THE STONE-ČECH COMPACTIFICATION

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ABSTRACT. The Stone-Čech Compactification serves as one of the major theorems in Point-Set Topology dealing with the construction of the largest compactification, denoted $\beta(X)$, of a completely regular topological space X. This compactification is characterized in the following: given any continuous (and bounded) map $f : X \to C$ from X to C, a compact Hausdorff space, f extends uniquely to another continuous map $g : \beta(X) \to C$ that equals f on X. In this paper, we will reiterate and define relevant definitions and concepts to build up to this theorem, utilizing and proving several other important theorems on the way, such as the Tychonoff Theorem, Urysohn's Lemma, and Urysohn's Metrization Theorem.

CONTENTS

1. INTRODUCTION AND MOTIVATION

This paper provides an introductory exploration of the topological principles relevant to the understanding of the Stone-Čech Compactification.

A compactification of the topological space X is the imbedding of X into a compact space called the compactification of X. A topological space can have multiple compactifications, changing with respect to its endowed topology. However, we tend to focus our study of imbeddings from general topological spaces into compact Hausdorff spaces as it is more interesting and useful. When studying compactifications on the topological space X, a basic question is often asked:

If Y is a compactification of X and f is a continuous function defined on X, what are the necessary condition(s) for there to exist a continuous extension of f to its compactification Y?

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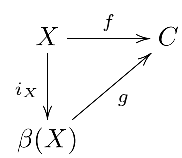
As shown throughout the paper, there are many possible compactifications that can be established on a topological space X. The Stone-Čech Compactification denoted $\beta(X)$, offers the maximal or largest compactification of X, preserving the continuous extension on X. This compactification of X is unique up to homeomorphism and admits a unique continuous extension of f from X to Y. It serves, somewhat, in contrast to the minimal compactification of X, namely the (Alexandroff) one-point compactification of X, given that X is not already a compact Hausdorff space.

The Stone-Čech Compactification states that: Given any completely regular space X, it admits a maximal compactification $\beta(X)$ such that any continuous map $f : X \to C$ from X into a compact Hausdorff space C extends uniquely to another continuous map $g : \beta(X) \to C$. The following diagram visually depicts the relationship between spaces $X, \beta(X)$, and C through the maps: i_X, f, and g where i_X is the inclusion map from X into $\beta(X)$.



2. Topological Spaces and Homeomorphisms

This paper assumes intermediate knowledge of Point-Set Topology, including the Subspace, Product, and Metric Topology, and properties of continuity, closed sets, and compactness. We will reiterate and define a few basic concepts needed for our understanding of the Stone-Čech Compactification.

Definition 2.1. Let X and Y be topological spaces; let the map $f : X \to Y$ be a bijection. If f and f^{-1} are both continuous, we say f is a *homeomorphism*.

A homeomorphism is especially important in Topology as it provides a correspondence between the topologies on X and Y. Any topological characteristic of X that is determined by its topology can be effectively carried over to the corresponding topological characteristic on Y through the function f, and vice versa. Homeomorphisms preserve topological structures and two spaces being homeomorphic can be considered as topologically equivalent. It is also worth mentioning that a homeomorphism is simultaneously an open and closed map.

We can generalize a homeomorphism by the following definition:

Definition 2.2. Let X and Y be topological spaces; let the map $f : X \to Y$ be an injective continuous map. Let Z be the image of X under f, namely $Z = f(X)$, considered as a subspace of Y. Then $f' : X \to Z$ is bijective, and if f' is a homeomorphism, we say that f is a *topological imbedding* of X into Y.

Note that restrictions of injective functions are injective and $f' : X \to f(X)$ is trivially surjective by the definition of a function.

Example 2.3. However, a bijective continuous function need not be a homeomorphism. Let the function f be defined by $f : \mathbb{R}_\ell \to \mathbb{R}$ be defined by the identity map i.e. $f(x) = x$. Here, $\mathbb{R}_\ell$ is the lower limit topology generated by the basis collection of all half-open intervals in the form of $[a, b)$ for all $a, b \in \mathbb{R}$ such that $a < b$. It follows that f is bijective and continuous, but f^{-1} is not continuous. To see why f^{-1} is not continuous, let $U = [a, b)$, which is open in $\mathbb{R}_\ell$. However $f(U) = [a, b)$ is not open in $\mathbb{R}$, as there exists no open set containing $\{a\}$ contained inside $f(U)$.

Homeomorphisms will play an important role in preserving topological properties. To reiterate, every property that can be defined in an arbitrary topological space is preserved under homeomorphisms. A topological property is defined to be a property such that it is preserved under a homeomorphism. Such properties are, but not limited to, metrizability, compactness, and connectedness, some of which we will touch on later.

3. Compactifications

To establish a compactification on a topological space X, we must first define the following relevant details that are crucial to the understanding of compactifications on spaces.

Definition 3.1. A topological space X is *Hausdorff* if for each pair of distinct points $x, y \in X$, there exists disjoint neighborhoods U_x and U_y of x and y, respectively.

Theorem 3.2. *Every finite point set in a Hausdorff space X is closed.*

Proof. It will suffice to show that every one-point set is closed. Let $x_0 \in X$, then for any $y \in X$ such that $y \neq x_0$, there exist disjoint neighborhoods of x_0 and y, respectively. Since, no points of X is a limit point of $\{x_0\}$, $\{x_0\}$ is closed, and the finite union of closed sets is closed; thus the proof is complete. □

The condition that finite point sets are closed is called the T_1 axiom. The T_1 axiom can be rephrased as given two disjoint points, each point has a neighborhood that doesn't contain the other point. Hausdorff spaces are commonly called T_2 or T_2 spaces. As shown above, every Hausdorff space is also T_1. Hausdorff spaces will be especially important as they allow us to obtain two disjoint open sets given any pair of disjoint points. Furthermore, when we require our topological space to be a compact Hausdorff space, we obtain many more useful properties. Compactness is useful as it allows us to generalize and represent (possibly infinite) spaces in terms of a finite collection of elements in the endowed topology. We will return to more T-axioms in Section 4.

When considering topological spaces with given properties, it is very important to ask the following question: Are these properties well-behaved with respect to the subspaces and products? More specifically, is the topological property of a topological space X preserved when considering a subspace A of X? In addition, we can also ask if X and Y are two topological spaces with a certain topological property, does $X \times Y$ necessarily inherit this property? These questions bring us to the following proposition:

Proposition 3.3. *Hausdorff spaces are well-behaved with respect to subspaces and products.*

Proof. Let X be a Hausdorff space and let Y be a subspace of X. Take $x, y \in Y$ such that $x \neq y$. Then there exist disjoint open sets U and V of X containing x

and y, respectively. The sets $U \cap Y$ and $V \cap Y$ serve as disjoint open sets in Y containing x and y, respectively.

Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of Hausdorff spaces and $X = \prod_{\alpha \in J} X_\alpha$. Take $\vec{x} = (x_\alpha)_{\alpha \in J}$ and $\vec{y} = (y_\alpha)_{\alpha \in J}$ to be distinct points in X. Since $\vec{x}$ and $\vec{y}$ are disjoint, there exists some index β such that $x_\beta \neq y_\beta$. Choose the disjoint open sets U and V of X_β containing x_β and y_β, respectively, then $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets of X containing $\vec{x}$ and $\vec{y}$, respectively. Here, π_β denotes the projection map from X into its β-th component. It's easy to see that π_β is a continuous open map. $\square$

Remark 3.4. The image of a Hausdorff space under a continuous function is not necessarily Hausdorff. Let $X = \{0, 1\}$ under the discrete topology and $Y = \{0, 1\}$ under the indiscrete topology. Let $f : X \to Y$ be the map sending 0 to 0 and 1 to 1. f is continuous and bijective (as you can check), but $f(X)$ is not a Hausdorff space because there does not exist disjoint open sets in Y containing $\{0\}$ and $\{1\}$, respectively.

Definition 3.5. A space X is *locally compact* at x if there exists a compact subspace C of X that contains a neighborhood U of x. If X is locally compact at every point of X, we say that X is locally compact.

It's easy to see that compact spaces are automatically locally compact, as for each point of $x \in X$, X serves as the compact subspace containing any neighborhood of x.

Example 3.6. The space $\mathbb{R}^n$ is locally compact. For each $\vec{x} \in \mathbb{R}^n$, there exists a compact subspace $C = [a_1, b_1] \times \cdots \times [a_n, b_n]$ containing the basis element $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$, which contains x.

Example 3.7. Consider $\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$. Then $\mathbb{R}^\omega$ is not locally compact. Suppose it is locally compact and take $\vec{x} \in \mathbb{R}^\omega$. Then there exists a compact subspace C that contains a neighborhood of x. Consider the following basis element in the product topology:

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \ldots$$

Taking the closure of B, we obtain the following:

$$\bar{B} = [a_1, b_1] \times \cdots \times [a_n, b_n] \times \mathbb{R} \times \cdots \times \mathbb{R} \times \ldots$$

Since $\bar{B}$ is closed and contained in a compact subspace C, $\bar{B}$ is compact, which is a clear contradiction.

Definition 3.8. If Y is a compact Hausdorff space and X is a dense subspace of Y, then Y is said to be a *compactification* of X. If $\{Y \setminus X\}$ consists of a single point, then Y is the *one-point compactification* of X. The point ∞ is commonly used to denote this single point and altogether, we have the compactification $Y = X \cup \{\infty\}$.

Theorem 3.9. *Let X be a topological space. Then X is locally compact Hausdorff space if and only if there exists a space Y satisfying the following conditions:*
(1) *X is a subspace of Y.*
(2) *The set $\{Y \setminus X\}$ consists of a single point.*
(3) *Y is a compact Hausdorff space.*

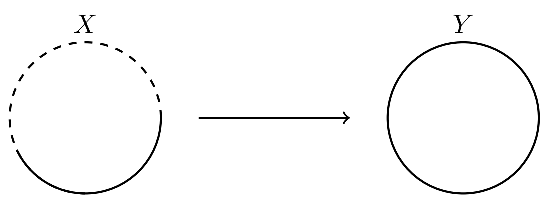
The compactification Y of X is uniquely constructed up to homeomorphism, i.e. if Y_1 and Y_2 are two one-point compactifications of X, there exists a homeomorphism f between Y_1 and Y_2.

Proof. Refer to [1] Chapter 3: Section 29 for the full proof. □

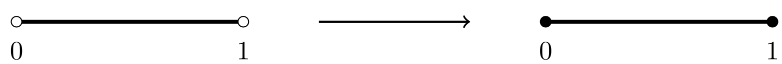
Corollary 3.10. *Let X be a locally compact Hausdorff space. Take a subspace A of X. If A is open or closed in X, then A is locally compact.*

Remark 3.11. It follows from Theorem 3.9 and Corollary 3.10 that every locally compact Hausdorff space is an open subspace or homeomorphic to an open subspace of a compact Hausdorff space through the one-point compactification.

Example 3.12. The following provides a visual example of a compactification on a topological space. Essentially, compactifications turn our initial topological space X into a nicer space Y with more useful properties. It is also easy to see that $X \subset Y$ and $\bar{X} = Y$.



Example 3.13. A simple compactification of $(0, 1)$ is the closed unit interval $[0, 1]$ constructed from adding $\{0\}$ and $\{1\}$ via the two-point compactification.



We shall see more examples of compactifications of $(0, 1)$ in the upcoming section when we introduce multiple ways to induce compactifications on topological spaces.

4. Countability and Separation Axioms

In this section, we will introduce the countability and separation axioms. These concepts or conditions help give important properties of topological spaces. Imposing these on a topological space allows one to prove stronger theorems in Topology. As we impose more conditions, we effectively restrict the overall range of topological spaces to which these theorems can be applied to. The countability axioms provide an important condition in Uysohn's Metrization Theorem and are necessary for future discussion of the conditions for a topological space to be metrizable. Similarly, the separation axioms are essential for Urysohn's Lemma and for constructing continuous functions that separate disjoint sets in a topological space. The separation axioms are commonly denoted by the letter T with a numerical subscript, indicating the strength of the axiom. The larger the numerical subscript the stronger the properties of the space. The notation for the separation axioms is derived from the German word: "*Trennungsaxiom*" or "separation axiom".

We have seen T_1 and T_2 spaces in previous sections, and now introduce new topological spaces with increasingly useful properties.

Definition 4.1. A space X has a *countable basis at x* if there is a countable collection denoted $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the element of $\mathcal{B}$. If X has a countable basis for all its points, we say that X is *first-countable*.

Definition 4.2. If a space X has a countable basis for its topology, then we say that X is *second-countable*.

Note that all second-countable spaces are first-countable because the countable basis for its topology serves as the countable basis for each point $x \in X$. It's important to observe that first-countable and second-countable spaces are also well-behaved with respect to the subspaces and products.

Example 4.3. $\mathbb{R}^n$ with the standard order topology admits a countable basis. It suffices to show that $\mathbb{R}$ admits a countable basis. Since $\mathbb{Q}$ is countable, then the collection:

$$\mathcal{B} = \{(r, s) \mid r, s \in \mathbb{Q}\}$$

serves as a countable basis as every point of $\mathbb{R}$ is contained in an interval of rational endpoints, and the intersection of two basis elements contains another basis element.

Definition 4.4. Suppose one point sets are closed in X. Then X is *regular* or T_3 if for each pair of a point $x \in X$ and closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. Similarly, X is *normal* or T_4 if for each pair of disjoint closed sets A and B, there exist disjoint open sets containing A and B, respectively.

It's easy to see that normal spaces are regular and regular spaces are Hausdorff.

Lemma 4.5. *Assume X is a topological space and one-point sets in X are closed, then:*

(1) *X is regular if and only if given $x \in X$ and a neighborhood U of x, there exists a neighborhood V of x such that $\bar{V} \subset U$.*

(2) *X is normal if and only if given a closed set A and an open set U containing A, there exists a open set V containing A such that $\bar{V} \subset U$.*

Regular spaces have many useful properties such as subspaces of regular spaces are regular and products of regular spaces are regular. However, these properties do not necessarily hold for all normal spaces. Therefore, we can impose another stronger condition to guarantee the hereditary property with the following:

Definition 4.6. X is said to be *completely normal* or T_5 if every subspace of X is normal.

It's trivial to see that completely normal spaces are normal, as X is, in fact, a subspace of itself.

Example 4.7. Every metrizable space and regular space equipped with a countable basis is a completely normal space. Note that every metrizable space and every regular space with a countable basis are normal, and since both types of spaces are well-behaved with respect to subspaces, their subspaces are normal. In Section 6, we prove that metrizable spaces are actually perfectly normal, a stronger condition than complete normality.

Proposition 4.8. *If X is compact Hausdorff, then X is normal.*

With the above properties on normal spaces established, we can now begin our first important theorem in topology called the Urysohn Lemma.

Theorem 4.9. ***(Urysohn Lemma).*** *Let X be a normal space, A and B be disjoint closed subsets of X, and $[a, b]$ be a closed interval in the real line. Then there exists a continuous map:*

$$f : X \to [a, b]$$

such that $f(x) = a, \forall x \in A$ and $f(y) = b, \forall y \in B$.

Proof. Refer to [1] Chapter 4: Section 33 for the full proof. □

The Urysohn Lemma proves the existence and constructs a continuous function on a normal space X, which lets us define a new way of separating spaces as shown in the following definition:

Definition 4.10. Let X be a topological space. If A and B are two subsets of X and if there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then we say that A and B can be *separated by a continuous function.*

However, we can strengthen the Urysohn Lemma by proving the converse. If we have a continuous function $f : X \to [0, 1]$ that separates the disjoint closed sets A and B of X, then $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint open sets containing A and B, respectively.

The Urysohn Lemma establishes yet another equivalent definition of a normal space. We can now summarize the expanded conditions for a topological space to be considered normal.

Definition 4.11. A topological space X is said to be *normal* if one of the following equivalent cases holds:

(1) For each pair of disjoint closed sets A and B, there exist disjoint open sets containing A and B, respectively, as stated in Definition 4.4.
(2) Given a closed set A and an open set U containing A, there exists a open set V containing A such that $\bar{V} \subset U$, as stated in (2) of Lemma 4.5.
(3) If A and B are two subsets of X and if there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, as stated in Definition 4.10.

Definition 4.12. X is *completely regular* or *Tychonoff* if one-point sets are closed in X and for each point x and closed set A not containing x, there is a continuous function: $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(A) = \{0\}$

By Urysohn Lemma, we have that normal spaces are completely regular. And completely regular spaces are regular, since the preimage of $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ under the continuous function f are disjoint open sets containing the A and x, respectively. It's important to note that the choices of the values 0 and 1 and the order for the assignment of values in the image of f are arbitrary.

Completely regular spaces will be essential in the later construction of the Stone-Čech Compactification. They also admit well-behaved properties with respect to subspaces and products, similar to regular spaces, but unlike normal spaces. Because of its relationship to regular and normal spaces, completely regular spaces are denoted as the $T_{3\frac{1}{2}}$ axiom.

To reiterate, we have the following separation axioms, listed in increasing order with respect to their strength: T_1, T_2 (Hausdorff), T_3 (regular), $T_{3\frac{1}{2}}$ (completely regular / Tychonoff), T_4 (normal), and T_5 (completely normal).

Now, we begin another important theorem that establishes the necessary conditions for a topological space to be metrizable. A topological space X is said to be metrizable if and only if there exists a metric $d : X \times X \to [0, \infty)$ such that d induces the endowed topology on X.

Theorem 4.13. ***(Urysohn Metrization Theorem)***. *Every regular space X with a countable basis is metrizable.*

Proof. Refer to [1] Chapter 4: Section 34 for the full proof. □

Here, it's worth noting that, showing metrizability is equivalent to imbedding X into a metrizable space Y. Since metrizability is defined in terms of a space's topology, it is preserved under homeomorphisms.

Definition 4.14. X is *locally metrizable* if each $x \in X$ has a neighborhood U_x that is metrizable in the subspace topology.

Proposition 4.15. *If X is compact Hausdorff and locally metrizable, then X is metrizable.*

Proof. Let $x \in X$ and U_x be the metrizable neighborhood of x. By Lemma 4.5, there exists a neighborhood V_x of x such that $\overline{V_x} \subset U_x$. $\overline{V_x}$ is compact and thus admits a countable basis, as every compact metrizable space admits a countable basis. Trivially, V_x also has a countable basis since $V_x \subset \overline{V_x}$. Then the collection $\{V_x\}_{x \in X}$ is an open cover of the compact Hausdorff space X, and thus admits a finite subcover $\{V_{x_i}\}_{i=1}^N$. The finite collection of countable bases for each V_{x_i} serves as a countable basis for X. It follows from Urysohn Metrization Theorem that X is metrizable. □

Proposition 4.16. *Let X be a compact Hausdorff space. Then X is metrizable if and only if X has a countable basis.*

Proof. Suppose X is metrizable, then for every $n \in \mathbb{N}$, define $\{B_d(x_n, \frac{1}{n})\}_{x \in X}$, where d is the metric on X. This collection is an open cover for X and, thus, admits a finite subcover. Let $x_1, \dots x_{N_n}$ be the centers of the elements in this finite subcover, which can be expressed as $\{B_d(x_{n,i}, \frac{1}{n})\}_{i=1}^{N_n}$. Let $\mathcal{B} = \bigcup_{n \in N} \{B_d(x_{n,i}, \frac{1}{n})\}_{i=1}^{N_n}$. It is clear that $\mathcal{B}$ is a countable basis for X.

Conversely, suppose X has a countable basis. By Proposition 4.8, X is normal and thus regular. By Urysohn Metrization Theorem, X is metrizable. □

Proposition 4.17. *Let X be a locally compact Hausdorff space. Then X is completely regular.*

Proof. Let Y be the one-point compactification of X. Y is compact Hausdorff, thus normal by Proposition 4.8, and also completely regular. Since completely regular spaces are well-behaved with respect to subspaces, it follows that X is completely regular. □

Theorem 4.18. ***(Imbedding Theorem)***. *Let X be a topological space where one-point sets are closed. Let $\{f_\alpha\}_{\alpha \in J}$ be an indexed collection of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the following condition: for each $x \in X$ and for each* *neighborhood U_x of x, there exists an index $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f(X \setminus U_x) = \{0\}$. Then the following function $F : X \to \mathbb{R}^J$ defined by:*

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

is an imbedding of X into $\mathbb{R}^J$.

Proof. Consider $\mathbb{R}^J$ endowed with the product topology. We can define a map $F : X \to \mathbb{R}^J$ as follows:

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x), \ldots)$$

Firstly, F is continuous because each of its components, f_n, is continuous, and $\mathbb{R}^J$ is equipped with the product topology. Furthermore, F is an injective map because given any $x \neq y$, we have that, by hypothesis, there exists an index $\alpha \in J$ such that $f_\alpha(x) > 0$ and $f_\alpha(y) = 0$ since the set $\{y\}$ is closed.

It remains to show that F^{-1} is continuous. Let $Z = F(X) \subset \mathbb{R}^J$. Take an open set U in X, then it suffices to show that $F(U)$ is open in $\mathbb{R}^J$. Choose $z_0 \in F(U)$, then there exists $x_0 \in X$ such that $F(x_0) = z_0$. By the hypothesis assumption, there exists an index $N \in J$ for which $f_N(x_0) > 0$ and $f_N(X \setminus U) = \{0\}$. Take the open set $(0, \infty)$ and define:

$$V = \pi_N^{-1}((0, \infty))$$

Since V is open in $\mathbb{R}^J$, $W = V \cap Z$ is open in Z by definition of the subspace topology.

Now we want to show the following relations: $z_0 \in W$ and $W \subset F(U)$. To start, $z_0 \in V$ because

$$\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$$

and trivially $z_0 \in Z$, therefore $z_0 \in W$.

Lastly, $W \subset F(U)$. Choose $y \in W$, then $y = F(x)$ for some $x \in X$. Since $y \in V$, we have that $\pi_N(y) \in (0, \infty)$ and

$$\pi_N(y) = \pi_N(F(x)) = f_N(x)$$

Since $f_n(x) > 0$, it follows that $x \in U$ because f_N vanishes outside of U. Altogether, we have $y = F(x) \subset F(U)$. Therefore F^{-1} is continuous and F is a homeomorphism, imbedding X into $\mathbb{R}^J$. □

5. The Tychonoff Theorem and the Stone-Čech Compactification

Here, we describe and prove two very important theorems in Topology with extensive use in mathematical analysis, geometry, and topology.

Before we begin, we need to establish properties of a maximal set with respect to certain properties. This lemma utilizes results from Zorn's Lemma.

Lemma 5.1. *Let X be a set and $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:*

(a) *Any finite intersection of elements in $\mathcal{D}$ is an element in $\mathcal{D}$*

(b) *If A is a subset of X, that intersects ever element of $\mathcal{D}$, then A is an element of $\mathcal{D}$.*

Theorem 5.2. ***(The Tychonoff Theorem)*** *The arbitrary product of compact spaces is compact with respect to the product topology.*

Proof. Let:

$$X = \prod_{\alpha \in J} X_\alpha$$

where each X_α is compact and J is an indexing set for an arbitrary collection. Proving the compactness of X is equivalent to showing that every collection $\mathcal{C}$ of closed sets of X with the finite intersection property has a nonempty intersection. Let $\mathcal{A}$ be an arbitrary collection of subsets of X having the finite intersection property. We will show that:

$$\bigcap_{A \in \mathcal{A}} \bar{A} \neq \phi \tag{5.3}$$

Choose a collection $\mathcal{D}$ of subsets of X that is maximal with respect to the finite intersection property and $\mathcal{D} \supset \mathcal{A}$. Showing $\bigcap_{D \in \mathcal{D}} \bar{D}$ is nonempty will prove 5.3 holds $\bigcap_{D \in \mathcal{D}} \bar{D} \subset \bigcap_{A \in \mathcal{A}} \bar{A}$.

Fix $\alpha \in J$, then let $\pi_\alpha : X \to X_\alpha$ be the projection map from X into its α-th component. Since $\mathcal{D}$ has the finite intersection property, the following collection

$$\{\pi_\alpha(D) \mid D \in \mathcal{D}\}$$

also inherits this property as $\bigcap_{i=1}^{N} D_i$ is nonempty and $\pi_\alpha\big(\bigcap_{i=1}^{N} D_i\big) \subset \bigcap_{i=1}^{N} \pi_\alpha(D_i)$ for every $N \in \mathbb{N}$. Since X_α is compact, we can choose a $x_\alpha \in X_\alpha$ such that:

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$$

Let $\vec{x} = (x_\alpha)_{\alpha \in J}$. It suffices to show that $\vec{x} \in \bar{D}$ for every $D \in \mathcal{D}$.

Since X has the product topology, $\vec{x}$ is contained in some subbasis element, $\pi_\beta^{-1}(U_\beta)$, where U_β is a neighborhood of x_β. From our above choice of x_β, there exists $y_\beta \in U_\beta \cap \pi_\alpha(D)$, thus $\vec{y} \in \pi_\beta^{-1}(U_\beta) \cap D$ for every $D \in \mathcal{D}$.

By Lemma 5.1(b), we have that every subbasis element containing $\vec{x}$ is an element in $\mathcal{D}$. And from (a) of Lemma 5.1(a), every basis element containing $\vec{x}$ is also an element in $\mathcal{D}$, since every basis element is formed through finite intersections of subbasis elements. Since $\mathcal{D}$ has the finite intersection property, every basis element, and thus every open set, containing $\vec{x}$ intersects every element of $\mathcal{D}$. Altogether, $\vec{x} \in \bar{D}$ for each $D \in \mathcal{D}$ and the proof is complete. □

The Tychonoff Theorem will be very important in the construction of a topological imbedding function that induces a compactification in the Stone-Čech Compactification, as constructed later in this section. Here, we reiterate the generalized definition of a compactification of a topological space.

Definition 5.4. A *compactification* of the topological space X is a compact Hausdorff space Y such that X is a dense subspace of Y.

It is important to note that compactifications are not necessarily uniquely determined by the initial topological space X.

Definition 5.5. Two compactifications Y_1 and Y_2 are *equivalent* if there exists a homeomorphism $f : Y_1 \to Y_2$ such that f equals the identity of X.

Example 5.6. Two compactifications can be homeomorphic without equaling the identity on X. Consider the following example: Let $X = [0,1] \times \{0,1\}$. Then the following are two compactifications of X:

$$Z_1 = (S^1 \times \{0\}) \cup ([0,1] \times \{1\})$$

$$Z_2 = ([0,1] \times \{0\}) \cup (S^1 \times \{1\})$$

A homeomorphism exists through the map $f : Z_1 \to Z_2$ defined by $f(x,y) = x \times (X \setminus y)$ but trivially f does not equal the identity on X.

Remark 5.7. X has a compactification Y if and only if X is completely regular. Observe that if Y is a compact Hausdorff space, it is completely regular, so X is, in fact, completely regular.

Before we prove the Stone-Čech Compactification, we must first establish a result that expands the possible compactifications on a topological space by allowing compactifications to arise from imbeddings into another topological space.

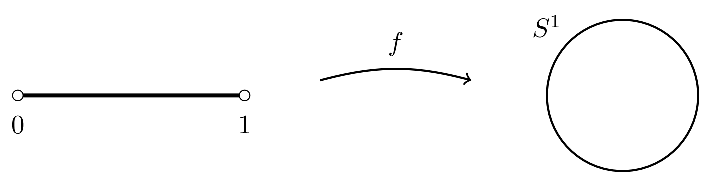
Lemma 5.8. *Let X be a topological space. Suppose that $h : X \to Z$ is an imbedding of X into a compact Hausdorff space Z. Then there exists a compactification Y of X such that there is an imbedding $H : Y \to Z$ that equals h on X. This compactification is uniquely determined up to equivalence and we say that this compactification Y of X is induced by the imbedding h.*

Now after the establishment of compactifications induced by imbeddings, we can return to the discussion of our previous example of compactifications on the open unit interval with the following two examples:

Example 5.9. Consider $S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$, or the unit circle in $\mathbb{R}^2$, and define the map $f : (0,1) \to S^1$ by:

$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$

Then the compactification induced by the imbedding f is equivalent to the one-point compactification of X because f adds the one-point, namely $(1,0)$, to the punctured unit circle. The following diagram provides a visual display of this compactification process:

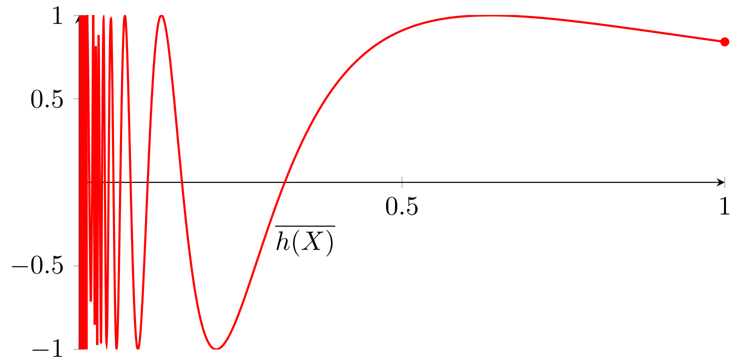


Example 5.10. Consider the function $h : (0,1) \to [-1,1]^2$ defined by:

$$h(x) = x \times \sin\left(\frac{1}{x}\right)$$

This graph is commonly referred to as the topologist's sine curve. Let $Y = \overline{h(X)}$. This compactification of X is obtained by adding a singular point and an entire vertical line on the y-axis from -1 to 1. Explicitly, this compactification can be expressed as the following:

$$Y = \overline{h(X)} = \left\{x \times \sin\left(\frac{1}{x}\right)\right\} \cup (1, \sin(1)) \cup \{(0,y) \mid y \in [-1,1]\}$$



As shown above, compactifications can arise from the addition of a singular point (Example 5.9) or even by adding unaccountably many points (Example 5.10).

Now, we are ready to explore more compactifications that serve as continuous extensions for a continuous function defined on our initial space. Here, we reiterate and go back to the initial question we asked in Section 1:

If Y is a compactification of X and f is a continuous function defined on X, what are the necessary conditions for there to exist a continuous extension of f to its compactification Y?

The following example will provide some additional insight into the motivation behind the Stone-Čech Compactification, which, in some sense, serves as the maximal or largest compactification of a topological space X, preserving a continuous extension of a continuous function defined on X.

Example 5.11. Using the map f in Example 5.9, f is a bounded continuous real-valued function and can be continuously extended to its compactification if and only if the following limits exist and are equal:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 1^-} f(x)$$

Theorem 5.12. ***(Stone-Čech Compactification)*** *Let X be a completely regular topological space. Then there exists a compactification Y of X such that every bounded continuous function $f : X \to \mathbb{R}$ extends uniquely to a continuous map $g : Y \to \mathbb{R}$.*

Proof. Let $\{f_\alpha\}_{\alpha\in J}$ be the indexed collection of all bounded continuous real-valued functions on X. Since each f_α is bounded, for each $\alpha \in J$, define the following closed interval I_α:

$$I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$$

Define the map $h : X \to \prod_{\alpha\in J} I_\alpha$ by $h(x) = (f_\alpha(x))_{\alpha\in J}$. By Tychonoff Theorem, $\prod_{\alpha\in J} I_\alpha$ is compact.

To show that h is an imbedding, fix $x \in X$ and fix a neighborhood U_x of x. Let $A = X\setminus U_x$, which is closed. Since X is completely regular, there exists a continuous function $f_\beta : X \to [0, 1]$ such that $f(x) = 1 > 0$ and $f(A) = f(X \setminus U_x) = \{0\}$. Since f_β is trivially bounded, it is, therefore, an element of the collection $\{f_\alpha\}_{\alpha\in J}$. It follows from the Imbedding Theorem that h is an imbedding of X.

Let Y be the compactification of X induced by the imbedding h. Then by Lemma 5.8 there exists an imbedding $H : Y \to \prod_{\alpha\in J} I_\alpha$ that equals h on X.

Now, given a bounded function f_β, we must show it extends uniquely to Y. Let $\pi_\beta : \prod_{\alpha\in J} I_\alpha \to I_\beta$ be the projection map of $\prod_{\alpha\in J} I_\alpha$ to its β-th component. Then $g = \pi_\beta \circ H$ is our desired continuous extension of f_β. g also equals f_β on X as:

$$\pi_\beta(H(x)) = \pi_\beta(h(x)) = \pi_\beta\big(\{f_\alpha\}_{\alpha\in J}\big) = f_\beta(x)$$

Altogether, g is our desired continuous extension of f from X to its compactification Y.

The uniqueness of this continuous extension can be shown in the following generalized lemma. □

Lemma 5.13. *Let X be a topological space and $f : X \to Z$ be a continuous map from X into a Hausdorff space Z. Then f admits at most one continuous extension, namely $g : \bar{X} \to Z$.*

Proof. Suppose g and g' are two continuous extensions of X such that $g, g' : \bar{X} \to Z$. Choose x such that $g(x) \neq g'(x)$, since there exists at least one $x \in X$ because $g \neq g'$. Then there exist disjoint open sets U and U' containing the points $g(x)$ and $g'(x)$, respectively. By continuity of g and g', we have a neighborhood V of x, such that $g(V) \subset U$ and $g(V) \subset U'$. V intersects X at some point y. Then $g(y) \in U$ and $g'(y) \in U'$. But since $y \in X$, we have that $g(y) = f(y) = g'(y)$, which contradicts the disjointedness of the open sets U and U'. □

This compactification satisfying the conditions stated in Theorem 5.12 is called the Stone-Čech Compactification of X, denoted by $\beta(X)$. The following theorem shows we can effectively replace the image $\mathbb{R}$ under our continuous function f with a compact Hausdorff space C.

Theorem 5.14. *Let X be a completely regular space and $\beta(X)$ be its Stone-Čech Compactification. Given any continuous map, $f : X \to C$ of X into a compact Hausdorff space C, f extends uniquely to a continuous map $g : \beta(X) \to C$.*

Proof. By Proposition 4.8, C is normal. C is also completely regular, and thus homeomorphic to a subset of $[0,1]^J \subset \mathbb{R}^J$ for some J. For simplicity, let us assume that $C \subset [0,1]^J$. Observe that for each component function f_α of f, f_α is a bounded continuous real-valued function defined on X. Because $\beta(X)$ is the Stone-Čech Compactification of X, we can continuously extend each f_α to the continuous map $g_\alpha : \beta(X) \to \mathbb{R}$. Now define the map $g : \beta(X) \to \mathbb{R}^J$ by $g(x) = (g_\alpha(x))_{\alpha\in J}$. Then g is continuous because each of its components g_α is continuous and $\mathbb{R}^J$ is endowed with the product topology.

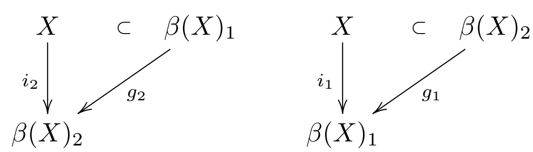
Finally, to replace the range of g with C, we use the fact that g is continuous and C is a compact Hausdorff space. Then the following relationship holds:

$$g(\beta(X)) = g(\bar{X}) \subset \overline{g(X)} = \overline{f(X)} \subset \bar{C} = C$$

□

Proposition 5.15. *Let X be a completely regular space. If $\beta(X)_1$ and $\beta(X)_2$ are two Stone-Čech Compactifications of X, then the compactifications $\beta(X)_1$ and $\beta(X)_2$ are equivalent, which implies that the Stone-Čech Compactification of X is unique up to homeomorphism.*

Proof. Let $i_2 : X \to \beta(X)_2$ be the inclusion map from X into $\beta(X)_2$. Since i_2 is a continuous map from X into a compact Hausdorff space $\beta(X)_2$, by Theorem 5.14, i_2 admits an extension to a continuous map $g_2 : \beta(X)_1 \to \beta(X)_2$. Define i_1 and g_1 similarly. Consider $g_1 \circ g_2 : \beta(X)_1 \to \beta(X)_1$ which sends $x \in X$ to $g_1(g_2(x)) = x$. Observe that $g_1 \circ g_2$ is a continuous extension of the identity map $Id_X : X \to X$. However, $Id_{\beta(X)_1}$ is also a continuous extension of Id_X. By Lemma 5.13, the composite map $g_1 \circ g_2$ is equal to the identity map $Id_{\beta(X)_1}$. Similarly, we have that the composite map $g_2 \circ g_1$ is equal to the identity map $Id_{\beta(X)_2}$. In summary, we have that $g_1 \circ g_2 = Id_{\beta(X)_1}$ and $g_2 \circ g_1 = Id_{\beta(X)_2}$. Therefore, $\beta(X)_1$ and $\beta(X)_2$ are homeomorphic and equal the identity on X, thus they are equivalent. The following diagram displays the relationship between the two Stone-Čech Compactifications of X and its inclusion and continuous maps:



□

Corollary 5.16. *Given an arbitrary compactification Y of X, there exists a continuous surjective closed map $h : \beta(X) \to Y$, that equals the identity on X.*

Proof. Let $i_X : X \to Y$ be the inclusion map from X into Y. Since Y is a compact Hausdorff space, we can compactify X via the Stone-Čech Compactification. Then there exists the following continuous extension $h : \beta(X) \to Y$. By definition, if $x \in X$, then $h(x) = i_X(x) = x$, which is precisely the identity on X.

Now we need to show that h is a closed surjective map. Take a closed set C of $\beta(X)$, then C is compact, and thus $g(C)$ is compact. Since Y is a compact Hausdorff space, $g(C)$ is closed. To prove surjectivity, we note that h equals i_X on X and $Y = \overline{i_X(X)} \subset \overline{h(\beta(X))} \subset Y$. Since $\beta(X)$ is compact, $h(\beta(X))$ is compact and, thus closed. Therefore $h(\beta(X)) = \overline{h(\beta(X))} = Y$. □

6. Implications of Conditions on Topological Spaces

In this section, we will visually summarize the relationships between the separation axioms, as well as the properties endowed on them. Through this paper, we have established most of the separation axioms notably: T_1, T_2 (Hausdorff), T_3 (regular), $T_{3\frac{1}{2}}$ (completely regular), T_4 (normal), and T_5 (completely normal). We have additionally introduced and expanded on many characteristics and properties these topological spaces might be endowed with such as, but not limited to, being: second-countable, metrizable (and locally), and compact (and locally). Here, we will introduce new definitions, reiterate previous relationships between topological spaces, as well as prove some new additional theorems.

Definition 6.1. X is *perfectly normal* or T_6 if X is normal and every closed set in X is a G_δ set. A set A is a G_δ set if A is the intersection of a countable collection of open sets of X.

Lemma 6.2. *Let X be a normal space and A be a subset of X. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for each $x \in A$ and $f(x) > 0$ for each $x \notin A$ if and only if A is a G_δ set.*

Theorem 6.3. *If X is perfectly normal, then X is completely normal.*

Proof. Suppose A and B are separated sets in X. Then there exists the following continuous functions: $f, g : X \to [0, 1]$ such that $f(\bar{A}) = g(\bar{B}) = 0$ and $f(X \setminus \bar{A}) > 0$ and $g(X \setminus \bar{B}) > 0$ by Lemma 6.2 because $\bar{A}$ and $\bar{B}$ are closed, thus G_δ sets. Let $h : X \to [-1, 1]$ be defined by $h = f - g$. If $x \in \bar{A}$, then $x \notin B$, which implies that $h(x) < 0$. Similarly, if $x \in \bar{B}$, then $x \notin A$, which implies that $h(x) > 0$. Take the open rays $(0, \infty)$ and $(-\infty, 0)$, then $h^{-1}(0, \infty)$ and $h^{-1}(-\infty, 0)$ are disjoint open sets containing A and B, respectively; thus every pair of separated sets yields disjoint open sets containing them.

Normality is shown in the following lemma. □

Lemma 6.4. *X is completely normal if and only if for every pair A and B of separated sets, there exist disjoint open sets containing them.*

Proof. Suppose X is completely normal, and let A and B be separated sets in X. Then consider the following subspace $S = X \setminus (\bar{A} \cap \bar{B})$. S is normal as it is a subspace of a completely normal space. Since $\bar{A}$ and $\bar{B}$ are closed in X, $A' = \bar{A} \cap S$ and $B' = \bar{B} \cap S$ are closed in S, through the subspace topology. Then there exist distinct neighborhoods U and V of A and B, respectively. Because S is open in X, it follows that U and V are open sets in X. Now, it remains to show that $A \subset U$ and $B \subset V$. Let $x \in A$, then $x \notin \bar{B}$, since A and B are separated sets. It follows that $x \in A' \subset U$ because $x \in \bar{A}$ and $x \notin \bar{A} \cap \bar{B}$. Similarly, $B \subset V$. Altogether, U and V are disjoint neighborhoods of A and B, respectively.

Conversely, let Y be a subspace of X and take C and D to be closed sets in Y. Then $C' = C \cap Y$ and $D' = D \cap Y$, where C' and D' are disjoint closed sets in X, thus separated. From the hypothesis, there exist disjoint open sets U and V containing C' and D', respectively. By definition of subspace topology, $U \cap Y$ and $V \cap Y$ are disjoint open sets containing C and D, respectively, thus Y is normal, and X is completely normal. □

Theorem 6.5. *Every metrizable space is perfectly normal.*

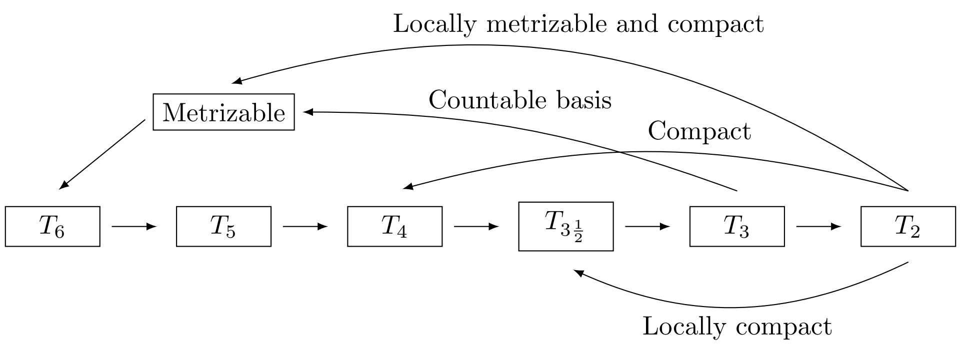
Proof. Let X be a metrizable space. X is normal because every metrizable space is normal. Take a closed set A in X and choose a point $x' \in X \setminus A$, then $\{x'\}$ is a closed set. Let $B = \{x'\}$. The following function satisfies the continuous function in Urysohn's Lemma:

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

It's easy to observe that $f(x) = 0$ for every $x \in A$, $f(x) = 1$ for $x \in B$, and $f(x) > 0$ for every $x \notin A \cup B$. Thus, f vanishes precisely on A. By Lemma 6.2, A is a G_δ set. Since X is normal and every closed set A is a G_δ set, we have that X is perfectly normal. □

In Theorem 6.3 and Theorem 6.5, we have actually strengthened the fact every metrizable space is completely normal, which was previously stated in Example 4.7. Because of Theorem 6.3, perfectly normal spaces are given the notation of the T_6 axiom to show their increased strength compared to completely normal or T_5 spaces. Moreover, here and in the following diagram, we heavily emphasize the importance and usefulness of a topological space being metrizable. Future studies of topology will bring you to the results of the Nagata-Smirnov Metrization Theorem, which is not mentioned in this paper but is located in [1] Chapter 6: Section 40.

The subsequent diagram presents a basic overview of the conditions on a topological space along with their corresponding implications. Each labeled box represents a topological space. The non-labeled arrows connecting different boxes denote the automatic implication of each topological space. The annotation associated with certain arrows indicates the necessary condition(s) that the initial topological space must possess in order for the implication to automatically follow. Throughout this paper, we have proved all the following implications:



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