# MINIMAL SURFACES FROM THE DIFFERENTIAL VIEWPOINT

TYLER GROSHONG

Abstract. A discussion of the existence and regularity of minimal surfaces, from an elliptic partial differential equation (PDE) perspective. We begin with the De Giorgi-Nash-Moser and Schauder theorems for the regularity of (weak) solutions to the divergence PDE $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$ where $a_{ij}$ is an elliptic matrix. We then use this theory to prove the existence and uniqueness of minimal surfaces on the ball with Lipschitz, and then with merely continuous, boundary data. We discuss Bernstein's method and a few interesting gradient estimates that follow for minimal surfaces, and we give a proof of a positive answer to Bernstein's problem in 2 dimensions.

## Contents



## 1. Introduction

If we define the function $L : \mathbb{R}^{d-1} \times \mathbb{R} \times \overline{A} \to \mathbb{R}$ (with $A \subset \mathbb{R}^{d-1}$) by

$$L(p, z, x) = \sqrt{1 + |p|^2} \tag{1.1}$$

then we have that the associated energy functional to this Lagrangian

$$J[u] := \int_A L(\nabla u(x), u(x), x)\,\mathrm{d}x = \int_A \sqrt{1 + |\nabla u|^2} \tag{1.2}$$

is the surface area functional for the graph of $u \in W^{1,\infty}(A) \subset H^1(A)$. By looking at the first variation of this functional (see, for instance, 8.1.2 from [1]) we have that a minimizer $u$ of surface area $J$ will solve the PDE

$$-\sum_{i=1}^{d-1}(L_{p_i}(\nabla u, u, x))_{x_i} = -\sum_{i=1}^{d-1}\left(\frac{\partial_i u}{\sqrt{1+|\nabla u|^2}}\right)_{x_i} = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

*Date*: August 2023.

Henceforth we call a function $u$ which satisfies

$$Mu := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{1.3}$$

a solution to the minimal surface equation (on the domain $A \subset \mathbb{R}^{d-1}$).

We will prove in this paper that minimal surfaces with bounded gradients are in fact smooth. Why should we expect this to be true? First of all, notice that if $|p|$ is small then $\sqrt{1+|p|^2} \approx |p|^2/2$, so if $u$ minimizes the surface area functional $L(\nabla u, u, x)$ from (1.1) over some region $A$ then we sort of have that $u$ minimizes the energy $\int_A |\nabla u|^2$, which means that it's close to being harmonic, and therefore smooth.

In Section 2 we look at two very important regularity results for the elliptic PDE problem $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, where $a_{ij}$ is a uniformly elliptic matrix. Namely, we prove the De Giorgi-Nash-Moser theorem (Theorem 2.1) for weak solutions to $\partial_i(a_{ij}\partial_j u) = 0$ with $L^2$ gradients, which says that $u$ is Hölder continuous. De Giorgi is a scale-invariant regularity result, in the sense that if we zoom in around a point we do not improve or otherwise change the result. The second famous theorem we look at, the Schauder theorem (Theorem 2.11), does vary with scale, in that it provides a bound on the Hölder norm of the gradient of a solution to $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$ in some ball, depending on the $L^2$ norm of $u$ (and the Hölder seminorm of $f$) in the ball of twice the radius. As we zoom closer and closer in around a point, the uniformly elliptic matrix $a_{ij}$ looks more like the identity, and the function $f$ looks more like a constant, so a solution $u$ to $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$ looks more like a harmonic function, and we can take advantage of this to prove such a theorem that depends on scale.

These theorems will allow us to prove the smoothness of minimal surfaces with bounded gradients. If $u : A \subset \mathbb{R}^{d-1} \to \mathbb{R}$ is a function such that the graph of $u$ is a minimal surface, and if we have that $\nabla u$ is bounded, then we can show that $u$ is a minimizer of the functional $J$ in (1.2) over the space of all Lipschitz functions $W^{1,\infty}(A)$. Therefore, as above, it solves the minimal surface equation (1.3), and by differentiating this equation we get that $u$ is a solution of $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$ for $a_{ij} \in C^\alpha(A, \mathbb{R}^{d\times d})$ uniformly elliptic and $f \in C^\alpha(A; \mathbb{R}^d)$. By the Schauder theorem (Theorem 2.11) this says that $u \in C^{1,\alpha}(A)$, which implies that $\nabla u \in C^\alpha(A)$. Then (1.3) says that the product of $D^2u$ with a nondegenerate $C^\alpha$ function is constant, so we must have that $D^2u$ is $C^\alpha$, so $u \in C^{2,\alpha}$. Then we repeat: $\nabla u$ is now $C^{1,\alpha}$, so (1.3) says that the product of $D^2u$ with a nondegenerate $C^{1,\alpha}$ function is constant, so $D^2u$ is $C^{1,\alpha}$, so $u \in C^{3,\alpha}$. Then $\nabla u$ is $C^{2,\alpha}$, and so on, thus proving that $u \in C^\infty$ by induction.

In Section 3 we prove a comparison principle for sub- and super-solutions to the minimal surface equation $Mu = 0$, and then give a proof of the existence and uniqueness of solutions to the minimal surface equation with Lipschitz boundary data $u = \varphi \in W^{1,\infty}$ on $\partial B_1$.

In Section 4, we look at the first and second variations of perimeter for minimal surface, and at Bernstein's method of introducing auxiliary functions to remove dependence on boundary data. From this, in Section 4.1 we derive two $L^\infty$ bounds for the gradient of minimal surfaces, and prove the existence and uniqueness of minimal surfaces on the ball with merely continuous boundary data. Then, in Section 4.2, we introduce Bernstein's problem, which asks when global minimal surfaces must be hyperplanes, and prove that minimal surfaces in $\mathbb{R}^2$ must be planar.

It is worth having an idea of why some of the main variational techniques for functional minimizers, such as those featured in this summer's REU papers by my peers Mr. Daniel Chen and Mr. Sammy Thiagarajan, do not directly apply to minimal surfaces. If $1 < p < \infty$ then we have that $L^p$ is reflexive, or equivalently that the unit ball in $L^p$ is weakly compact. These papers reference the fact that, for some functional $I : W^{1,p}(\Omega) \to \mathbb{R}$, if we have that we can bound the $L^p$ norm of the gradient

$$I[u] \geq \alpha \|\nabla u\|^p_{L^p(\Omega)} - \beta$$

(for some constants $\alpha, \beta$) then a sequence $\{u_k\} \subset W^{1,p}(\Omega)$ approaching the minimum will be bounded in $W^{1,p}(\Omega)$ and by the reflexivity of $L^p$ we can then extract a subsequence that converges in $W^{1,p}$. However, for the surface area functional (1.2) the best $p$ for which we can do this is $p = 1$ (since $\sqrt{1 + |p|^2}$ looks like $|p|^1$ for $|p|$ large), and $L^1$ is unfortunately not reflexive. So these methods will not exactly apply for the PDE $\partial_i(a_{ij}\partial_j u) = 0$ where $a_{ij} = (1 + |\nabla u|^2)^{-1/2}$.

All figures in the text are drawn by the author, after illustrations from [3].

## 2. Regularity

### 2.1. **De Giorgi-Nash-Moser Theorem.**

**Theorem 2.1.** *(De Giorgi-Nash-Moser) Let $u \in H^1(B_2)$ be a weak solution to $\partial_i(a_{ij}\partial_j u) = 0$, supposing only that the coefficient matrix $a_{ij}$ is measurable and uniformly elliptic (i.e., $\exists \lambda, \Lambda > 0$ such that $\lambda I \leq a_{ij} \leq \Lambda I$). Then there exists $\alpha > 0$ such that $u \in C^\alpha(B_1)$.*

We will prove Theorem 2.1 as follows: we begin by introducing the notion of a 'subsolution' to $\partial_i(a_{ij}\partial_j u) \geq 0$, which essentially says that any downward perturbation of $u$ increases the total energy $\int a_{ij}\partial_i u \partial_j u$, so even if $u$ is not a classical solution it will only be a worse candidate if we decrease it anywhere. We then prove an intermediate lemma (Lemma 2.4) that gives an $L^\infty$ bound on $B_{1/2}$ for nonnegative subsolutions in $B_1$, which relies on a combination of: **the Caccioppoli inequality**, which uses a subsolution's $L^2$ norm in $B_{r+\delta}$ (for some $\delta > 0$ small) to bound the *gradient*'s $L^2$ norm in $B_r$, sort of a reverse Sobolev inequality; **the Sobolev inequality**, which uses the $L^2$ bound on the gradient in $B_r$ to bound the $L^2$ norm of the function in $B_r$, so we've converted an $L^2$ bound on $B_{r+\delta}$ into a stronger $L^2$ bound on $B_1$; and finally **Chebychev's inequality**, which will allow us to put a $(1 - \epsilon)^k$ bound on the set where the function is greater than $C - 2^{-k}$, thus showing that the function is bounded by taking $k \to \infty$. This is all done in section 2.1.1, where we prove the $L^\infty$ bound in Lemma 2.4.

After having done that, in the Improvement of Oscillation section 2.1.2 we will prove a weak Harnack inequality that gives us a sense of how much time an $H^1$ function has to spend transitioning between values (i.e., if an $H^1$ function is 0 on at least this-much of the ball and 1 on at least this-much of the ball then it must be in between 0 and 1 on at least that-much of the ball, where that-much depends on its $H^1$ norm). Using this inequality and the $L^\infty$ bound on $B_{1/2}$ from Lemma 2.4, we can come up with an oscillation bound on nonnegative subsolutions in $B_{1/2}$. Through iteration of this oscillation bound we get the desired Hölder bound for Theorem 2.1. Finally, since it will be clear that weak solutions can be considered as subsolutions, and since a function has the same Hölder bound as its absolute value, we can consider a weak solution to $\partial_i(a_{ij}\partial_j u) = 0$ as a nonnegative subsolution to $\partial_i(a_{ij}\partial_j u) \geq 0$ and thus prove Theorem 2.1.

2.1.1. *Subsolutions and $L^\infty(B_1)$ bound.* Denote $E(u) := \int_\Omega a_{ij}\partial_i u \partial_j u$.

**Definition 2.2.** *(Subsolution) We say that $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is a subsolution to the equation $\partial_i(a_{ij}\partial_j u) \geq 0$ if for any $\varphi \in C^1(\Omega; \mathbb{R})$ with $\varphi \geq 0$ in $\Omega$ and $\varphi = 0$ on $\partial\Omega$ we have*

$$\int_\Omega a_{ij}\partial_i u \partial_j \varphi \leq 0$$

What motivates this definition? Notice that if we could integrate by parts—we can't, since we don't know that $a_{ij}\partial_j u$ is continuous, let alone differentiable,but if we could—we would get that $\int_\Omega \partial_j(a_{ij}\partial_i u)\varphi \geq 0$ for all test functions $\varphi$. For one thing, this tells us that if $u$ is a weak solution to $\partial_i(a_{ij}\partial_j u)$ then it is a subsolution to $\partial_i(a_{ij}\partial_j u) \geq 0$. More interestingly, we also can show that $u$ is a subsolution if and only if $E(u) \leq E(u + v)$ for all $v \in H_0^1(\Omega)$ with $v \leq 0$; in other words, every downward perturbation of a subsolution increases the energy. Therefore, a true minimizer must not result from a downward perturbation of a subsolution, so we have that a subsolution lies below every true minimizer, hence the name.

*Proof.* (downward perturbations of subsolutions increase energy) If $u$ is a subsolution, then the inequality in Definition 2.2 holds for all $\varphi \in H_0^1(\Omega)$. Let $u$ be a subsolution, and let $v \in H_0^1(\Omega)$ with $v \leq 0$. Then we have that $-v \geq 0$ with $v = 0$ on $\partial\Omega$, so $-v$ is an admissible test function for subsolutions which gives us

$$\begin{aligned}
E(u+v) - E(u) &= \int_\Omega a_{ij}\partial_i(u+v)\partial_j(u+v) - \int_\Omega a_{ij}\partial_i(u)\partial_j(u) \\
&= \int_\Omega a_{ij}(\partial_i(u)\partial_j(v) + \partial_i(v)\partial_j(u) + \partial_i(v)\partial_j(v)) \\
&= -\int_\Omega a_{ij}(\partial_i(u)\partial_j(-v) + \partial_i(-v)\partial_j(u)) + \int_\Omega a_{ij}\partial_i(v)\partial_j(v) \\
&\geq \int_\Omega a_{ij}\partial_i(v)\partial_j(v) \geq 0
\end{aligned}$$

so we have that $E(u) \leq E(u + v)$ for all $v \in H_0^1(\Omega)$ with $v \leq 0$.

On the other hand, let $E(u) \leq E(u + v)$ for all $v \in H_0^1(\Omega)$ with $v \leq 0$. Let $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$. Then

$$\begin{aligned}
0 &\leq \int_\Omega a_{ij}\partial_i(u-\varphi)\partial_j(u-\varphi) - \int_\Omega a_{ij}\partial_i(u)\partial_j(u) \\
&= 2\int_\Omega a_{ij}\partial_i(u)\partial_j(-\varphi) - \int_\Omega a_{ij}\partial_i(\varphi)\partial_j(\varphi) \\
&\leq 2\int_\Omega a_{ij}\partial_i(u)\partial_j(-\varphi)
\end{aligned}$$

and therefore $\int_\Omega a_{ij}\partial_i(u)\partial_j(\varphi) \leq 0$ for all $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$, so $u$ is a subsolution. Thus, we have that $u$ is a subsolution if and only if $E(u) \leq E(u+v)$ for all $v \in H_0^1(\Omega)$ with $v \leq 0$. ∎

As explained above, to obtain an $L^\infty(B_{1/2})$ bound for a nonnegative subsolution on $B_2$, we will need to make use of the Caccioppoli inequality:

**Lemma 2.3.** *(Caccioppoli inequality) Let $u \in H^1(B_{r+\delta})$ be a subsolution to the equation $\partial_i(a_{ij}\partial_j u) \geq 0$, with $u \geq 0$. Then there exists a constant $C > 0$ depending on the ellipticity constants $\lambda, \Lambda$ for $a_{ij}$ such that*

$$\|\nabla u\|_{L^2(B_r)} \leq \frac{C}{\delta}\|u\|_{L^2(B_{r+\delta})}$$

This will be instrumental in using the $L^2(B_2)$ norm of $u$ iteratively to obtain a finite $L^\infty(B_{1/2})$ norm.

*Proof.* (Caccioppoli inequality) Let $u \geq 0$ be a subsolution in $B_{r+\delta}$. Let $\eta : B_{r+\delta} \to [0,1]$ be a smooth cutoff function with $\eta = 1$ on $B_r$, $\eta = 0$ on $\partial B_{r+\delta}$, and $|\nabla\eta|_{L^\infty} \leq 2\delta^{-1}$. Then

$$\begin{aligned}\int_{B_{r+\delta}} \eta^2(\nabla u)^2 &\leq \frac{1}{\lambda}\int_{B_{r+\delta}} \eta^2 a_{ij}\partial_i u \partial_j u \\ &= \frac{1}{\lambda}\left[\int_{B_{r+\delta}} a_{ij}\partial_i u \partial_j(\eta^2 u) - 2\int_{B_{r+\delta}} u\eta a_{ij}\partial_i u\partial_j\eta\right]\end{aligned}$$

and since $\eta^2 u \in H^1_0(B_{r+\delta})$ with $\eta^2 u \geq 0$ we have that $\int_{B_{r+\delta}} a_{ij}\partial_i u\partial_j(\eta^2 u) \leq 0$ so

$$\begin{aligned}\int_{B_{r+\delta}} \eta^2(\nabla u)^2 &\leq \frac{-2}{\lambda}\int_{B_{r+\delta}} u\eta a_{ij}\partial_i u\partial_j\eta \\ &\leq \frac{2\Lambda}{\lambda}\int_{B_{r+\delta}} |u\eta\partial_i u\partial_j\eta| \\ &\leq \frac{2\Lambda}{\lambda}\left(\int_{B_{r+\delta}} u^2|\nabla\eta|^2\right)^{1/2}\left(\int_{B_{r+\delta}} \eta^2|\nabla u|^2\right)^{1/2}\end{aligned}$$

so we have that

$$\begin{aligned}\|\eta\nabla u\|^2_{L^2(B_{r+\delta})} &\leq \frac{2\Lambda}{\lambda}\|u\nabla\eta\|_{L^2(B_{r+\delta})}\|\eta\nabla u\|_{L^2(B_{r+\delta})} \\ \implies \|\nabla u\|_{L^2(B_1)} = \|\eta\nabla u\|_{L^2(B_1)} &\leq \|\eta\nabla u\|_{L^2(B_{r+\delta})} \\ &\leq \frac{2\Lambda}{\lambda}\|u\nabla\eta\|_{L^2(B_{r+\delta})} \\ &\leq \frac{4\Lambda}{\delta\lambda}\|u\|_{L^2(B_{r+\delta})}\end{aligned}$$

thus with $C := \frac{4\Lambda}{\lambda}$ we have that $\|\nabla u\|_{L^2(B_r)} \leq C\delta^{-1}\|u\|_{L^2(B_{r+\delta})}$. □

Now we are ready to prove the $L^\infty(B_{1/2})$ bound:

**Lemma 2.4.** *($L^\infty$ bound) Let $u \in H^1(B_2)$ be a nonnegative subsolution to the equation $\partial_i(a_{ij}\partial_j u) \geq 0$. Then $\|u\|_{L^\infty(B_1)} \leq C\|u\|_{L^2(B_2)}$ for some $C = C(d, \lambda, \Lambda)$.*

We will actually prove that there exists $\delta_0 = \delta_0(d, \lambda, \Lambda)$ such that if $\|u\|_{L^2(B_1)} \leq \delta_0$ then $\|u\|_{L^\infty(B_1)} \leq 1$. This is equivalent to Lemma 2.4, but we will actually use *this* statement of the lemma in the De Giorgi proof so we will not justify their equivalence.

To prove the existence of such a $\delta_0$, we will consider the setup where we let

$$\begin{aligned} \ell_k &= 1 - 2^{-k} \\ r_k &= 1 + 2^{-k} \\ u_k &= (u - \ell_k)_+ \\ a_k &= \|u_k\|_{L^2(B_{r_k})} \end{aligned}$$

Then we have that $u_k$ is a nonnegative subsolution, and $u_{k+1} \leq u_k$, and therefore $a_{k+1} \leq a_k$. Also, $a_0 = \|u\|_{L^2(B_2)}$. Therefore, if we can show that $a_0 \leq \delta_0$ implies $a_k \to 0$, then we will have shown that $u \leq 1$ on $B_1$ (this follows from the definition of $u_k$ above, since $\ell_k \to 1$), i.e. that $\|u\|_{L^2(B_2)} \leq \delta_0$ implies $\|u\|_{L^\infty(B_1)} \leq 1$.

*Proof.* For each $k \in \mathbb{N}$ let $\eta_k : B_2 \to \mathbb{R}$ be a smooth cutoff function such that $\eta_k = 1$ on $\overline{B_{r_k}}$ and $\eta_k = 0$ on $B_1 \setminus B_{r_{k-1}}$, and also $\|\nabla \eta_k\|_{L^\infty(B_2)} \leq 2^k$.

Let $V_k := \{x \in B_{r_{k-1}} : u_k(x) > 0\}$. Then, with $2^* := 2d/(d-2)$ the Sobolev conjugate of 2, we have by Hölder's inequality that

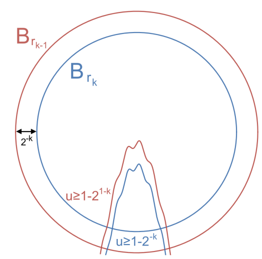


**Figure 1.** $\{u > 1-2^{-k}\} \cap B_k$ shrinks by Caccioppoli

$$\begin{aligned} \|u_k\|^2_{L^2(B_{r_k})} &\leq \|\eta_k u_k\|^2_{L^2(B_{r_{k-1}})} \\ &= \int_{B_{r_{k-1}}} \chi_{V_k} |\eta_k u_k|^2 \\ &\leq \|\chi_{V_k}\|_{L^{2^*/(2^*-2)}(B_{r_{k-1}})} \|\eta_k u_k^2\|_{L^{2^*/2}(B_{r_{k-1}})} \\ &= |V_k|^{2/d} \|\eta_k u_k\|^2_{L^{2^*}(B_{r_{k-1}})} \end{aligned} \tag{2.5}$$

Now, by definition of $u_k$ we have that $u_k > 0 \Leftrightarrow u_{k-1} > 2^{-(k+1)}$, so by Chebychev's inequality we have

$$|V_k| = |\{x \in B_{r_{k-1}} : u_{k-1} > 2^{-(k+1)}\}| \leq 2^{2(k+1)} \int_{B_{r_{k-1}}} |u_{k-1}|^2 \tag{2.6}$$

Furthermore, since $\eta_k u_k \in H^1_0(B_{r_{k-1}})$ we have by the Sobolev inequality that

$$\|\eta_k u_k\|_{L^{2^*}(B_{r_{k-1}})} \leq C \|\nabla(\eta_k u_k)\|_{L^2(B_{r_{k-1}})} \tag{2.7}$$

and finally by the Caccioppoli inequality that

$$\|\nabla(\eta_k u_k)\|_{L^2(B_{r_{k-1}})} \leq C \|u_k \nabla \eta_k\|_{L^2(B_{r_k})} \tag{2.8}$$

and thus

$$\begin{aligned} \|u_k\|_{L^2(B_{r_k})} &\leq |V_k|^{1/d} \|\eta_k u_k\|_{L^{2^*}(B_{r_k})} && \text{by (2.5)} \\ &\leq C 4^{(k+1)/2} \|u_{k-1}\|^{2/d}_{L^2(B_{r_{k-1}})} \|\nabla(\eta_k u_k)\|_{L^2(B_{r_{k-1}})} && \text{by (2.6) and (2.7)} \\ &\leq C 4^{(k+1)/d} \cdot 2^k \|u_{k-1}\|^{1+2/d}_{L^2(B_{r_{k-1}})} && \text{by (2.8)} \\ &= C 2^{k+2(k+1)/d} a^{1+2/d}_{k-1} \end{aligned}$$

Now, denote $n_k := k + 2(k+1)/d$, and $\epsilon := 2/d > 0$. By repeated application of this inequality we see that

$$\begin{aligned} a_k &\le C2^{n_k} a_{k-1}^{1+\epsilon} \le C^{1+(1+\epsilon)} 2^{n_k+(1+\epsilon)n_{k-1}} a_{k-2}^{(1+\epsilon)^2} \\ &\le C^{\sum_{i=0}^k (1+\epsilon)^i} 2^{\sum_{i=0}^k n_{k-i}(1+\epsilon)^i} a_0^{(1+\epsilon)^k} \\ &= \left[ C^{\sum_{i=0}^k (1+\epsilon)^{i-k}} 2^{\sum_{i=0}^k n_{k-i}(1+\epsilon)^{i-k}} a_0 \right]^{(1+\epsilon)^k} \\ &= \left[ C^{\sum_{i=0}^k (1+\epsilon)^{-i}} 2^{\sum_{i=0}^k n_i/(1+\epsilon)^i} a_0 \right]^{(1+\epsilon)^k} \end{aligned}$$

Also, we have that the following series converge

$$s_1 := \sum_{i=0}^{\infty} \frac{1}{(1+\epsilon)^i}, s_2 := \sum_{i=0}^{\infty} \frac{n_i}{(1+\epsilon)^i}$$

so the inequality

$$a_k \le \left[ C^{s_1} 2^{s_2} a_0 \right]^{(1+\epsilon)^k}$$

holds for all $k \ge 1$. Therefore choosing $a_0 < (C^{s_1} 2^{s_2})^{-1}$ we have that $a_k \to 0$ as $k \to \infty$, as desired. □

2.1.2. *Improvement of Oscillation.* In this section we prove for bounded $H^1$ subsolutions on the ball (i.e., subsolutions $u \in H^1(B_1)$ satisfying $0 \le u \le 1$) that we can find an explicit upper bound for $u$ that is strictly less than 1 as long as $|\{x \in B_1 : u(x) = 0\}|$ is large enough (Lemma 2.9). We prove this using the $L^\infty$ bound from Lemma 2.4, and then we also use Lemma 2.4 to ensure that the conditions of Lemma 2.9—that $u$ is bounded—are satisfied. By iteratively applying this lemma to a geometric series of nested balls between $B_1$ and $B_2$, we get the De Giorgi result Theorem 2.1.

Suppose that we are given constants $C, \delta_0, \delta_1 > 0$, and an $H^1$ function $u : B_1 \to [0,1]$ such that $\|u\|_{H^1(B_1)} \le C, |\{u = 0\}| \ge \delta_0, |\{u = 1\}| \ge \delta_1$. We must have that there is some $\epsilon = \epsilon(C, \delta_0, \delta_1) > 0$ such that $\epsilon < |\{0 < u < 1\}|$. For, suppose for the sake of contradiction that there exists $\{u_k\} \subset H^1(B_1)$ satisfying the same conditions, and such that additionally $|\{0 < u_k < 1\}| \to 0$ as $k \to \infty$. Then we have that $\{u_k\}$ is bounded in $L^2$ and therefore there is some subsequence (which we relabel $\{u_k\}$) satisfying $u_k \to u_\infty$ in $L^2$ for some $u_\infty \in L^2(B_1)$. Since we have that

$$|\{u_\infty = 0\}| \ge \delta_0 > 0, \qquad |\{u_\infty = 1\}| \ge \delta_1 > 0, \qquad |\{0 < u_\infty < 1\}| = 0$$

it must be the case that $u_\infty = \chi_E$ for some measurable $E \subset B_1$ with $\delta_1 \le |E| \le 1 - \delta_0$. But we have that $\chi_E$ cannot be in $H^1(B_1)$ since $\|\nabla \chi_E\|_{L^2(B_1)} = \infty$, which contradicts the fact that $\|u_k\|_{H^1(B_1)} \le C\sqrt{|B_2|}$ and $u_k \to \chi_E$ in $L^2$.

The observation that we can set a lower bound $\epsilon(C, \delta_0, \delta_1) < |\{0 < u < 1\}|$ is often referred to as the De Giorgi isoperimetric inequality. With this lower bound on $|\{0 < u < 1\}|$ we are now ready to prove the final lemma:

**Lemma 2.9.** *Let $u \in H^1(B_2)$ be a subsolution to the equation $\partial_i(a_{ij}\partial_j u) \ge 0$, with $0 \le u \le 1$. Then if $|\{x \in B_1 : u(x) = 0\}| \ge \delta_0 > 0$, we have that $\sup_{B_1} u \le 1 - \gamma = 1 - \gamma(d, \lambda, \Lambda, \delta_0)$.*

*Proof.* Set $w_k := 2^k[u-(1-2^{-k})]_+$. Then, since $0 \leq u \leq 1$ we have that $0 \leq w_k \leq 1$, and furthermore each $w_k$ is a subsolution. By the Caccioppoli inequality (Lemma 2.3) we have that $\|\nabla w_k\|_{L^2(B_1)} \leq C\|w_k\|_{L^2(B_2)} \leq C|B_2|$, so the sequence $\{w_k\}$ is uniformly bounded in $H^1$.

We have that $w_k = 0$ wherever $u = 0$, so $|\{w_k = 0\}| \geq \delta_0 > 0$. Furthermore, we have that

$$w_{k+1} = 2^{k+1}[u-(1-2^{-(k+1)})]_+ = 2(2^k[u-(1-2^{-k})]_+ + (2^{-1}-1))_+ = 2[w_k - 1/2]_+$$

so $\{0 < w_k < 1/2\}$ are disjoint for all $k \in \mathbb{N}$, and $\{0 < w_{k+1}\} \subset \{1/2 \leq w_k\}$. Therefore, we have that

$$|\{x \in B_1 : 1/2 \leq w_k(x)\}| \geq |\{x \in B_1 : 0 < w_{k+1}(x)\}| \geq \int_{B_1} w_{k+1}^2 \tag{2.10}$$

By Lemma 2.4 (see especially the note immediately following the statement), we have that there exists $\delta_1 > 0$ such that $\|u\|_{L^2(B_2)} \leq \delta_1$ implies $\|u\|_{L^\infty(B_1)} \leq 1$. Also, by the De Giorgi isoperimetric inequality, there exists some $\epsilon > 0$ such that $\|w_k\|_{H^1(B_1)} \leq C|B_2|$, $|\{w_k \leq 0\}| \geq \delta_0, |\{w_k \geq 1/2\}| \geq \delta_1^2$ implies $|\{0 < w_k < 1/2\}| > \epsilon$. Therefore if $\|w_{k+1}\|_{L^2(B_1)} \geq \delta_1$ we have by (2.10) that $|\{x \in B_1 : 1/2 \leq w_k(x)\}| \geq \delta_1^2$, and as stated in the second paragraph of this proof we have that $|\{w_k \leq 0\}| \geq \delta_0$, and as stated in the first paragraph we have $\|w_k\|_{H^1(B_1)} \leq C|B_2|$, so by the isoperimetric inequality we must have $|\{0 < w_k < 1/2\}| > \epsilon$. However, since all $\{0 < w_k < 1/2\}$ are disjoint, we must have that

$$\sum_{k=0}^{\infty} |\{0 < w_k < 1/2\}| \leq |B_2| < +\infty$$

so it is not possible to have that $|\{0 < w_k < 1/2\}| > \epsilon$ for all $k$, and therefore there must exist some $k_0 \in \mathbb{N}$ such that $\|w_{k_0}\|_{L^2(B_2)} < \delta_1$. By Lemma 2.4, we therefore have that

$$\|w_{k_0}\|_{L^\infty(B_1)} \leq C(d, \lambda, \Lambda)\|w_{k_0}\|_{L^2(B_2)} < C\delta_1$$

so by further shrinking $\delta_1$ (depending only on $C$, which depends only on $d, \lambda, \Lambda$) we can ensure that $\|w_{k_0}\|_{L^\infty(B_1)} \leq C\delta_1 \leq 1/2$. Thus,

$$\frac{1}{2} \geq w_{k_0} = 2^{k_0}[u - (1 - 2^{-k_0})] \implies u \leq 1 - 2^{-(k_0+1)}$$

so with $\gamma := 2^{-(k_0+1)}$, the proof is complete. □

Now we can iteratively apply Lemma 2.9 to get a $C^\alpha$ bound on $u$ in $B_1$, thus completing the proof of Theorem 2.1.

*Proof.* (De Giorgi-Nash-Moser) Since $u \in H^1(B_2)$ is a weak solution to $\partial_i(a_{ij}\partial_j u) = 0$, we have that $|u| \in L^2(B_2)$ and that $|u|$ is a nonnegative subsolution to the equation $\partial_i(a_{ij}\partial_j u) \geq 0$. Therefore, by Lemma 2.4, there exists $C = C(d, \lambda, \Lambda)$ such that $\|u\|_{L^\infty(B_1)} \leq C\|u\|_{L^2(B_2)}$.

Let $x_0 \in B_1$, and let $x \in B_1(x_0) \subset B_2$. There exists $k \in \mathbb{N}$ such that $2^{-k} \leq |x - x_0| < 2^{-(k-1)}$. By Lemma 2.9 we have for each $i \in \{1, ..., k\}$ that

$$\mathrm{osc}_{B_{2^{-i}}(x_0)} u \leq (1 - \gamma) \sup_{B_{2^{-(i-1)}}(x_0)} u$$

and therefore

$$|u(x) - u(x_0)| \leq \mathrm{osc}_{B_{2^{-k}}(x_0)} u \leq (1 - \gamma)^k \mathrm{osc}_{B_1(x_0)} u \leq (1 - \gamma)^k \|u\|_{L^\infty}(B_1)$$

With $\alpha := -\log_2(1-\gamma)$ we therefore have that $|u(x)-u(x_0)| \leq 2^{-k\alpha}\|u\|_{L^\infty}(B_1)$. Thus, since $2^{-k} \leq |x-x_0|$ we have that $|u(x)-u(x_0)| \leq |x-x_0|^\alpha \|u\|_{L^\infty}(B_1)$, so for all $x_0 \in B_1$ we have that

$$\sup_{x\in B_1(x_0)} \frac{|u(x)-u(x_0)|}{|x-x_0|} \leq \|u\|_{L^\infty}(B_1) \leq C\|u\|_{L^2}(B_2)$$

so $u \in C^\alpha(B_1)$. ☐

2.2. **Schauder Theorem.** The (first-order) Schauder theorem for Hölder regularity estimates of elliptic PDE solutions is the following:

**Theorem 2.11.** *Let $a_{ij} \in C^\alpha(B_1)$ be uniformly elliptic with ellipticity constants $\lambda, \Lambda$, and let $f \in C^\alpha(B_1; \mathbb{R}^d)$ be a Hölder-continuous vector field. Then if $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, we have the interior estimate*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq_{\{\lambda,\Lambda,d,[a_{ij}]_\alpha\}} \|u\|_{L^2(B_1)} + [f]_{C^\alpha(B_1)}$$

Our proof of Schauder's Hölder regularity theorem (Theorem 2.11) utilizes the following equivalent definition for $C^{1,\alpha}$ functions:

**Proposition 2.12.** *With $\Omega \subset \mathbb{R}^d$ open and $0 < \alpha < 1$, we have that $u \in C^{1,\alpha}(\Omega)$ if and only if for all $x \in \Omega$, there is some linear function $P_x(y) = a_x + \langle b_x, y-x\rangle$ (for $a_x \in \mathbb{R}, b_x \in \mathbb{R}^d$) and a constant $K$ not depending on $x$ such that*

$$\|u - P_x\|_{L^\infty(B_r(x))} \leq Kr^{1+\alpha}$$

Intuitively, one can recall the characterization of Lipschitz continuity in which we say that a function is Lipschitz continuous with constant $M$ if the graph of the function lies between the planes of slope $+M$ and $-M$ emanating from every point in its graph. A function being $C^{1,1}$ implies that at every point on its graph, there is a plane such that the function's graph lies between an upward and a downward parabola of uniformly bounded slope, oriented against the tangent plane. Notice that subtracting off a plane does not affect the elliptic divergence PDE $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, so with Proposition 2.12 we are squeezing the graph of $u$ between two parabolas of degree $1+\alpha$, emanating from each point $x \in \Omega$ (Fig. 2).

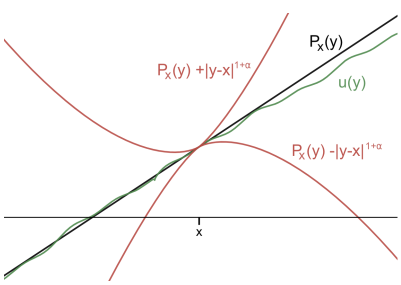


**Figure 2.** graph($u$) is bounded by a linear function $P_x$ plus $|y-x|^{1+\alpha}$ at each $x$ in domain $\Rightarrow u \in C^{1,\alpha}$

Proving Schauder regularity (Theorem 2.11) becomes a matter of proving that the conditions of Proposition 2.12 are satisfied in $B_{1/2}$ for solutions $u \in H^1(B_1)$ to the elliptic PDE $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, where $a_{ij}, f \in C^\alpha(B_1)$. We give a rough outline of the proof here. First, suppose that $u$ is harmonic in $B_1$. By a quadratic approximation for harmonic functions (see, e.g., Chapter 2 of [1]) we have that

$$|u(x) - \nabla u(0)\cdot x - u(0)| \leq C|x|^2$$

for all $x \in B_{1/2}$, where $C = \max\{\|\nabla u\|_{L^\infty(B_{1/2})}, \|D^2 u\|_{L^\infty(B_{1/2})}\}$. Set $r_0 := (2C)^{1/(\alpha-1)}$ and $C_0 := 2C$ and $b := \nabla u(0)$. Then we have that $|b| \leq C_0$ and for $r < r_0$

$$\operatorname{osc}_{B_r}(u(x) - \langle b, x\rangle) \le \sup_{x,y\in B_r} |(u(x) - \langle b, x\rangle) - u(0)| + |u(0) - (u(y) - \langle b, y\rangle)| \tag{2.13}$$

$$\le 2Cr^2 \le r^{1+\alpha} \tag{2.14}$$

by choice of $r_0$.

Now, dropping the assumption that $u$ is harmonic, we can zoom in very close to fixed points in $B_{1/2}$, and the farther we zoom in the closer $a_{ij}$ and $f$ are to being constant, so the closer $u$ is to being basically harmonic. That is, if $a_{ij}$ and $f$ are constant then $\partial_i f_i = 0$ and through change of basis and rescaling we can consider

$$\partial_i(a_{ij}\partial_j u)`=' \partial_i(\delta^i_j \partial_j u) = \Delta u = \partial_i f_i = 0$$

We zoom in as close as we need to each point until $a_{ij}, f$ are close enough to being constant that $u$ inherits the regularity from being harmonic on all very small balls. Specifically, we want to prove the following lemma, which is already shown to be satisfied for harmonic functions by (2.14).

**Lemma 2.15.** *Suppose $u$ is a solution so that $osc_{B_1} u \le 1$. Then there exist $r_0, \epsilon_0 \in (0, 1)$ and $C_0 > 0$ such that if*

$$osc_{B_1} a_{ij} < \epsilon_0, \quad osc_{B_1} f < \epsilon_0$$

*then there exists $b \in \mathbb{R}^d$ with $|b| \le C_0$ such that*

$$osc_{B_r}(u(x) - \langle b, x\rangle) \le r^{1+\alpha}$$

*for all $r \in (0, r_0)$.*

*Proof.* Suppose Lemma 2.15 does not hold. Then there exist sequences $\epsilon^k \to 0$ and $\{u^k\} \subset H^1(B_1)$ with coefficients $a^k_{ij}$ and vector fields $f^k \in C^\alpha$ where $\partial_j(a^k_{ij}u^k) = \partial_j f^k_j$ for all $k$, and such that $\operatorname{osc}_{B_1} f^k_i$,$\operatorname{osc}_{B_1} a^k_{ij} < \epsilon^k$ such that for all vectors $b \in B_{C_0}$ we have that there exists $r \in (0, r_0)$ such that

$$\operatorname{osc}_{B_r}(u^k - \langle b, x\rangle) > r^{1+\alpha} \tag{2.16}$$

By the oscillation assumption we have that $f^k$ and $a^k_{ij}$ are bounded functions converging in $C^0$ to constant functions $f^\infty$ and $a^\infty_{ij}$, respectively.

Denote $w^k := u^k - \frac{1}{|B_1|}\int_{B_1} u^k$, so $w^k$ has the same gradient and oscillation as $u^k$ and therefore is still a solution for each $a^k_{ij}, f^k$ and also still satisfies the inequality (2.16). By Poincaré's inequality for a ball ([1] 5.8, Theorem 2) there exists a constant $C = C(d)$ such that

$$\|w^k\|_{L^2(B_1)} \le C\|\nabla w^k\|_{L^2(B_1)} = C\|\nabla u^k\|_{L^2(B_1)} \tag{2.17}$$

and since $\partial_j(a^k_{ij}\partial_i u^k) = \partial_j(a^k_{ij}\partial_i w^k) = \partial_i f^k_i$ we have that

$$\|\nabla w^k\|^2_{L^2(B_1)} \le \frac{1}{\lambda}\int_{B_1} a^k_{ij}\partial_i w^k \partial_j w^k = \frac{1}{\lambda}\int_{B_1} f^k_i \partial_i w^k \le \frac{1}{\lambda}\|f^k\|_{L^2(B_1)}\|\nabla w^k\|_{L^2(B_1)}$$

so dividing by the common factor we see that

$$\|\nabla w^k\|_{L^2(B_1)} \le \frac{1}{\lambda}\|f^k\|_{L^2(B_1)}$$

Since $f^k \to f^\infty$ uniformly where $f^\infty$ is constant, we have that the right side of this inequality is bounded, so $\|\nabla w^k\|_{L^2}$ is uniformly bounded. Then, by (2.17) we have that $w^k$ are uniformly bounded in $H^1(B_1)$. Therefore there exists $w^\infty \in H^1(B_1)$ such that $w^k \rightharpoonup w^\infty$ weakly in $H^1(B_1)$.

Since $\partial_j(a^k_{ij}\partial_j w^k) = \partial_i f^k_i$ we have that each $w^k \in C^\beta(B_{1/2})$ by the De Giorgi theorem with nonzero right hand side, where $\beta$ depends only on $\lambda, \Lambda$, the dimension $d$, the radius $B_R = B_1$, and the degree $L^q = L^2$, which do not vary with $k$ (see Theorem 2.1, proved for zero right hand side, or Chapter 8 of [2] for nonzero right hand side). Specifically,

$$\|w^k\|_{C^\beta(B_{1/2})} \leq C\|w^k\|_{L^2(B_1)} + \|f^k\|_{L^\infty(B_1)}$$

Since $w^k$ is uniformly bounded in $L^2(B_1)$ and $f^k \to f^\infty$ uniformly with $f^\infty$ constant, we have that $w^k$ is uniformly bounded in $C^\beta(B_{1/2})$, so by the Arzelà-Ascoli theorem for Hölder functions we have that there exists $w^\infty \in C(B_{1/2})$ such that $w^k \to w^\infty$ uniformly after passing to a subsequence.

Furthermore, we have that $w^\infty$ is harmonic: for, let $v \in C^\infty_c(B_{1/2})$ be an arbitrary test function. Then we have that

$$\begin{aligned}\int a^\infty_{ij}\partial_i w^\infty \partial_j v &= \int a^\infty_{ij}\partial_i w^\infty \partial_j v + \int (a^k_{ij} - a^k_{ij})\partial_i w^\infty \partial_j v + \int a^k_{ij}(\partial_i w^k - \partial_i w^k)\partial_j v \\ &= \int (a^\infty_{ij} - a^k_{ij})\partial_i w^\infty \partial_j v + \int a^k_{ij}(\partial_i w^\infty - \partial_i w^k)\partial_j v + \int a^k_{ij}\partial_i w^k \partial_j v\end{aligned}$$

and the first term goes to zero since $a^k_{ij} \to a^\infty_{ij}$ uniformly and the second term goes to zero by combining $a^k_{ij} \to a^\infty_{ij}$ uniformly with $\nabla w^k \rightharpoonup \nabla w^\infty$ weakly (recall that $w^k \rightharpoonup w^\infty$ weakly in $H^1(B_1)$). The third term goes to zero since $\int a^k_{ij}\partial_i w^k \partial_j v = \int f^k_j \partial_j v \to \int f^\infty_j \partial_j v = 0$ since $f^k \to f^\infty$ uniformly, $f^\infty$ constant, and $v$ compactly supported in $B_{1/2}$. Thus $\partial_j(a^\infty_{ij}\partial_i w^\infty) = 0$ weakly. Since $a^\infty_{ij}$ is a constant uniformly elliptic matrix we can consider after a linear change of variables that $\partial_j(\delta^i_j \partial_i w^\infty) = \Delta w^\infty = 0$ weakly, and since $w^\infty \in C^0(B_{1/2})$ we in fact have that $w$ is harmonic (see problem 2.8 of [2]). But this contradicts the fact that we know the lemma to hold for harmonic functions from (2.14). Thus, no such sequences $\epsilon^k, u^k, a^k_{ij}, f^k$ exist, concluding the proof of Lemma 2.15 . □

Recalling that the Schauder inequality (Theorem 2.11) depends on scale, we can for each $x \in B_{1/2}$ construct a sequence $\{b_k\} \subset \mathbb{R}^d$ with $|b_k| \to 0$ such that, for $r_0, \epsilon_0, C_0 > 0$ as in Lemma 2.15, we have

$$\partial_i\left(a_{ij}(r^k_0 x)\partial_j\left(\frac{1}{r_0^{k(1+\alpha)}}[u(r^k_0 x) - \langle b_k, r^k_0 x\rangle]\right)\right) = \partial_i f_i(r^k_0 x) \tag{2.18}$$

which is a bit ugly to look at, but is just saying that a scaled-down $u$ satisfies the same elliptic PDE for a scaled-down $a_{ij}$ and scaled-down $f$. Applying Lemma 2.15 for each $k$ we have that

$$\operatorname{osc}_{B_{r^k_0}}(u - \langle b_k, x\rangle) \leq r_0^{k(1+\alpha)}$$

which implies that $u$ satisfies the conditions of Proposition 2.12 for each point in $B_{1/2}$, which thus implies that $u \in C^{1,\alpha}(B_{1/2})$, as stated in Theorem 2.11.

## 3. Existence and Uniqueness of Minimal Surface Equation Solutions on the Unit Ball

We say that $u \in C^{0,1}(B_1)$ is a subsolution of the minimal surface equation $Mu = 0$ (as in (1.3)) if, with $J$ denoting the surface area functional $J(u) = \int_{B_1} \sqrt{1+|\nabla u|^2}$, we have that $J(u-\psi) \geq J(u)$ for all $\psi \in C_c^\infty(B_1)$ with $\psi \geq 0$. We say that $u$ is a supersolution if $J(u+\psi) \geq J(u)$ always holds under the same conditions. Similarly to when they were introduced in Section 2.1.1, a subsolution (resp. supersolution) is not necessarily a classical solution, but perturbing it downwards (resp. upwards) anywhere will only increase the surface area. It turns out that we can derive a comparison principle between continuous supersolutions and subsolutions to the minimal surface functional $J$:

**Theorem 3.1.** *Let $u$ be a supersolution and $v$ a subsolution on the unit ball $B_1$. Then*

$$\sup_{B_1}(v-u) = \sup_{\partial B_1}(v-u)$$

Which can be proved easily after proving the following lemma:

**Lemma 3.2.** *Let $u$ be a supersolution and $v$ be a subsolution on the unit ball $B_1$ such that $u \geq v$ on $\partial B_1$. Then $u \geq v$ on $B_1$.*

*Proof.* (Lemma 3.2) Set $U := \{v > u\}$. Then since $u$ and $v$ are continuous we have that $U$ is open. Suppose for the sake of contradiction that $|U| \neq 0$. Set

$$\tilde{u} := \begin{cases} u & \text{on } B_1 \setminus U \\ v & \text{on } U \end{cases}, \text{ and } \tilde{v} := \begin{cases} v & \text{on } B_1 \setminus U \\ u & \text{on } U \end{cases}$$

Then $u \leq \tilde{u}$ and $\tilde{v} \leq v$, with equality on $\partial B_1$. Since $u$ is a supersolution and $v$ a subsolution, and since $v > u$ on $U$, we have also that $0 \leq J(u) \leq J(\tilde{u})$, and $0 \leq J(v) \leq J(\tilde{v})$ (with $J$ and $L$ as in (1.1)-(1.2)). Therefore

$$\begin{aligned} J(u) \leq J(\tilde{u}) &\Rightarrow \int_{B_1 \setminus U} L(\nabla u) + \int_U L(\nabla v) \leq \int_{B_1 \setminus U} L(\nabla u) + \int_U L(\nabla u) \\ &\Rightarrow \int_U L(\nabla v) \leq \int_U L(\nabla u) \end{aligned}$$

and through the same line of reasoning with $J(v) \leq J(\tilde{v})$ we see that $\int_U L(\nabla u) \leq \int_U L(\nabla v)$, so

$$\int_U L(\nabla u) = \int_U L(\nabla v)$$

However, by the strict convexity of $L$ this leads to a contradiction. Let $w := (u+v)/2$. Then we have that $w = u + \psi$ for some $\psi \in C_0^{0,1}(U)$ with $\psi \geq 0$ (note that $\psi = 0$ on $\partial U$ since $u = v$ on $\partial U$) and therefore since $u$ is a supersolution we have that

$$\int_U L(\nabla u) \leq \int_U L(w)$$

but by strict convexity we have that

$$\int_U L(w) < \frac{1}{2}\int_U L(\nabla u) + \frac{1}{2}\int_U L(\nabla v) = \int_U L(\nabla u)$$

with the last equality coming from having proved $\int_U L(\nabla u) = \int_U L(\nabla v)$. This implies that $\int_U L(\nabla u) < \int_U L(\nabla u)$, a contradiction. Thus, $|U| = 0$, so $u \geq v$ on $B_1$. □

*Proof.* (Theorem 3.1) For $x \in \partial B_1$ we have that

$$v(x) = (u + v - u)(x) \leq u(x) + \sup_{y \in \partial B_1} (v - u)(y)$$

Also, $u(\cdot) + \sup_{y \in \partial B_1}(v - u)(y)$ is a subsolution, so by the comparison principle (Lemma 3.2) we have that

$$v(x) \leq u(x) + \sup_{y \in \partial B_1} (v - u)(y)$$
$$\implies (v - u)(x) \leq \sup_{y \in \partial B_1} (v - u)(y)$$

for all $x \in B_1$, as desired. □

**Theorem 3.3.** *Let $\varphi : \partial B_1 \to \mathbb{R}$ be Lipschitz. Then there exists a unique $u : B_1 \to \mathbb{R}$ such that $\operatorname{div}(\nabla u/\sqrt{1 + |\nabla u|^2}) = 0$. Furthermore, $u$ is smooth on the interior of $B_1$.*

*Proof.* Let $J$ and $L$ be as in (1.1)-(1.2). First, suppose that $u \in W^{1,\infty}(B_1)$ is a minimizer of $J$. Let $m = \sup_{\partial B_1} |\nabla \varphi|$. Then we have that $\varphi(y) - m|x - y| \leq u(x) \leq \varphi(y) + m|x - y|$ for all $y \in \partial B_1, x \in B_1$ (Fig. 3). Therefore for each $y \in \partial B_1$ there exist tangent planes $v_y^-$ to the cone $\varphi(y) - m|x - y|$ and $v_y^+$ to the cone $\varphi(y) + m|x - y|$ such that $v_y^- \leq u \leq v_y^+$ in $B_1$. Since $v_y^-, v_y^+$ are affine they are also solutions to the minimal surface equation, so by the comparison principle (Theorem 3.1) we must have for each $y \in \partial B_1$ that

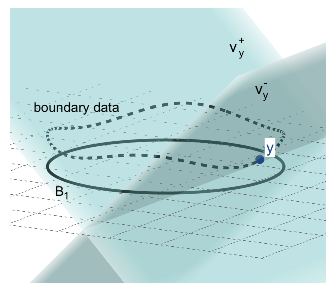


**Figure 3.** Affine bounds on graph

$$\sup_{x \in B_1} \frac{|u(x) - u(y)|}{|x - y|} \leq \sup_{x \in B_1} \frac{\max\{|v_y^+(x) - v_y^+(y)|, |v_y^-(x) - v_y^-(y)|\}}{|x - y|} = m \tag{3.4}$$

Now, using Theorem 3.1 we can show that this bounds the gradient of $u$ on $B_1$. Let $x_1, x_2 \in B_1$, and let $v = x_2 - x_1$. Then we have that $x \mapsto u(x+v)$ is a minimizer of $J$ on $W^{1,\infty}(B_{x_2}^{x_1})$, where $B_{x_2}^{x_1} = \{x \in \mathbb{R}^d : x + v \in B_1\}$. Then since $x_1 \in B_1 \cap B_{x_2}^{x_1}$ we have that this intersection is nonempty and that both $u$ and $u(\cdot + v)$ minimize $J$ on $W^{1,\infty}(B_1 \cap B_{x_2}^{x_1})$, and therefore by Theorem 3.1 that their difference is achieved on the boundary, i.e. that there must exist $z \in \partial(B_1 \cap B_{x_2}^{x_1})$ such that

$$|u(x_1) - u(x_2)| = |u(x_1) - u(x_1 + v)| \leq |u(z) - u(z + v)|$$

and therefore, since one of $z$ or $z + v$ is in $\partial B_1$, we have that

$$\sup_{x,y \in B_1} \frac{|u(x) - u(y)|}{|x - y|} \leq \sup_{x \in B_1, y \in \partial B_1} \frac{|u(x) - u(y)|}{|x - y|}$$

But the left hand side is precisely $\|\nabla u\|_{L^\infty(B_1)}$, so combining with (3.4) we have that $\|\nabla u\|_{L^\infty(B_1)} \leq m$.

Now, if we set

$$\mathcal{A}_{m+1} := \{u \in W^{1,\infty}(B_1) : u = \varphi \text{ on } \partial B_1, \|\nabla u\|_{L^\infty} \leq m + 1\}$$

then we have that $\mathcal{A}_{m+1}$ is closed under uniform limits, and also that for all $u \in \mathcal{A}_{m+1}$, $J[u]$ is bounded below by $|B_1|$, so there exists a minimizer $u^{m+1} \in \mathcal{A}_{m+1}$. But we've just shown that $u^{m+1}$ must satisfy

$$\|\nabla u^{m+1}\|_{L^\infty(B_1)} \leq m < m+1$$

This in fact implies that $u^{m+1}$ is a minimizer over all of $\{u \in W^{1,\infty}(B_1) : u = \varphi \text{ on } \partial B_1\}$. To see this, let $w \in \mathcal{A} \setminus \mathcal{A}_m$, so $\|\nabla w\|_{L^\infty} > m$. We can show using the convexity of $J$ that $J(u^{m+1}) \leq J(w)$, by choosing $t \in [0,1]$ so that $tw+(1-t)u^{m+1} \in \mathcal{A}^m$, which is possible because of these two strict inequalities $\|\nabla u^{m+1}\|_{L^\infty} < m$ and $\|\nabla w\|_{L^\infty} > m$. Choose

$$t \in \left(0, \frac{m - \|\nabla u^{m+1}\|_{L^\infty}}{\|\nabla w\|_{L^\infty} - \|\nabla u^{m+1}\|_{L^\infty}}\right)$$

Then, we have that

$$\|\nabla(tw + (1-t)u^{m+1})\|_{L^\infty} \leq t\|\nabla w\|_{L^\infty} + (1-t)\|\nabla u^{m+1}\|_{L^\infty} < m$$

by our choice of $t \in (0,1)$. Therefore, $tw + (1-t)u^{m+1} \in \mathcal{A}_m$, so

$$J(u^{m+1}) \leq J\left(tw + (1-t)u^{m+1}\right) \leq tJ(w) + (1-t)J(u^{m+1}) \tag{3.5}$$

with the first inequality from the fact that $u^{m+1}$ is a minimizer on $\mathcal{A}_m$, and the second inequality from the convexity of $J$. From (3.5) we have that $J(u^{m+1}) \leq J(w)$, so $J(u^{m+1})$ is a minimizing minimal surface.

Uniqueness follows from the comparison principle (Theorem 3.1). □

## 4. Bernstein's Method and Problem

4.1. **Bernstein's Method & Interior Gradient Estimates.** We've just proved (Theorem 3.3) that a unique smooth solution exists for Lipschitz boundary data, but this is not so satisfying since Lipschitz is such a strong condition. If we have merely continuous boundary data $\varphi : \partial B_1 \to \mathbb{R}$ then we might think about mollifying the boundary data to get smooth solutions, but for all we know now, as we take the mollifier $\varphi * \rho_\epsilon$ to the identity $\epsilon \to 0$ we might have that $|\nabla u| \to \infty$ on the interior, which prevents us from extracting a solution for merely continuous boundary data.

Bernstein's method involves multiplying a solution $u$ by some auxiliary cutoff function $\eta$ that is zero on $\partial B_1$, and choosing $\eta$ specifically in a way that lets us say something interesting about $u$. Using this method, we will be able to derive a bound on the gradient of solutions to mollified boundary data $\varphi * \rho_\epsilon$ that depends only on $\|\varphi\|_{L^\infty(\partial B_1)}$, thus allowing us to extract a convergent subsequence and therefore a smooth solution to the minimal surface equation with continuous boundary data.

If $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is harmonic (where $\Omega$ is open, with $\partial\Omega \in C^1$) we have that $u$ achieves its maximum on the boundary $\partial\Omega$. We can show furthermore that $|\nabla u|^2$ is subharmonic by looking at its Laplacian:

$$\begin{aligned} \Delta|\nabla u|^2 &= 2(\partial_i\partial_j u)^2 + 2\partial_j u \partial_j \partial_i^2 u \\ &= 2\left[|D^2 u|^2 + \partial_j u(\partial_j \Delta u)\right] \\ &= 2|D^2 u|^2 \geq 0 \end{aligned} \tag{4.1}$$

where $D^2u$ is the Hessian of $u$. Since $|\nabla u|^2$ is subharmonic we also have the same maximum principle, i.e. that $\max_\Omega |\nabla u|^2 = \max_{\partial\Omega} |\nabla u|^2$.

The next two interior gradient estimates are proved by the method of Bernstein:

**Theorem 4.2.** *There exists a constant $C > 0$ such that for all harmonic functions $u : B_1 \to \mathbb{R}$ we have*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}$$

**Theorem 4.3.** *Let $u \in C^3(B_1)$ satisfy the minimal surface equation $Mu = 0$ (as in (1.3)). Then, there exists a constant $C = C(\|u\|_{L^\infty(B_1)})$ depending on the maximum of $u$ in the unit ball such that*

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)})$$

*Proof.* (Theorem 4.2) Let $w := |\nabla u|^2$, and let $\eta \in C_c^\infty(B_1)$. Then, we have that

$$\begin{aligned}\Delta(\eta^2 w) &= 2w(|\nabla\eta|^2 + \eta\Delta\eta) + 4\eta\nabla\eta\cdot\nabla w + \eta^2\Delta w \\ &= 2w(|\nabla\eta|^2 + \eta\Delta\eta) + 8\eta\nabla\eta\cdot(D^2u)^\top\nabla u + 2\eta^2|D^2u|^2\end{aligned}$$

and by Cauchy-Schwartz we have that

$$8\Big(|\eta||D^2u|\Big)\Big(|\nabla\eta||\nabla u|\Big) \leq 2\Big(|\eta||D^2u|\Big)^2 + 8\Big(|\nabla\eta||\nabla u|\Big)^2$$

and therefore

$$\begin{aligned}\Delta(\eta^2 w) &\geq 2w(|\nabla\eta|^2 + \eta\Delta\eta) - 2\eta^2|D^2u|^2 - 8|\nabla\eta|^2|\nabla u|^2 + 2\eta^2|D^2u|^2 \\ &= 2w\Big(|\nabla\eta|^2 + \eta\Delta\eta - 4|\nabla\eta|^2\Big)\end{aligned}$$

so with $C := 2(3|\nabla\eta|^2 - \eta\Delta\eta)$ depending only on $\eta$ we have that $\Delta(\eta^2 w) \geq -C|\nabla u|^2$.

Now considering $\eta^2 w + \frac{C}{2}u^2$, we have that

$$\begin{aligned}\Delta(\eta^2 w + \frac{C}{2}u^2) &\geq \frac{C}{2}\Delta(u^2) - C|\nabla u|^2 \\ &= Cu\Delta u + (C - C)|\nabla u|^2 = 0\end{aligned}$$

and therefore $\eta^2 w + \frac{C}{2}u^2$ is subharmonic. Let $\eta$ be a smooth nonnegative function with $\eta \equiv 1$ on $B_{1/2}$ and $\eta = 0$ on $\partial B_1$. Then we have by the maximum principle for subharmonic functions that

$$\begin{aligned}\|w\|_{L^\infty(B_{1/2})} &\leq \left\|\eta^2 w + \frac{C}{2}u^2\right\|_{L^\infty(B_{1/2})} \\ &\leq \left\|\eta^2 w + \frac{C}{2}u^2\right\|_{L^\infty(B_1)} \\ &= \left\|\eta^2 w + \frac{C}{2}u^2\right\|_{L^\infty(\partial B_1)} \\ &= \frac{C}{2}\|u^2\|_{L^\infty(\partial B_1)} \leq \frac{C}{2}\|u^2\|_{L^\infty(B_1)}\end{aligned}$$

where the first inequality comes from the fact that $w \geq 0$ and $u^2 \geq 0$. Thus,

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq \sqrt{\frac{C}{2}}\|u\|_{L^\infty(B_1)}$$

where $C = 2(3|\nabla\eta|^2 - \eta\Delta\eta)$ does not depend on $u$. □

*Proof.* (Theorem 4.3) Set

$$v := \sqrt{1+|\nabla u|^2}, \qquad \nu := \frac{\nabla u}{v}, \qquad g^{ij} := \delta^i_j - \nu^i\nu^j$$

then we have that

$$\partial_i v = \partial_i\partial_k u\nu^k$$

$$\partial_i\partial_j v = \partial_i\partial_j\partial_k u\nu^k + \frac{\partial_i\partial_k u\partial_j\partial_k u - \partial_i\partial_k u\partial_j\partial_\ell u\nu^k\nu^\ell}{v} \tag{4.4}$$

Also, if through a change of coordinates we ensure that $D^2u$ is diagonal, the second term of (4.4) is given by

$$\partial_i^2 u\partial_j^2 u(\delta^i_j - \nu^i\nu^j)$$

which is positive since $(\nu^i)^2 \leq |\nu| \leq 1$. Therefore,

$$g^{ij}\partial_i\partial_j v \geq g^{ij}\partial_i\partial_j\partial_k u\nu^k \tag{4.5}$$

and differentiating the equation for $u$ with respect to $x_k$ we obtain

$$g^{ij}\partial_i\partial_j\partial_k u\nu^k = \frac{2}{v}g^{ij}\partial_i v\partial_j v$$

and therefore by (4.5) that

$$g^{ij}\partial_i\partial_j v \geq \frac{2}{v}g^{ij}\partial_i v\partial_j v > 0 \tag{4.6}$$

where the second inequality follows from (4.4) and the reasoning that follows. Thus, we have that $v$ is a (nonnegative) subsolution.

Now, Bernstein's method relies on the step of multiplying $v$ by a cutoff function $\nu(x,u) : B_1 \times \mathbb{R} \to \mathbb{R}$. Let $w : B_1 \to \mathbb{R}$ be defined by $w(x) := \nu(x,u(x))v(x)$. Then, since $v$ is a subsolution and $\nu$ will be a nice cutoff, we have by the maximum principle for subsolutions (Theorem 3.1) that $w$ achieves a maximum on the interior of $B_1$. At this interior maximum, we must have that

$$0 = \partial_i w = \partial_\eta v + \eta\partial_v$$

and furthermore that the Hessian of $w$

$$\partial_i\partial_j w = \partial_i\partial_j\eta v + \partial_i\eta\partial_j v + \partial_j\eta\partial_i v + \eta\partial_i\partial_j v = \partial_i\partial_j\eta v + \eta\left(\partial_i\partial_j v - w\frac{\partial_i v\partial_j v}{v}\right)$$

is nonpositive at the maximum.

Since $g^{ij}\partial_i\partial_j v \geq \frac{2}{v}g^{ij}\partial_i v\partial_j v$ by (4.6) we have that this gives

$$\begin{aligned} 0 &\geq g^{ij}\partial_i\partial_j w \\ &= g^{ij}\partial_i\partial_j\eta v + \eta\left(g^{ij}\partial_i\partial_j v - \frac{2}{v}g^{ij}\partial_i\partial_j v\right) \\ &\geq g^{ij}\partial_i\partial_j\eta \end{aligned}$$

so if we shift $U$ by a constant so that $u \geq 0$ on $B_1$ and set

$$\eta(x,u(x)) := e^{C(2A(1-|x|^2)-u(x))_+} - 1$$

where $C$ is a large constant depending on $A$, we have that $\eta$ is a compact perturbation with $\eta(0) \neq 0$. By computing $g^{ij}\partial_i\partial_j\eta$ for this specific $\eta$ we get that, at the maximum,

$$g^{ij}\partial_i\partial_j w \geq \frac{1}{1+|\nabla u|^2}\left(|\nabla u|^2 - A|\nabla u \cdot x| - \frac{A}{C}(1+|\nabla u|^2)\right)$$

Since $|x \cdot \nabla u| \leq \frac{\nabla u}{|\nabla u|} \cdot \nabla u = |\nabla u|$, we get that

$$g^{ij}\partial_i\partial_j w \geq \frac{(1-A/C)|\nabla u|^2 - A|\nabla u| - A/C}{1+|\nabla u|^2}$$

so with $|\nabla u|$ very large we would get, with $C > A$, that $g^{ij}\partial_i\partial_j w > 0$ in a neighborhood of the maximum. But this implies that $w$ is a subsolution in a neighborhood of its maximum, which is a contradiction. Therefore we have that $|\nabla u|$ cannot be too large; specifically, we have that $\eta(0, A)v(0) \leq \max(\eta v) \leq C(A)$, as desired. □

Using the estimate of Theorem 4.3 we can prove the existence of a unique solution to the minimal surface equation on $B_1$ for merely continuous boundary data:

**Theorem 4.7.** *Let $\varphi \in C^0(\partial B_1)$. Then there exists a unique function $u : B_1 \to \mathbb{R}$ such that $u$ solves the minimal surface equation $\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2}) = 0$ and such that $u = \varphi$ on $\partial B_1$.*

*Proof.* Let $\rho^\epsilon \in C^\infty(\mathbb{R}^d)$ be a standard mollifier, and denote $\varphi^\epsilon := \varphi * \rho^\epsilon$. Then for all $\epsilon > 0$ we have that $\varphi^\epsilon$ is smooth, so by Theorem 3.3 we have that there exists a unique $u^\epsilon : B_1 \to \mathbb{R}$ that minimizes $J(u) := \int_{B_1} \sqrt{1+|\nabla u|^2}$ and that $u^\epsilon$ is smooth on the interior. Then since $u^\epsilon$ is certainly both a super- and sub-solution for all $\epsilon > 0$, we have by the comparison principle in Theorem 3.1 that for all $\alpha, \beta > 0$

$$\|u^\alpha - u^\beta\|_{L^\infty(B_1)} = \|u^\alpha - u^\beta\|_{L^\infty(\partial B_1)} = \|\varphi^\alpha - \varphi^\beta\|_{L^\infty(\partial B_1)} \tag{4.8}$$

and since $\varphi^\epsilon \to \varphi$ uniformly as $\epsilon \to 0$ we have therefore that $u^\epsilon$ is uniformly bounded for all $\epsilon > 0$. Let $K \subset \operatorname{Int}(B_1)$ be compact. Then by Theorem 4.3 and the fact that $u^\epsilon$ is uniformly bounded there exists a constant $C_K$ such that $\|\nabla u^\epsilon\|_{L^\infty(K)} \leq C_K$ for all $\epsilon > 0$, and therefore we have that $(1+|\nabla u^\epsilon|^2)^{-1/2}$ is uniformly elliptic on $K$, bounded between $(1+C_K^2)^{-1/2}$ and 1. Therefore by the Schauder Theorem (Theorem 2.11) we have the estimate

$$\|u^\epsilon\|_{C^{1,\alpha}(K)} \leq C\|u^\epsilon\|_{L^2(B_1)} \leq C\sqrt{|B_1|}\|u^\epsilon\|_{L^\infty(B_1)}$$

Then by the higher-order Schauder theorem (since a uniform $C^{1,\alpha}$ estimate for $u^\epsilon$ gives a uniform $C^{1,\alpha}$ estimate for the coefficient $\sqrt{1+|\nabla u^\epsilon|^2}$) we have a uniform $C^{2,\alpha}$ estimate

$$\|u^\epsilon\|_{C^{2,\alpha}(K)} \leq C'\|u^\epsilon\|_{L^\infty(B_1)}$$

Therefore the $u^\epsilon$ are bounded in $C^{2,\alpha}_{loc}$, which is sufficient to pass to a sequence $\epsilon_k \to 0$ such that $u^{\epsilon_k}$ converges in $C^{2,\alpha}_{loc}$, and therefore converges in $C^2_{loc}$ to some $u^0$. Convergence in $C^2_{loc}$ is enough to guarantee that $u^0$ inherits solving the minimal surface equation $\operatorname{div}(\nabla u^0/\sqrt{1+|\nabla u^0|^2}) = 0$, and that $u^0$ is smooth. Since $u^0$ solves the minimal surface equation, we have the same comparison principle as in (4.8) and can say also that $u^0 = \varphi$ on $\partial B_1$, as desired. □

4.2. **The Bernstein Problem in $\mathbb{R}^2$ .** The Bernstein problem asks, 'are all globally defined minimal graphs necessarily hyperplanes?' Results from Bernstein himself in the second decade of the 20th century, along with results from papers by Fleming, De Giorgi, Almgren, and Simons in the 1960s showed that the answer is **yes** in $\mathbb{R}^d$ for $1 \leq d \leq 8$, i.e. that a globally-defined function $u : \mathbb{R}^{d-1} \to \mathbb{R}$ such that graph$(u) \subset \mathbb{R}^d$ is a minimal surface is necessarily a degree-one polynomial if $d \leq 8$. A paper by Bombieri, De Giorgi, & Giusti in 1969 proved the converse, that in fact the answer to the Bernstein problem is **no** for $d \geq 9$, i.e. that for each $d \geq 9$ there exists $u : \mathbb{R}^{d-1} \to \mathbb{R}$ where graph$(u) \subset \mathbb{R}^d$ is minimal but not a hyperplane.

A logical approach to the Bernstein problem is to look at the curvature of a minimal surface $\Sigma$. Since an entire graph with zero curvature must be a plane, if we could bound the curvature of a minimal surface $\Sigma \subset \mathbb{R}^d$ this might allow us to argue that it is a hyperplane. As it turns out, using a variational technique we can prove the following lemma:

**Lemma 4.9.** *Let $u : \mathbb{R}^d \to \mathbb{R}$ solve the minimal surface equation $Mu = 0$ on the whole space (as in (1.3)). Assume further that $u \in C^2$. Then for all $\varphi \in C_c^\infty(\Sigma)$ we have the following interior gradient estimate:*

$$\int_\Sigma \left( \sum_{i=1}^d \kappa_i^2 \right) \varphi^2 \leq \int_\Sigma |\nabla \varphi|^2$$

*where $\kappa_i$ are the principal curvatures of $\Sigma$.*

If we perturb a surface $\Sigma$ by $\epsilon \varphi \nu(x)$, where $\epsilon > 0$ is small, $\varphi \in C_c^\infty(\Sigma)$, and $\nu(x)$ is the normal to $\Sigma$ at $x \in \Sigma$, then since $\Sigma$ is a minimal surface we have that

(4.10)
$$\frac{\partial}{\partial \epsilon}\left[ \int_{\Sigma + \epsilon\varphi\nu(x)} \sqrt{1 + |\nabla u|^2} \right]_{\epsilon=0} = 0, \text{ and } \frac{\partial^2}{\partial \epsilon^2}\left[ \int_{\Sigma + \epsilon\varphi\nu(x)} \sqrt{1 + |\nabla u|^2} \right]_{\epsilon=0} \geq 0$$

We prove Lemma 4.9 by calculating these derivatives and plugging in $\epsilon = 0$.

*Proof.* (Lemma 4.9) The change of variables $(x \mapsto x + \epsilon\varphi\nu(x))$ in $\mathbb{R}^d$ has determinant $\sqrt{1 + \epsilon^2 |\nabla \varphi|^2} \prod_{i=1}^d (1 - \epsilon\varphi\kappa_i)$, and expanding the second factor

$$\prod_{i=1}^d (1 - \epsilon\varphi\kappa_i) = 1 - \epsilon\varphi \sum_{i=1}^d \kappa_i + \frac{1}{2}\epsilon^2\varphi^2 \left( \left( \sum_{i=1}^d \kappa_i \right)^2 - \sum_{i=1}^d \kappa_i^2 \right) + O(\epsilon^3)$$

Denoting the mean curvature $H = \frac{1}{d}\sum_{i=1}^d \kappa_i$ and the sum of squares of the principle curvatures $c^2 = \sum_{i=1}^d \kappa_i^2$ we can differentiate the change of variables determinant

$$\begin{aligned}
&\frac{\partial}{\partial \epsilon}\left[ \sqrt{1 + \epsilon^2|\nabla\varphi|^2} \prod_{i=1}^d (1 - \epsilon\varphi\kappa_i) \right] \\
&= \frac{\partial}{\partial \epsilon}\left[ \sqrt{1 + \epsilon^2|\nabla\varphi|^2} \left( 1 - d\epsilon\varphi H + \frac{1}{2}\epsilon^2\varphi^2(d^2H^2 - c^2) + O(\epsilon^3) \right) \right] \\
&= \frac{\epsilon|\nabla\varphi|^2}{\sqrt{1 + \epsilon^2|\nabla\varphi|^2}} \left( 1 - d\epsilon\varphi H + \frac{1}{2}\epsilon^2\varphi^2(d^2H^2 - c^2) + O(\epsilon^3) \right) \\
&\qquad + \sqrt{1 + \epsilon^2|\nabla\varphi|^2} \Big( - d\varphi H + \epsilon\varphi^2(d^2H^2 - c^2) + O(\epsilon^2) \Big)
\end{aligned} \tag{4.11}$$

so considering the first variational equality in (4.10) we must have that (4.11) is zero at $\epsilon = 0$, so

$$-d\varphi H = 0 \implies H = 0 \tag{4.12}$$

Then, differentiating (4.11) again with respect to $\epsilon$, we get

$$\begin{aligned}
&\frac{\partial^2}{\partial \epsilon^2}\left[\sqrt{1+\epsilon^2|\nabla\varphi|^2}\prod_{i=1}^{d}(1-\epsilon\varphi\kappa_i)\right] \\
&= \left(\frac{|\nabla\varphi|^2}{\sqrt{1+\epsilon^2|\nabla\varphi|^2}} - \frac{\epsilon^2|\nabla\varphi|^4}{(1+\epsilon|\nabla\varphi|^2)^{3/2}}\right) \\
&\quad \cdot \left(1 - d\epsilon\varphi H + \frac{1}{2}\epsilon^2\varphi^2(d^2H^2 - c^2) + O(\epsilon^3)\right) \\
&\quad + \frac{\epsilon|\nabla\varphi|^2}{\sqrt{1+\epsilon^2|\nabla\varphi|^2}}\left(-d\varphi H + \epsilon\varphi^2(d^2H^2 - c^2) + O(\epsilon^2)\right) \\
&\quad + \frac{\epsilon|\nabla\varphi|^2}{\sqrt{1+\epsilon^2|\nabla\varphi|^2}}\left(-d\varphi H + \epsilon\varphi^2(d^2H^2 - c^2) + O(\epsilon^2)\right) \\
&\quad + \sqrt{1+\epsilon^2|\nabla\varphi|^2}\left(\varphi^2(d^2H^2 - c^2) + O(\epsilon)\right)
\end{aligned} \tag{4.13}$$

and considering now the second variational inequality in (4.10) we must have that (4.13) is nonnegative at $\epsilon = 0$. Combined with the fact that $H = 0$ from (4.12), this tells us that

$$|\nabla\varphi|^2 - \varphi^2 c^2 \geq 0$$

□

This gives us immediately that the Bernstein problem has an affirmative answer in $d = 2$. For it is the case that in $\mathbb{R}^2$ we can define a sequence of functions $\{\varphi_N\} \subset C_c^\infty(\mathbb{R}^3)$ such that $\varphi_N \to 1$ pointwise and $|\nabla\varphi_N| \to 0$ uniformly, and such that $\text{supp}(\varphi_N) \to \mathbb{R}^3$. Combined with Lemma 4.9, this lets us bound $\int_\Sigma c^2$ above by an arbitrarily small number, thus implying that $\Sigma$ has zero curvature. This proof is originally due to [4].

**Theorem 4.14.** *(Bernstein's problem in $\mathbb{R}^2$) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a solution to the minimal surface equation on all of $\mathbb{R}^2$. Then $u$ is a plane, i.e. $u(x, y) = \alpha x + \beta y + \gamma$ for constants $\alpha, \beta, \gamma \in \mathbb{R}$.*

*Proof.* Denote $\Sigma := \text{graph}(u)$. For $R > 0$ define $\varphi_R : \mathbb{R}^3 \to \mathbb{R}$ by

$$\varphi_R(x) := \begin{cases} 1 & |x|^2 \leq R \\ 2 - 2\log(|x|)/\log R & R < |x|^2 \leq R^2 \\ 0 & R^2 < |x|^2 \end{cases}$$

Then, with $\nabla_\Sigma$ denoting the gradient with respect to the surface $\Sigma$, we have that

$$\Big|\nabla_\Sigma |x|\Big| \leq \Big|\nabla |x|\Big| = 1$$

since the distance to the origin $|x|$ has derivative 1 where the tangent plane to $\Sigma$ intersects the origin, and is less than 1 if $\Sigma$ has any wrinkles or bends and thus slows our distance from the origin when travelling along its surface. With this, since $\varphi_R$ is radially symmetric with

$$\frac{\partial}{\partial|x|}\varphi_R = \begin{cases} 0 & |x|^2 \leq R \text{ or } R^2 < |x|^2 \\ \frac{-2}{|x|\log R} & \text{otherwise} \end{cases}$$

we have that

$$|\nabla_\Sigma \varphi_R| = \left|\frac{\partial}{\partial|x|}\varphi_R\right| \cdot \left|\nabla_\Sigma |x|\right| \leq \frac{2}{|x|\log R} \tag{4.15}$$

Therefore by Lemma 4.9 we have that

$$\begin{aligned}
\int_{B_{\sqrt{R}}\cap\Sigma} c^2 &\leq \int_\Sigma \varphi_R^2 c^2 && (\varphi_R \equiv 1 \text{ on } B_{\sqrt{R}}) \\
&\leq \int_\Sigma |\nabla_\Sigma \varphi_R|^2 && (\text{by Lemma 4.9}) \\
&\leq \frac{4}{(\log R)^2}\int_{B_R\cap\Sigma} |x|^{-2} && (\text{by (4.15)}) \\
&\leq \frac{4}{(\log R)^2}\sum_{i=(\log R)/2}^{\log R}\int_{B_{e^i}\setminus B_{e^{i-1}}\cap\Sigma} |x|^{-2} \\
&\leq \frac{4}{(\log R)^2}\sum_{i=(\log R)/2}^{\log R} 2\pi e^2 \leq \frac{4\pi e^2}{\log R}
\end{aligned}$$

(We assume $\log R, (\log R)/2 \in \mathbb{N}$ for simplicity's sake, one could just take integer parts). Thus, taking $R \to \infty$ we get that $\int_\Sigma c^2 = 0$, so $\Sigma$ is planar. □

## Acknowledgements

Thank you to Peter May for organizing the REU and providing detailed feedback on all of the papers. Thank you to David Bowman, for an entire year of chalkboard explanations and swift, thorough emails. Thank you also to the 61st Street Farmers Market, Anne Sexton, my parents, George F. Handel, and Holden's Breville espresso machine.

## References

[1] Evans, Lawrence C. *Partial Differential Equations: Second Edition.* American Mathematical Society. 1998.

[2] Gilbarg, David and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer Berlin, Heidelberg. 2001.

[3] Mooney, Connor. *Notes on Elliptic Partial Differential Equations.* $https://www.math.uci.edu/\sim mooneycr/$. 2012.

[4] Simon, L. *The minimal surface equation*, Geometry, V, Encyclopaedia of Mathematical Sciences 90. Springer, Berlin. 1997.