ON EQUIVARIANT S-MODULES AND ORTHOGONAL SPECTRA

YANBO CHEN

Contents

1. Introduction

This article explores two distinct infrastructures addressing a longstanding issue: S-modules developed by Elmendorf, Kriz, Mandell, and May, and orthogonal spectra developed by Mandell and May. Both approaches offer various advantages. As the pioneering model in the closed symmetric monoidal category, constructing S-modules poses challenges, particularly evident in the semi-product construction. However, its standard model category ensures that all objects are fibrant. On the other hand, orthogonal spectra, while not the initial model from a diagrammatic approach (with symmetric spectra by Hovey, Shipley, and Smith taking that honor), combines the strengths of both S-modules and symmetric spectra. It is straightforward to construct, leveraging the convenience of starting from the category of finite-dimensional inner product spaces, itself a closed symmetric monoidal category, allowing for a symmetric and commutative smash product.

The article also delves into transitioning from standard cases to equivariant settings, introducing group actions of a compact Lie group G. The focus lies on providing a survey under the equivariant assumption. Both equivariant models are constructed in a coordinated fashion, guiding the creation of sphere objects associated with representations. Additionally, their model categories are direct and natural, defining weak equivalences in the realm of normal "stable equivalences." Ultimately, the model categories are Quillen equivalent, granting flexibility in choosing the contextually relevant model based on the crucial information presented at the survey's conclusion.

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2. Equivariant S-modules

2.1. **The construction of the symmetric monoidal category.** As we have seen in the Introduction, we need categories of spectra that are functors from symmetric monoidal categories to symmetric monoidal categories, in order to get a good internal smash product. Since we focus on the equivariant version of spectra, we should take the action of a group G into consideration. Here we give the construction of the universe $\mathscr{U}$ in terms of the direct sum of some G-representations.

Construction 2.1. A universe $\mathscr{U}$ for the group G contains countably infinite dimensional orthogonal representations of G, such that

(1) The trivial representation $\mathbb{R} \in \mathscr{U}$.
(2) Once a finite dimensional representation $V \subset \mathscr{U}$, then the infinite direct sum of V with itself also belong to $\mathscr{U}$.

Throughout this paper, we work with a complete, or genuine universe, i.e. $\mathscr{U}$ contains all of G's irreducible representations. We will explain the convenience to do so in the chapter equivariant orthogonal spectra. Now we can give the definition of G-prespectrum and G-spectrum by taking source from the universe and target from the based G-spaces.

Definition 2.2. (1) A **G-prespectrum** X consists of based G-spaces $X(V)$ for indexing G-spaces $V \subset \mathscr{U}$ and structure maps $\sigma : \Sigma^{W-V} X(V) \to X(W)$ for $V \subset W$ which is a based G-map, and σ is an identity once $V = W$. The evident transitivity diagram must commute when $V \subset W \subset Z$, i.e.

$$\begin{tikzcd}
\Sigma^{Z-V} X(V) = \Sigma^{Z-W}(\Sigma^{W-V} X(V)) \arrow[rr, "\sigma_{V,Z}"] \arrow[dr, "\Sigma^{Z-W}\sigma_{V,W}"'] & & X(Z) \\
& \Sigma^{Z-W} X(W) \arrow[ur, "\sigma_{W,Z}"'] &
\end{tikzcd}$$

commutes. A map $f : X \to Y$ of prespectra consists of based maps $f(V) : X(V) \to Y(V)$ that commute with the structure maps σ; f is a G-map if each $f(V)$ is a G-map between G-spaces.

(2) A **G-spectrum** is a G-prespectrum whose adjoint structure G-maps $\tilde{\sigma} : X(V) \to \Omega^{W-V} X(W)$ are homeomorphisms of G-spaces.

We denote the category of G-prespectra and G-spectra by $\mathscr{P}_G$ and $\mathscr{S}_G$. There is a spectrification G-functor $L : \mathscr{P}_G \to \mathscr{S}_G$ which is left adjoint to the forgetful functor $l : \mathscr{S}_G \to \mathscr{P}_G$. We can see this construction easily in the case that each adjoint structure map $\sigma_{V,W} : E(V) \to \Omega^{W-V} E(W)$ for the G-prespectrum E is an inclusion, we then define the spectrification LE of E to be

$$LE(V) = \operatorname{colim}_{V \subset W} \Omega^{W-V} E(W).$$

To underline the change of universes under the external smash product, consider the category $\mathscr{S}_G U$ of G-spectra indexed on the universe U. Write U^j for the direct sum of j copies of U. It's easy to find for positive integer j we have a external smash product $\wedge : (\mathscr{S}_G U)^j \to \mathscr{S}_G U^j$, explicitly,

$$(E_1 \wedge ... \wedge E_j)(V_1 \wedge ... \wedge V_j) = E_1(V_1) \wedge ... \wedge E_j(V_j).$$

Since our aim is to have a closed symmetric monoidal category, we should have internal smash product with its adjoint function functor. Without loss of generality, we need only define the internal smash product for two G-spectra, then the condition of finite internal product is given inductively. The idea is, since U is complete, then U^2 is isomorphic to U. If we have linear isometry f from U^2 to U, then f is an isometric isomorphism, and we can give one internal type smash product $f_*(E_1 \wedge E_2)$ explicitly by

$$f_*(E_1 \wedge E_2)(W) = (E_1 \wedge E_2)(f^{-1}(W))$$

The problem is, this kind of internal smash product depends on the choice of linear isometry. To deal with this problem, we consider all the linear isometries.

Construction 2.3.
(1) Define the set $\mathscr{I}(U, U')$ to be the set of linear isometries between G-universes U and U'. Here we do not need the isometries to be G-linear, but have G-actions on $\mathscr{I}(U, U')$ be conjugation.
(2) Given a space A and a map $\alpha : A \to \mathscr{I}(U, U')$. For a spectrum E indexed on the G-universe U, the semi-product $A \ltimes E$ is a spectrum indexed on the G-universe U'.

Here the space A with the map α coarsely gives the union of the internal type smash products as f we described above. If we let $\mathscr{L}(i) = \mathscr{I}(U^i, U)$ and consider the union of smash products

$$\mathscr{L}(2) \ltimes (E_1 \wedge E_2)$$

which is too big to be associative. As a result we have to quotient something. One problem is, we expect the single action from $\mathscr{L}(1)$ to E_i (if it exists) to be an isomorphism of U_i indexed on the same G-universe, but it actually includes a lot more. Then for the embedding $\mathscr{L}(1) \times \mathscr{L}(1)$ into $\mathscr{L}(2)$, we calculate the action of $\mathscr{L}(1)$ on each E_i respectively for repeatedly too many times. In conclusion, we choose the spectra which have the $\mathscr{L}(1)$ action on them, for such spectrum E, we denote $\mathscr{L}(1) \ltimes E$ by $\mathbb{L}E$. Before giving the normal definition of internal smash product we need, let us clear some properties and notions.

Property 2.4. The sign $\mathbb{L}$ specified by the $\mathscr{L}(1)$ action is actually a monad in the category of G-spectra indexed on the G-universe U. The product

$$\mu : \mathbb{L}\mathbb{L}E \cong (\mathscr{L}(1) \times \mathscr{L}(1)) \ltimes E \to \mathscr{L}(1) \ltimes E = \mathbb{L}E$$

is induced by the compositional product $\mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(1)$. The unit

$$\eta : E \cong \{1\} \ltimes E \to \mathscr{L}(1) \ltimes E = \mathbb{L}E$$

is induced by the evident inclusion $\{1\} \to \mathscr{L}(1)$. Both the product map and unit map are G-maps. It's easy to check that $\mathbb{L}$ is associative and unital which verify it to be a monad.

Property 2.5. A spectrum with an action of $\mathscr{L}(1)$, which we denote it by an $\mathbb{L}$-spectrum, specified by the G-map $\xi : \mathbb{L}M \to M$, is actually an $\mathbb{L}$-algebra. This is equivalent to say the following diagrams commutes :

$$\begin{tikzcd} \mathbb{L}\mathbb{L}M \arrow[r, "\mu"] \arrow[d, "\mathbb{L}\xi"'] & \mathbb{L}M \arrow[d, "\xi"] \\ \mathbb{L}M \arrow[r, "\xi"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mu"] \arrow[dr, "="'] & \mathbb{L}M \arrow[d, "\xi"] \\ & M \end{tikzcd}$$

And the morphism between the $\mathbb{L}$-algebra are G-maps $f : M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{L}M \arrow[r, "\mathbb{L}f"] \arrow[d, "\xi_M"'] & \mathbb{L}N \arrow[d, "\xi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Property 2.6. The linear isometries G-operad $\mathscr{L}(j)$ comes together with the structure map

$$\gamma : \mathscr{L}(k) \times \mathscr{L}(j_1) \times ... \times \mathscr{L}(j_k) \to \mathscr{L}(j_1 + ... + j_k)$$
$$\gamma(g; f_1, ..., f_k) = g \cdot (f_1 \oplus ... \oplus f_k).$$

Thus we have map $\gamma : \mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1) \to \mathscr{L}(2)$ which can be regarded as a right action of $\mathscr{L}(1) \times \mathscr{L}(1)$ on $\mathscr{L}(2)$.

Definition 2.7. Mimic the tensor product, we define the internal product of two $\mathbb{L}$-spectra E_1 and E_2 to be the coequalizer of

$$(\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)) \ltimes (E_1 \wedge E_2) \underset{\mathscr{L}(2)\ltimes(\xi\wedge\xi)}{\overset{\gamma\ltimes(E_1\wedge E_2)}{\rightrightarrows}} \mathscr{L}(2) \ltimes (E_1 \wedge E_2).$$

In the diagram the upper arrow is given by the right action we mentioned above, while the lower one is given by the structure map of $\mathbb{L}$-spectra. We denote the smash product defined in this way by $E_1 \wedge_{\mathscr{L}} E_2$.

Theorem 2.8. *The smash product of $\mathbb{L}$-spectra is associative. Formally, there is a natural isomorphism of $\mathbb{L}$-spectra*

$$(M \wedge_{\mathscr{L}} N) \wedge_{\mathscr{L}} P \cong M \wedge_{\mathscr{L}} (N \wedge_{\mathscr{L}} P).$$

Proof. Since for algebra over a monad we have a split coequalizer

$$\mathbb{L}\mathbb{L}E \underset{\mu}{\overset{\mathbb{L}\xi}{\rightrightarrows}} \mathbb{L}E \longrightarrow E,$$

we get a conclusion as in the tensor product of rings that $E \cong \mathscr{L}(1) \ltimes_{\mathscr{L}(1)} E$ for any $\mathbb{L}$-spectrum E. Then

$$\begin{aligned}(M \wedge_{\mathscr{L}} N) \wedge_{\mathscr{L}} P &\cong \mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} (\mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} (M \wedge N) \wedge (\mathscr{L}(1) \ltimes_{\mathscr{L}(1)} P)) \\ &\cong (\mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} \mathscr{L}(2) \times \mathscr{L}(1)) \ltimes_{\mathscr{L}(1)^3} (M \wedge N \wedge P) \\ &\cong \mathscr{L}(3) \ltimes_{\mathscr{L}(1)^3} M \wedge N \wedge P\end{aligned}$$

and similar for the calculation of $M \wedge_{\mathscr{L}} (N \wedge_{\mathscr{L}} P)$ with the same result. The last step of the calculation uses a result from Hopkins, that for $i, j \geq 1$,

$$\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1) \times \mathscr{L}(i) \times \mathscr{L}(j) \underset{\gamma\times id}{\overset{id\times\gamma^2}{\rightrightarrows}} \mathscr{L}(2) \times \mathscr{L}(i) \times \mathscr{L}(j) \xrightarrow[\gamma]{} \mathscr{L}(i+j)$$

is a split coequalizer of spaces. Therefore,

$$\mathscr{L}(i+j) \cong \mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} \mathscr{L}(i) \times \mathscr{L}(j)$$

$\square$

And we also have the smash product is natural commutative, derived from the canonical isomorphism

$$\mathscr{L}(2) \ltimes M \wedge N \cong \mathscr{L}(2) \ltimes t_*(M \wedge N) \cong \mathscr{L}(2) \ltimes N \wedge M$$

in which the map t is the transposition isomorphism between U^2. In fact, if we substitute $\mathscr{L}(2)$ by $\mathscr{L}(2) \times \mathscr{L}(1) \times \mathscr{L}(1)$ we get a similar isomorphism, and then we make a comparison between two diagrams of coequilizers.

Recall that we need a closed symmetric monoidal category, now we have the commutativity and associativity, but not unital as we have not found a suitable unit with respect to the internal smash product for $\mathbb{L}$-spectra. In fact, we cannot find such a unit for all $\mathbb{L}$-spectra. However, there is a candidate. For the sphere spectrum S_G indexed in the G-universe U, it sends $V \subset U$ to the one point compactification S^V.

Theorem 2.9. *Let M be an $\mathbb{L}$-spectrum, then there is a natural map of $\mathbb{L}$-spectra $\lambda : S_G \wedge_{\mathscr{L}} M \to M$*

Lemma 2.10. *There is an infinite suspension functor Σ^∞ from the category of G-spaces to the category of G-spectra which is left adjoint to the infinite loop functor Ω^∞ which takes the zero space of G-spectra. Since a based G-space can be regarded as a G-spectrum indexed on the zero universe $\{0\}$, we can rewrite $\Sigma^\infty X = \mathscr{L}(0) \ltimes X$.*

Proof of 2.9. We start with the the special case when $M = \mathbb{L}X$ for some spectrum X. Then λ is given by the map

$$\begin{aligned} S_G \wedge_{\mathscr{L}} \mathbb{L}X &= \mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} (\mathscr{L}(0) \ltimes S^0) \wedge (\mathscr{L}(1) \ltimes X) \\ &\cong (\mathscr{L}(2) \times_{\mathscr{L}(1)^2} \mathscr{L}(0) \times \mathscr{L}(1)) \ltimes (S^0 \wedge X) \\ &\to \mathscr{L}(1) \ltimes X = \mathbb{L}X \end{aligned}$$

The last arrow uses the structure map of linear isometries G-operads. For the general case M, the map is just induced by a map of coequilizer diagrams

$$\begin{tikzcd} S_G \wedge_{\mathscr{L}} \mathbb{L}\mathbb{L}M \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & S_G \wedge_{\mathscr{L}} \mathbb{L}M \arrow[r] \arrow[d] & S_G \wedge_{\mathscr{L}} M \arrow[d] \\ \mathbb{L}\mathbb{L}M \arrow[r, shift left] \arrow[r, shift right] & \mathbb{L}M \arrow[r] & M \end{tikzcd}$$

$\square$

To make the candidate of unit become a real unit, we shrink the scope of G-spectra we focus.

Definition 2.11. Define an S_G-module to be an $\mathbb{L}$-spectrum M which is unital in the sense that

$$\lambda : S_G \wedge_{\mathscr{L}} M \to M$$

is an isomorphism. Let $\mathscr{M}_G$ denote the full subcategory of $\mathscr{S}_G[\mathbb{L}]$ (the category of $\mathbb{L}$-spectra).

Remark 2.12. We define the category $G\mathscr{M}$ at the same time. Its objects are the same as $\mathscr{M}_G$'s but its morphisms are the G-fixed points of the arrows in $\mathscr{M}_G$. Notice $\mathscr{M}_G$ is enriched over $G\mathscr{T}$ while $G\mathscr{M}$ is enriched over $\mathscr{T}$. We define the G-fixed category $G\mathscr{M}$ because we need a category of S_G-modules with G-equivariant map to define a model structure. The notation is similar for other categories with G-actions.

The last step for constructing a closed symmetric monoidal category is to build a reasonable smash product and function spectra (as a functor), in which the former one is left adjoint to the latter one. In fact, although we have chosen the S_G-modules as the right category, we have the smash product and function spectra in the category of $\mathbb{L}$-spectra, and they already have the good adjunction property. We will list the result below, and then modify them to be the smash product and function spectra in $\mathscr{M}_G$.

Theorem 2.13. *Let M,N and P be $\mathbb{L}$-spectra. There is a function $\mathbb{L}$-spectrum functor $F_{\mathscr{L}}(M, N)$, contravariant in M and covariant in N, such that*

$$G\mathscr{S}[\mathbb{L}](M \wedge_{\mathscr{L}} N, P) \cong G\mathscr{S}[\mathbb{L}](M, F_{\mathscr{L}}(N, P)).$$

Definition 2.14. For S_G-modules M and N, define

$$M \wedge_G N = M \wedge_{\mathscr{L}} N,\ F_G(M,N) = S_G \wedge_{\mathscr{L}} F_{\mathscr{L}}(M,N).$$

Lemma 2.15. *The functor $S_G \wedge_{\mathscr{L}} (-) : G\mathscr{S}[\mathbb{L}] \to G\mathscr{M}$ is adjoint to the inclusion $l : G\mathscr{M} \to G\mathscr{S}[\mathbb{L}]$.*

Proof. Let M be an G-module and N be an $\mathbb{L}$-spectrum. Given a map $f : M \to S_G \wedge_{\mathscr{L}} N$ of S_G-modules, we can have a map $\lambda \circ f : M \to M$ between $\mathbb{L}$-spectra. Conversely, given a map $g : M \to N$ between $\mathbb{L}$-spectra, we can have a map $(id \wedge g) \circ \lambda^{-1} : M \to S \wedge_{\mathscr{L}} N$ of S_G-modules. From the naturality of λ, the diagram

$$\begin{tikzcd} M \arrow[r, "\lambda^{-1}"] \arrow[d, "\lambda\circ f"'] \arrow[dr, "f"] & S \wedge_{\mathscr{L}} M \arrow[d, "id\wedge(\lambda\circ f)"] \\ N & S \wedge_{\mathscr{L}} N \arrow[l, "\lambda"'] \end{tikzcd}$$

commutes. Using it it's easy to see our construction s are inverse bijections. □

Theorem 2.16. *The category $G\mathscr{M}$ is a closed symmetric monoidal category due to the adjunction*

$$G\mathscr{M}(M \wedge_G N, P) \cong G\mathscr{M}(M, F_G(N,P))$$

for S_G-modules M,N, and P.

Proof. Combine the 2.15 and the adjunction of $\mathbb{L}$-spectra, we have natural isomorphisms

$$\begin{aligned} G\mathscr{M}(M, F_G(N,P)) &= G\mathscr{M}(M, S_G \wedge_{\mathscr{L}} F_{\mathscr{L}}(N,P)) \\ &\cong G\mathscr{S}[\mathbb{L}](lM, F_{\mathscr{L}}(N,P)) \\ &\cong G\mathscr{S}[\mathbb{L}](M \wedge_{\mathscr{L}} N, P) \\ &\cong G\mathscr{M}(M \wedge_G N, P) \end{aligned}$$

□

2.2. **The model structure of S_G-modules.** As in the general case, to define a model structure on $G\mathscr{M}$, we have to point out the explicit classes of fibrations, cofibrations and weak equivalences. Then it's formal to verify the axioms of model category by using the classes of maps we define at the beginning. It's more convenient to give these classes for $G\mathscr{M}$, since we have very special "cell" objects like those in the CW complexes theory, and we can use them to reformulate the classes of cofibrations and acyclic cofibrations, which is very direct.

2.2.1. *Weak Equivalence.*

Definition 2.17. Write $\pi(E,E')_G$ for the set of homotopy classes of maps $E \to E'$ in $G\mathscr{S}$. Then for $H \subset G, n \in \mathbb{Z}$ and $E \in G\mathscr{S}$, define the homotopy group

$$\pi_n^H(E) = \pi(G/H_+ \wedge S_G^n, E)_G.$$

Or equivalently, since we always have $E(0)$ is homeomorphism to $\Omega^V E(V)$ from the definition of G-spectra, so

$$\begin{aligned}\pi_n^H(E) &= \pi(G/H_+ \wedge S_G^n, E)_G \\ &\cong \mathrm{colim}_V\, \pi(G/H_+ \wedge S_G^n(V), E(V)) \\ &\cong \mathrm{colim}_V\, \pi(G/H_+ \wedge S^n \wedge S^V, E(V)) \\ &\cong \mathrm{colim}_V\, \pi(G/H_+ \wedge S^n, \Omega^V E(V)) \\ &\cong \mathrm{colim}_V\, \pi(G/H_+, E(0)) \\ &= \pi_n^H(E(0)).\end{aligned}$$

Similarly we have $\pi_{-n}^H(E(\mathbb{R}^n))$. The homotopy groups of an S_G-module are the homotopy groups of its underlying G-spectrum.

Now we have the definitions of homotopy groups in the context of G-spectra and S_G-modules, it's natural to say that a morphism between two G-spectra (or S_G-modules) is a weak equivalence if it induces isomorphisms between all homotopy groups with n and H varying.

2.2.2. *q-fibrations.* For the q-fibrations, here we use the special notation q to avoid some ambiguities, it actually denotes the real fibrations in our model structure. We start from the category of Serre fibrations (q-fibrations) in the category of spectra, that is, a Serre fibration of spectra is a map that satisfies the RLP with respect to the set of all inclusions

$$i_0 : \Sigma_q^\infty CS^n \to \Sigma_q^\infty CS^n \wedge I_+.$$

For the equivariant case, since the sphere objects are different, we adjust that for G-spectra, a Serre fibration is a map satisfying the RLP with respect to the set of all inclusions

$$i_0 : \Sigma_V^\infty (G/H_+ \wedge CS^n) \to \Sigma_V^\infty (G/H_+ \wedge CS^n) \wedge I_+.$$

Apply monad $\mathbb{L}$ on $G\mathscr{S}$ and combine the adjunction

$$G\mathscr{S}[\mathbb{L}](\mathbb{L}X, A) \cong G\mathscr{S}(X, A), \tag{2.18}$$

the q-fibrations in $\mathscr{S}_G[\mathbb{L}]$ are the maps that satisfies the RLP with respect to the set of inclusions

$$\mathbb{L}i_0 : \mathbb{L}\Sigma_V^\infty (G/H_+ \wedge CS^n) \to \mathbb{L}\Sigma_V^\infty (G/H_+ \wedge CS^n) \wedge I_+.$$

These maps deserve to be called Serre fibrations of $\mathbb{L}$-algebras. Up to now, we get the Serre fibrations in different categories step by step, they are very close to the original Serre fibrations in the category of spectra, since we just use the very simple adjunction in which the right adjoint is exactly the forgetful functor. However, for S_G-modules the story is a little different, since if we combine two adjunctions we get a new adjuction

$$G\mathscr{M}(S_G \wedge_{\mathscr{L}} \mathbb{L}X, M) \cong G\mathscr{S}[\mathbb{L}](\mathbb{L}X, F_{\mathscr{L}}(S_G, M)) \cong G\mathscr{S}(X, F_{\mathscr{L}}(S_G, M))$$

in which the right adjoint is not forgetful functor! As a result, when we use this adjunction to define the Serre fibrations in the category of $G\mathscr{M}$, we want somehow to define them to be the maps satisfying RLP with respect to

$$S_G \wedge_{\mathscr{L}} \mathbb{L}i_0 : S_G \wedge_{\mathscr{L}} \mathbb{L}\Sigma_V^\infty (G/H_+ \wedge CS^n) \to S_G \wedge_{\mathscr{L}} \mathbb{L}\Sigma_V^\infty (G/H_+ \wedge CS^n) \wedge I_+.$$

as we have done for $\mathbb{L}$-spectra. This definition corresponds to the maps between function spectra to be Serre fibrations in the category of G-spectra. To be more specific, we have the theorem

Theorem 2.19. *The category $G\mathscr{M}$ is a model category with weak equivalence created in $G\mathscr{S}$. Its q-fibrations are the Serre fibrations of S_G-modules, which are maps $f : M \to N$ of S_G-modules such that*

$$F(id, f) : F_{\mathscr{L}}(S_G, M) \to F_{\mathscr{L}}(S_G, N)$$

is a Serre fibration of spectra.

Here we need to remind, $G\mathscr{M}$'s weak equivalences are created in $G\mathscr{S}$ means the maps are weak equivalences in $G\mathscr{M}$ if they are weak equivalences when regarded as maps in $G\mathscr{S}$. Furthermore, there is a conclusion saying that for any $\mathbb{L}$-spectrum M, $\lambda : M \to F_{\mathscr{L}}(S_G, M)$ is a weak equivalence of $\mathbb{L}$-spectra. Then from the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\lambda_M"] \arrow[d, "f"'] & F_{\mathscr{L}}(S_G, M) \arrow[d, "F(id,f)"] \\ N \arrow[r, "\lambda_N"'] & F_{\mathscr{L}}(S_G, N), \end{tikzcd}$$

f is a weak equivalence if and only if $F_{\mathscr{L}}(id, f)$ is, which means although the definition of a morphism in $G\mathscr{M}$ to be a q-fibration is not itself when being regarded as a map in $G\mathscr{S}$, the weak equivalences are unchanged. And this is why for $G\mathscr{M}$ we do not need to define its homotopy groups differently from homotopy groups in $G\mathscr{S}$.

2.2.3. *q-cofibrations.* Since we have a very explicit description for weak equivalences and q-fibrations, we can define the q-cofibrations in $G\mathscr{M}$ to be the maps which satisfy the LLP with respect to the acyclic q-fibrations (the maps which are q-fibrations and weak equivalences at the same time).

2.2.4. *The proofs of model structure theorems.* We introduce following theorem illustrating the construction of model category of $G\mathscr{M}$.

Theorem 2.20. *The category $G\mathscr{S}$ is a model category with respect to the weak equivalences and Serre fibrations. If $\mathbb{T} : G\mathscr{S} \to G\mathscr{S}$ is a monad that preserves reflexive coequalizers and satisfies the "Cofibration Hypothesis", then $G\mathscr{S}$ creates a model structure in $G\mathscr{S}[\mathbb{T}]$.*

The settings of this theorem is complex, and we need some explanations.

Remark 2.21 (Cofibration Hypothesis)**.** Consider we have a functor $\mathbb{T} : \mathscr{C} \to \mathscr{C}[\mathbb{T}]$, being a left adjoint, perserves coproducts. And we have "sphere objects" in $\mathscr{C}$. For example, they are $G/H_+ \wedge S^n$ in the category of G-spaces, and $\sum_V^\infty (G/H_+ \wedge S^n)$ in $G\mathscr{S}$, and so on. We consider a pushout in $\mathscr{C}[\mathbb{T}]$ of the general form

$$\begin{tikzcd} \mathbb{T}E \arrow[r] \arrow[d] & A \arrow[d, "i"] \\ \mathbb{T}CE \arrow[r] & B \end{tikzcd}$$

where E is a wedge of sphere objects. The Cofibration Hypothesis means that the map i is a cofibration of G-spectra or S_G-modules, depending on the ground category.

This setting mimics the fact in the category of pointed topological spaces that the inclusions of subcomplexes are cofibrations. Once the hypothesis is satisfied in the ground category, we have the inclusions of the relative cell $\mathbb{T}$-algebras are q-cofibrations. In fact, these maps generate all the q-cofibrations in our model category of $\mathbb{T}$-algebras, i.e. a map is a q-cofibration if and only if it is a retract of a relative cell $\mathbb{T}$-algebra. The maps $\mathbb{T}E \to \mathbb{T}CE$ are q-cofibrations is proved in 2.28, and the inclusions of relative cell $\mathbb{T}$-algebras form a generator set of q-cofibrations is shown in 2.30.

Remark 2.22. The condition for $\mathbb{L}$ to be a continuous monad that preserves reflexive coequalizers is to make category $G\mathscr{S}[\mathbb{T}]$ have all indexed colimits, thus we can do pushout work with no worry.

Remark 2.23. $G\mathscr{S}$ creates a model structure on $G\mathscr{S}[\mathbb{T}]$ means that $G\mathscr{S}[\mathbb{T}]$ has a forgetful functor to $G\mathscr{S}$, and $G\mathscr{S}[\mathbb{T}]$ is a model category in which a map is a weak equivalence or q-fibration if it is a weak equivalence or q-fibration when regarded as a map in $G\mathscr{S}$.

Remark 2.24. We care about the model structure of $G\mathscr{M}$ in this section. Actually, all the proofs play equally well if we change the ground category from $G\mathscr{S}$ to $G\mathscr{M}$. And if we let $\mathbb{T}$ to be the identity monad, we get simply the model structures of $G\mathscr{S}$ and $G\mathscr{M}$.

We then prove 2.20 by checking the axioms of model category verbatimly.
First, to prove "2 out of 3" is trivial.
Second, the retract axiom is also not hard, and we just prove for the case of q-fibrations.

$$\begin{tikzcd}
\mathbb{T}\Sigma^\infty_V(G/H_+ \wedge CS^n) \arrow[r] \arrow[d, "\mathbb{T}i_0"'] & E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[r, "h"] \arrow[d, "p_2"'] & E_1 \arrow[d, "p_1"'] \\
\mathbb{T}\Sigma^\infty_V(G/H_+ \wedge CS^n) \wedge I_+ \arrow[r] \arrow[urr] & B_1 \arrow[r, "g"] & B_2 \arrow[r, "k"] & B_1
\end{tikzcd}$$

In the diagram map $p_1 : E_1 \to B_1$ is a retract of a q-fibration $p_2 : E_2 \to B_2$. The red arrow is induced by the RLP of p_2 with respect to the $\mathbb{T}i_0$ class. Then its composition with h gives the RLP of p_1 with respect to $\mathbb{T}i_0$.

Third, we define the q-cofibrations as the maps satisfying LLP with respect to the acyclic q-fibrations. So we only have to check that the q-fibrations satisfy the RLP with respect to the acyclic q-fibrations. This will be proved later. Fourth, the essential part of the proof, is about showing that arbitrary maps factor appropriately. The proof depends on an important conclusion called "small object argument". In the following context, let $\mathscr{C}$ be either $G\mathscr{S}$ or $G\mathscr{M}$.

Definition 2.25. Define a finite pair of G-spectra to be a pair of the form $(\Sigma^\infty_V(G/H_+ \wedge B), \Sigma^\infty_V(G/H_+ \wedge A))$, where B is a finite based CW complex, A is a subcomplex, and $\dim(V) \geq 0$. Define a finite pair of $\mathbb{L}$-spectra to a pair obtained by applying $\mathbb{L}$ to be a finite pair of spectra, and similarly for S_G-modules.

Lemma 2.26 (Small Object Argument)**.** *Let $\mathscr{F}$ be a set of maps in $\mathscr{C}[\mathbb{T}]$, each of which is of the form $\mathbb{T}E \to \mathbb{T}F$ for some pair (F, E) in $\mathscr{C}$. Then any map $f : X \to Y$ in $\mathscr{C}[\mathbb{T}]$ factors as a composite*

$$X \xrightarrow{i} X' \xrightarrow{p} Y,$$

where p satisfies the RLP with respect to each map in $\mathscr{F}$ and i satisfies the LLP with respect to any map that satisfies the RLP with respect to each map in $\mathscr{F}$.

Sketch Proof. Let $X = X_0$, we construct a "tower" of X_i's and take the colimit, getting the "inclusion" and "projection" which are our i and p respectively. We construct a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "i_0"] \arrow[d, "f=p_0"'] & X_1 \arrow[r] \arrow[d, "p_1"] & \dots \arrow[r] & X_n \arrow[r, "i_n"] \arrow[d, "p_n"] & X_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & \dots \\
Y \arrow[r, "id"] & Y \arrow[r] & \dots \arrow[r] & Y \arrow[r, "id"] & Y \arrow[r] & \dots
\end{tikzcd}$$

as follows. Suppose inductively we have constructed p_n. Consider all maps from a map in $\mathscr{F}$ to p_n. Each such map fits in a commutative diagram

$$\begin{tikzcd}
\mathbb{T}E \arrow[r, "\alpha"] \arrow[d] & X_n \arrow[d, "p_n"] \\
\mathbb{T}F \arrow[r] & Y.
\end{tikzcd}$$

We collect all such diagram and add them together, then take the pushout of the diagram, denote it X_{n+1}. Then there is a induced map p_{n+1} from X_n to Y. It's easy to see $p_{n+1} \circ i_n = p_n$, where $i_n : X_n \to X_{n+1}$ is the canonical inclusion.

$$\begin{tikzcd}
\coprod \mathbb{T}E \arrow[r, "\Sigma\alpha"] \arrow[d] & X_n \arrow[d, "i_n"] \arrow[ddr, "p_n"] & \\
\coprod \mathbb{T}F \arrow[r] \arrow[drr] & X_{n+1} \arrow[dr, dashed, "p_{n+1}"] & \\
 & & Y.
\end{tikzcd}$$

Let $X' = \operatorname{colim} X_n$, and then we also get canonical inclusion $i : X \to X'$, and $p : X' \to Y$ is got by passage to colimit from the p_n's. Now we have to show that i and p satisfy the properties correspondingly. Firstly, each i_n and therefore i satisfies LLP with respect to the maps that satisfy the RLP with respect to maps in $\mathscr{F}$. To show p satisfies RLP with respect to maps in $\mathscr{F}$, we need to find a map $g : \mathbb{T}F \to X'$ to make the diagram commutes

$$\begin{tikzcd}
\mathbb{T}E \arrow[r, "\alpha"] \arrow[d, "i"'] & X' \arrow[d, "p"] \\
\mathbb{T}F \arrow[r] \arrow[ur, dashed, "g"] & Y.
\end{tikzcd}$$

From the Cofibration Hypothesis, we know that X' is the colimit of sequences of q-cofibrations. We can see a conclusion in [] III.1.7 saying that the natural map

$$\operatorname{colim} \mathscr{C}[\mathbb{T}](\mathbb{T}E, X_n) \to \mathscr{C}[\mathbb{T}](\mathbb{T}E, \operatorname{colim} X_n)$$

is bijective. Thus the map $\alpha : \mathbb{T}E \to X'$ factors through some X_k, and $\mathbb{T}F$ maps into X_{n+1} as in the former commutative diagram. Then compositing with inclusion from X_{n+1} to X', we get the required $g : \mathbb{T}F \to X'$.

□

With this important conclusion, we can let $\mathscr{F}$ be different classes of maps which induce different factorizations. We will iterately use the adjunction 2.18, and we would not analyse the chasing process repeatedly.

Theorem 2.27. *Any map $f : X \to Y$ in $\mathscr{C}[\mathbb{T}]$ factors as $p \circ i$, where i is an acyclic q-cofibration that satisfies the LLP with respect to any q-fibration and p is a q-fibration.*

Sketch. Let $\mathscr{F}$ be the set of pairs obtained by letting (B,A) run through all pairs of spaces $(G/H_+ \wedge CS^n \wedge I_+, G/H_+ \wedge CS^n \wedge \{0\}_+)$ where $n \geq 0$. We can see that a map is a q-fibration if and only if it satisfies the RLP with respect to every map in $\mathscr{F}$, and every map in $\mathscr{F}$ is an acyclic q-cofibrations. These are direct from the definitions. Then use 2.26 to factor f. Then p is a q-fibration and i satisfies LLP with respect to all q-fibrations and thus is a q-cofibration. Use the fact that the free functor $\mathbb{T}$ and its adjoint forgetful functor preserve homotopies, we know each i_n is a deformation retract and so is i. Thus i is an acyclic q-cofibration. $\square$

Theorem 2.28. *Any map $f : X \to Y$ in $\mathscr{C}[\mathbb{T}]$ factors as $p \circ i$, where i is a q-cofibration and p is an acyclic q-fibration.*

Sketch. Let $\mathscr{F}$ be the set of pairs $(G/H_+ \wedge CS^n, G/H_+ \wedge S^n)$ where $n \geq 0$. This time we can check easily a map of $\mathbb{T}$-algebra is an acyclic q-fibration if and only if it satisfies the RLP with respect to all of the maps in $\mathscr{F}$ (the reason is similar to $\pi_n(X, A) = 0$ if and only if every map in $\pi_n(X, A)$ is homotopic to a map which maps D^n into A rel S^{n-1}). This also proves the inclusions $\mathbb{T}E \to \mathbb{T}CE$ we mentioned in 2.21 are q-cofibrations. Then use 2.26 directly. $\square$

To complete constructing a model category, we only have to prove the RLP of q-fibrations with respect to acyclic q-cofibrations.

Theorem 2.29. *The q-fibrations satisfy the RLP with respect to the acyclic q-cofibrations.*

Sketch. If there is a map from any acyclic q-cofibration $f : E \to F$ to a q-fibration $h : A \to B$, we have a commutative diagram

$$\begin{tikzcd}
E' \arrow[d, "p"'] & E \arrow[l, "i"'] \arrow[r] \arrow[d, "f"] & A \arrow[d, "h"] \\
F \arrow[r, equal] & F \arrow[r] \arrow[ul, dashed, "g"'] & B
\end{tikzcd}$$

in which we apply 2.26 to f such that i is an acyclic q-cofibration and p is a q-fibration. Since f and i are both weak equivalences, so is p. The red dotted arrow is induced by the fact that f satisfies LLP with respect to acyclic q-fibration p. Then we get f is in fact a retract of i, so f satisfies LLP with respect to q-fibrations as i does.

$\square$

Remark 2.30. Use the same technique, we can prove that a map of $\mathbb{T}$-algebras is a q-cofibration if and only if it is a retract of a relative cell $\mathbb{T}$-algebra.

We have prove more that everything needed to construct the model category of $G\mathscr{M}$ or just $G\mathscr{S}$. To make life easier, or more clear, we restate the settings in a neater way.

Definition 2.31 (Sphere Objects)**.**
(1) A generalized sphere G-spectrum is a G-spectrum of the form $\Sigma_V^\infty(G/H_+ \wedge S^n)$, where V is an indexing G-space V in the G-universe $U, n \geq 0$, and $H \subset G$. Write $S_G^{-V} = \Sigma_V^\infty S^0$ as the desuspention with indexing G-space V.
(2) A generalized sphere S_G-module is an S_G-module of the form $\mathbb{F}E \equiv S_G \wedge_{\mathscr{L}} \mathbb{L}E$ where E is a generalized sphere G-spectrum.

Definition 2.32 (Generating (acyclic) Cofibrations)**.**
(1) A generalized generating q-cofibration in $G\mathscr{M}$ is a map of the form $E \to CE = E \wedge I_+$, where E is a generalized sphere object and CE is the cone on E.
(2) A generalized generating acyclic q-cofibration is a map of the from $i_0 : CE \to CE \wedge I_+$, where E is a generalized sphere object.

Theorem 2.33 (Model Category Settings)**.** *Here we consider the category $G\mathscr{S}$ and $G\mathscr{M}$ of G-spectra and S_G-modules, then*

(1) A map in either category is a weak equivalence if it induces an isomorphism on all homotopy groups π_n^H.

(2) A map is a generalized q-cofibration if it is a retract of a relative cell G-complex defined in terms of generalized generating q-cofibrations.

(3) A map is a restricted q-fibration if it satisfies RLP with respect to all generalized generating acyclic q-cofibrations.

Remark 2.34. Actually, 2.33 specifies a convenient way to describe a model category by giving the weak equivalences, the generating set of cofibrations and the generating set of acyclic cofibrations. In such a setting, a map is an (acyclic) fibration if it satisfies RLP with respect to each map in the generating set of (cofibrations) acyclic cofibrations; a map is an (acyclic) cofibration if it is a retract of a relative cell complex in terms of generating (acyclic) cofibrations.

In next chapter, we will also give the model structure of equivariant orthogonal spectra by specifying the generating sets of cofibrations and acyclic cofibrations. We will not give the detailed proof there, as the verification for it to be a model structure is similar to this chapter: they are in fact verifying that *Kan Recognition Theorem* holds for the generating sets we provide.

Theorem 2.35 (Kan Recognition Theorem)**.** *Let C be a category with all small limits and colimit and W a class of maps satisfying 2-out-of-3. If I and J are sets of maps in C such that*

(1) both I and J permit the small object argument;

(2) $LLP(RLP(J)) \subset LLP(RLP(I)) \cap W$;

(3) $RLP(I) \subset RLP(J) \cap W$;

(4) one of the following holds

1) $LLP(RLP(I)) \cap W \subset LLP(RLP(J))$;

2) $RLP(J) \cap W \subset RLP(I)$,

then there is the stucture of a cofibrantly generated model category on C with

1. weak equivalences $W_C := W$;

2. generating cofibrations I;

3. generating acyclic cofibrations J.

In the practice, to give a cofibratly generating model structure for equivariant S-modules, we introduce the generating sets of acyclic cofibrations and cofibrations in 2.27 and 2.28, respectively. Then, (2) in 2.35 holds automatically by definition; 2.26 states the fundamental model of "small object argument" for our setting, combining with 2.27 and 2.28 we prove (1) and (4) in 2.35. At last, (3) in 2.35 is proved in 2.28. This makes the statements in 2.33 clear.

3. EQUIVARIANT ORTHOGONAL SPECTRA

3.1. **Basic constructions.** Following the publication of [1], Mandell, May, Schwede, and Shipley proposed an alternative approach to establish a well-behaved symmetric monoidal category. They introduced various types of spectra as modifications of "diagram spectra," essentially framing them as categories of functors mapping from distinct special symmetric monoidal categories to the pointed topological spaces category. This chapter will present equivariant orthogonal spectra as an exemplary model of diagram spectra, demonstrating its straightforward and convenient applicability in construction.

Recall that the most technical step of building a symmetric monoidal category is constructing the product. Our important motivation to overcome it comes from *Day Convolution Theorem.*

Theorem 3.1 (Day Convolution Theorem)**.** *Let $(\mathscr{J}, \oplus, 0)$ be a small symmetric monoidal category enriched over a cocomplete closed symmetric monoidal category $(\mathscr{V}, \otimes, 1)$. Then the enriched functor category $[\mathscr{J}, \mathscr{V}]$ is closed symmetric monoidal.*

Here, the $\oplus$ and $\otimes$ are product of the two symmetric monoidal categories, respectively. If X, Y are two functors from $\mathscr{J}$ to $\mathscr{V}$, then the dotted arrow in the following picture representing the new commutative and symmetric product (we denote it by "$\wedge$" temporarily) of the category $[\mathscr{J}, \mathscr{V}]$ comes from the *Left Kan Extension*.

$$\begin{tikzcd} \mathscr{J} \times \mathscr{J} \arrow[r, "X\times Y"] \arrow[dr, "\oplus"'] & \mathscr{V} \times \mathscr{V} \arrow[r, "\otimes"] & \mathscr{V} \\ & \mathscr{J} \arrow[ur, dashed, "\wedge"'] & \end{tikzcd}$$

Explicitly, the day convolution product $\wedge$ is given by the **coend** of the functor

$$\mathscr{J}(- \oplus -, C) \otimes X(-) \otimes Y(-) : P_{/C}^{op} \times P_{/C} \to \mathscr{V}.$$

Here C is a fixed object in $\mathscr{J}$, $P_{/C}$ is the category of pairs over C, i.e. objects $(A, B) \in \mathscr{J} \times \mathscr{J}$ such that there is a morphism from $A \oplus B$ to C. Then for any C in $\mathscr{J}$, we have

$$(X \wedge Y)(C) = \int_{P_{/C}} \mathscr{J}(A \oplus B, C) \otimes X(A) \otimes Y(B).$$

Here $\mathscr{J}(A \oplus B, C)$ is viewed as a pointed topological space since $\mathscr{J}$ is enriched over $\mathscr{V}$. In our settings $\mathscr{V}$ is cocomplete and $P_{/C}$ is small, so the coend is the coequalizer of

$$\coprod_{(A,B)\to(A',B')} \mathscr{J}(A' \oplus B', C) \otimes X(A) \otimes Y(B) \rightrightarrows \coprod_{(A,B)\in P_{/C}} \mathscr{J}(A \oplus B, C) \otimes X(A) \otimes Y(B)$$

As a corollary, it's easy to calculate directly with the help of the evident adjuction

$$[\mathscr{J}, \mathscr{V}](X \wedge Y, Z) \cong [(\mathscr{J} \times \mathscr{J}), \mathscr{V}](X \overline{\wedge} Y, Z \circ \oplus).$$

in which $\overline{\wedge}$ is the evident external symmetric product. Now we can introduce the explicit roles of $\mathscr{J}$ and $\mathscr{V}$ in our construction of equivariant orthogonal spectra. First, we also want the new spectra to go out of a G-universe, and output a G-topological space. Second, we should give the elements in the G-universe a categorical structure, such that the category is enriched over the category of topological G-spaces. Inspired by these ideas, we give the following definitions of equivariant orthogonal spectra.

Definition 3.2. The indexing category $\mathscr{J}_G$ is the topological G-category whose objects are finite dimensional real orthogonal representations V of G. Let $O(V, W)$ denote the Stiefel manifold of (possibly nonequivariant) orthogonal embeddings $V \to W$. For each such embedding we have an orthogonal complement $W - V$, giving us a vector bundle over $O(V, W)$. The morphism object $\mathscr{J}_G(V, W)$ is its Thom space, which is a pointed G-space.

It's easy to verify that $\mathscr{J}_G$ is a symmetric monoidal category enriched over $\mathscr{T}_G$. Since it has

(1) Composition map $\mathscr{J}_G(V, W) \wedge \mathscr{J}_G(U, V) \to \mathscr{J}_G(U, W)$ which is in fact composed in $\mathscr{J}_G$ by the formula $(g, y) \circ (f, x) = (g \circ f, g(x) + y)$.
(2) Direct sum on arrows: $\oplus : \mathscr{J}_G(V, W) \wedge \mathscr{J}_G(V', W') \to \mathscr{J}_G(V \oplus V', W \oplus W')$. specified by the direct sum of spaces and embeddings $(f, x) \oplus (g, y) = (f \oplus g, x \oplus y)$.

Definition 3.3. An orthogonal G-spectrum E is a functor $\mathscr{J}_G \to \mathscr{T}_G$. We will denote its value on V by $E(V)$ and the category of orthogonal G-spectra $\mathscr{J}_G\mathscr{T}$.

As a reminder, here $\mathscr{T}_G$ is the category of pointed G-spaces and all (not necessarily equivariant) continuous pointed maps. Using the functoriality of orthogonal G-spectra, we can easily get the structure maps for orthogonal G-spectra. For two finite dimensional G-representations V and W, we have a nature map

$$\epsilon_{V,W} : \mathscr{J}_G(V,W) \wedge E(V) \to E(W).$$

This structure map contains more information than the usual "Whitehead's" structure map, as we can see in a degenerated case.

Remark 3.4. Given the definition of an orthogonal G-spectrum E, if G is trivial, we get E a functor $\mathscr{J} \to \mathscr{T}$, where $\mathscr{J}$ is the topological category of finite dimensional orthogonal vector spaces with morphism spaces as before. Such vector spaces are determined by their dimensions, so the structure maps degenerate to those maps like

$$\epsilon_{n,n+1} : \mathscr{J}(n,n+1) \wedge E(n) \to E(n+1).$$

Recall that we also have another definition for the spectra, in which version the structure map is like

$$\epsilon_n : \Sigma E(n) \to E(n+1).$$

This structure map actually chooses an orthogonal embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$, and we smash $E(n)$ with the one-point compactification of the orthogonal complement. However, for our orthogonal G-spectra, the structure maps factor through $\mathscr{J}(n,n+1) \wedge_{O(n)} E(n)$ and they turn out amount to a family of maps $\Sigma E(n) \to E(n+1)$ parameterized by all orthogonal embeddings $\mathbb{R}^n \to \mathbb{R}^{n+1}$. This makes equivariant orthogonal spectra a coordinate free construction.

Definition 3.5. For an object V of $\mathscr{J}_G$, define the orthogonal G-spectrum V^* represented by V by $V^*(W) = \mathscr{J}_G(V,W)$. In particular, $0^* = S^G$. Define the *shift desuspension spectrum functors* $F_V : \mathscr{T}_G \to \mathscr{J}_G\mathscr{T}$ and the evaluation functors $Ev_V : \mathscr{J}_G\mathscr{T} \to \mathscr{T}_G$ by $F_V(A) = V^* \wedge A$ and $Ev_V(X) = X(V)$. Then F_V and Ev_V are left and right adjoint:

$$\mathscr{J}_G\mathscr{T}(F_V A, X) \cong \mathscr{T}_G(A, Ev_V X).$$

This is a direct result of *enriched Yoneda lemma*.

Notation 3.6. Let $\Sigma^\infty = F_0$ and $\Omega^\infty = Ev_0$. These are the suspension orthogonal G-spectrum and zeroth space functors. Note that $\Sigma^\infty A = S_G \wedge A$. Similarly, let $\Sigma_V^\infty = F_V$ and $\Omega_V^\infty = Ev_V$; we let $S^{-V} = \Sigma_V^\infty S^0$ and call it the canonical $(-V)$-sphere.

Then, we can introduce the "tautological representation" of orthogonal G-spectra, and give a specific picture of its smash product (that is associative and commutative as the product of the symmetric monoidal category $\mathscr{J}_G\mathscr{T}$).

Theorem 3.7. *For an orthogonal G-spectrum E, we have an evaluation map $V^* \wedge E(V) \to E$. We can compile all these maps and express the spectrum E in the form of the coend of $\mathbb{D}$ and E where $\mathbb{D}$ is the functor $\mathscr{J}_G \to \mathscr{J}_G\mathscr{T}$ specified by $\mathbb{D}V = V^*$. In other words, we have a natural isomorphism*

$$\mathbb{D} \otimes_{\mathscr{J}_G} E = \int^{V \in sk\mathscr{J}_G} V^* \wedge E(V) \longrightarrow E$$

Since the category $\mathscr{J}_G$ is small, the coend is exactly the coequalizer in the following diagram

$$\bigvee_{V,W \in sk\mathscr{J}_G} W^* \wedge \mathscr{J}_G(V,W) \wedge E(V) \rightrightarrows \bigvee_{V \in sk\mathscr{J}_G} V^* \wedge E(V) \longrightarrow \mathbb{D} \otimes_{\mathscr{J}_G} E.$$

Or we can say that a orthogonal G-spectrum E is the homotopy colimit of $V^* \wedge E(V)$ when V runs through all elements in $sk\,\mathscr{J}_G$.

Proof. We first assert that the set $\mathscr{J}_G\mathscr{T}(X,Y)$ of morphisms between two orthogonal G-spectra is the equalizer in the category of based spaces displayed in the diagram

$$\mathscr{J}_G\mathscr{T}(X,Y) \longrightarrow \prod_d \mathscr{T}_G(X(d),Y(d)) \underset{\tilde{\nu}}{\overset{\tilde{\mu}}{\rightrightarrows}} \prod_{\alpha:d\to e} \mathscr{T}_G(X(d),Y(e))$$

where the products run over the objects and morphisms of $\mathscr{J}_G$. For $f = (f_d)$, the αth component of $\tilde{\mu}(f)$ is $Y(\alpha) \circ f_d$ and the αth component of $\tilde{\nu}(f)$ is $f_e \circ X(\alpha)$.

$$\begin{tikzcd} X(d) \arrow[r, "f_d"'] \arrow[d, "X(\alpha)"'] & Y(d) \arrow[d, "Y(\alpha)"] \\ X(e) \arrow[r, "f_e"'] & Y(e). \end{tikzcd}$$

The identification of these two maps is equivalent to the commutativity of the above diagram, which means f is a morphism between X and Y, i.e. $f \in \mathscr{J}_G\mathscr{T}(X,Y)$. Furthermore, use the adjunction in 3.5, we can get

$$\prod_d \mathscr{T}_G(X(d),Y(d)) \cong \prod_d \mathscr{J}_G\mathscr{T}(d^* \wedge X(d),Y) \cong \mathscr{J}_G\mathscr{T}(\bigvee_d d^* \wedge X(d),Y),$$

$$\prod_{\alpha:d\to e} \mathscr{T}_G(X(d),Y(e)) \cong \prod_{d,e} \mathscr{T}_G(\mathscr{J}_G(d,e) \wedge X(d),Y(e)) \cong \mathscr{J}_G\mathscr{T}(\bigvee_{d,e} e^* \wedge \mathscr{J}_G(d,e) \wedge X(d),Y).$$

Since Y is chose arbitrarily, we get X is the coequalizer of the following diagram

$$\bigvee_{d,e} e^* \wedge \mathscr{J}_G(d,e) \wedge X(d) \rightrightarrows \bigvee_d d^* \wedge X(d) \longrightarrow X.$$

□

Thus if we write $E = \mathrm{hocolim}_V\, V^* \wedge E(V)$ for an orthogonal G-spectrum E, and the same for other orthogonal G-spectra. Then we can define the smash product of E and F as the coequalizer of

$$\begin{array}{c} \bigvee_{V,V',W,W'} (W \oplus W')^* \wedge \mathscr{J}_G(V,W) \wedge \mathscr{J}_G(V',W') \wedge E(V) \wedge F(V') \\ \downdownarrows \\ \bigvee_{V,V'} (V \oplus V')^* \wedge E(V) \wedge F(V') \end{array} .$$

The process is actually the same as we introduce at the beginning of this chapter, i.e. the construction using 3.1. It says the smash product of two orthogonal G-spectra is some kind of "colimit" with inputs running through all objects and morphisms in the category $\mathscr{J}_G$.

3.2. **Model Category of Equivariant Orthogonal Spectra.** The final model structure we need for equivariant orthogonal spectra is *positive stable model structure*. We shall introduce the *level model structure*, and then "stabilize" it to get *stable model structure*. Finally, we restrict *stable model structure* to get the *positive stable model structure*. One important reason we did not do this for the model structure of equivariant S-modules is, the model structure of equivariant S-modules is itself stable, and we did not need adjust our generating (acyclic) cofibration sets to avoid sphere spectrum being fibrant. We will explain later.

$$\begin{array}{c} \textit{level model structure} \\ \big\downarrow{\scriptstyle stabilize} \\ \textit{stable model structure} \\ \big\downarrow{\scriptstyle restrict} \\ \textit{positive stable model structure} \end{array}$$

3.2.1. *Level Model Structure of Orthogonal G-spectra.* As what we did when introducing the model structure, the most convenient way to give a model category is pointing out its generating cofibrations and generating acyclic cofibrations. We will write the both sets for the level model structure of equivariant orthogonal spectra directly, and this time we will not give the specific proof.

Definition 3.8. We will give the generating sets of (acyclic) cofibrations of G-spaces here in convenience of translating them to be generating sets of (acyclic) cofibrations of orthogonal G-spectra later. They are intuitive inclusions.

(1) Let I be the set of cell inclusions
$$i : (G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$$
where $n \geq 0$ (S^{-1} being empty) and H runs through the (closed) subgroups of G.

(2) Let J be the set of inclusions
$$j : (G/H \times D^n)_+ \to (G/H \times D^n \times I)_+$$
and observe that each such map is the inclusion of a G-deformation retract.

Definition 3.9. Define FI to be the set of all maps $F_V i$ with $V \in sk\mathscr{J}_G$ and $i \in I$. Define FJ to be the set of all maps $F_V j$ with $V \in sk\mathscr{J}_G$ and $j \in J$. It's convenient to write them as $FI = \{S^{-V}\} \wedge I_+$ and $FJ = \{S^{-V}\} \wedge J_+$.

Theorem 3.10. *The level equivalence, generating q-cofibrations and generating acyclic q-cofibrations for the level model structure of equivariant orthogonal spectra are as follows,*

(1) For a map $f : X \to Y$ between orthogonal G-spectra, f is a level equivalence if each map $f(V) : X(V) \to Y(V)$ of G-spaces is a weak equivalence.
(2) The set $FI = \{S^{-V}\} \wedge I_+$ is the generating q-cofibrations.
(3) The set $FJ = \{S^{-V}\} \wedge J_+$ is the generating acyclic q-cofibrations.

Once we have generating sets, we can formulate the whole model category clearly.

(1) The level fibrations are the maps that satisfy the RLP with respect to FJ or, equivalently, with respect to retracts of relative FJ-cell complexes.
(2) The level acyclic fibrations are the maps that satisfy the RLP with respect to FI or, equivalently, with respect to retracts of relative FI-cell complexes.
(3) The q-cofibrations are the retracts of relative FI-cell complexes
(4) The level acyclic q-cofibrations are the retracts of relative FJ-cell complexes.

3.2.2. *Stable Model Structure of Orthogonal G-spectra.* The stabilization of level model structure is realized by *Bousfield Localization.* The idea is, suppose we have a model category $\mathscr{M}$ (like the level model structure in last section), then we

(1) Enlarge the class of weak equivalences in some way.
(2) Keep the same class of cofibrations as before.

(3) Define fibrations in terms of right lifting properties with respect to the newly defined trivial cofibrations. The class of trivial fibrations remains unaltered.

The enlargement of weak equivalences in step(1) above is due to *Bousfield Localization*. As a result, we have more weak equivalences and more acyclic cofibrations in the new category, and thus less fibrations.

Definition 3.11 (Notions about Localization)**.** Suppose we have a model category $\mathscr{N}$ which is enriched and bitensored(it means we can smash or map out of objects in $\mathscr{M}$ to objects in $\mathscr{N}$ to get new objects in $\mathscr{N}$) over another model category $\mathscr{M}$, and S is a set of morphisms in $\mathscr{N}$. Then

(1) An object Z of $\mathscr{N}$ is called **S-local** if for each $f : A \to B$ in S, the map
$$f^* : \mathscr{N}(B, Z) \to \mathscr{N}(A, Z)$$
is a weak equivalence in $\mathscr{M}$.

(2) A morphism $g : X \to Y$ in $\mathscr{N}$ is an **S-equivalence** if for each S-local object Z the map
$$g^* : \mathscr{N}(Y, Z) \to \mathscr{N}(X, Z)$$
is a weak equivalence in $\mathscr{M}$.

We now apply these notions in the context of equivariant orthogonal spectra. Explicitly, we already have level model structure which is enriched and bitensored over $G\mathscr{T}$. Then we define the set S to be natural maps
$$\lambda_{V,W} : F_{V\oplus W}S^W = S^{-V\oplus W} \wedge S^W \to S^{-V} = F_V S^0$$
where $V, W \in \mathscr{J}_G$, it's adjoint to the map
$$S^W \to (F_V S^0)(V \oplus W) \cong O(V \oplus W)_+ \wedge_{O(W)} S^W$$
which sends w to $e \wedge w$ where e is the identity element of $O(V \oplus W)$. The maps are important for the reason: $\lambda^*_{v,w}$'s are exactly the adjoints of the structure maps of orthogonal spectra.

Theorem 3.12. *For any orthogonal G-spectrum X,*
$$\lambda^*_{V,W} : \mathscr{J}_G\mathscr{T}(F_V S^0, X) \to \mathscr{J}_G\mathscr{T}(F_{V\oplus W}S^W, X)$$
coincides with the adjoint of X's structure map $\tilde{\sigma} : X(V) \to \Omega^W X(V \oplus W)$ under the canonical homeomorphisms
$$X(V) \cong \mathscr{T}_G(S^0, X(V)) \cong \mathscr{J}_G\mathscr{T}(F_V S^0, X)$$
and
$$\Omega^W X(V \oplus W) \cong \mathscr{T}_G(S^W, X(V \oplus W)) \cong \mathscr{J}_G\mathscr{T}(F_{V\oplus W}S^W, X).$$

Proof. Let $X = F_V S^0$, then $\tilde{\sigma}$ can be identified as
$$\begin{tikzcd} \mathscr{J}_G\mathscr{T}(F_V S^0, F_V S^0) \arrow[r, "\tilde{\sigma}"] \arrow[d, dashed, "\cong"] & \mathscr{J}_G\mathscr{T}(F_{V\oplus W}S^W, F_V S^0) \arrow[d, dashed, "\cong"] \\ \mathscr{T}_G(S^0, F_V S^0(V)) \arrow[r] & \mathscr{T}_G(S^W, F_V S^0(V \oplus W)) \end{tikzcd}$$
where the vertical ones are due to adjunctions. It's easy to see, the map in $\mathscr{T}_G(S^W, F_V S^0(V \oplus W))$ sending w to $e \wedge w$ corresponds to the identity map in $\mathscr{J}_G(V, V) \cong \mathscr{T}_G(S^0, F_V S^0(V))$. Then we can see $\lambda_{V,W}$ is the image of the identity map on the left above, which explains our coincidence. □

Corollary 3.13 (S-local Objects)**.** *The S-local objects in $\mathscr{J}_G\mathscr{T}$ are the orthogonal G-spectra whose adjoints of structure maps are weak equivalences, i.e. classically defined G-Ω-spectra.*

Corollary 3.14 (S-equivalences)**.** *A map $g : X \to Y$ is an S-equivalence if*

$$g^* : \mathscr{J}_G\mathscr{T}(Y, Z) \to \mathscr{J}_G\mathscr{T}(X, Z)$$

is a weak equivalence for every G-Ω-spectrum Z, i.e g induces an isomorphism in every generalized cohomology theory. Evidently, the level equivalences are all S-equivalences.

Then there is an important theorem that tells us, the S-equivalences coincides with our "classically defined" stable equivalences.

Definition 3.15 (π_*-equivalences)**.** For subgroups H of G and integers q, define the homotopy groups $\pi_q^H(X)$ of an orthogonal G-spectrum X by

$$\pi_q^H(X) = \mathrm{colim}_V\, \pi_q^H(\Omega^V X(V)) \ if\ q \geq 0,$$

where V runs over the indexing G-spaces in U, and

$$\pi_{-q}^H(X) = \mathrm{colim}_{V \supseteq \mathbb{R}^q}\, \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \ if\ q > 0.$$

A map $f : X \to Y$ of orthogonal G-spectra is a π_**-isomorphism* if it induces isomorphisms on all homotopy groups.

Remark 3.16. We can see that, we do not need to do localization to S-modules because their underlying spectra are G-Ω-spectra. The homotopy groups defined this way are stabilized themselves.

Theorem 3.17. *A map of orthogonal G-spectra is a π_*-equivalence if and only if it's an S-equivalence(classically defined stable equivalence).*

Sketch Proof.
(1) π_*-isomorphism induces S-equivalence. Define $RX = F(F_1S^1, X)$ and find $R^nX = F(F_nS^n, X)$. The map $\lambda_1 : F_1S^1 = S^{-1} \wedge S^1 \to S^0$ induces $\lambda_1^* : X \to RX$, and we have a telescope from iterating this map. Take $QX = \mathrm{colim}_n\, R^nX$.
It's easy to find $\pi_q(QX(m)) \cong \pi_{q-m}(X)$. For the special case that E is a G-Ω-spectrum, the natural map $\iota : E \to QE$ is a level equivalence. From the naturality of the functor $Q(-)$, for any orthogonal G-spectrum X and G-Ω-spectrum E, [X,E] is naturally a retract of [QX,E].
The functor $Q(-)$ also translates the π_*-isomorphisms into level equivalences, so if $f : X \to Y$ is a π_*-equivalence, $f^* : [Y, E] \to [X, E]$ is a retract of the isomorphism $Qf^* : [QY, E] \to [QX, E]$ and is therefore an isomorphism. For more details, see Proposition 8.8, [2].
(2) S-equivalence induces π_*-isomorphism See the end of chapter 9 in [2].

□

Now we have successfully enlarge the weak equivalences from level equivalences to π_*-equivalences. As we claim at the beginning of this chapter, we should also describe the enlarged generating set of acyclic cofibrations. In the new model category, denote the new "weak equivalences"(the S-equivalences or π_*-equivalences) by $\mathcal{W}'$ while the old "weak equivalences"(the level equivalences) by $\mathcal{W}$. Denote the new generating set of cofibrations by $\mathcal{I}'$ while the old generating set of cofibrations by $\mathcal{I} = FI$. Actually, we do not change this set, so $\mathcal{I}' = \mathcal{I}$. Denote the enlarged generating set of acyclic cofibrations by $\mathcal{J}'$, while the old generating set of acyclic cofibrations (level acyclic cofibrations) by $\mathcal{J} = FJ$. We can write

$$\mathcal{J}' = \mathcal{J} \cup \mathcal{J}''$$

where $\mathcal{J}''$ is some non-empty set. The most important thing to verify is, the new $\mathcal{W}'\mathcal{I}'$ and $\mathcal{J}'$ satisfy the 2.35 in which the crucial parts are usually (3) and (4). Before write down the specific $\mathcal{J}''$, we now introduce some knowledge about G-topological model category.

Suppose we have general model categories $\mathscr{C}_G$ and $G\mathscr{C}$ where the latter one has G-equivariant morphisms but the former one does not. We suppose there is a given model category on $G\mathscr{C}$, and $\mathscr{C}_G$ is enriched over the category of G-spaces $\mathscr{T}_G$. For maps $i : A \to X$ and $p : E \to B$ in $G\mathscr{C}$, define $\mathscr{C}_G(i^*, p_*)$ to be

$$\mathscr{C}_G(X, E) \to \mathscr{C}_G(A, E) \times_{\mathscr{C}_G(A,B)} \mathscr{C}_G(X, B)$$

which is the map of G-spaces induced by the universal property of the pullback.

$$\begin{tikzcd}
\mathscr{C}_G(X,E) \arrow[dr, "\mathscr{C}_G(i^*,p_*)"] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "i^*"'] & & \\
& \mathscr{C}_G(A,E) \times_{\mathscr{C}_G(A,B)} \mathscr{C}_G(X,B) \arrow[r] \arrow[d] & \mathscr{C}_G(X,B) \arrow[d, "i^*"] \\
& \mathscr{C}_G(A,E) \arrow[r, "p_*"] & \mathscr{C}_G(A,B)
\end{tikzcd}$$

Definition 3.18. A model category $G\mathscr{C}$ is G-topological if both

(1) The map $\mathscr{C}_G(i^*, p_*)$ is a Serre fibration of G-spaces when i is a q-cofibration and p is a q-fibration.
(2) The map $\mathscr{C}_G(i^*, p_*)$ is a weak equivalence of G-spaces when either i or p is a weak equivalence.

Remark 3.19. The model category of equivariant S-modules we constructed in the first chapter and the level model category of equivariant orthogonal spectra are both G-topological.

The pair (i, p) has the lifting property if and only if $G\mathscr{C}(i^*, p_*)$ (passage to equivariant G-category of $\mathscr{C}_G$) is surjective.

Construction 3.20. There are two pairs of analogues of $\mathscr{C}_G(i^*, p_*)$ above.

(1) For a map $i : A \to B$ of based G-spaces and a map $j : X \to Y$ in $G\mathscr{C}$, passage to pushout gives a map
$$i\Box j : (A \wedge Y) \cup_{A\wedge X} (B \wedge X) \to B \wedge Y$$
and passage to pullbacks gives a map
$$F_\Box(i, j) : F(B, X) \to F(A, X) \times_{F(A,Y)} F(B, Y),$$
where $\wedge$ and F denote the tensor and cotensor in $\mathscr{C}_G$.
(2) In the closed symmetric monoidal category $\mathscr{C}_G$ with product $\wedge_{\mathscr{C}}$ and internal function functor $F_{\mathscr{C}}$. Then, for maps $i : X \to Y$ and $j : W \to Z$ in $G\mathscr{C}$, passage to pushouts gives a map
$$i\Box j : (Y \wedge_{\mathscr{C}} W) \cup_{X\wedge_{\mathscr{C}} W} (X \wedge_{\mathscr{C}} Z) \to Y \wedge_{\mathscr{C}} Z,$$
and passage to pullbacks gives a map
$$F_\Box(i, j) : F_{\mathscr{C}}(Y, W) \to F_{\mathscr{C}}(X, W) \times_{F_{\mathscr{C}}(X,Z)} F_{\mathscr{C}}(Y, Z).$$

And we have a lemma whose corollary helps a technical step in verifying the new generating set $\mathcal{J}'$.

Lemma 3.21. *Let $i : A \to B$ be a map of based G-spaces and let $j : X \to Y$ and $p : E \to F$ be maps in $G\mathscr{C}$. Then there are natural isomorphisms of G-maps*

$$\mathscr{C}_G((i\Box j)^*, p_*) \cong \mathscr{T}_G(i^*, \mathscr{C}_G(j^*, p_*)_*) \cong \mathscr{C}_G(j^*, F_\Box(i,p)_*).$$

Corollary 3.22. *Take the natural isomorphisms above and pass to G-fixed points, using 3.2.2, we know $(i\Box j, p)$ has the lifting property in $G\mathscr{C}$ if and only if $(i, \mathscr{C}_G(j^*, p_*))$ has the lifting property in $G\mathscr{T}$.*

Now we can describe the set $\mathcal{J}''$ as all maps of the form $i\Box\lambda_{V,W}$. Where $i \in I$ defined in 3.8 and $\lambda_{V,W} : F_{V\oplus W}S^W \to F_V S^0$. Here v and W are taken from $sk\mathscr{J}_G$. Now

$$\mathcal{J}' = FJ \cup \{i\Box\lambda_{V,W}\}.$$

To verify all criterions in 2.35, we need to figure out, when does a map between orthogonal G-spectra satisfy RLP with respect to $\mathcal{J}'$?

Theorem 3.23. *A map $p : E \to B$ satisfies RLP with respect to $\mathcal{J}'$ if and only if p is a level fibration and the diagram*

$$\begin{array}{ccc} EV & \xrightarrow{\tilde{\sigma}} & \Omega^W E(V\oplus W) \\ {\scriptstyle pV}\downarrow & & \downarrow{\scriptstyle \Omega^W p(V\oplus W)} \\ BV & \xrightarrow{\tilde{\sigma}} & \Omega^W B(V\oplus W) \end{array}$$

is a homotopy pullback for all V and W, i.e the map induced from EV to the pullback $BV \times_{\Omega^W B(V\oplus W)} \Omega^W E(V\oplus W)$ is a weak equivalence.

Proof. The map p has the RLP with respect to $\mathcal{J}' = FJ \cup \mathcal{J}''$ means it is a level fibration and has the RLP with respect to all maps of form $i\Box\lambda_{V,W}$. From 3.22 we know $(i\Box\lambda_{V,W})$ has lifting property if and only if $(i, \mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*))$ has lifting property for all $i \in I$, i.e if and only if $\mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*)$ is a acyclic Serre fibration of G-spaces.

$\mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*)$ is in fact a Serre fibration. This is because $\lambda_{V,W}$ is a q-cofibration and p is a fibration. Since the level model category is G-topological, $\mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*)$ is a Serre fibration. Then, the map p has the RLP with respect to all maps of form $i\Box\lambda_{V,W}$ if and only if $\mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*)$ is a weak equivalence. Since we have

$$\mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*) : \mathscr{J}_G\mathscr{T}(F_V S^0, E) \to \mathscr{J}_G\mathscr{T}(F_V S^0, B) \times_{\mathscr{J}_G\mathscr{T}(F_{V\oplus W}S^W, B)} \mathscr{J}_G\mathscr{T}(F_{V\oplus W}S^W, E)$$

and use the adjunction, we have $\mathscr{J}_G\mathscr{T}(\lambda^*_{V,W}, p_*)$ is isomorphic to the map

$$EV \to BV \times_{\Omega^W B(V\oplus W)} \Omega^W E(V\oplus W)$$

which is also induced by the universal property of pullback.

□

With this theorem we can verify 2.35 directly. We will not prove them one by one, but will show the crucial one: (4)2), which states that $RLP(\mathcal{J}') \cap \mathcal{W}' \subset RLP(\mathcal{I}')$. We can translate this into a theorem.

Theorem 3.24. *A stable equivalence (S- or π_*-equivalence) $p : E \to B$ that satisfies the RLP with respect to $\mathcal{J}'$, then p is a level acyclic fibration.*

Sketch Proof. From 3.23 we know p is a level fibration, so we only need to prove p is a level equivalence. Suppose ΛE is the homotopy colimit of orthogonal G-spectrum E which is given by $\Lambda E(W) = \mathrm{colim}_V \Omega^V E(V \oplus W)$. It is obviously an G-Ω-spectrum. We now take the diagram in 3.23 and pass it to colimit, we get

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & \Lambda E \arrow[d, "\Lambda p"] \\ B \arrow[r] & \Lambda B \end{tikzcd}$$

is also a homotopy pullback. The functor Λ can translate the stable equivalence into level equivalence, i.e Λp is a level equivalence. Then compare the long exact sequence (see [2] Theorem 6.9) we know that $p : E \to B$ is also a level equivalence. □

Up to now we have verified $\mathcal{W}'$, $\mathcal{I}'$ and $\mathcal{J}'$ are stable equivalences, generating set of q-cofibrations and generating set of acyclic q-cofibrations of the stable model category of orthogonal G-spectra, respectively.

3.2.3. *Positive Stable Model Structure of Orthogonal G-spectra.* Let us modify the positive stable model structure of orthogonal spectra by specifying.

Definition 3.25. We specify a new model category on the basic of stable model category of orthogonal G-spectra as follows.

(1) For map $f : X \to Y$ of orthogonal G-spectra, it is a weak equivalence if it is a π_*-equivalence.
(2) The set of generating cofibrations is
$$\{G_+ \wedge_H S^{-W} \wedge (S^{n-1}_+ \to D^n_+) : n \geq 0, H \subseteq G\}$$
where W ranges over all representations of all subgroups H of G with $W^H \neq 0$.
(3) The set of generating acyclic cofibrations is
$$\{G_+ \wedge_H S^{-W} \wedge (D^n_+ \to D^n_+ \wedge I) : n \geq 0, H \subseteq G\}$$
where W ranges over all representations of all subgroups H of G with $W^H \neq 0$.

The restriction $W^H \neq 0$ is aim at driving sphere spectrum non-cofibrant. And we have two reasons to do so:

(1) As the last theorem we will tell, we want a monoidal Quillen equivalence between equivariant orthogonal spectra and S-modules. The left adjoint we produce goes from equivaraint orthogonal spectra to S-modules. If this is a Quillen equivalence then it will preserve cofibrant objects. However, in the stable model structure on equivariant orthogonal spectra, the unit is cofibrant, which would imply its image is also cofibrant. Since the comparison is supposed to be monoidal, this would imply the S-module S_G is cofibrant. This is known to be false, so such a Quillen equivalence does not exist. The positive model structure of orthogonal spectra has different cofibrations, in particular, the unit is no longer cofibrant.
(2) The positive model structure is also built for equivariant commutative orthogonal spectra. If we keep using the standard stable model category, since the forgetful functor preserves fibrations and weak equivalences, the free/forgetful adjunction is a Quillen adjunction, which is to say that the free functor preserves cofibrations and acyclic cofibrations. A theorem of Ken Brown (see [4] Proposition 1.2.5) says that the free functor preserves weak

equivalences between cofibrant objects. Even take a look at the non-equivariant case when the group G is trivial, and we have weak equivalences

$$F_V S^V \to S$$

for $V^G \neq 0$. Take the free functor, we get

$$\bigvee_n (F_V S^V)^n/\Sigma_n \to \bigvee_n S^n/\Sigma_n$$

where (n) means smash power and Σ_n is the symmetric group. On the right, the n-th smash power of S is S with trivial action by the symmetric group, so $S^n/\Sigma_n = S$. We also have an important lemma (see [3] Lemma 8.4) which says

$$((E\Sigma_n)_+ \wedge (F_V S^V)^n)/\Sigma_n \to (F_V S^V)^n/\Sigma_n$$

is a weak equivalence. For example for $n = 2$, we get $\Sigma^\infty RP^\infty_+$, which is not equivalent to S. So the free functor does not preserve the weak equivalnce $F_V S^V \to S$ and if we want to have the free forgetful functor be a Quillen adjunction, we can't have both of these be cofibrant.

For the definitions of Quillen adjoint pairs and the way to verify whether they are Quillen equivalence, see the appendix in [2]. And we have a theorem, which says that the model category of equivariant S-modules and orthogonal spectra are Quillen equivalent. It means their homotopy theory are "the same", and we can use any of them in the context we need. For example, if we want all objects to be fibrant, we can use S-modules. If we want all the objects to be treated as CW-spectra, we can use orthogonal spectra...

Theorem 3.26. *There is a strong symmetric monoidal functor* $\mathbb{N} : G\mathscr{J}\mathscr{S} \to G\mathscr{M}$ *and a lax symmetric monoidal functor* $\mathbb{N}^\sharp : G\mathscr{M} \to G\mathscr{J}\mathscr{S}$ *such that* $(\mathbb{N}, \mathbb{N}^\sharp)$ *is a Quillen equivalence between* $G\mathscr{J}\mathscr{S}$ *and* $G\mathscr{M}$.

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