\sum_{E\in\mathcal{E}}\mu(E)\log\mu(E).$$

where the convention is that $0\log 0=0$.

If our probability measure μ is only supported on one element of the partition $\mathcal{E}$, then $H(\mu,\mathcal{E})=0$. However, if μ behaves more chaotically, the entropy increases, though we always have the bound $H(\mu,\mathcal{E})\leq\log k$ if μ is supported on k elements of $\mathcal{E}$, with equality if μ is uniform on these k elements.

To prove the result, Hochman uses properties of entropy with respect to self-similar measures which arise from IFSs.

Definition 2.5. If we have a probability vector $(p_i)_{i\in\Lambda}$ and an IFS Φ, then the self-similar measure associated with Φ and (p_i) is the unique probability measure defined on the Borel sets that satisfies

$$\mu=\sum_{i\in\Lambda}p_i\cdot(\mu\circ\varphi_i^{-1})$$

.

To prove that Δ_n goes to 0 super-exponentially, we can approximate μ, and then examine the entropy of this approximation at smaller scales, i.e. using smaller partitions of $\mathbb{R}$. Then, it is possible that there is "excess entropy" from a smaller scale partition compared to a discrete partition. Then, we can get super-exponential condensation by controlling this excess entropy.

3. A corollary for algebraic parameters

Following Hochman's main result is a corollary which confirms the dimension drop conjecture, in the special case where the parameters of the IFS are algebraic numbers.

Theorem 3.1. *If X is the attractor of an IFS with algebraic parameters, that is, r_i and a_i are algebraic numbers for each contraction, then there are exact overlaps if and only if* $\dim X < \dim_{\mathrm{sim}} X$.

This is the statement of the dimension drop conjecture restricted to IFS's with algebraic parameters. The proof of this corollary relies on an algebraic lemma, which we will prove first.

Lemma 3.2. *If A is a set of algebraic numbers, then there exists some constant $0 < s < 1$ such that for any any polynomial expression x of these numbers with degree n and $x \neq 0$, then $x \geq s^n$.*

Proof. (Following the proof of Hochman, see [5], Lemma 5.10). Let α be an algebraic integer such that $A \subset \mathbb{Q}(\alpha)$, i.e. all elements of A can be expressed as polynomials with rational coeffieicnets in α. Now, by multiplying by integers, we can make every element of A expressible as an integer polynomial in α, and since our set is finite, these polynomials must have degree bounded by some d and bounded coefficients.

Thus, we can write all $y \in A$ as follows:

$$y = \sum_{k=0}^{d} a_k \alpha^k$$

where $a_k \in \mathbb{Z}$ and the a_k are bounded. Now, substituting, we can write any polynomial expression x of degree n in elements of A as

$$x = \sum_{k=0}^{dn} n_k \alpha^k$$

where the n_k come from expanding our expression for y, so they are still integers and still bounded, so we can say $|n_k| \leq N$ for some N.

Now, let $\alpha_2, ..., \alpha_d$ be the algebraic conjugates of α, writing $\alpha_1 := \alpha$, and let $\sigma_1, ..., \sigma_d$ be the automorphisms of $\mathbb{Q}(\alpha)$, with $\sigma_i \alpha := \alpha_i$. Being automorphisms, the σ_i fix all the rationals, but interchange the α_i. But what do the σ_i do to our polynomial x? By a fact from Galois theory, if $x \neq 0$, then $\prod_{i=1}^{d} \sigma_i(x)$ is a nonzero integer. Now we can establish an inequality for x. Since all nonzero integers have absolute value greater than or equal to 1,

$$1 \leq \left| \prod_{i=1}^{d} \sigma_i(x) \right| = \prod_{i=1}^{d} \left| \sum_{k=0}^{dn} \sigma_i(n_k \alpha^k) \right| = |x| \cdot \prod_{i=2}^{d} \left| \sum_{k=0}^{dn} n_k \alpha_i^k \right|.$$

We can distribute the σ_i in this way because they are automorphisms, so they respect the structure of $\mathbb{Q}(\alpha)$, and we can take out the x because the first term in the product is simply our expression for x, since $\alpha_1 = \alpha$.

Finally, by the triangle inequality and some simple bounds,

$$1 \leq |x| \ \cdot \prod_{i=2}^{d} \sum_{k=0}^{dn} |n_k||\alpha_i|^k \leq |x| \cdot \prod_{i=2}^{d} N dn |\alpha_i|^{dn} \leq |x| (N dn \alpha_{\max}^{dn})^d,$$

so now,

$$|x| \geq ((Ndn)^{-\frac{d}{n}} \alpha_{\max}^{-d^2})^n,$$

which gives the lower bound for x if $x \neq 0$. □

We can now prove Theorem 3.1 using the lemma.

Proof. Examining the cylinders for our IFS, if $\mathbf{i}, \mathbf{j} \in \Lambda^n$, then the cylinders $\varphi_{\mathbf{i}}$ and $\varphi_{\mathbf{j}}$ are just compositions of the linear contractions of the IFS. Thus $\varphi_{\mathbf{i}}(0)$ and $\varphi_{\mathbf{j}}(0)$ are both polynomials of degree n in r_i and a_i, implying that Δ_n is also such a polynomial.

By the lemma, either $\Delta_n = 0$ or $\Delta_n \geq s^n$ for some constant s. If $\Delta_n = 0$, we know what this means: exact overlaps. But if $\Delta_n \neq 0$,

$$\Delta_n \geq s^n$$
$$\log \Delta_n \geq n \log s$$
$$-\frac{1}{n} \log \Delta_n \leq -\log s.$$

This means that $\lim_{n\to\infty}(-\frac{1}{n}\log\Delta_n) \neq \infty$, so by the contrapositive of Theorem 2.3, $\dim X = \dim_{\mathrm{sim}} X$. Thus, for the case of algebraic parameters in the IFS, there are exact overlaps if and only if $\dim X < \dim_{\mathrm{sim}} X$. □

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