AN INTRODUCTION TO THE FUNDAMENTAL GROUP

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Abstract. The fundamental group of a topological space categorizes the structure of loops present in the space. It provides essential information about the shape of a topological space; in particular, it is instrumental in studying its holes. In this paper, we will define the fundamental group and demonstrate some of its essential properties and applications, including a proof of the Brouwer fixed point theorem. Moreover, we will discuss the Seifert-Van Kampen theorem, which provides a tool to compute fundamental groups. Finally, as an application of the Seifert-Van Kampen theorem, we will compute the fundamental groups of two topological spaces: the three-dimensional sphere and the figure-eight.

Contents

1. Paths and homotopies

Definition 1.1. A **pointed topological space** (X, x_0) is a topological space equipped with a point $x_0 \in X$. The point x_0 is called the basepoint of X.

Definition 1.2. A **continuous map between two topological spaces** (X, x_0) and (Y, y_0) is a continuous map $f\colon (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

From this point onwards, we will assume that all spaces are topological.

Definition 1.3. A **path** in a space X is a continuous map $f\colon [0, 1] \to X$. The points $f(0) = x_0$ and $f(1) = x_1$ are called the **endpoints** of the path. The **inverse path** of the path f, denoted by f^{-1}, is given by $f^{-1}(t) = f(1 - t)$. A **loop** is a path with equal endpoints.

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Definition 1.4. A space X is called **path-connected** if, for any two points $x, y \in X$, there exists a path in X with x and y as endpoints.

Definition 1.5. (Product of paths) Let $f\colon [0,1] \to X$ and $g\colon [0,1] \to X$ be two paths in a space X. We define the **product** of f and g, denoted by $f \star g$, by

$$(f \star g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Note that $f \star g$ is a path.

Remark 1.6. Observe that the path product operation is associative.

Definition 1.7. A **path homotopy** in a space X is a continuous map $F\colon [0,1] \times [0,1] \to X$ such that

$$F(0,t) = F(0,0) \text{ and } F(1,t) = F(1,0)$$

for all $t \in [0,1]$.

Definition 1.8. Let f and g be two paths in a space X that share the same endpoints; that is, such that $f(0) = g(0)$ and $f(1) = g(1)$. We will say f and g are **homotopic** (and write $f \sim g$) if there exists a path homotopy $F\colon [0,1] \times [0,1] \to X$ satisfying

$$F(s,0) = f(s) \text{ and } F(s,1) = g(s)$$

for all $s \in [0,1]$. We will call such a homotopy F as a homotopy between f and g.

Intuitively, two paths are homotopic if one can be continuously deformed into the other. We may interpret the values $F(s,0)$ as the initial path, which is deformed into $F(s,1)$, the second path. Now, given three paths f, g, and h, one may observe that if f and g are homotopic and so are g and h, then f is homotopic to h. This observation is made precise by the following proposition:

Proposition 1.9. *Homotopy defines an equivalence relation on paths.*

Proof. Let X be a space. We must prove that the homotopy relation on paths in X is reflexive, transitive, and symmetric.

Reflexivity: Given a path f in X, we define the constant homotopy as the homotopy $F_c\colon [0,1] \times [0,1] \to X$ given by $F_f(s,t) = f(s)$ for all $s, t \in [0,1]$. Using the constant homotopy, it holds that $f \sim f$, by Definition 1.5. Thus, $\sim$ is reflexive.

Symmetry: Let f and g be two paths in X such that $f \sim g$ and let F be a homotopy between f and g. We will define the reverse homotopy F^{-1} as the homotopy satisfying $F^{-1}(s,t) = F(s, 1-t)$ for all $s, t \in [0,1]$. By Definition 1.5, F^{-1} is a homotopy between g and f, which implies $g \sim f$.

Transitivity: Let f, g, and h be three paths in X such that $f \sim g$ and $g \sim h$ and let F and G be homotopies between f and g and between g and h, respectively. We will define the homotopy H by

$$H(s,t) = \begin{cases} F(s,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(s,2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

By Definition 1.5, H is a homotopy between f and h. Because $\sim$ is symmetric, it holds that $h \sim f$. $\square$

Definition 1.10. Given a path f in a space X, we denote the **homotopy class** of f by $[f]$.

2. The Fundamental group

In this section, we will introduce the fundamental group, which describes the loops contained in a topological space up to homotopy.

2.1. **Definition.**

Lemma 2.1. *Let (X, x_0) be a pointed space and let f_1, g_1, f_2, and g_2 be loops based at x_0 such that $f_1 \sim f_2$ and $g_1 \sim g_2$. Then, $f_1 \star g_1 \sim f_2 \star g_2$.*

Proof. Let F be a homotopy between f_1 and f_2 and G be a homotopy between g_1 and g_2. Let $H\colon [0,1] \times [0,1] \to X$ be a continuous map given by

$$H(s,t) = \begin{cases} F(2s,t) & \text{if } 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

The continuity of H is given by the continuity of F and G. Now, note that $H(s,0) = F(2s,0) = f_1(2s)$ for $s \in [0, \frac{1}{2}]$. Moreover, $H(s,0) = G(2s-1,0) = g_1(2s-1)$ for $s \in [\frac{1}{2}, 1]$. Thus, by Definition 1.5,

$$H(s,0) = (f_1 \star g_1)(s).$$

Similarly,

$$H(s,1) = (f_2 \star g_2)(s).$$

It follows that H is a homotopy between $f_1 \star g_1$ and $f_2 \star g_2$, and so $f_1 \star g_1 \sim f_2 \star g_2$. $\square$

Definition 2.2. Let (X, x_0) be a pointed space and let f and g be loops based at x_0. Define the **product of the homotopy classes** of f and g by

$$[f] \cdot [g] = [f \star g].$$

Remark 2.3. By Lemma 2.1, Definition 2.2 is well-defined. That is, the product of two loop homotopy classes is independent of the choice of loop in each class.

Definition 2.4. Let (X, x_0) be a pointed space. The **fundamental group** of (X, x_0), denoted by $\pi_1(X, x_0)$, is the group whose set contains all homotopy classes of loops based at x_0 and whose operation is the product between two homotopy classes, as described in Definition 2.2.

Theorem 2.5. *Let (X, x_0) be a pointed space. Then, $\pi_1(X, x_0)$ is a group.*

Proof. We begin by defining the identity loop as the loop $id\colon [0,1] \to X$ defined by $id(t) = x_0$ for all $t \in [0,1]$. The identity element of $\pi_1(X, x_0)$ is thus defined as $[id]$. We must now prove that $\pi_1(X, x_0)$ satisfies each of the group axioms.

Identity: Let γ be a loop based at x_0. We want to show that

$$[\gamma] \cdot [id] = [id] \cdot [\gamma] = [\gamma].$$

By Definition 2.2,

$$[id] \cdot [\gamma] = [id \star \gamma].$$

Now, let $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(s,t) = \begin{cases} x_0 & \text{if } 0 \leq s \leq \frac{t}{2} \\ \gamma(\frac{s-\frac{t}{2}}{1-\frac{t}{2}}) & \text{if } \frac{t}{2} \leq s \leq 1 \end{cases}.$$

Note that F is a homotopy between γ and $id \star \gamma$. Therefore,

$$[id] \cdot [\gamma] = [id \star \gamma] = [\gamma].$$

Similarly, $[\gamma] \cdot [id] = [\gamma]$.

Inverse: Let γ be a loop based at x_0. We will define the inverse of γ as the inverse loop γ^{-1}. Now, we must show that

$$[\gamma] \cdot [\gamma^{-1}] = [id].$$

Let $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(s,t) = \begin{cases} \gamma(2st) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \gamma^{-1}(2(s-\frac{1}{2})t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}.$$

Note that F is a homotopy between id and $\gamma \star \gamma^{-1}$. Therefore,

$$[\gamma \star \gamma^{-1}] = [id].$$

Because $[\gamma \star \gamma^{-1}] = [\gamma] \cdot [\gamma^{-1}]$, we have that $[\gamma] \cdot [\gamma^{-1}] = [id]$, as desired.

Associativity: Let γ_1, γ_2, and γ_3 be loops based at x_0. In order to demonstrate associativity for $\pi_1(X, x_0)$, we must show that

$$([\gamma_1] \cdot [\gamma_2]) \cdot [\gamma_3] = [\gamma_1] \cdot ([\gamma_2] \cdot [\gamma_3]).$$

By the associativity of the product of paths operation ($\star$), we have

$$\begin{aligned} &[(\gamma_1 \star \gamma_2) \star \gamma_3] = [\gamma_1 \star (\gamma_2 \star \gamma_3)] \\ \implies &[\gamma_1 \star \gamma_2] \cdot [\gamma_3] = [\gamma_1] \cdot [\gamma_2 \star \gamma_3] \\ \implies &([\gamma_1] \cdot [\gamma_2]) \cdot [\gamma_3] = [\gamma_1] \cdot ([\gamma_2] \cdot [\gamma_3]). \end{aligned}$$

Hence, associativity holds. $\square$

Definition 2.6. A space X is called **simply connected** if it is path connected and, for all points $x_0 \in X$, the fundamental group of $\pi_1(X, x_0)$ is trivial. That is, $\pi_1(X, x_0) = 0$.

2.2. Examples of fundamental groups.

Example 2.7. The fundamental group of $\mathbb{R}^n$, based at 0, is the trivial group. That is, $\pi_1(\mathbb{R}^n, 0) = 1$.

In order to check this fact, we may take $F(s,t) = f(s) \cdot t$ as a homotopy between any loop f based at 0 and the constant loop at 0. Then, it follows that any loop based at 0 is homotopic to the constant loop at 0, and so $\pi_1(\mathbb{R}^n, 0)$ is trivial.

Definition 2.8. The n-th dimensional disk, denoted my D^n, is defined by $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Similarly, the $(n+1)$-th dimensional sphere, denoted by S^n, is defined by $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

Example 2.9. The fundamental group of the n-dimensional disk, based at 0, is the trivial group. That is, $\pi_1(D^n, 0) = 1$.

This fact follows from a similar argument as the one made in Example 2.6. By taking $F(s,t) = f(s) \cdot t$ as a homotopy between a loop f based at 0 and the constant loop at 0, we may note that any loop in D^n is homotopic to the constant loop, and so $\pi_1(D^n, 0) = 1$.

Example 2.10. The fundamental group of the circle, based at 0, is isomorphic to the group of integers with addition as operation. That is,

$$\pi_1(S^1, 0) \cong (\mathbb{Z}, +).$$

From this result, we may observe that the homotopy classes of loops in S^1 are precisely those containing the loops formed by revolving clockwise in the circle an integer number of times. The fundamental group of the circle is instrumental in computing fundamental groups of other more complex spaces.

A demonstration of this statement can be found in [1], Chapter 1, Section 1.1.

2.3. **Properties.**

We now turn our attention to discussing two important properties of fundamental groups.

Lemma 2.11. *Let X be a path-connected space and let x_0, y_0 be two points in X. Then, $\pi_1(X, x_0) \cong \pi_1(X, y_0)$.*

Proof. Let $\rho\colon [0,1] \to X$ be a path with $\rho(0) = y_0$ and $\rho(1) = x_0$. Note that, for any loop γ based at x_0, the path

$$\rho \star \gamma \star \rho^{-1}$$

is a loop based at y_0. Using this observation, we will define $\phi\colon \pi_1(X, x_0) \to \pi_1(X, y_0)$ by

$$\phi([\gamma]) = [\rho \star \gamma \star \rho^{-1}],$$

for any loop γ based at x_0. We will now show that ϕ is well-defined and, furthermore, that it induces and isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$.

Part 1: We begin by showing that ϕ is well-defined. Let γ be a loop based at x_0 that is homotopic to another loop γ'. We wish to show that

$$\rho \star \gamma \star \rho^{-1} \sim \rho \star \gamma' \star \rho^{-1}.$$

Let $F\colon [0,1] \times [0,1] \to X$ be a homotopy between γ and γ'. For each $t \in [0,1]$, define $F_t\colon [0,1] \to X$ by $F_t(s) = F(s,t)$. Then, for any $t \in [0,1]$, F_t is a loop based at x_0. Next, for each $t \in [0,1]$, define $H_t\colon [0,1] \to X$ by

$$H_t = \rho \star F_t \star \rho^{-1}.$$

Finally, let $H\colon [0,1] \times [0,1] \to X$ be defined by $H(s,t) = H_t(s)$. Note that H is continuous since H_t is continuous for each $t \in [0,1]$. Moreover,

$$H(s,0) = H_0(s) = (\rho \star \gamma \star \rho^{-1})(s)$$

and

$$H(s,1) = H_1(s) = (\rho \star \gamma' \star \rho^{-1})(s).$$

Thus, H is a homotopy between $\rho \star \gamma \star \rho^{-1}$ and $\rho \star \gamma' \star \rho^{-1}$. It follows that ϕ is well-defined.

Part 2: We will now show that ϕ induces an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$. Let $\psi\colon \pi_1(X, y_0) \to \pi_1(X, x_0)$ be defined by

$$\psi([\gamma]) = [\rho^{-1} \star \gamma \star \rho],$$

for each loop γ based at y_0. Note that ψ is well-defined for analogous reasons as the ones shown in Part 1. We now claim that $\phi \circ \psi = id$, where id denotes the identity homomorphism. In order to show this, let γ be a loop based at y_0. Then,

$$\begin{aligned}
\phi(\psi([\gamma])) &= \phi([\rho^{-1} \star \gamma \star \rho]) \\
&= [\rho \star (\rho^{-1} \star \gamma \star \rho) \star \rho^{-1}] \\
&= [(\rho \star \rho^{-1}) \star \gamma \star (\rho \star \rho^{-1})], \text{ by the associativity of } \star.
\end{aligned}$$

We now claim that $\rho \star \rho^{-1}$ is homotopic to id, the identity loop. Hence, by Lemma 2.1,

$$[(\rho \star \rho^{-1}) \star \gamma] = [id \star \gamma] = [\gamma].$$

Thus,

$$\phi(\psi([\gamma])) = [(\rho \star \rho^{-1}) \star \gamma \star (\rho \star \rho^{-1})] = [\gamma \star (\rho \star \rho^{-1})] = [\gamma].$$

It follows that $\phi \circ \psi = id$. Similarly, $\psi \circ \phi = id$. Therefore, ϕ induces an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$, and so $\pi_1(X, x_0) \cong \pi_1(X, y_0)$. □

According to Lemma 2.11, the fundamental group of any path-connected space is independent of the choice of basepoint, up to isomorphism. When convenient, we will refer to the fundamental group of a path-connected space X as $\pi_1(X)$.

Theorem 2.12. *(Homomorphism induced by continuous map) Let $f\colon (X, x_0) \to (Y, y_0)$ be a continuous map between two pointed spaces. Let $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be defined by $f_*([\gamma]) = [f \circ \gamma]$ for any loop γ in X based at x_0. Then, f_* is a well-defined homomorphism between $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.*

Proof. **Part 1**: We begin by proving that f_* is well-defined. Let $\gamma\colon [0,1] \to X$ be a loop in X based at x_0. Note that $f \circ \gamma$ is a loop in Y based at y_0 since, by definition, f is continuous and $f(x_0) = y_0$. Now, let γ' be a loop in X such that $\gamma \sim \gamma'$. We wish to show that $f \circ \gamma \sim f \circ \gamma'$. For this, let $F\colon [0,1] \times [0,1] \to X$ be a homotopy between γ and γ' and define $H\colon [0,1] \times [0,1] \to Y$ by

$$H(s,t) = f(F(s,t)).$$

Since f and F are both continuous, H is also continuous. Moreover,

$$H(s,0) = f(F(s,0)) = f(\gamma(s))$$

and

$$H(s,1) = f(F(s,1)) = f(\gamma'(s)).$$

Hence, H is a homotopy between $f \circ \gamma$ and $f \circ \gamma'$, which implies $f \circ \gamma \sim f \circ \gamma'$. We may therefore conclude that f_* is well-defined.

Part 2: We now wish to show that f_* is a homomorphism. Let γ_1 and γ_2 be two loops in X based at x_0. We want to prove that $f_*([\gamma_1] \cdot [\gamma_2]) = f_*([\gamma_1)]) \cdot f_*([\gamma_2])$. Note that

$$\begin{aligned} f_*([\gamma_1] \cdot [\gamma_2]) &= f_*([\gamma_1 \star \gamma_2]), \text{ by Definition 2.2} \\ &= [f \circ (\gamma_1 \star \gamma_2)], \text{ by the definition of } f_*. \end{aligned}$$

Moreover, we also have that

$$\begin{aligned} f_*([\gamma_1]) \cdot f_*([\gamma_2]) &= [(f \circ \gamma_1)] \cdot [(f \circ \gamma_2)], \text{ by the definition of } f_* \\ &= [(f \circ \gamma_1) \star (f \circ \gamma_2)], \text{ by Definition 2.2} \\ &= [f \circ (\gamma_1 \star \gamma_2)], \text{ by Definition 1.5.} \end{aligned}$$

Thus, $f_*([\gamma_1] \circ [\gamma_2]) = f_*([\gamma_1]) \circ f_*([\gamma_2])$, which implies f_* is a homomorphism. □

2.4. **Application: Brouwer's fixed point theorem.**

By utilizing the previously shown results on the fundamental groups of the circle and the disk, we are able to demonstrate Brouwer's fixed point theorem.

Theorem 2.13. *(Brouwer's fixed point theorem) Let $f\colon D^2 \to D^2$ be a continuous map. Then, there exists a point $x \in D^2$ such that $f(x) = x$.*

Proof. Assume, for the sake of contradiction, that no such point exists. Then, we may define the map $g\colon D^2 \to S^1$ as follows: for each $x \in D^2$, let $g(x)$ be the point of intersection between the edge of the disk (the circle, S^1) and the ray starting at x and moving in the direction of $f(x)$.

We now claim that g is continuous. Indeed, if the point x is moved by a small distance, then, by continuity, $f(x)$ is moved by a small distance and, consequently, the same holds true for $g(x)$. Now, observe that we have the following commutative diagram:

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[dr, "j"'] & D^2 \arrow[d, "g"] \\ & S^1 \end{tikzcd}.$$

In the diagram, $i\colon S^1 \to D^2$ is the homomorphism induced by the inclusion of S^1 into D^2 and $j\colon S^1 \to S^1$ is the identity. The commutativity of the diagrams holds since, for any point x at the edge of the disk, $g(x) = x$.

Now, let $i_*\colon \pi_1(S^1) \to \pi_1(D^2)$ and $g_*\colon \pi_1(D^2) \to \pi_1(S^1)$ be the homomorphisms induced by i and g, respectively, as defined in Theorem 2.12. Let γ be a loop in S^1. We then have

$$\begin{aligned} g_*(i_*([\gamma])) &= g_*([i \circ \gamma]), \text{ by the definition of } i_* \\ &= g_*([\gamma]) \\ &= [g \circ \gamma], \text{ by the definition of } g_* \\ &= [\gamma], \text{ since } \gamma \in S^1. \end{aligned}$$

Therefore, we have the following commutative diagram:

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[dr, "j_*"'] & \pi_1(D^2) \arrow[d, "g_*"] \\ & \pi_1(S^1) \end{tikzcd},$$

where $j_*\colon \pi_1(S^1) \to \pi_1(S^1)$ is the identity. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) \cong 1$, the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_*"] \arrow[dr, "j_*"'] & 1 \arrow[d, "g_*"] \\ & \mathbb{Z} \end{tikzcd}$$

Note, however, that the identity homomorphism cannot be realized as a composition of a homomorphism from $\mathbb{Z}$ to the trivial group and a homomorphism from the trivial group to $\mathbb{Z}$. We have therefore arrived at a contradiction, which implies there exists a point $x \in D^2$ such that $f(x) = x$. □

3. Computing fundamental groups

In this section, we will discuss the Seifert-Van Kampen theorem, which provides a method to compute fundamental groups of more complex topological spaces. In order to state the theorem, we must first define free products on groups.

3.1. Free products.

Definition 3.1. Let G and H be groups. A **word** in G and H is an element of the form

$$f_1 f_2 \cdots f_n$$

such that each f_i is an element of either G or H. A **reduction operation** in a word $f_1 f_2 \cdots f_n$ consists of performing the following operation: if two consecutive elements f_i and f_{i+1} are elements of a same group, remove f_i and f_{i+1} and replace them by $f_i f_{i+1}$. A **reduced word** is a word in which no reduction operation can be performed.

Remark 3.2. Equivalently, a reduced word in G and H is a word $f_1 f_2 \cdots f_n$ whose elements alternate between elements of G and elements of H such that, for any two consecutive elements f_i and f_{i+1}, f_i and f_{i+1} are not elements of the same group.

Definition 3.3. Given two groups G and H, the **free product** of G and H, denoted $G \star H$, is the group whose set contains all reduced words in G and H and whose operation consists of concatenation followed by reduction operations, until the resulting element is a reduced word.

Note that the free product is, indeed, a group: the empty word may be taken as the identity element. The inverse of a word $f_1 f_2 \cdots f_n$ may be defined as $f_n^{-1} f_{n-1}^{-1} \cdots f_1^{-1}$, since performing an operation between both words results in the empty word. Finally, associativity holds by the definition of the operation of $G \star H$.

Example 3.4. The free product $\mathbb{Z} \star \mathbb{Z}$ consists of all words of the form

$$x^{a_1} y^{b_1} x^{a_2} y^{b_2} \cdots .$$

3.2. The Seifert-Van Kampen theorem.

Throughout this section, let X, U, and V be path connected spaces with a common point x_0 such that $X = U \cup V$, $U \cap V$ is path-connected, and U and V are open in X. Define the inclusion homomorphisms i_u, i_v, j_u, and j_v as follows:

- $i_u : U \to X$ as the inclusion of U into X;
- $i_v : V \to X$ as the inclusion of V into X;
- $j_u : U \cap V \to U$ as the inclusion of $U \cap V$ into U;
- $j_v : U \cap V \to V$ as the inclusion of $U \cap V$ into V.

Furthermore, let i_{u*}, i_{v*}, j_{u*}, and j_{v*} be the homomorphisms between fundamental groups induced by each respective inclusion as defined in Theorem 2.12.

Theorem 3.5. *(Seifert-Van Kampen) Let $\phi\colon \pi_1(U, x_0) \star \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphism induced by i_{u*} and i_{v*}. That is, for a given word $u_1 v_1 \cdots u_n v_n$ in the free product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$,*

$$\phi(u_1 v_1 \cdots u_n v_n) = i_{u*}(u_1) i_{v*}(v_1) \cdots i_{u*}(u_n) i_{v*}(v_n).$$

Then, the following are true:

- *ϕ is surjective.*
- *The kernel of ϕ is the minimal normal subgroup N generated by all elements of the form $(j_{u*}(e)j_{v*}(e)^{-1})$, for all $e \in \pi_1(U \cap V, x_0)$.*

By the surjectivity of the homomorphism ϕ and the First isomorphism theorem, we obtain the following corollary:

Corollary 3.6. *The following holds:*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) \star \pi_1(V, x_0)/N.$$

The Seifert-Van Kampen theorem provides a method to compute the fundamental group of a space by decomposing it into simpler subspaces whose fundamental groups are already known. While a proof of the theorem will not be presented in this article, a full demonstration may be seen in [1], Chapter 1, Section 1.2.

3.3. **Applications of the Seifert-Van Kampen theorem.**
In this section, we will use the Seifert-Van Kampen theorem to compute the fundamental groups of the three-dimensional sphere (S^2) and of the figure-eight.

Example 3.7. The fundamental group of the three-dimensional sphere, S^2, is the trivial group. That is,

$$\pi_1(S^2) = 1.$$

In order to apply Van Kampen's theorem, we will decompose the sphere into its northern and southern hemispheres (denoted U and V, respectively) such that their intersection ($U \cap V$) is nonempty, as shown in the image below:

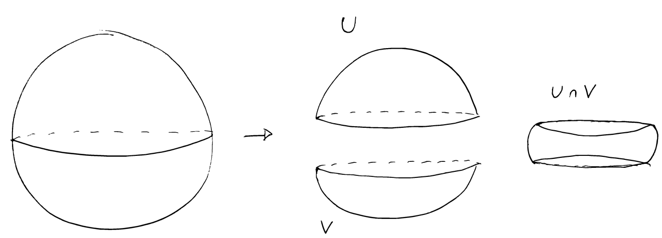

FIGURE 1. The sphere, U, V, and $U \cap V$

Note that the intersection of both hemispheres is an annulus. Therefore, U, V, and $U \cap V$ are path-connected. We now claim that both U and V are isomorphic to D^2. Hence,

$$\pi_1(U) = \pi_1(V) = 1.$$

By Van-Kampen's theorem, we have that $\pi_1(S^2)$ is generated by the free product of $\pi_1(U)$ and $\pi_1(V)$. Since both are trivial, we may conclude that $\pi_1(S^2)$ is also trivial.

Example 3.8. The fundamental group of the figure-eight is isomorphic to $\mathbb{Z} \star \mathbb{Z}$.

In this example, we will decompose the figure-eight into U and V as shown in the figure below, such that $U \cap V$ has shape similar to that of the letter "x":

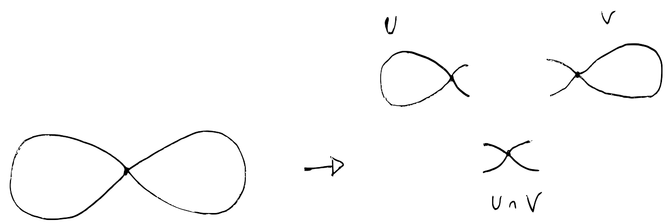

FIGURE 2. The figure-eight, U, V, and $U \cap V$

Next, observe that U, V, and $U \cap V$ are path connected. Moreover, we claim that U and V are homotopic to S^1 (indeed, the figure-eight is formed by joining two circles through a single point) and that $U \cap V$ is homotopic to a point. Thus,

$$\pi_1(U) \cong \pi_1(V) \cong \mathbb{Z}$$

and

$$\pi_1(U \cap V) = 1.$$

By the Seifert-Van Kampen theorem, then, $\pi_1(U \cup V) \cong \mathbb{Z} \star \mathbb{Z}/1 = \mathbb{Z} \star \mathbb{Z}$.

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References

[1] Allen Hatcher. Algebraic Topology. Cambridge University Press. 2001.
[2] Si Li, Yu Qiu. Introduction to Algebraic Topology. https://sili-math.github.io/AT2020/Algebraic-Topology.2020.pdf