VECTOR BUNDLES AND STIEFEL-WHITNEY CLASSES

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Abstract. This paper introduces the basics of vector bundles, including induced bundles, Whitney products, and bundles over projective spaces, before defining Stiefel-Whitney classes based on their axioms. It then uses Stiefel-Whitney classes as a tool to prove a number of interesting results involving projective space. It then introduces Stiefel-Whitney numbers, and uses those to state the basic idea of cobordism classes. The reader is assumed to have a basic understanding of manifolds, vector spaces, and algebraic topology including cohomology classes.

Contents

1. Vector Bundles

Definition 1.1. For a given **base space** B, the **vector bundle over** B, ξ, is defined to include a **total space**, $E(\xi)$, and a continuous **projection map**, $\pi :\ E \to B$, such that for every $b \in B$, $\pi^{-1}(b)$ has the structure of a vector space called the **fiber over** b, or $F_b(\xi)$.

A vector bundle must also fulfill the condition of **local triviality**: for every $b \in B$, there must exist a neighborhood $U \subseteq B$ of b and a homeomorphism $f :\ U \times\ \mathbb{R}^n \to \pi^{-1}(U)$ for some nonnegative integer n such that for every $b \in U$, $x \mapsto f(b, x)$ represents an isomorphism between vector spaces.

If it is possible to meet this condition with $U = B$, the ξ is called a **trivial bundle**.

Definition 1.2. For a smooth manifold M, the **tangent bundle**, τ_M, is the one whose total space includes all points (x, v), such that $x \in M$ and v is tangent to M at x, with projection map $\pi(x, v) = x$. If this is a trivial bundle then M is **parallelizable**.

Definition 1.3. A **Euclidean vector bundle** is a vector bundle ξ and a continuous function $\chi : E(\xi) \to \mathbb{R}$ such that χ is positive definite and quadratic when restricted to any fiber of the bundle.

Definition 1.4. Consider two bundles, ξ and η, that share the same base space. They are **isomorphic**, $\xi \cong \eta$, if there is a homeomorphism $f : E(\xi) \to E(\eta)$ that when restricted to any fiber $F_b(\xi)$ is an isomorphism into the vector space of $F_b(\eta)$.

We now want to consider in what situations bundles are isomorphic.

Lemma 1.5. *For ξ, η defined as above, if $f : E(\xi) \to E(\eta)$ is continuous and maps each $F_b(\xi)$ isomorphically into $F_b(\eta)$, then f is a homeomorphism.*

Proof. Pick arbitrary $b_0 \in B$. There exists a neighborhood U in which local triviality is met for both ξ and η. Then $g : U \times (R)^n \to F_u(\xi)$ and $h : U \times (R)^n \to F_u(\eta)$ exist and are homeomorphisms.

Now, say $h^{-1}(f(g(b,x))) = (b,y)$ (as f does not move in the base space). Clearly, y varies continuously with x, as this relationship is an isomorphism. Then, since we can likewise see $g^{-1} \circ f^{-1} \circ h$ to be continuous, $h^{-1} \circ f \circ g$ is a homeomorphism because it is bijective and has continuous inverse. Also, since f^{-1} sends $F_b(\eta)$ to $F_b(\xi)$, it is also continuous, and so since we saw above that g and h are homeomorphisms, f itself is a homeomorphism. □

Now, we will consider ways that might generate bundles from other bundles

Definition 1.6. Given a bundle ξ and any space B_1, a map $g : B_1 \to B$ generates an **induced bundle** $g^*\xi$ over B_1 with total space $E_1 \subset B_1 \times E$ containing all (b,e) such that $g(b) = \pi(e)$ and projection map $\pi_1(b,e) = b$.

It can be seen that the function $\hat{g}(b,e) = e$ isomorphically takes each fiber $F_b(g^*\xi)$ into the fiber $F_{g(b)}(\xi)$. This alludes to a more general relationship between bundles.

Definition 1.7. A **bundle map** from η to ξ is any continuous function $h : E(\eta) \to E(\xi)$ such that each fiber $F_b(\eta)$ is brought isomorphically into $F_{b_0}(\xi)$ for some $b_0 \in B(\xi)$.

Lemma 1.8. *If $h : E(\eta) \to E(\xi)$ is a bundle map, and $\bar{h}$ the corresponding map between base spaces, then $\eta \cong \bar{h}^*\xi$.*

Proof. Define a function $f : E(\eta) \to E(\bar{h}^*\xi)$ such that $f(e) = (\pi_\eta(e), h(e))$. (That by definition $\bar{h}(\pi_\eta(e)) = \pi_\xi(h(e))$ proves that this is an accurate codomain.) Then, f is continuous, as its components are by definition, and takes each $F_b(\eta)$ isomorphically into $F_b(\bar{h}^*\xi)$, then Lemma 1.5 means that f is a homeomorphism, and so the definition of isomorphism is satisfied. □

We will now develop another way to generate new vector bundles from other bundles.

Definition 1.9. Two bundles ξ_1, ξ_2 have a **Cartesian product** $\xi_1 \times \xi_2$, a bundle with the total space $E(\xi_1) \times E(\xi_2)$ and projection map $\pi_1 \times \pi_2(e_1, e_2) = (\pi_1(e_1), \pi_2(e_2))$.

If these two bundles share the same base space B, and $d : B \to B \times B$ denotes the diagonal embedding, we can define the **Whitney sum** of the two bundles, $\xi_1 \oplus \xi_2 = d^*(\xi_1 \times \xi_2)$.

Lemma 1.10. *For a bundle ξ, let η_1, η_2 be bundles such that for every $b \in B$, $F_b(\xi)$ has $F_b(\eta_1)$ and $F_b(\eta_2)$ as vector subspaces and is equal to their direct sum. Then $\xi \cong \eta_1 \oplus \eta_2$.*

Proof. We know that $\eta_1 \oplus \eta_2$ includes the points (b, e_1, e_2) such that $d(b) = \pi_1 \times \pi_2(e_1, e_2)$, i.e., $\pi_1(e_1) = \pi_2(e_2) = b$. We can define a function $f : E(\eta_1 \oplus \eta_2) \to E(\xi)$ such that $f(b, e_1, e_2) = e_1 + e_2$. It is obvious that f is continuous, and f is an isomorphism because for any $b \in B$, $F_b(\xi)$ will be spanned by all sums of elements of $F_b(\eta_1), F_b(\eta_2)$. Then, by Lemma 1.5, f is a homeomorphism, and the proof follows. □

In general, we can call a bundle ξ a sub bundle of η is they share a base space B, and for every $b \in B$, $F_b(\xi)$ is a vector subspace of $F_b(\eta)$. This begs the question of under what conditions a subbundle has an associated bundle with which its Whitney sum is isomorphic to the original.

Definition 1.11. If η is a Euclidean vector bundle (with function χ), a subbundle ξ has an **orthogonal complement** $\xi^\perp$ with the same base space, defined that $F_b(\xi^\perp)$ includes all points $w \in F_b(\eta)$ such that for every $v \in F_b(\xi)$,

$$v \cdot w := \frac{1}{2}(\chi(v + w) - \chi(v) - \chi(w)) = 0.$$

Lemma 1.12. *For ξ a subbundle of η, $\xi \oplus \xi^\perp \cong \eta$.*

Proof. Following from Lemma 1.10, we need only to show $F_b(\eta)$ is the direct sum of $F_b(\xi)$ and $F_b(\xi^\perp)$. It is known as a property of vector subspaces that every vector in $F_b(\eta)$ can be decomposed into a part parallel to $F_b(\xi)$ and a part orthogonal to it. Then, observing the above construction, this means it is the sum of a vector in $F_b(\xi)$ and $F_b(\xi^\perp)$. □

2. Projective Space

Definition 2.1. The **real projective space** P^n is a quotient space of $S^n \subset \mathbb{R}^{n+1}$ that maps together all pairs $x, -x$ on S^n.

Alternatively, this is the set of lines through the origin in $\mathbb{R}^{n+1}$, each of which intersects S^n on a set of this form.

Definition 2.2. The **canonical line bundle** over P^n, γ_n^1, has total space $E \subseteq P^n \times \mathbb{R}^{n+1}$ of all pairs $(\pm x, v)$ such that v is a scalar multiple of x and $\pi(\pm x, v) = \pm x$.

Lemma 2.3. *γ_n^1 as defined above meets the condition of local triviality (and thus is, in fact, a bundle).*

Proof. For open $U \subset S^n$ containing no pair of points $-x, x$, there corresponds a set U' in P^n. A neighborhood of this sort can be created around any point in P^n. Then, we can have a homeomorphism $f : U' \times \mathbb{R} \to \pi^{-1}(U')$ such that $f(\pm x, t) = (\pm x, tx)$. Clearly this represents an isomorphism into the vector space, thus satisfying local triviality. □

We will now approach the question of if γ_n^1 is trivial.

Definition 2.4. A **nowhere zero cross-section** of a bundle ξ is a continuous function $s : B(\xi) \to E(\xi)$ such that for every $b \in B$, $s(b)$ is a nonzero vector in $F_b(\xi)$.

Lemma 2.5. *The bundle γ_n^1 defined above is nontrivial for every $n \geq 1$.*

Proof. We can first consider that if γ_n^1 were trivial, then there would be a homeomorphism $f : P^n \times \mathbb{R} \to E(\gamma_n^1)$, and thus a nowhere zero cross-section $s(b) = f(b, 5)$ ($|s(b)| \geq 5 > 0$). We can therefore prove this lemma by showing that there is not nowhere zero cross-section.

Consider that, for any cross section s, the composition $S^n \to P^n \xrightarrow{s} E(\gamma_n^1)$ takes each $x \in S^n$ to $(\pm x, t(x)x) \in E(\gamma_n^1)$. As the cross-section depends only on the point in P^n, we have $t(x) = -t(-x)$, so by the Intermediate Value Theorem, $t(x_0) = 0$ for some $x_0 \in S^n$. Then, $s(\pm x_0) = (\pm x, 0)$ means that s cannot be a nowhere zero cross-section. □

3. Stiefel-Whitney Classes

Definition 3.1. The **Stiefel-Whitney classes** of a vector bundle ξ are a sequence $w_i(\xi)_{i=0,1,2,...}$ of the singular cohomology groups of B with coefficients in $\mathbb{Z}/2$, fully characterized by four axioms.

Axiom 3.2. $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2)$ such that $w_0(\xi) = 1$ and $w_i(\xi) = 0$ for $i > n$, where n is as seen in the condition of local triviality to be the maximum dimension of any fiber.

Axiom 3.3. If $f : B(\xi) \to B(\eta)$ is the restriction to $B(\xi)$ of a bundle map between ξ and η, then $w_i(\xi) = f^* w_i(\eta)$. (* here refers to pullback in the cohomology sense.)

Axiom 3.4. If ξ and η are bundles over the same base space, then $w_k(\xi \oplus \eta) = \sum_{i=0}^k w_i(\xi) w_{k-i}(\eta)$. (The product operation here is the cup product, $\smile$.)

Axiom 3.5. $w_1(\gamma_1^1)$ is nonzero.

We can define a ring $H^\Pi(B; \mathbb{Z}/2)$ of all formal infinite series $a = a_0 + a_1 + ...$, where $a_i \in H^i(B; \mathbb{Z}/2)$. This ring will have product operation

$$ab = (a_0 b_0) + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + ...$$

Definition 3.6. The **total Stiefel-Whitney class** of ξ is the element $w(\xi) \in H^\Pi(B; \mathbb{Z}/2)$ such that $w(\xi) = 1 + w_1(\xi) + ... + w_n(\xi) + 0 + 0 + ...$

We can now state a few very short consequences of these axioms.

Lemma 3.7. *If ξ and η are bundles such that $\xi \cong \eta$, then $w_i(\xi) = w_i(\eta)$.*

Proof. Given $\xi \cong \eta$, the function $f : B(\xi) \to B(\eta)$ is a homeomorphism when defined as in Axiom 3.3. Thus, f^* is the identity, so $w_i(\xi) = f^* w_i(\eta) = w_i(\eta)$. □

Lemma 3.8. *If ξ is a trivial vector bundle, then $w_i(\xi) = 0$ for $i > 0$.*

Proof. If ξ is trivial, then

$$E(\xi) \xrightarrow{f^{-1}} B(\xi) \times \mathbb{R}^n \xrightarrow{k} \mathbb{R}^n,$$

where $k(b, x) = x$ and f is as in condition of triviality, represents a bundle map from ξ to a bundle over a single point.

Then $g : B(\xi) \to x$ can be taken as in Axiom 3.3 for some point x, and so g^* brings classes to zero, and thus for $i > 0$, $w_i(\xi) = g^* w_i(x) = 0$. □

Lemma 3.9. *If ξ is trivial, then $w_i(\xi \oplus \eta) = w_i(\eta)$.*

Proof. By Axiom 3.4 and the above result,

$$w_i(\xi \oplus \eta) = \sum_{j=0}^{i} w_j(\xi) w_{i-j}(\eta) = 1 w_i(\eta) + \sum_{j=1}^{i-1} 0 w_{i-j}(\eta) = w_i(\eta).$$

□

4. Projective Space and Stiefel-Whitney Classes

The remainder of the paper will use Stiefel-Whitney classes to prove some results about projective spaces.

Lemma 4.1. *The canonical line bundle γ_n^1 has total Stiefel-Whitney class $w(\gamma_n^1) = 1 + a$.*

Proof. The inclusion map $f : P^1 \to P^n$ is the restriction to P^1 of a bundle map from γ_1^1 to γ_n^1. By Axiom 3.5, $f^* w_1(\gamma_n^1) = w_1(\gamma_1^1) \neq 0$, so it must be true that $w_1(\gamma_n^1) = a$. Then, Axiom 3.2 fixes all other classes. □

It is obvious that γ_n^1 is the subbundle of a trivial bundle ξ with total space $P^n \times \mathbb{R}^{n+1}$. Then let $\gamma^\perp$ denote the orthogonal complement of γ_n^1 in ξ.

Lemma 4.2. $w(\gamma^\perp) = 1 + a + a^2 + ... + a^n$

Proof. Given that $\gamma_n^1 \oplus \gamma^\perp$ is trivial, then by Axiom 3.4, $w(\gamma_n^1) w(\gamma^\perp) = 1$. Then, given that $w(\gamma_n^1) = 1 + a$,

$$w(\gamma^\perp) = (1 + a)^{-1} = 1 + a + ... + a^n.$$

□

For two bundles ξ, η over the same base space B, we can define the bundle $Hom(\xi, \eta)$ over B such that for every $b \in B$, F_b consists of all linear transformations from $F_b(\xi)$ to $F_b(\eta)$.

Lemma 4.3. *The tangent bundle of P^n is isomorphic to the bundle $Hom(\gamma_n^1, \gamma^\perp)$.*

Proof. Consider the function $f : S^n \to P^n$ such that $f(x) = \pm x$. This induces a function $df : TS^n \to TP^n$ that gives (x, v) and $(-x, -v)$ the same image. TP^n, then, can be considered the set of pairs $(x, v), (-x, -v)$ such that $x \cdot x = 1, x \cdot v = 0$. Such a pair defines a map from the line containing x into the orthogonal n-plane in $\mathbb{R}^{n+1}$ according to the value of v. Thus, at each point $\pm x$ of P^n, the tangent space is isomorphic to the set of transformations from the fiber of γ_n^1 at that point to the fiber of $\gamma^\perp$. Then, that $\tau_{P^n} \cong Hom(\gamma_n^1, \gamma^\perp)$ follows. □

Lemma 4.4. *The tangent bundle of P^n has total Stiefel-Whitney class $w(\tau_{P^n}) = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + ... + \binom{n+1}{n} a^n$.*

Proof. Define η to be the trivial line bundle over P^n. Consider that the bundle $Hom(\gamma_n^1, \gamma_n^1)$ is trivial based on the function $f : P^n \times \mathbb{R} \to E$ where $f(\pm x, t)$ is the map that multiplies by t. Then, applying Lemma 4.3,

$$\tau_{P^n} \oplus \eta \cong Hom(\gamma_n^1, \gamma^\perp) \oplus Hom(\gamma_n^1, \gamma_n^1).$$

According to Lemma 1.10, this is isomorphic to the bundle whose fibers are direct sums of maps from γ_n^1 to $\gamma^\perp$ and maps from γ_n^1 to γ_n^1. These, we see, are just maps from γ_n^1 to $\gamma^\perp \oplus \gamma_n^1$, so it follows that

$$\tau_{P^n} \oplus \eta \cong Hom(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \cong Hom(\gamma_n^1, \xi),$$

where ξ is defined as in Lemma 4.2.

However, it follows from their being trivial bundles that

$$\xi = \underbrace{\eta \oplus ... \oplus \eta}_{n+1}.$$

Thus, as above,

$$\tau_{P^n} \oplus \eta \cong \underbrace{Hom(\gamma_n^1, \eta) \oplus ... \oplus Hom(\gamma_n^1, \eta)}_{n+1}.$$

Now, the fact that $\gamma_n^1 \cong Hom(\gamma_n^1, \eta)$ proves that

$$\tau_{P^n} \oplus \eta \cong \underbrace{\gamma_n^1 \oplus ... \oplus \gamma_n^1}_{n+1}.$$

Thus,

$$w(\tau_{P^n}) = w(\tau_{P^n} \oplus \eta) = \underbrace{w(\gamma_n^1) \oplus ... \oplus w(\gamma_n^1)}_{n+1} = (1+a)^{n+1},$$

which along with the binomial theorem completes the proof. $\square$

Lemma 4.5. *A projective space P^n can be parallelizable only if $n+1$ is a power of 2.*

Proof. In mod 2, $(a+b)^2 = a^2 + b^2 + 2ab = a^2 + b^2$. Thus, $(1+a)^{2^k} = 1 + a^{2^k}$. If, then, $n+1 = 2^k$,

$$w(\tau_{P^n}) = (1+a)^{n+1} = 1 + a^{n+1} = 1$$

(because Axiom 3.2 requires $w_{n+1}(P^n) = 0$). However, if $n+1 = b \cdot 2^k$ for $b > 1$ odd, then

$$w(\tau_{P^n}) = (1 + a^{2^k})^b = 1 + b \cdot a^{2^k} + ...,$$

and because $2^k < n+1$ and b is odd, $w(\tau_{P^n}) \neq 1$. Thus, $w(\tau_{P^n}) = 1$ if and only if $n+1$ is a power of 2, and the lemma follows from Lemma 3.8. $\square$

Definition 4.6. Nowhere zero cross-sections $s_1, ..., s_n$ are **nowhere dependent** if for every $b \in B$, $s_1(b), ..., s_n(b)$ are linearly independent vectors. (The nowhere zero condition becomes redundant here.)

This allows us to relate projective space to the existence of real division algebras.

Lemma 4.7. *If there exists a bilinear product operation $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisors, then P^{n-1} is parallelizable.*

Proof. If $b_1, ..., b_n$ are the standard basis vector of $\mathbb{R}^n$, then because p is without zero divisors, we can take the formula $v_i(p(y, b_1) = p(y, b_i)$ to define a linear transformation $v_i : \mathbb{R}^n \to \mathbb{R}^n$. Imagine there were a point $x = p(y, b_1)$ such that $\sum_i \lambda_i v_i(x) = 0$. Then, $p(y, \sum_i \lambda_i b_i) = 0$, so $\lambda_i = 0$ for every i. Thus, $v_1(x), ..., v_n(x)$ are linearly independent for $x \neq 0$.

We then see that $v_1(x) = x$, so $v_2, ..., v_n$ yield $n-1$ linearly independent vectors in the $(n-1)$ plane orthogonal to x, and thus (because this can be done at *any* $x \neq 0$), $n-1$ nowhere dependent cross-sections of the bundle $Hom(\gamma_{n-1}^1, \gamma^\perp) \cong \tau_{P^{n-1}}$. Now, define a function

$$f : P^{n-1} \times \mathbb{R}^{n-1} \to E(\tau_{P^{n-1}})$$

such that

$$f(\pm x, z) = \sum_{i=1}^{n-1} v_{i+1}(\pm x) z_i.$$

f is a continuous map that takes fibers isomorphically between the trivial bundle over P^{n-1} and $\tau_{P^{n-1}}$. Thus, by Lemma 1.5, $\tau_{P^{n-1}}$ is isomorphic to a trivial bundle, thus trivial itself. $\square$

It follows from Lemma 4.5 that for such a division algebra to exist, n must be a power of 2. Now, we will prove one last fact about projective spaces.

Definition 4.8. A smooth map between manifolds is an **immersion** if the Jacobian $df_x : T_xM \to T_{f(x)}N$ is an injective mapping at every $x \in M$.

Lemma 4.9. *If P^{2^k} can be immersed on $\mathbb{R}^n$, then $n \geq 2^{k+1} - 1$.*

Proof. If there is an immersion $f : P^{2^k} \to \mathbb{R}^n$, then $f^*\tau_{\mathbb{R}^n}$ has $\tau_{P^{2^k}}$ as a subbundle. Then, by Lemma 1.12, $f^*\tau_{\mathbb{R}^n} \cong \tau_{P^{2^k}} \oplus \tau^\perp$. Thus, $\tau^\perp$ must have Stiefel-Whitney classes of 0 for $i > n - 2^k$. However, since $\tau_{\mathbb{R}^n}$ is evidently trivial, $w(\tau_{P^{2^k}})w(\tau^\perp) = 1$. Given that

$$w(\tau_{P^{2^k}}) = (1+a)^{2^k+1} = 1 + a + a^{2^k}$$

(as can be derived from Lemma 4.4), it follows that

$$w(\tau^\perp) = 1 + a + a^2 + ... + a^{2^k-1}.$$

Therefore, $w_i(\tau^\perp) = 0$ only if $i > 2^k - 1$, and so it must be true that $n \geq 2 \cdot 2^k - 1$ for these previous conditions to be met. $\square$

5. Stiefel-Whitney Numbers and Cobordism Classes

Consider M to be a compact, smooth, n-dimensional manifold.

Definition 5.1. We can define M to have **fundamental homology class** $\mu_M \in H_n(M; \mathbb{Z})$ such that for any point $x \in M$, the isomorphism $\rho_x : H_n(M) \to H_n(M, M - x)$ is such that $\rho_x(\mu_M) = \mu_x$ is one of the two possible generators of $H_n(M, M - x; \mathbb{Z})$. We then call μ_x a **local orientation** at x.

For our purposes, we can change the coefficients so that $\mu_M \in H_n(M; \mathbb{Z}/2)$.

Definition 5.2. For any cohomology class $v \in H^n(M; \mathbb{Z}/2)$, the **Kronecker index**, $v[M] \in \mathbb{Z}/2$, is defined as the output of v acting on μ_M, or $\langle v, \mu_M \rangle$.

Now, take nonnegative integers $r_1, r_2, ..., r_n$ such that $\sum_{i=1}^n r_i i = n$. For a vector bundle ξ, each such set corresponds to a monomial class

$$w_1(\xi)^{r_1} w_2(\xi)^{r_2} \cdots w_n(\xi)^{r_n} \in H^n(B(\xi); \mathbb{Z}/2).$$

Definition 5.3. A **Stiefel-Whitney number** of M is a value

$$w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}[M] \in \mathbb{Z}/2$$

for a monomial as constructed above.

Two manifolds, M, M', are considered to have the same Stiefel-Whitney numbers when

$$w_1^{r_1} \cdots w_n^{r_n}[M] = w_1^{r_1} \cdots w_n^{r_n}[M']$$

for every suitable monomial. (That the classes are of the tangent bundle is implied here.)

Deriving from Lemma 4.4, we can compute all Stiefel-Whitney numbers of a projective space P^n. For example, if n is a power of 2, $w(\tau_{P^n}) = 1 + a + a^n$, so $w_1^n[P^n]$ and $w_n[P^n]$ are nonzero, but all other Stiefel-Whitney numbers are zero.

On the other hand, if n is off, i.e. $n = 2k-1$, then $w(\tau_{P^n}) = (1+a)^{2k} = (1+a^2)^k$, so $w_j(\tau_{P^n}) = 0$ for all odd j. However, every monomial, then, must contain some factor of odd dimension. Thus, each of these monomials must multiply to zero, and therefore every Stiefel-Whitney number of P^n must be odd.

We will now prove an important result involving Stiefel-Whitney numbers.

Theorem 5.4. *If M is the boundary of a smooth compact (n+1)-manifold K, then every Stiefel-Whitney number of M is zero.*

Proof. We can construct a fundamental homology class $\mu_K \in H_{n+1}(K, M; \mathbb{Z}/2)$ such that the boundary homomorphism

$$\partial : H_{n+1}(K, M) \to H_n(M)$$

maps μ_K to μ_M. Then, by the definition of cohomology classes, we can note that for any $v \in H^n(M)$, $\langle v, \partial\mu_K\rangle = \langle \delta v, \mu_K\rangle$, where δ denotes the coboundary homomorphism from $H^n(M)$ to $H^{n+1}(K, M)$.

Consider now the bundle τ_K restricted to M. τ_M is clearly a subbundle of this bundle. Further, because M represents the boundary of K, it can be seen that $\tau_M^\perp \cong \xi$, where ξ is a trivial line bundle. (Simply, the portion of $\tau_K|_M$ not in τ_M will be that which points out, away from the boundary.) Thus, it follows that $w(\tau_K|M) = w(\tau_M)$.

Taking i^* as the restriction homomorphism, it follows from the exact sequence

$$H^n(K) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(K, M)$$

that $\delta(w_1^{r_1} \cdots w_n^{r_n}) = 0$, as the class is kept the same through the restriction. Therefore,

$$w_1^{r_1} \cdots w_n^{r_n}[M] = \langle w_1^{r_1} \cdots w_n^{r_n}, \mu_M\rangle = \langle w_1^{r_1} \cdots w_n^{r_n}, \partial\mu_K\rangle = \langle \delta w_1^{r_1} \cdots w_n^{r_n}, \mu_K\rangle = 0.$$

□

We will also state the converse; for a proof see [1].

Theorem 5.5. *If all Stiefel-Whitney numbers of M are zero, then M is the boundary of some smooth compact manifold.*

For example, then projective space P^n is the boundary of some manifold for every n odd.

We can introduce one more concept based on these results.

Definition 5.6. Smooth compact n-manifolds M_1, M_2 are said to share a **cobordism class** is their disjoint union is the boundary of a smooth compact (n+1)-manifold.

Thus, manifolds belong to the same cobordism class if and only if all of their Stiefel-Whitney numbers are equal.

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